The following solutions only provide one way of computing each limit. In many cases, there are alternate methods and you are encouraged to consider them (they might be easier).

1. First, we factor out an x and apply some limit laws to get

$$\lim_{x\to\infty} x - \ln(x) = \lim_{x\to\infty} x\left(1 - \frac{\ln(x)}{x}\right) \stackrel{?}{=} \lim_{x\to\infty} x \cdot \left(1 - \lim_{x\to\infty}\frac{\ln(x)}{x}\right).$$

The "?" is because we don't yet know that the second limit exists. We check that it exists by using L'Hospital's rule to compute it (why are we allowed to use L'Hospital's rule?) as follows:

$$\lim_{x\to\infty}\frac{\ln(x)}{x} \stackrel{\text{L'H}}{=} \lim_{x\to\infty}\frac{\frac{1}{x}}{1} = 0.$$

So we have

$$\lim_{x\to\infty} x - \ln(x) = \lim_{x\to\infty} x\cdot(1-0) = \infty.$$

2. We compute

$$\begin{aligned}\lim_{x\to 0}\frac{\sin(2x)}{\tan(3x)} &= \lim_{x\to 0}\frac{2\sin(2x)}{2x}\cdot\frac{3x\cos(3x)}{3\sin(3x)}\\ &= 2\lim_{x\to 0}\frac{\sin(2x)}{2x}\cdot\frac{1}{3}\lim_{x\to 0}\frac{3x}{\sin(3x)}\cdot\lim_{x\to 0}\cos(3x)\\ &= 2(1)\cdot\frac{1}{3}(1)\cdot 1\\ &= \frac{2}{3}.\end{aligned}$$

3. We compute

$$\begin{aligned}\lim_{x\to 0} x^2\sec(2x)\cot(3x) &= \lim_{x\to 0}\frac{x^2}{\cos(2x)}\cdot\frac{3x\cos(3x)}{3x\sin(3x)}\\ &= \lim_{x\to 0}\frac{x}{3\cos(2x)}\cdot\lim_{x\to 0}\frac{3x}{\sin(3x)}\\ &= \frac{0}{3}\cdot 1\\ &= 0.\end{aligned}$$

4. We compute

$$\begin{aligned}\lim_{x\to 4}\frac{x^2-6x+8}{\sqrt{x}-2} &= \lim_{x\to 4}\frac{(x-4)(x-2)}{\sqrt{x}-2}\\ &= \lim_{x\to 4}\frac{(\sqrt{x}-2)(\sqrt{x}+2)(x-2)}{\sqrt{x}-2}\\ &= \lim_{x\to 4}(\sqrt{x}+2)(x-2)\\ &= 8.\end{aligned}$$

5. The best way to approach this problem is to begin by factoring the denominator. We get

$$\lim_{x\to 4}\frac{x^3-4x^2+x-4}{x^2-2x-8}=\lim_{x\to 4}\frac{x^3-4x^2+x-4}{(x-4)(x+2)}.$$

From this, we see that if we can cancel the $(x-4)$ factor in the denominator, the limit should be easy to compute by continuity. In order to cancel the $(x-4)$ factor in the denominator, we must see that it is a factor of the numerator. This can be done with polynomial long division (or you can factor by grouping). Either way, we get

$$\begin{aligned}\lim_{x\to 4}\frac{x^3-4x^2+x-4}{x^2-2x-8}&=\lim_{x\to 4}\frac{(x-4)(x^2+1)}{(x-4)(x+2)}\\&=\lim_{x\to 4}\frac{x^2+1}{x+2}\\&=\frac{17}{6}.\end{aligned}$$

6. We compute

$$\begin{aligned}\lim_{t\to 1}\frac{\sqrt{10-t}-3}{1-t}\cdot\frac{\sqrt{10-t}+3}{\sqrt{10-t}+3}&=\lim_{t\to 1}\frac{(10-t)-9}{(1-t)(\sqrt{10-t}+3)}\\&=\lim_{t\to 1}\frac{1-t}{(1-t)(\sqrt{10-t}+3)}\\&=\lim_{t\to 1}\frac{1}{\sqrt{10-t}+3}\\&=\frac{1}{6}.\end{aligned}$$

