1. (12 points; 3 points each) Differentiate the following functions. You need not simplify your answers.

   (a) $y = \dfrac{x^2}{x^3 - 1}$

   Solution. The quotient rules gives us

$$y' = \frac{2x(x^3-1) - x^2(3x^2)}{(x^3-1)^2}.$$

□

   (b) $y = \tan^{-1}(\ln x) + \ln(\tan^{-1} x)$

   Solution. Applying the chain rule to each term gives us

$$y' = \frac{1}{1 + (\ln(x))^2} \cdot \frac{1}{x} + \frac{1}{\tan^{-1}(x)} \cdot \frac{1}{1+x^2}.$$

   NOTE: $\tan^{-1}(x) \neq \dfrac{1}{\tan(x)}$. □

   (c) $y = \dfrac{1}{\sqrt[3]{x + \sqrt{x}}}$

   Solution. It may help to rewrite the function as $y = \left(x + x^{1/2}\right)^{-1/3}$. The chain rule gives us

$$y' = -\frac{1}{3}\left(x + x^{1/2}\right)^{-4/3}\left(1 + \frac{1}{2}x^{-1/2}\right).$$

□

   (d) $y = \left[\cos\left(\dfrac{1}{x}\right)\right]^x$

   Solution. We will use logarithmic differentiation.

$$\begin{aligned}
\ln(y) &= x \ln\left(\cos\left(\frac{1}{x}\right)\right) \\
\frac{y'}{y} &= \ln\left(\cos\left(\frac{1}{x}\right)\right) + \frac{x}{\cos\left(\frac{1}{x}\right)}\left(-\sin\left(\frac{1}{x}\right)\right)\left(-\frac{1}{x^2}\right) \\
\frac{y'}{y} &= \ln\left(\cos\left(\frac{1}{x}\right)\right) + \frac{\tan\left(\frac{1}{x}\right)}{x} \\
y' &= \left[\ln\left(\cos\left(\frac{1}{x}\right)\right) + \frac{\tan\left(\frac{1}{x}\right)}{x}\right]\left[\cos\left(\frac{1}{x}\right)\right]^x
\end{aligned}$$

   NOTE: you didn't need to make the simplifications from line 2 to line 3; however, you do need to make sure your answer is in terms of x. □

2. (10 points; 2 points each) Evaluate the following limits.

(a) $\lim_{x\to 0} \frac{\sin^2(x)}{\cos(x)-1}$

Solution. Observe that $\sin^2(0) = 0$ and $\cos(0) - 1 = 0$. So we can use L'Hopital's Rule to get

$$\lim_{x\to 0} \frac{\sin^2(x)}{\cos(x)-1} = \lim_{x\to 0} \frac{2\sin(x)\cos(x)}{-\sin(x)} = \lim_{x\to 0} -2\cos(x) = -2,$$

where we evaluate the last limit using continuity. □

(b) $\lim_{x\to 0} \frac{\sin^2 x}{\cos(3x)-1}$

Solution. Observe that $\sin^2(0) = 0$ and $\cos(0) - 1 = 0$. So we can use L'Hopital's Rule to get

$$\lim_{x\to 0} \frac{\sin^2(x)}{\cos(3x)-1} = \lim_{x\to 0} \frac{2\sin(x)\cos(x)}{-3\sin(3x)}.$$

Now observe that $2\sin(0)\cos(0) = 0$ and $-3\sin(0) = 0$. So we can use L'Hopital's Rule once more to get

$$\lim_{x\to 0} \frac{2\sin(x)\cos(x)}{-3\sin(3x)} = \lim_{x\to 0} \frac{2(\cos^2(x)-\sin^2(x))}{-9\cos(3x)} = -\frac{2}{9},$$

where we evaluated the last limit using continuity. □

(c) $\lim_{x\to 0^-} \frac{|x|}{x-2|x|}$

Solution. Since $x \to 0^-$, we know that $x < 0$. So $|x| = -x$. This gives us

$$\lim_{x\to 0^-} \frac{|x|}{x-2|x|} = \lim_{x\to 0^-} \frac{-x}{x+2x} = -\frac{1}{3},$$

where we evaluated the last limit by continuity after canceling x in the numerator and denominator. □

(d) $\lim_{x\to\infty} \frac{\ln(\ln(x))}{e^{\sqrt{x}}}$

Solution. Observe that $\lim_{x\to\infty} \ln(\ln(x)) = \infty$ and $\lim_{x\to\infty} e^{\sqrt{x}} = \infty$. So we can use L'Hopital's Rule to get

