Worksheet week 11
Answers (problem 5 corrected)

Math 124 **Final Examination** **Spring 2005**

Print Your Name	Signature
Student ID Number	**Quiz Section**
Professor's Name	**TA's Name**

!!! READ...INSTRUCTIONS...READ !!!

1. Your exam contains 9 questions and 12 pages; PLEASE MAKE SURE YOU HAVE A COMPLETE EXAM.
2. The entire exam is worth 100 points. Point values for problems vary and these are clearly indicated. You have 2 hours and 50 minutes for this final exam.
3. Make sure to ALWAYS SHOW YOUR WORK; you will not receive any partial credit unless all work is clearly shown. If in doubt, ask for clarification. Make sure to do your own work on the exam.
4. There is plenty of space on the exam to do your work. If you need extra space, use the back pages of the exam and clearly indicate this.
5. You are allowed one 8.5 $\times$ 11 sheet of handwritten notes (both sides). Graphing calculators are NOT allowed; scientific calculators are allowed. Make sure your calculator is in radian mode.

Problem	Total Points	Score
1	12	
2	10	
3	10	
4	10	
5	12	

Problem	Total Points	Score
6	10	
7	12	
8	10	
9	14	
Total	100	

1. (12 points; 3 points each) Differentiate the following functions. You need not simplify your answers.

(a) $y = \dfrac{x^2}{x^3 - 1}$

log dif:

$$y' = \frac{x^2}{x^3-1}\left(\frac{2}{x} - \frac{3x^2}{x^3-1}\right)$$

(b) $y = \tan^{-1}(\ln x) + \ln(\tan^{-1} x)$. $= \arctan(\ln x) + \ln(\arctan x)$

$$y' = \frac{1}{(\ln x)^2+1} \cdot \frac{1}{x} + \frac{1}{\arctan x} \cdot \frac{1}{x^2+1}$$

2. (10 points; 2 points each) Evaluate the following limits.

(a) $\lim_{x \to 0} \frac{\sin^2 x}{\cos(x) - 1}$ $= -2$

(b) $\lim_{x \to 0} \frac{\sin^2 x}{\cos(3x) - 1}$ $= -\frac{2}{9}$

(c) $\lim_{x \to 0^-} \frac{|x|}{x - 2|x|}$ $= -\frac{1}{3}$

(d) $\lim_{x \to \infty} \frac{\ln \ln x}{e^{\sqrt{x}}}$ $= 0$

(e) $\lim_{x \to 5^+} \frac{4 - x}{x^2 - 25}$ $= -\infty$

1.(continued)

(c) $y = \dfrac{1}{\sqrt[3]{x+\sqrt{x}}}$.

log. diff.

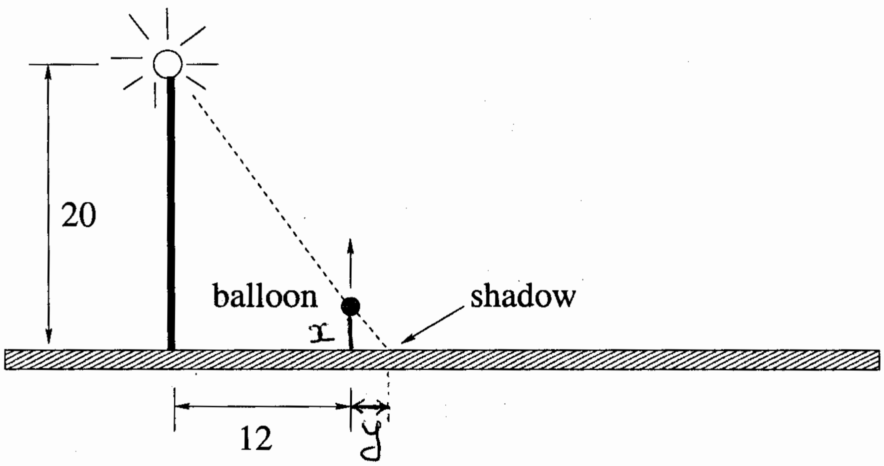
$$y' = \frac{1}{\sqrt[3]{x+\sqrt{x}}}\left(-\frac{1}{3}\,\frac{1}{(x+\sqrt{x})}\left(1+\frac{1}{2\sqrt{x}}\right)\right)$$

(d) $y = \left[\cos\left(\dfrac{1}{x}\right)\right]^x$.