7. We compute

$$\lim_{t\to 0}\frac{\frac{t}{t+1}-t}{t}=\lim_{t\to 0}\frac{1}{t+1}-1=0.$$

8. We compute

$$\lim_{m\to 0}\frac{\frac{1}{m^2}-\frac{1}{m}}{\frac{1}{m^2}}=\lim_{m\to 0}1-m=0.$$

9. We compute

$$\begin{aligned}
\lim_{x\to 0^+} \frac{(\ln(x))^2}{e^{1/x}} &\overset{\text{L'H}}{=} \lim_{x\to 0^+} \frac{\dfrac{2\ln(x)}{x}}{-\dfrac{1}{x^2}e^{1/x}} \\
&= \lim_{x\to 0^+} \frac{2\ln(x)}{-\dfrac{1}{x}e^{1/x}} \\
&\overset{\text{L'H}}{=} \lim_{x\to 0^+} \frac{\dfrac{2}{x}}{\dfrac{1}{x^3}e^{1/x} + \dfrac{1}{x^2}e^{1/x}} \\
&= \lim_{x\to 0^+} \frac{2}{e^{1/x}\left(\dfrac{1}{x^2} + \dfrac{1}{x}\right)} \\
&= 0.
\end{aligned}$$

10. We compute

$$\begin{aligned}
\lim_{x\to 0} \frac{1-\cos(x)}{x} \cdot \frac{1+\cos(x)}{1+\cos(x)} &= \lim_{x\to 0} \frac{1-\cos^2(x)}{x(1+\cos(x))} \\
&= \lim_{x\to 0} \frac{\sin^2(x)}{x(1+\cos(x))} \\
&= \lim_{x\to 0} \frac{\sin(x)}{x} \cdot \lim_{x\to 0} \frac{\sin(x)}{1+\cos(x)} \\
&= 1 \cdot 0 \\
&= 0.
\end{aligned}$$

11. We compute

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin\left(\frac{x}{5}\right)}{\sin(3x)} &= \lim_{x\to 0} \frac{\frac{1}{5}\sin\left(\frac{x}{5}\right)}{\frac{x}{5}} \cdot \frac{3x}{3\sin(3x)} \\
&= \frac{1}{5}\lim_{x\to 0} \frac{\sin\left(\frac{x}{5}\right)}{\frac{x}{5}} \cdot \frac{1}{3}\lim_{x\to 0} \frac{3x}{\sin(3x)} \\
&= \frac{1}{5}(1) \cdot \frac{1}{3}(1) \\
&= \frac{1}{15}.
\end{aligned}$$

12. We begin by combining the two terms into a single fraction:

$$\lim_{x\to 1^+} \frac{x}{x-1} - \frac{1}{\ln(x)} = \lim_{x\to 1^+} \frac{x\ln(x) - (x-1)}{(x-1)\ln(x)}.$$

This is an indeterminate of the form $\frac{0}{0}$. So we compute

$$\lim_{x\to 1^+} \frac{x\ln(x)-(x-1)}{(x-1)\ln(x)} \overset{\text{L'H}}{=} \lim_{x\to 1^+} \frac{\ln(x)+1-1}{\ln(x)+1-\frac{1}{x}} \overset{\text{L'H}}{=} \lim_{x\to 1^+} \frac{\frac{1}{x}}{\frac{1}{x}+\frac{1}{x^2}} = \frac{1}{2}.$$

13. This is an indeterminate of the form ∞^0. So we set $y = (\tan(x))^{\cos(x)}$ and compute

$$\begin{aligned}
\lim_{x\to(\pi/2)^-} \ln(y) &= \lim_{x\to(\pi/2)^-} \cos(x)\ln(\tan(x)) \\
&= \lim_{x\to(\pi/2)^-} \frac{\ln(\tan(x))}{\sec(x)} \\
&\overset{\text{L'H}}{=} \lim_{x\to(\pi/2)^-} \frac{\cot(x)\sec^2(x)}{\sec(x)\tan(x)} \\
&= \lim_{x\to(\pi/2)^-} \frac{\sec(x)}{\tan^2(x)} \\
&= \lim_{x\to(\pi/2)^-} \frac{\cos(x)}{\sin^2(x)} \\
&= 0.
\end{aligned}$$