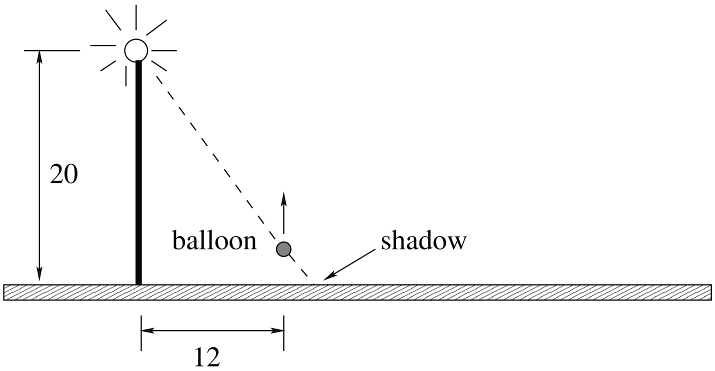
$$\lim_{x\to\infty} \frac{\ln(\ln(x))}{e^{\sqrt{x}}} = \lim_{x\to\infty} \frac{\frac{1}{x\ln(x)}}{\frac{e^{\sqrt{x}}}{2\sqrt{x}}} = \lim_{x\to\infty} \frac{2\sqrt{x}}{x\ln(x)e^{\sqrt{x}}} = \lim_{x\to\infty} \frac{2}{\sqrt{x}\ln(x)e^{\sqrt{x}}} = 0,$$

where we use the fact that the denominator in the last limit is going to ∞ while the numerator remains constant. □

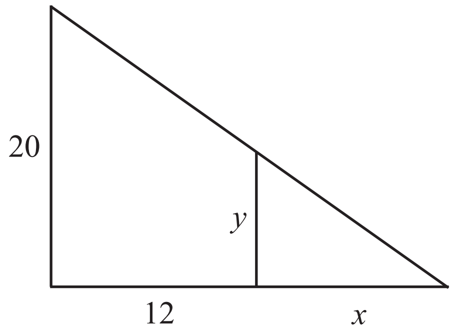
(e) $\lim_{x\to 5^+} \frac{4-x}{x^2-25}$

Solution. Observe that $\lim_{x\to 5^+} 4-x=-1$ and $\lim_{x\to 5^+} x^2-25=0$. So there is a non-removable discontinuity at $x=5$. We can factor the denominator to get $(x+5)(x-5)$. From this we see that as x gets close to 5 from the right (i.e., $x>5$), the denominator will be a very small positive number. Meanwhile, the numerator will approach -1. Therefore, the limit is $-\infty$. □

3. A small helium balloon is rising at the rate of 8 ft/sec, a horizontaldistance of 12 feet from a 20 ft. lamppost. At what rate is the shadow of the balloon moving along the ground when the balloon is 5 feet above the ground?



Solution. Let y represent the height of the balloon and x represent the horizontal distance from the balloon to it's shadow as indicated in the figure below:



Then $\frac{dy}{dt}=8$ ft/sec. We are asked to find $\left.\frac{dx}{dt}\right|_{y=5}$. The chain rule gives us

$$\frac{dx}{dt}=\frac{dy}{dt}\cdot\frac{dx}{dy}.$$

So we need an equation that relates x and y. From the figure, we see that we have similar triangles. This gives us

$$\frac{x}{y}=\frac{x+12}{20},$$

which we can solve for x in terms of y to get

$$x=\frac{12y}{20-y}.$$

Then

$$\frac{dx}{dy} = \frac{12(20-y)+12y}{(20-y)^2}.$$

Therefore,

$$\left.\frac{dx}{dt}\right|_{y=5} = \frac{12(20-5)+12(5)}{(20-5)^2}\cdot 8 = \frac{128}{15} \text{ ft/sec.}$$

□

4. The point $(1/4, 1/4)$ lies on the curve

$$(3x^2+y^2)^{3/2} = 3x^2 - y^2.$$

(a) (8 pts) Using linear approximation, find the approximate value of the number z near $1/4$ such that $(z, 0.23)$ is a point on this curve.

