

Answers to practice problems for Midterm I  
Math 125, Sections C&D  
October, 2007

For the actual midterm you may use one  $8 \times 11.5$  sheet of *handwritten* notes (one-side only). You may use your “simple” scientific calculator on the exam. No books, printed notes or graphing calculators. You have to show ALL YOUR WORK to get full credit.

This is intended for practice and is much longer than the actual midterm. There are sample midterms linked to our class webpage; you can find even more old midterms at the unified MATH 125 website.

**Practice Problems.**

1. Find  $f$  if  $f'(x) = 1 - 3\sqrt{x} + e^{-x}$  and  $f(1) = \frac{1}{e} - 1$ .

**Answer.**  $\boxed{f(x) = x - 2x^{\frac{3}{2}} - e^{-x} + \frac{2}{e}}$ .

2. Evaluate  $\int_0^2 (1 + \sqrt{4 - x^2}) dx$  by interpreting it in terms of areas.

**Answer.**  $\int_0^2 (1 + \sqrt{4 - x^2}) dx = \int_0^2 1 dx + \int_0^2 \sqrt{4 - x^2} dx = \boxed{2 + \pi}$ .

3. If  $\int_{-1}^7 h(x) dx = 10$  and  $\int_{-1}^3 h(x) dx = 5$ , find  $\int_3^7 4h(x) dx$ .

**Answer.**  $\int_3^7 4h(x) dx = 4 \int_3^7 h(x) dx = 4(\int_{-1}^7 h(x) dx - \int_{-1}^3 h(x) dx) = 4(10 - 5) = \boxed{20}$ .

4. Compute

(a)  $\int_0^2 \frac{d}{dx}(2 \cos x^4) dx$

**Answer.**  $\int_0^2 \frac{d}{dx}(2 \cos x^4) dx = \int_0^2 (2 \cos x^4)' dx$ . Make a substitution:  
 $u = 2 \cos x^4$ . Then  $du = (2 \cos x^4)' dx$ . Hence,

$$\int (2 \cos x^4)' dx = \int du = u + C = 2 \cos x^4 + C$$

Now plug in limits for  $x$  to compute the definite integral :

$$\int_0^2 (2 \cos x^4)' dx = 2 \cos x^4 \Big|_0^2 = \boxed{2 \cos(16) - 2}.$$

(b)  $\frac{d}{dx} \int_0^2 2 \cos t^4 dt$

**Answer.** The derivative of  $\int_0^2 2 \cos t^4 dt$  is  $\boxed{0}$  because  $\int_0^2 2 \cos t^4 dt$  is a constant.

(c)  $\frac{d}{dx} \int_0^{2x} 2 \cos t^4 dt$

**Answer.** For this problem we use FTC I and the chain rule.

$$\frac{d}{dx} \int_0^{2x} 2 \cos t^4 dt = 2 \cos(2x)^4 \bullet (2x)' = \boxed{4 \cos(16x^4)}.$$

5. Evaluate the following integrals

(a)  $\int_0^{\frac{1}{2}} \frac{5}{\sqrt{1-x^2}} dx = 5(\arcsin \frac{1}{2} - \arcsin 0) = 5(\frac{\pi}{6} - 0) = \boxed{\frac{5\pi}{6}}.$

(b)  $\int \frac{\sin(\ln x)}{x} dx = \boxed{-\cos(\ln x) + C}$  (Hint:  $u = \ln x$ .)

(c)  $\int \frac{x^5}{2 + \frac{1}{3}x^6} dx = \boxed{\frac{1}{2} \ln(6 + x^6) + C}$  (Hint:  $u = x^6$  or  $u = 2 + \frac{1}{3}x^6$ .)

(d)  $\int_0^1 \frac{x^2}{\sqrt{1+x^3}} dx = \frac{1}{3} \int_1^2 \frac{du}{\sqrt{u}} = \frac{2}{3} u^{\frac{1}{2}} \Big|_1^2 = \boxed{\frac{2}{3}(\sqrt{2} - 1)}$  (Hint:  $u = 1 + x^3$ ,  
 new limits: lower  $1 + 0^3 = 1$ , upper  $1 + 1^3 = 2$ .)

