Name ______________________________ Section __________

# Practice Midterm I Answer Key

Math 124, Sections C, D

April, 2006

Note: you must use limits to find derivatives and slopes of tangent lines. No credit will be given for using shortcuts you might have learned outside of the classroom (not to worry: you'll have plenty of opportunity to use these shortcuts later in the course).

Note: the actual midterm is "no notes, books or graphing calculators". You may use your "simple" scientific calculator on the exam. You have to show ALL YOUR WORK on the exam to get credit.

1. Here is the picture of the graph of a distance function $y = f(x)$. Distance is in feet, time is in seconds.

Solution: We can solve this problem by just looking at the plot or writing down the multipart function to answer the questions. It is not necessary to write down the multipart functions but it may help to figure out some of the limits.

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le 10 \\ 20 - x & \text{if } 10 < x < 30 \\ x - 40 & \text{if } 30 \le x \end{cases} \qquad f'(x) = \begin{cases} 1 & \text{if } 0 \le x \le 10 \\ -1 & \text{if } 10 < x < 30 \\ 1 & \text{if } 30 \le x \end{cases}$$

(a) $f'(7) = 1$

(b) $\lim_{x \to 0} f(x + 20) = 0$

(c) $\lim_{x\to 0} \dfrac{f(x+20)}{x} = -1$

(d) $\lim_{x\to 0} \dfrac{f(x+20)-20}{x-20} = 1$

(e) The average velocity on the time interval [0,40] = 0

$$v_{ave} = \frac{f(40)-f(0)}{40} = 0$$

(f) The maximum velocity on the time interval [0,40] =1

(g) Let $g(x) = \frac{x}{x+1}$ and $h(x) = f\left(g(x)\right)$. Find $\lim_{t\to 10} h(t)$
Plugging in $\frac{x}{x+1}$ into $f(x)$ we can find out that $\lim_{t\to 10} h(t) = 10/11$

2. The picture above is the the graph of a function $y = f(x)$. Determine at which points the function is discontinuous.

3. Evaluate the following limits.
(a) $\lim_{x\to 1} \frac{x^2-3x+2}{x-1} = \lim_{x\to 1} \frac{(x-1)(x-2)}{x-1} = -1$
(b) $\lim_{x\to 0} \frac{\sqrt{x+1}-\sqrt{1-x}}{x} = 1$
(c) $\lim_{x\to 0} \frac{\tan x}{x} = 1$
(d) $\lim_{t\to 3} \frac{t^3-9t}{t^2-9} = 3$
(e) $\lim_{\theta\to 0} \frac{\sin^2\theta}{\theta^2} = \left(\lim_{\theta\to 0} \frac{\sin\theta}{\theta}\right)^2 = 1$
(f) $\lim_{\theta\to 0} \frac{\sin^2\theta}{\theta^3} = \infty$
(g) $\lim_{h\to 2} \frac{\frac{1}{h}-\frac{1}{h-2}}{h-2} = -\infty$
(h) $\lim_{x\to 2} \frac{x^2-4}{3x^2-2x-8} = 2/5$
(i) $\lim_{x\to\infty} \frac{\sqrt{17x^4-1}}{x^2+x} = \sqrt{17}$

Solution: If we let $\lim_{x\to\infty} \frac{\sqrt{17x^4-1}}{x^2+x} = L$ and square both sides of the equation we can get the equation to something we can deal with algebraically,

$$L^2 = \left(\lim_{x\to\infty} \frac{\sqrt{17x^4-1}}{x^2+x}\right)^2 = \lim_{x\to\infty} \frac{17x^4-1}{x^4+3x^3+x^2} = 17.$$

Now if we look back at the original definition of $L$, we see that $L^2 = 17$, therefore $L = \sqrt{17} = \lim_{x\to\infty} \frac{\sqrt{17x^4-1}}{x^2+x}$.

(j) $\lim_{x\to-\infty} \frac{\sqrt{17x^4-1}}{x^2+x} = \sqrt{17}$ (see above)

(k) $\lim_{x\to 0} x^3 \cos\frac{1}{x} = 0$ (squeeze theorem with $-x^3 < x^3\cos\frac{1}{x} < x^3$)

4. Evaluate the following limit:

$$\lim_{h\to 0} \frac{(2+h)^2-4}{h}$$

What is the function for which this limit will compute the slope of the tangent line at the point $x = 2$?

Solution:

$$\lim_{h\to 0} \frac{(2+h)^2-4}{h} = \lim_{h\to 0} \frac{(4+4h+h^2-4}{h} = \lim_{h\to 0} 4+h = 4$$

This is the slope of the tangent line of the function $x^2$.

5. (a) Graph the function $2\ln(t+1)-1$, compute the domain and range. Label carefully intermediate steps.

