Math 125D **Midterm II Solutions** **November 15, 2007**

1. [**9 points total**] Evaluate the following integral:

$\int \frac{2e^x}{e^{2x} - 3e^x + 2}\, dx$

Solution. Take $u = e^x$, $du = e^x dx$.

$\int \frac{2e^x}{e^{2x} - 3e^x + 2}\, dx = \int \frac{2}{u^2 - 3u + 2}\, du.$

We can factor the denominator as

$$u^2 - 3u + 2 = (u - 2)(u - 1).$$

Using partial fractions, we compute

$\int \frac{2}{u^2 - 3u + 2}\, du = 2 \int \frac{1}{u - 2} - \frac{1}{u - 1}\, du = 2 \ln \left| \frac{e^x - 2}{e^x - 1} \right| + C$

2. [**9 points total**] Evaluate the following definite integral:

$\int_0^{\pi/6} \cos^2(3x) \sin^3(3x)\, dx$

Solution. Write $\int_0^{\pi/6} \cos^2(3x) \sin^3(3x)\, dx = \int_0^{\pi/6} \cos^2(3x) \sin^2(3x) \sin(3x)\, dx$ and take $u = \cos(3x)$. Then $du = -\frac{1}{3} \sin(3x)\, dx$. The integral becomes $-\frac{1}{3} \int_1^0 u^2(1 - u^2)\, du = \frac{1}{3} \int_1^0 u^2 - u^4\, du = \frac{2}{45}$

3. [**9 points total**] Evaluate the following integral:

$\int \frac{x^3}{\sqrt{x^2 - 1}}\, dx$

Solution. Make a u-substitution $u = x^2 - 1$. Then $du = 2xdx$.

$$\int \frac{x^3}{\sqrt{x^2 - 1}}\, dx = \frac{1}{2} \int \frac{u + 1}{\sqrt{u}}\, du = \frac{1}{2} \int u^{1/2} + u^{-1/2}\, du = \frac{1}{3} u^{3/2} + u^{1/2} =$$

$$\frac{(x^2 - 1)^{3/2}}{3} + \sqrt{x^2 - 1}$$

Other possible approaches: take $u = \sqrt{x^2 - 1}$ or do a trig substitution $u = \sec\theta$.

4. [**10 points total**] Consider the region in the first quadrant bounded by the curves $y = \frac{1}{4}x^2$, $x = 0$, and $y = 4$. This region is rotated about the y axis to create a three dimensional solid. Suppose we have a tank with the shape of that solid, oriented so that the y-axis is perpendicular to the ground, the origin is at the bottom of the tank, and units are in meters. If the tank is filled with water, how much work is required to pump all of the water to the top of the tank? (Recall that the mass density of water is 1000 kg/m^3 and the acceleration due to gravity is 9.8 m/sec^2.)

Solution. $W = \int_0^4 9.8 \cdot 1000 \cdot \pi 4y(4-y)\ dy = 418133\pi = 1313605$ J.

5. [**13 points total**] Let $\mathcal{R}$ be the region in the first quadrant bounded by the curves $y = \ln x$, the x-axis and the vertical line $x = 2$ (see picture on the board). Consider the solid of revolution obtained by rotating $\mathcal{R}$ around the x-axis.

(a) [**9 points**] Compute the volume of this solid of revolution. Indicate clearly whether you are using washers or shells. Sketch a typical rectangle to be rotated to obtain a washer/shell (depending on the method you are using).

Solution. Using disks, we get $V = \pi \int_1^2 \ln^2 x\ dx$. To evaluate this integral, one can use integration by parts with $u = \ln^2 x$ and $dv = dx$. Then $du = 2\ln x \frac{1}{x}/dx$ and $v = x$. We get $\pi \int_1^2 \ln^2 x\ dx = \pi x \ln^2 x \Big|_1^2 - 2\pi \int_1^2 \ln x\ dx$. To integrate $\ln x$ we use integration by part again with $u = \ln x, dv = dx$. The end result is $\pi(x\ln^2 x - 2x\ln x + 2x)\Big|_1^2 = (2\ln^2(2) - 4\ln 2 + 2)\pi \simeq 0.188\pi$.

If we use shells, we irst need to solve for x: $x = e^y$. The limits for y are 0 and $\ln 2$. We have

$$V = \int_0^{\ln 2} 2\pi y(2-e^y)\ dy = 2\pi \int_0^{\ln 2} 2y - 2\pi \int_0^{\ln 2} ye^y/dy = 2\pi \ln^2(2) - 2\pi \int_0^{\ln 2} ye^y/y$$

. To integrate ye^y we use integration by parts with $u = yanddv = e^y\ dy$. The end result is the same as for the disks method.

(b) [**4 points**] Approximate the volume of the same solid of revolution using Trapezoidal rule with $n = 4$. Leave your answer in the sum form.

Solution. Here, we apply Trapezoidal rule to the integral that we set up in (a). Let's use $V = \pi \int_1^2 \ln^2 x\ dx$. The interval is $[1, 2]$, so the points are $1, 5/4, 3/2, 7/4, 2$ and $\Delta x = 1/4$. We get

$$T_n = \frac{1}{8}\pi(2\ln^2(5/4) + 2\ln^2(3/2) + 2\ln^2(7/4) + \ln^2(2)) \simeq 0.191$$