Name ______________________________ Section ________

MIDTERM II

Math 124, Section C

May 16, 2006

Problem	Total Points	Score
1	12	
2	12	
3	12	
4	12	
5	12	
Total	60	
6(Bonus)	2	

- No book, notes or graphing calculators are allowed. You may use a scientific calculator.

- Show all your work to get full credit.

- Read instructions for each problem CAREFULLY.

- Check your work!

1. (12pts) Find the following derivatives. You do not have to simplify.

   (a) [4pts] $f(x) = \tan(\frac{x^4}{\sqrt[4]{17x^3+1}})$

   Answer. $f(x)' = \sec^2(\frac{x^4}{\sqrt[4]{17x^3+1}}) \cdot \frac{4x^3\sqrt[4]{17x^3+1} - \frac{51x^6}{4(17x^3+1)^{3/4}}}{\sqrt{17x^3+1}}$

   Equivalently,

   $f(x)' = \sec^2(\frac{x^4}{\sqrt[4]{17x^3+1}}) \cdot (\frac{4x^3}{\sqrt[4]{17x^3+1}} - \frac{51x^6}{4(17x^3+1)^{5/4}})$

   (b) [4pts] $f(x) = x^{\cos x}$

   Answer. $f(x)' == x^{\cos x}(\sin x \ln x + \frac{\cos x}{x})$

   (c) [4pts] $y = arccos(t)$

   Answer. $\frac{d^2y}{dt^2} = -\frac{t}{\sqrt{(1-t^2)^3}}$

2. (12pts) A (spherical) snowball is rolling down a snow covered hill in such a way that its radius is changing at the rate of 3 cm/min. Determine the rate of change of the volume of the snowball when the radius is 4 cm. Include units.

   (You may use the formula for the volume of a sphere of radius r: $V = \frac{4}{3}\pi r^3$.)

   Answer. Differentiating the formula $V = \frac{4}{3}\pi r^3$ we get

   $$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

   Plug in $\frac{dr}{dt} = 3$, $r = 4$. Answer: $\frac{dV}{dt} = 192\pi$ cm^3/min.

3. (12pts) Consider the curve given by the equation

   $$y^2 = (x+1)(x^2 - 1/2)$$

   Use implicit differentiation to answer the following questions:

   [8pts] Find all values of x such that the tangent line to the curve at the point (x, y) is horizontal. How many such points are on the curve?

   Note: you do not have to compute the values of y.

   Answer. Differentiating implicitly, we get

   $$\frac{dy}{dx} = \frac{3x^2 + 2x - \frac{1}{2}}{2y}$$

   The tangent line is horizontal when $3x^2 + 2x - \frac{1}{2} = 0$. Solving for x using the quadratic formula, we get two solutions:

   $$x_1 = \frac{-2+\sqrt{10}}{6} \approx 0.194, \quad x_2 = \frac{-2-\sqrt{10}}{6} \approx -0.86$$

   Plugging in back to the curve, we get $(x_1+1)(x_1^2 - 1/2) = -0.55 < 0$. So, there is no y corresponding to x_1, i.e. x_1 does not give a point on the curve. For $x_2 = -0.86$, we get $(x_2+1)(x_2^2 - 1/2) = 0.033$. This is positive, and we get **TWO** y-coordinates corresponding to x_2. The points where the tangent line is horizontal are $(-0.86, \sqrt{0.033})$ and $(-0.86, -\sqrt{0.033})$.

[4pts] Find all points (x, y) on the curve where the tangent line is vertical.

Answer. Solve $y = 0$ since this is the denominator in $\frac{dy}{dx}$. We get $x = -1$, $x = \pm 1/\sqrt{2}$. The numerator is not 0 at these three points. We get that the tangent line is vertical at three points, all on the x-axis:

$$(-1, 0), \quad (\frac{1}{\sqrt{2}}, 0), \quad (-\frac{1}{\sqrt{2}}, 0)$$

4. [6pts] Find an equation of the tangent line to graph of the function $y = \sqrt[4]{x}$ at the point $(16, 2)$.

Answer. $y' = \frac{1}{4x^{3/4}}$. $y'(16) = 1/32$. Equation of the tangent line:

$$y = \frac{1}{32}(x - 16) + 2$$

[4pts] Using linear approximation, estimate $\sqrt[4]{17}$.

Answer. $\sqrt[4]{17} \approx 1/32 + 2 = 2.031$

[2pts] Is your estimate below or above the actual value? Give a short graphical explanation.

Answer. The estimate is larger than the actual value since the tangent line is above the graph.

5. A particle starts moving at the time $t = 0$. Its position at the time t is given by the parametric equations

$$x(t) = \frac{2t}{t^2 + 1}, \quad y(t) = \frac{t^2 - 1}{t^2 + 1}$$

[1pt] Find the coordinates of the position of the particle at the time $t = 2$.

Answer. $(\frac{4}{5}, \frac{3}{5})$

[3pts] Compute horizontal and vertical derivatives of the particle

Answer. $\frac{dx}{dt} = \frac{2(1-t^2)}{(1+t^2)^2}$, $\frac{dy}{dt} = \frac{4t}{(1+t^2)^2}$.

[3pts] Compute $\frac{dy}{dx}$ as a function of t.

Answer. $\frac{dy}{dx} = \frac{2t}{1-t^2}$

[3pts] Find an equation of the tangent line to the trajectory at the time $t = 2$.

Answer. When $t = 2$, we have $\frac{dy}{dx} = -\frac{4}{3}$. Equation: $y - \frac{3}{5} = -\frac{4}{3}(x - \frac{4}{5})$.

[2pts] Show that the tangent line at the point $(x(2), y(2))$ is perpendicular to the line connecting the origin with the point $(x(2), y(2))$.

Answer. The line through $(0, 0)$ and $(x(2), y(2))$ has the slope $m = \frac{y(2)}{x(2)} = \frac{3}{4}$. The slope of the tangent line is $-\frac{4}{3} = -\frac{1}{m}$. Hence, these two lines are perpendicular.

6. (2pt) Bonus. Sketch and name the parametric curve in problem 5. Justify your answer. FULL CREDIT ONLY.

Answer. The curve is a semicircle. Compute $x(t)^2 + y(t)^2 = (\frac{2t}{t^2+1})^2 + (\frac{t^2-1}{t^2+1})^2 = \frac{4t^2}{(t^2+1)^2} + \frac{t^4-2t^2+1}{(t^2+1)^2} = \frac{4t^2+t^4-2t^2+1}{(t^2+1)^2} = \frac{t^4+2t^2+1}{(t^2+1)^2} = \frac{(t^2+1)^2}{(t^2+1)^2} = 1$. Thus, $(x(t), y(t))$ is a point on a unit circle. We only get half of the circle because the particle starts at $t = 0$.