log. diff.

$$y' = \left(\cos\left(\frac{1}{x}\right)\right)^x\left(\ln\left(\cos\left(\frac{1}{x}\right)\right) + \frac{\tan\left(\frac{1}{x}\right)}{x}\right)$$

3. (10 points) A small helium balloon is rising at the rate of 8 ft/sec, a horizontal distance of 12 feet from a 20 ft. lamppost. At what rate is the shadow of the balloon moving along the ground when the balloon is 5 feet above the ground?



$$\frac{dx}{dt} = 8$$

Similar triangles: $\dfrac{y}{x} = \dfrac{y+12}{20} \Rightarrow$

$$20y - xy = 12x \quad *$$

$$y = \frac{12x}{20-x} \; ; \quad x = 5 \Rightarrow y = 4$$

Diff $(*)$: $\quad 20\dfrac{dy}{dt} - y\dfrac{dx}{dt} - x\cdot\dfrac{dy}{dt} = 12\dfrac{dx}{dt}$

Plug in $\dfrac{dx}{dt} = 8, \; x = 5, \; y = 4$

$$20 \cdot \frac{dy}{dt} - 4 \cdot 8 - 5\frac{dy}{dt} = 12 \cdot 8 \Rightarrow$$

$$15\frac{dy}{dt} = 16 \cdot 8 \Rightarrow \frac{dy}{dt} = \frac{128}{15}$$

4. (10 points)

The point $(1/4, 1/4)$ lies on the curve

$$(3x^2 + y^2)^{3/2} = 3x^2 - y^2.$$

(a) (8 pts) Using linear approximation, find the approximate value of the number z near $1/4$ such that $(z, 0.23)$ is a point on this curve.

$\frac{d}{dx}$: $\quad \frac{3}{2}(3x^2+y^2)^{\frac{1}{2}}\left(6x + 2y \cdot \frac{dy}{dx}\right) = 6x - 2y \cdot \frac{dy}{dx}$

$x = y = \frac{1}{4}$:

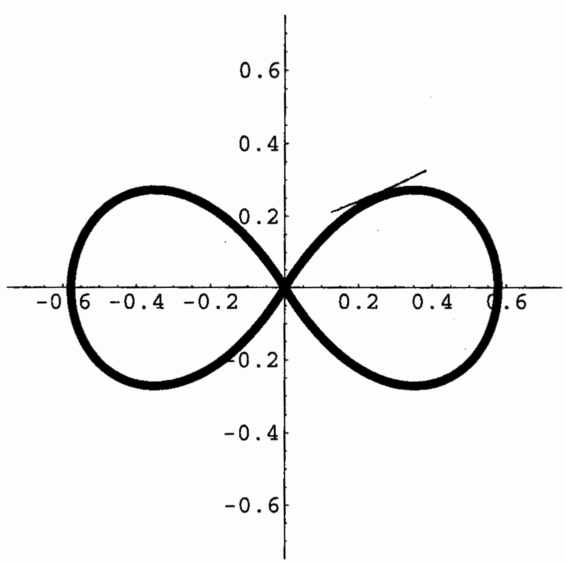
$\frac{3}{2}\left(3 \cdot \frac{1}{16} + \frac{1}{16}\right)^{\frac{1}{2}}\left(\frac{6}{4} + \frac{2}{4} \cdot \frac{dy}{dx}\right) = 6 \cdot \frac{1}{4} - 2 \cdot \frac{1}{4} \cdot \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{3}{7}$ tangent line: $y - \frac{1}{4} = \frac{3}{7}\left(x - \frac{1}{4}\right)$

approxim: $0.23 - \frac{1}{4} = \frac{3}{7}\left(z - \frac{1}{4}\right) \Rightarrow$