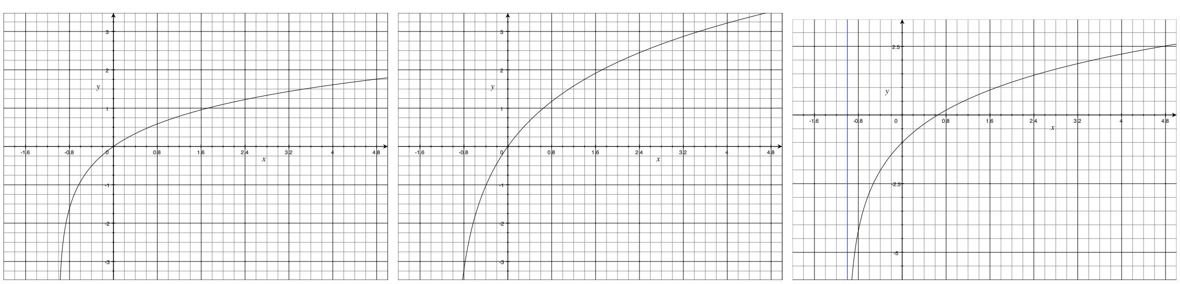

Figure 1: From left to right, $\ln t+1$, $2\ln t+1$, and $2\ln t+1-1$

(b) What are the vertical asymptotes? $t = -1$

(c) Where is this function continuous? $-1 < t$

(d) What is the rate of change of $2\ln t+1-1$ at the point $t = 0$. Use limits and Problem 6 from Homework assignment 3 to answer this question.
Solution: We need to calculate the tangent at $t = 0$:

$$\lim_{h\to 0} \frac{2\ln h+1-1-2\ln 1+1}{h} = \lim_{h\to 0} 2\frac{\ln h+1}{h} = 2$$

6. For the following functions, determine horizontal and vertical assymptotes if they exist. For each function state where the function is continuous.

Solution: Find where the denominator goes to 0 for the vertical asymptotes and calculate $\lim_{x\to\pm\infty} f(x)$ for the horizontal asymptotes.

(a) $f(x) = \frac{x^2+1}{x^2-1} = \frac{x^2+1}{(x+1)(x-1)}$.

Vertical Asymptotes: $x = \pm 1$
Horizontal Asymptotes: $f(x) = 1$
Continuous everywhere except at $x = \pm 1$

(b) $f(x) = \frac{x}{1+x}$.

Vertical Asymptote: $x = -1$
Horizontal Asymptotes: $f(x) = 1$
Continuous everywhere except at $x = -1$

(c) $f(x) = \sqrt{x+1} - \sqrt{x}$.

No vertical asymptotes
Horizontal Asymptotes: $f(x) = 0$
Continuous for $x \geq 0$

7. Let $P(t)$ denote the population of a community of squirrels at time $t$ in years after the year 1999. Suppose that

$$P(t) = \frac{4000}{1 + Ca^t},$$

where $C$ and $a$ are positive constants and $0 < a < 1$. Suppose also that the population of squirrels in 1999 ($t = 0$) was 1000, and the population of squirrels now ($t = 5$) is 2000.

(a) Determine the constants $C$ and $a$.

$$1000 = P(0) = \frac{4000}{1 + C} \Rightarrow 4 = 1 + C \Rightarrow C = 3$$

$$2000 = P(5) = \frac{4000}{1 + 3a^5} \Rightarrow 1 = 3a^5 \Rightarrow -\ln 3 = 5 \ln a \Rightarrow \ln 3^{-1/5} = \ln a \Rightarrow a = 3^{-1/5}$$

(b) When will the population of squirrels be 2500?

$$2500 = \frac{4000}{1 + 3^{1-t/5}} \Rightarrow 8/5 - 1 = 3^{1-t/5} \Rightarrow \ln 3/5 = (1 - t/5)\ln 3 \Rightarrow$$
$$\frac{\ln 3 - \ln 5}{\ln 3} = 1 - t/5 \Rightarrow t = 5(1 - \frac{\ln 3 - \ln 5}{\ln 3}) = \frac{5 \ln 5}{\ln 3}$$

(c) What is $\lim_{t\to\infty} P(t)$?

Solution: Since $\lim_{t\to\infty} 3^{1-t/5} = 0$, we know that $\lim_{t\to\infty} \frac{1}{1+3^{1-t/5}} = 1$ so:

$$\lim_{t\to\infty} \frac{4000}{1+3^{1-t/5}} = 4000$$

8. The price of a stock is modeled by a function $v(t) = Ae^{rt}$ of exponential type, where $A$ and $r$ are constants. Here, $t$ is time in months after January 28, 2003. Assume the stock price is \$200 on January 28, 2003 and the price was \$50 on January 28, 2001 (i.e. exactly 24 months ago).

(a) Find a formula for $v(t)$. Give EXACT answers for $A$ and $r$.