(e)  $\int (x^3 - 2) \cdot \sin(x^4 - 8x + 3) dx = \boxed{-\frac{1}{4} \cos(x^4 - 8x + 3) + C}$  (Hint:  $u = x^4 - 8x + 3$ .)

(f)  $\int_{-3}^{-1} \frac{1}{5 + x^2 + 4x} dx = \arctan(x + 2) \Big|_{-3}^{-1} = \arctan(1) - \arctan(-1) = \frac{\pi}{4} - (-\frac{\pi}{4}) = \boxed{\frac{\pi}{2}}$ . (Hint: Complete the square  $5 + 4x + x^2 = 1 + (x + 2)^2$ , then make substitution  $u = x + 2$ .)

(g)  $\int_{-1}^1 \frac{2 \sin^3 x}{3 + 2x^2 + 5x^8} dx = \boxed{0}$ . (Hint: Use symmetry. The integrand is an *odd* function.)

6. Use the Midpoint Rule with  $n = 3$  to approximate  $\int_0^\pi \sin^2 x dx$ . (Leave  $\pi$  in your answer.)

**Answer.**  $\Delta x = \frac{\pi}{3}$ , the midpoints are  $x_1 = \frac{\pi}{6}$ ,  $x_2 = \frac{\pi}{2}$ ,  $x_3 = \frac{5\pi}{6}$ .  
 $M_3 = \frac{\pi}{3}(\sin^2(\frac{\pi}{6}) + \sin^2(\frac{\pi}{2}) + \sin^2(\frac{5\pi}{6})) = \frac{\pi}{3}(\frac{1}{4} + 1 + \frac{1}{4}) = \boxed{\frac{\pi}{2}}$ .

7. Let  $g(x) = \int_0^{\sin x} \sqrt[3]{1 - t^2} dt$ . Compute  $g'(x)$ ,  $g'(\pi)$  and  $g(\pi)$ .

**Answer.** To compute  $g'(x)$  we use FTC I + chain rule:

$$g'(x) = \sqrt[3]{1 - \sin^2 x} \cdot \cos x.$$

We can simplify the answer using the fundamental trig identity  $\sin^2 x + \cos^2 x = 1$ . It implies  $1 - \sin^2 x = \cos^2 x$ . Hence,

$$g'(x) = \sqrt[3]{\cos^2 x} \cdot \cos x = \boxed{\sqrt[3]{\cos^5 x}}.$$

To compute  $g'(\pi)$ , plug in  $x = \pi$ :

$$g'(\pi) = \sqrt[3]{\cos^5(\pi)} = \sqrt[3]{(-1)^5} = \boxed{-1}$$

To compute  $g(\pi)$ , we plug in  $x = \pi$  into the original integral. Then we use that  $\sin \pi = 0$ .

$$g(\pi) = \int_0^{\sin \pi} \sqrt[3]{1 - t^2} dt = \int_0^0 \sqrt[3]{1 - t^2} dt = \boxed{0}$$

8. A particle moves along a line with velocity function  $v(t) = t^2 - 4t$ , where  $v$  is measured in meters per second. Find (a) the displacement and (b) the distance traveled by the particle during the time interval  $[0, 6]$  seconds.

**Answer.** a). Displacement  $= \int_0^6 v(t) dt = \int_0^6 (t^2 - 4t) dt = \boxed{0\text{m}}$ .

b). Distance  $=$

$$\int_0^6 |v(t)| dt = \int_0^4 (4t - t^2) dt + \int_4^6 (t^2 - 4t) dt = \left(2t^2 - \frac{1}{3}t^3\right)\Big|_0^4 + \left(\frac{1}{3}t^3 - 2t^2\right)\Big|_4^6 = \boxed{\frac{64}{4}\text{m}}.$$

9. Let  $\mathcal{R}$  be the region in the first quadrant bounded by the curves  $y = \cos\left(\frac{\pi x}{2}\right)$  and  $y = 1 - x$ .

