LECTURE NOTES ON "THE MATHEMATICS BEHIND ESCHER'S PRINTS: FORM SYMMETRY TO GROUPS"

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1. Introduction to symmetries of the plane

1.1. **Rigid motions of the plane.** Examples of four plane symmetries (using a primitive model of the Space needle):

(1) Bilateral (reflection)
(2) Rotational
(3) Translational
(4) Glide symmetry (=reflection + rotation)

Applying plane symmetries to a fixed object we can create lots of wallpaper patterns (tessellations). Now watch Fries and Ghost examples.

All these symmetries have a common property: they do not change distance between objects.

Definition 1.1. A *rigid motion* (= an isometry) is a motion that preserves distance.

$$\left.\begin{array}{l}\text{Reflection (Refl)}\\ \text{Rotation (Rot)}\\ \text{Translation (Tr)}\\ \text{Glide Reflection (Gl)}\end{array}\right\} \text{(rigid) motions of the plane.}$$

Theorem 1.2. *The list of the four motions above contains ALL rigid motions of the plane.*

This poses a question: what if we do a rotation and then a reflection? By the theorem it should again be a rotation or a reflection or a translation of a glide reflection. Which one? Let's investigate this in the first exercise set.

We shall denote the "composition" of two symmetries by a circle: $\circ$. For example, what is

$$\text{Rotation} \circ \text{Rotation} = ?$$

1.2. **Orientation.**

Definition 1.3. A rigid motion is orientation preserving if it does not flip the plane over. Otherwise, it is orientation reversing.

Orientation preserving	Orientation reversing
Rotation (Rot)	Reflection (Refl)
Translation (Tr)	Glide reflection ($\text{Gl} = \text{Refl} \circ \text{Tr}$)
$+1$	-1

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When we compose symmetries, "orientations" multiply. Let's make a table for "compositions":

Composition	Orientation
$\mathrm{Tr} \circ \mathrm{Tr} = \mathrm{Tr}$	$+$
$\mathrm{Rot} \circ \mathrm{Rot} = \mathrm{Rot} \setminus \mathrm{Tr}$	$+$
$\mathrm{Rot} \circ \mathrm{Tr} = \mathrm{Rot} \setminus \mathrm{Tr}$	$+$
$\mathrm{Tr} \circ \mathrm{Rot} = \mathrm{Rot} \setminus \mathrm{Tr}$	$+$
$\mathrm{Rot} \circ \mathrm{Refl} = \mathrm{Refl} \setminus \mathrm{Gl}$	$-$
$\mathrm{Refl} \circ \mathrm{Rot} = \mathrm{Refl} \setminus \mathrm{Gl}$	$-$
$\mathrm{Refl} \circ \mathrm{Refl} = \mathrm{Rot} \setminus \mathrm{Tr}$	$+$

We do not put glide reflection in this table because it is a composition in itself.

Remark 1.4. Another useful "invariant" or characteristic of a symmetry is fixed points.

Symmetry	Fixed points
Translation	none
Glide reflection	none
Rotation	one
Reflection	line

Noting fixed points can be an easy path towards figuring out which symmetry you are dealing with.

2. Groups

Let's note a few observations about symmetries of the plane:

(1) We can "combine" them - take a composition of two symmetries
(2) There is a distinguished rigid motion - namely, staying still and doing nothing
(3) For any rigid motion there is an "undoing" - doing the same motion "backwards"
(4) Order matters!

Definition 2.1. A *group* is a set G on which the law of composition is defined: for any $a, b \in G$, we have $a \circ b \in G$. Moreover, it satisfies the following rules:

(1) (Identity element) There exists an element $e \in G$ such that for any $a \in G$, $a \circ e = e \circ a = a$.
(2) (Associativity) For any $a, b, c \in G$, we have
$$a \circ (b \circ c) = (a \circ b) \circ c.$$
(3) (Inverse) For any $a \in G$ there exists an element $a^{-1} \in G$ such that $a \circ a^{-1} = a^{-1} \circ a = e$.

Definition 2.2. A group G is called *abelian* (after the French mathematician Abel) if it satisfies the commutativity law: $a \circ b = b \circ a$ for any $a, b \in G$.