Solution. First, note that we are being asked to estimate the x-coordinate of a point on the curve. So we will view x as a function of y and compute the derivative at $(1/4, 1/4)$ implicitly:

$$\frac{3}{2}(3x^2+y^2)^{1/2}(6xx'+2y) = 6xx' - 2y$$

$$\frac{3}{2}\left(3\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^2\right)^{1/2}\left(6\left(\frac{1}{4}\right)x'+2\left(\frac{1}{4}\right)\right) = 6\left(\frac{1}{4}\right)x' - 2\left(\frac{1}{4}\right)$$

$$x' = \frac{7}{3}.$$

So the tangent line to the curve at the point $(1/4, 1/4)$ is

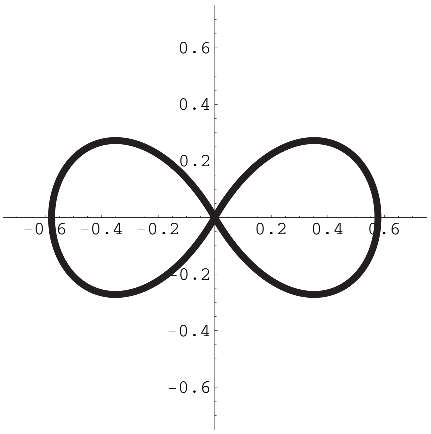
$$x = \frac{7}{3}\left(y-\frac{1}{4}\right)+\frac{1}{4}.$$

Using this tangent line to approximate z gives us

$$z \approx \frac{7}{3}(0.23-0.25)+0.25 \approx 0.203.$$

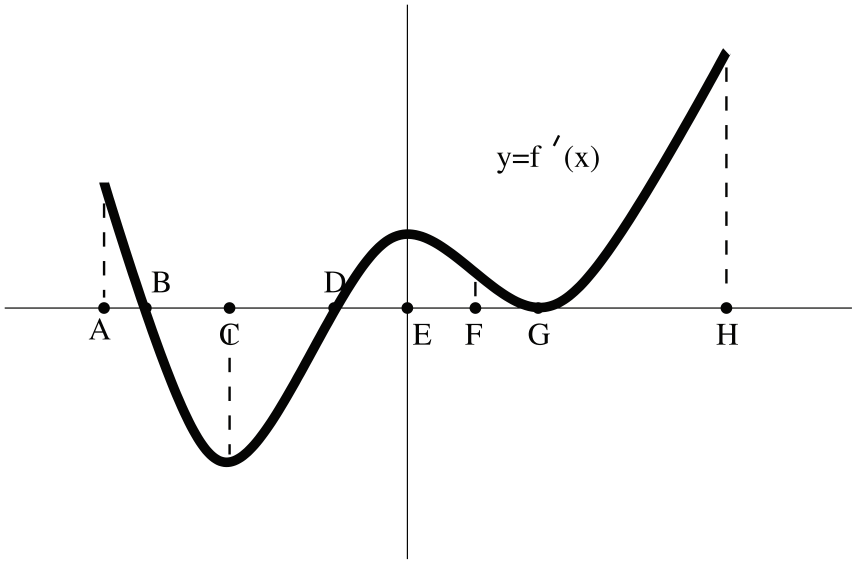
□

(b) (2 pts) Using the drawing of the curve below, determine whether the actual number z is bigger or smaller than the number found using linear approximation:



Solution. Notice that the curve is concave down near the point $(1/4, 1/4)$. Sketching the tangent line at this point shows that the tangent line lies to the "left" of the curve near the point $(1/4, 1/4)$. Therefore, the actual value of z is larger than the number we found using the linear approximation. □

5. (12 points) The function f has domain (A, H) and the graph of f' is as shown. Each part is worth 3 points; there is no partial credit on this problem.



(a) Give the open interval(s) over which f is increasing.

Solution. Recall that f is increasing whenever f' is positive. So looking at the graph, we see that f is increasing on (A, B) and (D, H). □

(b) Give the open interval(s) over which f is concave downward.