Then

$$\lim_{x\to(\pi/2)^-} y = \lim_{x\to(\pi/2)^-} e^{\ln(y)} = e^0 = 1.$$

14. This is an indeterminate of the form $\frac{\infty}{\infty}$. So we apply L'Hosipital's rule to get

$$\lim_{x\to\infty} \frac{\ln(e^{-x}+e^{-2x})}{x} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{-e^{-x}-2e^{-2x}}{e^{-x}+e^{-2x}}.$$

At this point, we may recognize that this new limit is an indeterminate of the form $\frac{\infty}{\infty}$ and apply L'Hospital's rule once more. This would give us

$$\lim_{x\to\infty} \frac{-e^{-x}-2e^{-2x}}{e^{-x}+e^{-2x}} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{e^{-x}+4e^{-2x}}{-e^{-x}-2e^{-2x}},$$

which is again an indeterminate of the form $\frac{\infty}{\infty}$. At this point, we should realize that applying L'Hosiptal's rule again will not make things look any better. So we should go back to the previous step and instead of applying L'Hospital's rule the second time, we need to look for an alternate way of computing the limit. In fact, we see that we can factor e^{-x} from the numerator and denominator. This gives us

$$\begin{aligned}\lim_{x\to\infty}\frac{-e^{-x}-2e^{-2x}}{e^{-x}+e^{-2x}} &= \lim_{x\to\infty}\frac{e^{-x}(-1-2e^{-x})}{e^{-x}(1+e^{-x})}\\ &= \lim_{x\to\infty}\frac{-1-2e^{-x}}{1+e^{-x}}\\ &= -1.\end{aligned}$$

The moral here is that L'Hospital's rule does not always work, even when the indeterminate is of the correct form.

15. To simplify things a bit, we should rewrite the second factor using the laws of logarithms:

$$\lim_{x\to\infty} e^{-x}(\ln(2x+3)-2\ln(x)) = \lim_{x\to\infty} e^{-x}\ln\left(\frac{2x+3}{x^2}\right).$$

This is an indeterminate of the form $0\cdot\infty$ so we rewrite it and compute as follows:

$$\begin{aligned}\lim_{x\to\infty} e^{-x}\ln\left(\frac{2x+3}{x^2}\right) &= \lim_{x\to\infty}\frac{\ln\left(\frac{2x+3}{x^2}\right)}{e^x}\\ &\overset{\text{L'H}}{=} \lim_{x\to\infty}\frac{\frac{x^2}{2x+3}\cdot\frac{2x^2-2x(2x+3)}{x^4}}{e^x}\\ &= \lim_{x\to\infty} -\frac{2(x+3)}{x(2x+3)e^x}\cdot\frac{\frac{1}{x}}{\frac{1}{x}}\\ &= \lim_{x\to\infty} -\frac{2+\frac{3}{x}}{(2x+3)e^x}\\ &= 0\end{aligned}$$

16. First note that

$$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) \neq \lim_{x\to 0} x\cdot\lim_{x\to 0}\sin\left(\frac{1}{x}\right).$$

The reason is that the second limit does not exist. It just oscillates between 1 and -1. No matter how close x gets to zero, there are always two values closer to zero where $\sin\left(\frac{1}{x}\right)$ is 1 and -1. (Can you figure out why this is true?) The way to approach this problem is to look at $\sin\left(\frac{1}{x}\right)$ and observe that

$$\left|\sin\left(\frac{1}{x}\right)\right| \leq 1$$

for all values of x (except zero of course). Multiplying both sides by $|x|$ gives us

$$\left|x\sin\left(\frac{1}{x}\right)\right| \leq |x|.$$

This means that

$$-x \leq x \sin\left(\frac{1}{x}\right) \leq x.$$

Then the Squeeze Theorem gives us

$$\lim_{x\to 0} -x \leq \lim_{x\to 0} x \sin\left(\frac{1}{x}\right) \leq \lim_{x\to 0} x.$$

Since the limits on the left and right are both zero, the middle limit must also be zero.