The composition is also called the "group law" or simply "the operation". It can be denoted by many symbols: $\circ$, $+$, $*$, etc.

Example 2.3. (1) $\mathbb{Z}$ - yes, $+$, no $*$

(2) $\mathbb{N}$ - no
(3) $\mathbb{R}$ - yes +, yes *
(4) - other examples with two operations? Rationals; complex, Quaternions (to show up in Max Lieblich's class)

Definition 2.4. A group G is called *finite* if G is a finite set (has only finitely many elements); otherwise it is infinite. For a finite group G, the *order* of G is the number of elements of G. We say that the order of an infinite group is ∞.

Rigid motions of the plane form a *group of rigid motions*. This group is NON-ABELIAN (see exercise 4,5). This group is VERY BIG though. Definitely not finite.

For relatively small groups, one way to present them is to write a multiplication table. If two groups have the same multiplication tables (that is, have the same number of elements that multiply in the "same" way) then they are "isomorphic". That is, you can map elements of one group to the elements of the other such that multiplication does not change.

Example 2.5. Find all groups of order two. Solution - make a multiplication table.

2.1. **Dihedral groups.** We shall start by studying some small groups of symmetries. Many finite groups have names; some have several names because they show up in different disguises. Some names are quite eloquent. For example, the "biggest" (in a certain sense) finite group is called the Monster. It has

$$808017424794512875886459904961710757005754368000000000 \sim 8 \times 10^{53}$$

elements. Its baby brother is called the baby Monster, that one is much smaller at

$$4154781481226426191177580544000000 \sim 4 \times 10^{33}$$

elements. Of course, one does not really try to write down a multiplication table for those. They are studied by the means of *Representation theory* which is in essence the way we look at the group of symmetries of the plane: we don't think of it abstractly but rather we think how it "acts" on the plane.

Definition 2.6. Dihedral group D_n ($n \geq 3$) is a group of symmetries of a regular n-sided polygon.

Exercise set on D_3.

This is the end of Lecture I.

Lecture II starts with the multiplication table for D_3 and Exercise 1 from the first homework.

Cancellation Law in a group. Let $a, b, c \in G$. We have

$$ab = ac \Leftrightarrow b = c$$

Now we shall study the dihedral group D_n. We'll do the next exercise set as we go. First, the order of D_n is $2n$: it has the identity element, $n-1$ rotations and n reflections.

Elements of D_n: let ρ be the rotation (counterclockwise) by $2\pi/n$. Then all the other rotations have the form

$$e, \rho, \rho \circ \rho = \rho^2, \rho^3, \dots, \rho^{n-1}$$

Observation: $\rho^n = e$.

Definition 2.7. An element $a \in G$ has order n if n is the smallest non-negative integer such that $a^n = e$.

Hence, ρ has order n.

Now, reflections in D_n. Let r be any reflection (situate our n-gon on the plane so that r is the reflection through the x axis).

Lemma 2.8. *Any other reflection has the form* $\boxed{\text{Rot} \circ r}$ *(that is,* $\rho^i \circ r$ *for some* i*,* $0 \leq i \leq n-1$*).*

Proof. This is the first exercise. □

Definition. (Informal) We say that a group is generated by two elements a, b if any element of the group can be written as a product of a's and b's.

Corollary 2.9. *The dihedral group* D_n *has order* $2n$ *and is generated by two elements:* ρ *and* r*, of orders* n *and* 2 *respectively. Any element of* D_n *can be written as* $\rho^i r$ *or* ρ^i *for* $0 \leq i \leq n-1$*. That is,*

$$D_n = \{e, \rho, \rho^2, \dots, \rho^{n-1}, r, \rho r, \rho^2 r, \dots, \rho^{n-1} r\}$$

Now do exercise 2 from the first set.

To be able to do such calculations effectively, we need "multiplication rules" or "relations" (an issue arising constantly for non-abelian groups). We already have two easy relations: $\rho^n = e, r^2 = e$. In the next exercise you need to prove one more relation that will allow you to "permute" r past ρ and get everything to the form as in the Corollary above effectively. Do exercise 3 and if you have time left, 4, from the first set.