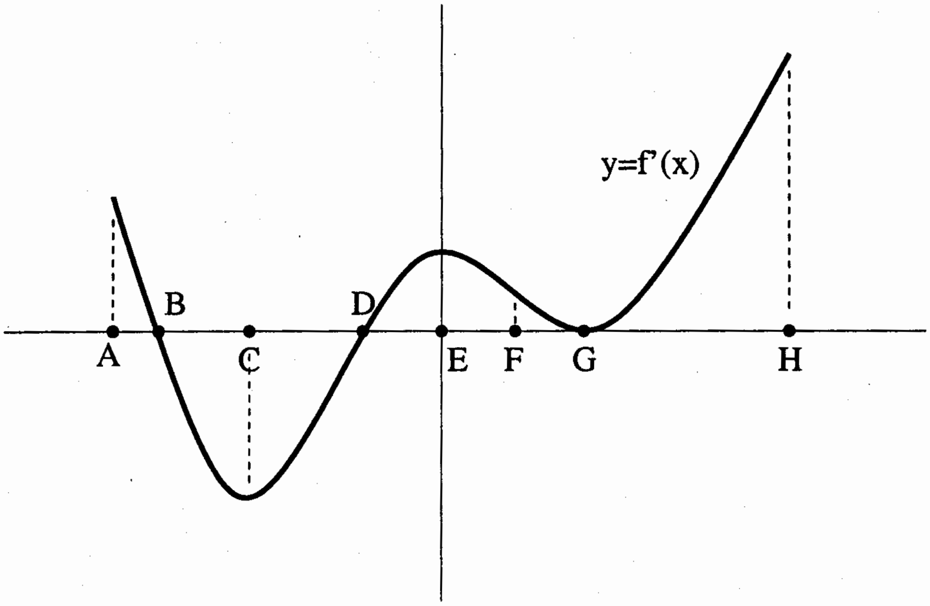
$z \approx 0.203$

(b) (2 pts) Using the drawing of the curve below, determine whether the actual number z is bigger or smaller than the number found using linear approximation:



bigger

5. (12 points) The function f has domain (A, H) and the graph of f' is as shown. Each part is worth 3 points; there is no partial credit on this problem.



(a) Give the open interval(s) over which f is increasing.

(AB), (DH)

(b) Give the open inteval(s) over which f is concave downward.

(AC), (EG)

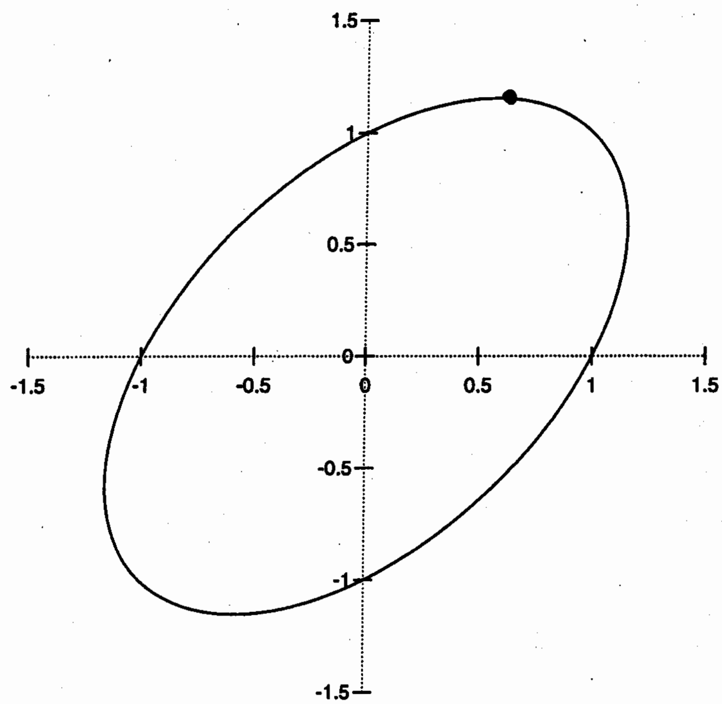
(c) At which point(s) does f have a local minimum?

D

(d) At which point(s) does f change concavity?

C, E, G

6. (10 points) Find the point on the ellipse $x^2 + y^2 - xy = 1$ with the largest y coordinate.



$$y' = \frac{y - 2x}{2y - x}$$

$y' = 0$ if $y = 2x$

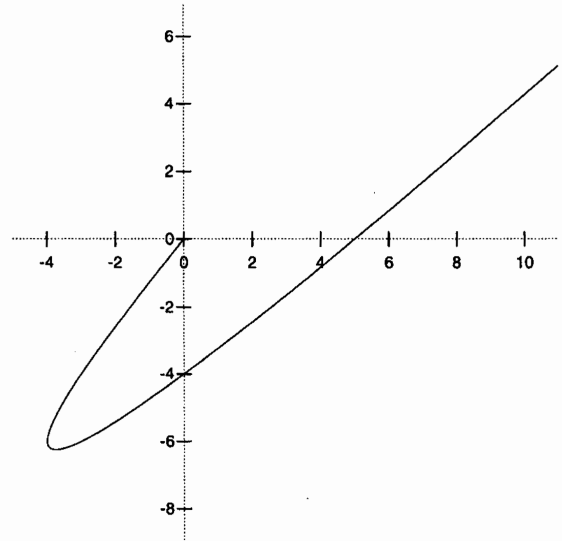
Plug into $x^2 + y^2 - xy = 1$:

