NAME ________________________ SECTION ______________

Practice Midterm II Solution Key

Math 124, Sections C,D

Week 7, Spring Quarter 2006

- No book, notes or graphing calculators are allowed. You may use a scientific calculator.

- Show all your work to get full credit unless the problem instructs otherwise

- Read instructions for each problem CAREFULLY.

- Check your work!

1. Calculate derivatives of the following functions. Use logariphmic differentiation when appropriate. You need not carry out any algebra simplification. BOX YOUR FINAL ANSWER. We will only grade the answer you box.

(a) $y = \dfrac{40t^5 - \sqrt{t}}{t^4 + 1}$ $\qquad \dfrac{dy}{dt} = \dfrac{\left[(200t^4 - 1/(2\sqrt{t}))(t^4 - 1)\right] - \left[4t^3(40t^5 - \sqrt{t})\right]}{(t^4 - 1)^2}$

(b) $f(x) = \sec^2(x) - \tan^2(x),$ $\qquad f'(x) = 0$

(c) $f(x) = \sin^2(e^{2x^2-3x})$ $\qquad f'(x) = 2\sin(e^{2x^2-3x})\cos(e^{2x^2-3x})e^{2x^2-3x}(4x-3)$

(d) $f(x) = \dfrac{(x^4-1)(x+2)^3}{e^{x^2}\sqrt{x^3-4}}$ $\qquad f'(x) = f(x)\left\{\dfrac{4x^3}{x^4-1} + \dfrac{3}{x+2} - 2x - \dfrac{3x^2}{2(x^3-4)}\right\}$

(e) $f(x) = 2^{x^{arccos\, x}}$ $\qquad f'(x) = 2^{x^{\arccos(x)}} x^{\arccos(x)} \left(-\dfrac{\ln(x)}{\sqrt{1-x^2}} + \dfrac{\arccos(x)}{x}\right)\ln(2)$

2. An object is moving along number line and its location at time t seconds is given by the function $d(t) = 3t^3 - 5t^2 + 2t + 1$ cm.

   (a) What are the velocity and acceleration of the object at time t? Include UNITS.

   Velocity: $d'(t) = 9t^2 - 10t + 2$ cm/s
   Acceleration: $d''(t) = 18t - 10$ cm/s^2

   (b) What is the maximum acceleration of the object during the time interval $[0, 1]$? Explain.

   At $t = 1$ the acceleration is a maximum with $d''(t) = 8$.

3. An object is moving with parametric equations

$$\begin{aligned} x(t) &= e^{-t}\sin(\pi t), \\ y(t) &= \cos(\pi t) \end{aligned}$$

The location of the object at time t seconds is $P(t) = (x(t), y(t))$ and the path followed during the first 2 seconds is pictured below. Units on the axes are centimeters (cm).

   (a) Find the horizontal and vertical velocity of the object at time t.

$$\begin{aligned} x'(t) &= -e^{-t}\sin(\pi t) + \pi e^{-t}\cos(\pi t) \\ y'(t) &= -\pi\sin(\pi t) \end{aligned}$$

   (b) Find the slope of the curve when the object is located at $P(1/2)$. Then find the equation of the tangent line at this point.

$$\frac{dy}{dx} = \frac{-\pi\sin(\pi t)}{-e^{-t}\sin(\pi t) + \pi e^{-t}\cos(\pi t)}$$

   At $t = 1/2$ we have $\frac{dy}{dx} = \pi e^{1/2}$. Equation for the tangent is $y(x) = \pi(e^{1/2}x - 1)$.

   (c) Is the object moving faster at time $t = 0$ or at time $t = 1$? Explain.
   Speed:

$$\sqrt{(x'(t))^2 + (y'(t))^2} = \sqrt{(-e^{-t}\sin(\pi t) + \pi e^{-t}\cos(\pi t))^2 + \pi^2\sin^2(\pi t)}$$

   Evaluating at $t = 0, 1$ gives,

$$\begin{aligned} t = 0 \qquad & \sqrt{\pi^2} = \pi, \\ t = 1 \qquad & \sqrt{\pi^2 e^{-2}} = \pi/e, \end{aligned}$$

   which means that the object is moving faster at $t = 0$.

(d) Find the locations where the tangent line to the path is vertical.
Answer.

$$t_1 = \frac{\arctan \pi}{\pi} \quad \Rightarrow \quad \left(e^{-t_1} \frac{\pi}{\sqrt{\pi^2+1}}, \frac{1}{\sqrt{\pi^2+1}} \right)$$

$$t_2 = \frac{\arctan \pi}{\pi} + 1 \quad \Rightarrow \quad \left(-e^{-t_2} \frac{\pi}{\sqrt{\pi^2+1}}, -\frac{1}{\sqrt{\pi^2+1}} \right)$$

4. The equation $x^2 + xy + y^3 = 2$

(a) The point $P = (0, \sqrt[3]{2})$ is on the curve. Find an equation of the tangent line to the curve at the point P.

Answer. $\frac{dy}{dx} = \frac{-2x-y}{x+3y^2}$
Then the tangent line equation at $P = (0, \sqrt[3]{2})$ is

$$y = -\frac{x}{3\sqrt[3]{2}} + \sqrt[3]{2}$$

(b) Let Q be the point on the curve whose x-coordinate is 0.1. Using linear approximation at P, estimate the y-coordinate of Q. In other words, if $Q = (0.1, y_o)$, estimate y_o using linear approximation. Leave your estimate in exact form.
Answer. $y_o \approx -\frac{0.1}{3\sqrt[3]{2}} + \sqrt[3]{2}$

5. Below is a picture of a portion of the graph of the equation: $\sin(x + 2y) = 2x\cos(y)$.

(a) Find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{2\cos y - \cos(x+2y)}{2\cos(x+2y) + 2x\sin y}$$

(b) Write the equation of the tangent line to this curve at the origin $(0, 0)$ and sketch this tangent line in the picture.

$$y = \frac{x}{2}$$

6. Find the equation of the tangent line to the curve

$$\sin x + \cos y = \sin x \cos y$$

at the point $(\pi, \pi/2)$.
Answer.

$$\frac{dy}{dx} = \frac{\cos(x)(\cos(y) - 1)}{\sin(y)(\sin(x) - 1)}$$

$$y = -x + \frac{3\pi}{2}$$

7. Gravel is being dumped from a conveyor belt at a rate of 30 ft^3/min, and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high? (Note: The volume of a cone of height h having a base of radius r is given by the formula: $V = \frac{1}{3}\pi r^2 h$.)

   Rewrite the volume equation to $V = \frac{1}{12}\pi h^3$ and implicitly differentiate to get,

$$\frac{dV}{dt} = \frac{\pi}{4}h^2\frac{dh}{dt},$$

   solve for $\frac{dh}{dt}$,

$$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt},$$

   and evaluate with $h = 10$ ft and $\frac{dV}{dt} = 30\ \text{ft}^3/\text{min}$ to get $\frac{dh}{dt} = \frac{6}{5\pi}$ ft/min.