(a) Sketch  $\mathcal{R}$ . (**Hint:** What is  $\cos 0$ ?  $\cos \frac{\pi}{2}$ ?)

(b) Find the area of  $\mathcal{R}$ . **Answer.** The area is found by computing the definite integral

$$\int_0^1 \left(\cos\left(\frac{\pi x}{2}\right) - (1 - x)\right) dx = \frac{\sin\left(\frac{\pi x}{2}\right)}{\frac{\pi}{2}} - x + \frac{x^2}{2}\Big|_0^1 = \boxed{\frac{2}{\pi} - \frac{1}{2}}.$$

(c) Set up (but **DO NOT EVALUATE**) an integral expression for the volume of the solid generated when  $\mathcal{R}$  is rotated about the  $x$ -axis.

**Answer.** Try to do a rough sketch. You should get a conical shape situated around the  $x$ -axis. The cone is curved on the outside and straight inside.

We use the method of “washers”. We rotate around  $x$ -axis, hence, we slice perpendicularly the  $x$ -axis. The slicing plane will, therefore, move along the  $x$ -axis. Hence, the integrating variable will be  $x$ .

The limits are the same as in b): from 0 to 1. The area of each washer is the area of the big disk ( $A_{big} = \pi R^2$ ) minus the area of the small disk ( $A_{small} = \pi r^2$ ). The radius of the big disk is  $R = \cos^2\left(\frac{\pi x}{2}\right)$  (the “top” function), the radius of the small disk is  $r = 1 - x$  (the “bottom function”).

We get

$$V = \int_0^1 \pi(\cos^2\left(\frac{\pi x}{2}\right) - (1 - x)^2) dx.$$

(d) Set up (but **DO NOT EVALUATE**) an integral expression for the volume of the solid generated when  $\mathcal{R}$  is rotated about the line  $y = 1$ .

**Answer.** First, try to sketch a rough picture. You should get a conical shape with a hole inside situated around the axis  $y = 1$ . It is now straight

on the outside but curved inside. We use washers again but now the function  $y = \cos\left(\frac{\pi x}{2}\right)$  is closer to the axis so it gives the small radius  $r = 1 - \cos\left(\frac{\pi x}{2}\right)$  (the radius is the distance from the axis to the function), and the function  $y = 1 - x$  is farther away so it gives the big radius  $R = 1 - (1 - x) = x$ .

$$V = \int_0^1 \pi(x^2 - (1 - \cos\left(\frac{\pi x}{2}\right))^2) dx.$$

10. Let  $\mathcal{R}$  be the region in the first quadrant bounded by the curves  $y = x^3$  and  $y = 2x - x^2$ .

- (a) Sketch  $\mathcal{R}$ .
- (b) Compute the area  $\mathcal{R}$ .

**Answer.** Find the intersection point of the two graphs:  $x^3 = 2x - x^2$ . Solve  $x^3 + x^2 - 2x = x(x^2 + x - 2) = x(x - 1)(x + 2)$ . Hence, there are two intersection points in the first quadrant:  $x = 0, 1$ . You should see from the sketch that  $2x - x^2$  is the “top” function on the interval  $[0, 1]$ .

$$Area = \int_0^1 ((2x - x^2) - x^3) dx = x^2 - \frac{x^3}{3} - \frac{x^4}{4} \Big|_0^1 = \boxed{\frac{5}{12}}.$$

- (c) Compute the volume obtained by rotating  $\mathcal{R}$  around the  $x$ -axis.

**Answer.**

$$\begin{aligned} Volume &= \int_0^1 \pi((2x - x^2)^2 - (x^3)^2) dx = \\ &= \pi \int_0^1 (4x^2 - 4x^3 + x^4 - x^6) dx = \frac{4\pi}{3}x^3 - x^4 + \frac{1}{5}x^5 - \frac{1}{7}x^7 \Big|_0^1 = \\ &= \pi(4/3 - 1 + 1/5 - 1/7) = \boxed{\frac{41\pi}{105}} \end{aligned}$$