$x^2 + 4x^2 - 2x^2 = 1 \Rightarrow x = \pm \frac{1}{\sqrt{3}}$

maximum $x = \frac{1}{\sqrt{3}}$, $y = \frac{2}{\sqrt{3}}$

7. (12 points) A particle starts moving at time $t = 0$. Its position at time $t \geq 0$ is given by

$$x(t) = t^2 - 4t, \quad y(t) = t^2 - 5t.$$



(a) Express $\dfrac{dy}{dx}$ in terms of t.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t-5}{2t-4}$$

(b) For which values of t does the tangent line to the curve at $(x(t), y(t))$ pass through the point $(0, -8)$?

tangent line: $y - y(t) = \dfrac{2t-5}{2t-4}(x - x(t))$

$x = 0,\ y = -8$ $\qquad -8 - (t^2 - 5t) = \dfrac{2t-5}{2t-4} \cdot (-(t^2-4t))$

Solve for t:

$$t = +8 \pm 4\sqrt{2}$$

(c) When is the particle heading directly toward the point $(0, -8)$?

$$t = 8 - 4\sqrt{2}$$

8. (10 points) The product of two positive numbers is 100. How small can the sum of one of the numbers plus the square of the other number be? (Make sure to show your work and justify your answer.)

$$xy = 100$$

Find min/~~max~~ $y + x^2$, $x, y > 0$

$$y = \frac{100}{x}$$

$$f(x) = y + x^2 = \frac{100}{x} + x^2$$

$$f'(x) = -\frac{100}{x^2} + 2x = 0$$

$$+\frac{100}{x^2} = 2x$$

$$100 = 2x^3$$

$$\sqrt[3]{50} = x \quad (\text{since } x > 0)$$

$$f''(x) = \frac{200}{x^3} + 2 > 0 \Rightarrow x = \sqrt[3]{50} \text{ is } \underline{\text{minimum}}$$

$$y = 2 \cdot \sqrt[3]{50^2}$$

$$y + x^2 = 2 \cdot \sqrt[3]{50^2} + \left(\sqrt[3]{50}\right)^2 = \boxed{3 \cdot \sqrt[3]{50^2} \approx 40.7}$$

9. (14 points) Let $f(x)$ be the function

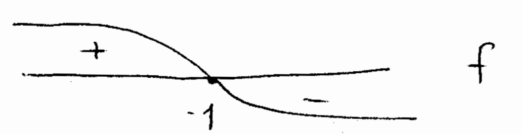
$$f(x) = (x+2)e^{-x}$$

(a) Find the zeros of $f(x)$, i.e., the values of x at which $f(x) = 0$.

$x = -2$

(b) List the intervals on which $f(x)$ is increasing. List the intervals on which $f(x)$ is decreasing.

$f' = -(x+1)e^{-x}$



$x < -1$ increasing

$x > -1$ decreasing

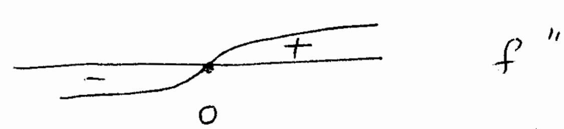
(c) Find all local maxima and local minima of $f(x)$.

$x = -1$ local max

no local min

(d) List the intervals on which $f(x)$ is concave up. List the intervals on which $f(x)$ is concave down.

$f'' = xe^{-x}$



$x > 0$ concave up

$x < 0$ concave down

9.(continued)

(e) Find all inflection points of $f(x)$.

$x = 0$ inflection point

(f) Find the global maximum and global minimum of $f(x)$ on the interval $[-2.5, 6]$.

max $x = -1$ $f(-1) = e$

min $x = -2.5$ $f(-2.5) = -0.5 \cdot e^{2.5}$

(g) Graph $f(x)$ on $[-2.5, 6]$ using the grid below.