Solution. Recall that f is concave downward when f'' is negative and f'' is negative when f' is decreasing. So looking at the graph, we see that f is concave downward on (A, C) and (E, G). □

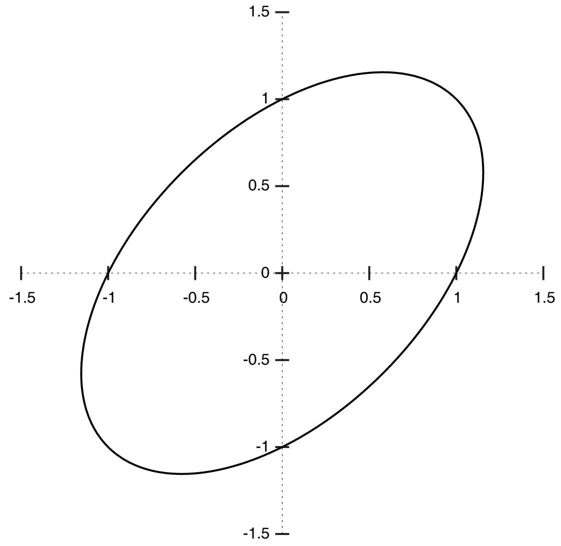
(c) At which point(s) does f have a local minimum?

Solution. Recall that f has a local minimum whenever it changes from decreasing to increasing (i.e., whenever f' changes from negative to positive). So looking at the graph, we see that f has a local minimum at D. □

(d) At which point(s) does f change concavity?

Solution. Recall that f changes concavity when f'' changes signs, which happens when f' changes from decreasing to increasing or from increasing to decreasing. So looking at the graph, we see that f changes concavity at C, E, and G. □

6. (10 points) Find the point on the ellipse $x^2+y^2-xy=1$ with the largest y coordinate.



Solution. We are trying to maximize the y-coordinate, so we begin by computing y'.

$$2x + 2yy' - y - xy' = 0$$
$$y' = \frac{y-2x}{2y-x}.$$

Next, we determine the critical points on the curve. These are the points where y' is zero or undefined. Observe that $y' = 0$ when the numerator is zero, which happens when $y = 2x$. Substituting this into the equation for the curve gives us

$$x^2 + 4x^2 - 2x^2 = 1$$
$$x = \pm\frac{1}{\sqrt{3}} \quad \Rightarrow \quad y = \pm\frac{2}{\sqrt{3}}.$$

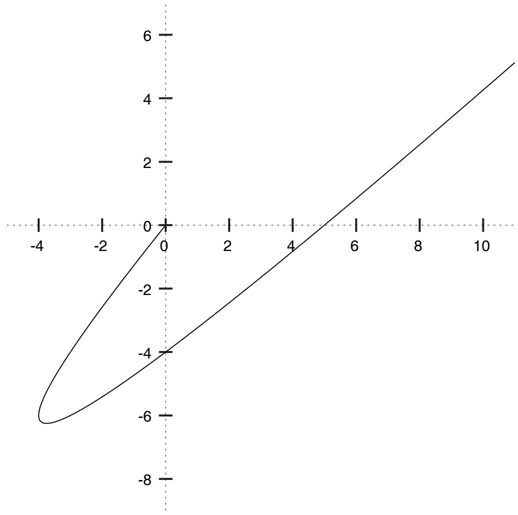
Now observe that y' is undefined when the denominator is zero, which is when $y = \dfrac{x}{2}$. Substituting this into the equation for the curve gives us

$$x^2 + \frac{x^2}{4} - \frac{x^2}{2} = 1$$
$$x = \pm\frac{2}{\sqrt{3}} \quad \Rightarrow \quad y = \pm\frac{1}{\sqrt{3}}.$$

These four points correspond to where the tangent line to the curve is horizontal or vertical. So by inspection of the graph, we see that the largest y-coordinate occurs at the point $\left(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$. □

7. (12 points) A particle starts moving at time $t = 0$. Its position at time $t \geq 0$ is given by

$$x(t) = t^2 - 4t, \;\; y(t) = t^2 - 5t.$$



(a) Express $\frac{dy}{dx}$ in terms of t.

Solution. First, we compute $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

$$\frac{dx}{dt} = 2t - 4$$
$$\frac{dy}{dt} = 2t - 5.$$

By the chain rule, we have

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t-5}{2t-4}.$$

□

(b) For which values of t does the tangent line to the curve at $(x(t), y(t))$ pass through the point $(0, -8)$?

Solution. So the tangent line to the curve at the point $(x(t), y(t))$ is given by

$$y = \frac{2t-5}{2t-4}(x - t^2 + 4t) + t^2 - 5t.$$

If the line passes through $(0, -8)$ we substitute into the equation of the tangent line and solve for t to get

$$-8 = \frac{2t-5}{2t-4}(-t^2 + 4t) + t^2 - 5t$$
$$-8(2t-4) = (2t-5)(-t^2+4t) + (t^2-5t)(2t-4)$$
$$0 = -t^2 + 16t - 32$$
$$t = 8 \pm 4\sqrt{2}.$$

□

(c) When is the particle heading directly toward the point $(0, -8)$?