Solution: From $v(0) = 200$ we know that $A = 200$. For $r$ we do the following:

$$v(-24) = 50 = 200e^{-24r} \quad \Rightarrow \quad -\ln 4 = -24r \quad \Rightarrow \quad r = \frac{\ln 4}{24}.$$

We then get the final equation to be,

$$v(t) = 200e^{\frac{\ln 4}{24}t}$$

(b) What is the average rate of change in the stock price over the past 24 months? Give and EXACT answer and include units.

Solution: Just like when we calculate $v_{ave}$, we must use a formula for the slope using the two endpoints, in this case $t = 0$ and $t = -24$:

$$v_{ave} = \frac{v(0) - v(-24)}{0 - (-24)} = \frac{200 - 50}{24} = 150/24 = \$6.25$$

(c) How long does it take the stock price to triple? (Round your answer to the nearest month.)

Solution: Set the equation to 3 times the initial amount of 200 and solve for $t$ or $3A$

$$v_3(t) = 600 = 200e^{\ln 4/24t} \quad \Rightarrow \quad \ln 3 = \ln 4/24t \quad \Rightarrow \quad t = \frac{24 \ln 3}{\ln 4} \approx 19 \text{ months}$$

9. (a) Find the equation of the tangent line to the curve $y(x) = x^3 - x$ at the point where $x = r$. For which values of $r$ is this tangent line horizontal?

Solution: Use our definition of the derivative...

$$\begin{aligned}\lim_{h\to 0}\frac{y(r+h)-y(r)}{h} &= \lim_{h\to 0}\frac{(r+h)^3 - r - h - r^3 + r}{h} = \lim_{h\to 0}\frac{r^3+3r^2h+3rh^2+h^3-h-r^3}{h}\\ &= \lim_{h\to 0}\frac{3r^2h+3rh^2+h^3-h}{h} = \lim_{h\to 0} 3r^2+3rh+h^2-1 = 3r^2-1\end{aligned}$$

(b) Find an equation of the tangent line to $y = e^x \sin x$ at the point $(0,0)$.

Solution: Again find the derivative using limits and the product rule,

$$\lim_{h\to 0}\frac{e^{h+0}\sin h + 0 - e^0 \sin 0}{h} = \lim_{h\to 0}\frac{e^h \sin h}{h}.$$

Now use the product rule and the fact that $\lim_{x\to 0}\frac{\sin h}{h} = 1$ and we find that the tangent line has the equation,

$$m = \left(\lim_{h\to 0} e^h\right)\left(\lim_{h\to 0}\frac{\sin h}{h}\right) = 1 \quad \Rightarrow \quad y = x$$

10. The height $y(t)$ (in feet at time t seconds) of a ball thrown vertically upwards is given by $y(t) = -16t^2 + 128t + 25$. Find the velocity of the ball at time $t = 1$. Find the velocity of the ball when it hits the ground.

Solution: We need to find the velocity of the ball (derivative) at an arbitrary $t$ value,

$$\begin{aligned}y'(t) &= \lim_{h\to 0}\frac{y(t+h)-y(t)}{h} = \lim_{h\to 0}\frac{-16(t+h)^2+128(t+h)+25-16t^2-128t-25}{h}\\ &= \lim_{h\to 0}\frac{-16(t^2+2th+h^2)+128(t+h)+25-16t^2-128t-25}{h}\\ &= \lim_{h\to 0}\frac{-32th-16h^2)+128h}{h} = \lim_{h\to 0} -32t-16h)+128 = 128-32t.\end{aligned}$$

We then know that at $t = 1$ the velocity of the ball is $y'(1) = 128 - 32 = 96$. In order to find the velocity of the ball when it hits the ground we must first figure out at what time the ball hits the ground. We do this by solving $y(t) = 0$ using the quadratic formula,

$$\begin{aligned}t &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \quad \text{where } at^2+bt+c=0\\ y(t) &= 0 = -16t^2+128t+25 \quad \Rightarrow \quad t = \frac{-128 \pm \sqrt{128^2+4\cdot 16\cdot 25}}{-2\cdot 16} \approx [8.19, -0.19].\end{aligned}$$

We only want the positive time (the negative time has no physical relevance) and we then evaluate $y'(t)$ at $t = 8.19$ to find $y'(8.19) \approx -134$ ft/sec. Note that the velocity is negative in this case (i.e. it is falling downward). The equation for the position of the ball represents an object falling with initial velocity $v_0 = 128$ ft/sec, meaning it was initially heading upwards, and an initial position of 25 ft. In general, the equation for any object under uniform acceleration (gravity in this case) has the form $y(t) = \frac{1}{2}at^2 + v_0 t + y_0$, where $a$ is the acceleration, $v_0$ is the initial velocity, and $y_0$ is the initial position. Later on this class and others we will see how this position equation is related to the equation for velocity and acceleration.