Solution. Looking at the graph, we see that the particle is heading directly toward the point $(0, -8)$ when $t = 8 - 4\sqrt{2}$. □

8. (10 points) The product of two positive numbers is 100. How small can the sum of one of the numbers plus the square of the other number be? (Make sure to show your work and justify your answer.)

Solution. Let x represent one of the positive numbers. Then the other number is $\frac{100}{x}$. So we can express the desired sum as a function of x:

$$S(x) = \frac{100}{x} + x^2.$$

We seek to minimize S.

$$\begin{aligned} S'(x) &= -\frac{100}{x^2} + 2x \\ 0 &= -\frac{100}{x^2} + 2x \\ \frac{100}{x^2} &= 2x \\ 50 &= x^3 \\ x &= \sqrt[3]{50}. \end{aligned}$$

So $x = \sqrt[3]{50}$ is the only critical number of S. Observe that $S'(x) < 0$ when $x < \sqrt[3]{50}$ and $S'(x) > 0$ when $x > \sqrt[3]{50}$. By the first derivative test for absolute extreme values, $S(\sqrt[3]{50}) = \frac{100}{\sqrt[3]{50}} + \sqrt[3]{50^2}$ is the absolute minimum value of $S(x)$. □

9. (14 points) Let $f(x)$ be the function

$$f(x) = (x + 2)e^{-x}$$

(a) Find the zeros of $f(x)$, i.e., the values of x at which $f(x) = 0$.

Solution. Since e^{-x} is never zero, it follows that $f(x) = 0$ only when $x + 2 = 0$. So the zero of $f(x)$ is $x = -2$. □

(b) List the intervals on which $f(x)$ is increasing. List the intervals on which $f(x)$ is decreasing.

Solution. First, we compute $f'(x)$.

$$f'(x) = e^{-x} - (x + 2)e^{-x} = e^{-x}(-x - 1).$$

Observe that $f'(x) = 0$ when $x = -1$. Since $f'(x)$ is defined everywhere, $x = -1$ is the only critical number of $f(x)$. Since $e^{-x} > 0$ for all values of x, it is easy to see that $f'(x) > 0$ when $x < -1$ and $f'(x) < 0$ when $x > -1$. So $f(x)$ is increasing on $(-\infty, -1]$ and decreasing on $[-1, \infty)$. □

(c) Find all local maxima and local minima of $f(x)$.

Solution. From part (b), we apply the first derivative test for local extrema and see that $f(x)$ has a local maximum value of $f(-1) = e$ and no local minima. □

(d) List the intervals on which $f(x)$ is concave up. List the intervals on which $f(x)$ is concave down.

Solution. For the concavity of $f(x)$, we look at $f''(x)$.

$$f''(x) = -e^{-x}(-x-1) - e^{-x} = xe^{-x}.$$

Since e^{-x} is never zero, we see that $f''(x) = 0$ only when $x = 0$. We check to see that $f''(x) > 0$ when $x > 0$ and $f''(x) < 0$ when $x < 0$. Applying the second derivative test for concavity, we see that $f(x)$ is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$. □

(e) Find all inflection points of $f(x)$.

Solution. Recall that a point of inflection is a point where $f(x)$ changes concavity. From part (d), we see that $f(x)$ has a point of inflection at the point $(0, 2)$. □

(f) Find the global maximum and global minimum of $f(x)$ on the interval $[-2.5, 6]$.

Solution. In part (b) we found that the only critical number was $x = -1$, which is in the interval $[-2.5, 6]$. To find the global maximum and global minimum of $f(x)$ on the closed interval $[-2.5, 6]$, it suffices to compare the values of $f(x)$ at the endpoints and at the critical numbers in the interval:

$$f(-2.5) = -\frac{e^{5/2}}{2} \approx -6.09125$$
$$f(-1) = e \approx 2.71828$$
$$f(6) = \frac{8}{e^6} \approx 0.01983.$$

So the global maximum on $[-2.5, 6]$ is $f(-1) = e$ and the global minimum on $[-2.5, 6]$ is $f(-2.5) = -\frac{e^{5/2}}{2}$. □

(g) Graph $f(x)$ on $[-2.5, 6]$ using the grid below.