In conclusion:

Theorem 2.10. *The group* D_n *is given by the following generators and relations:*

$$D_n = \{\rho, r \mid \rho^n = e, r^2 = e, r\rho = \rho^{n-1} r\}$$

3. Maps and subgroups

Definition 3.1. Let G, H be two groups. We say that a map

$$f : H \to G$$

is a *group homomorphism* if

(1) $f(e_H) = e_G$
(2) $f(ab) = f(a)f(b)$ for any $a, b \in H$

Recall the group of order two from last time. Call it $\mathbb{Z}_2 = \{+1, -1\}$. Denote by

$$\mathbb{M}$$

the group of rigid motions of the plane.

Example 3.2. We have a group homomorphism

$$f : \mathbb{M} \to \mathbb{Z}_2$$

that sends

$$\text{isometry} \mapsto \text{orientation}$$

Definition 3.3. We say that H is a subgroup of G, $H \subset G$, if it is a subset of G such that

(1) $e \in H$
(2) If $a, b \in H$, then $a \circ b \in H$ (closed under composition)
(3) If $a \in H$, then $a^{-1} \in H$.

Now do the second exercise set for today on subgroups of D_3 and D_4.

Question 1. Is D_3 a subgroup of D_4?

Theorem 3.4. *(Lagrange) If H is a subgroup of G then the order of $|H|$ divides the order of $|G|$.*

Definition 3.5. A group homomorphism $f : H \to G$ is an *isomorphism* if it is surjective and injective.
Two groups H and G are isomorphic if there exists a group isomorphism $f : H \to G$.

Definition 3.6. A group is called*cyclic* if it is generated by one element.

Notation: $C_n = \langle a \,|\, a^n = e \rangle$ is a cyclic group of order n.
Note: $C_2 = \mathbb{Z}_2$
We finish the lecture with the discussion of the subgroup lattice of D_3.

This is the end of Lecture II.

Description of $\mathbb{M}$ via generators and relations. This group needs to get organized! Let

ρ_θ be the rotation by θ (counterclockwise) around the origin, $0 \le \theta < 2\pi$
$t_{a,b}$ - translation by the vector (a, b), where $a, b \in \mathbb{R}$.
r - reflection through the x-axis.

Theorem 3.7. *The group of rigid motions of the plane is generated by the symmetries ρ_θ, $t_{a,b}$, and r.*

Orders: r - order 2, $t_{a,b}$ - infinite, ρ_θ has finite order if and only if $\theta/\pi \in \mathbb{Q}$.

4. Plane symmetries, linear transofmrations, and matrices

Matrix multiplication: $\begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix} \begin{bmatrix} \mathrm{e} & \mathrm{g} \\ \mathrm{f} & \mathrm{h} \end{bmatrix} =$

Matrix-vector multiplication $\begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix} \begin{bmatrix} \mathrm{x} \\ \mathrm{y} \end{bmatrix} =$

Geometrically, multiplication by a matrix $A = \begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix}$ sends a vector (x, y) to the vector $(ax + by, cx + dy)$. Hence, a matrix A defines a *linear transformation*[1] of the plane.

Example 4.1. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Compute $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathrm{x} \\ \mathrm{y} \end{bmatrix} = \begin{bmatrix} \mathrm{y} \\ \mathrm{x} \end{bmatrix}$. Hence, A defines the following transformation:

$$(x, y) \leftrightarrow (y, x).$$

Geometrically, this is the reflection along the line $x = y$.

Question 2. Is any linear transformation a rigid motion?

Observation. Any linear transformation is determined by where it sends the basis vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

In the first exercise set today we practice matrix multiplication and identify some linear transformations geometrically.

Theorem 4.2. *Compositions of linear transformations of the plane correspond to matrix multiplication.*

Recall that a counter-clockwise rotation around the origin by an angle θ is described by the matrix

$$\rho_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The theorem has the following amusing corollary

Example 4.3. $\rho_\theta \circ \rho_\phi = \rho_{\phi+\theta}$ by geometric considerations. On the other hand,

$$\rho_\theta \circ \rho_\phi = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} \cos\phi\cos\theta - \sin\phi\sin\theta & -(\cos\phi\sin\theta + \cos\theta\sin\phi) \\ \cos\phi\sin\theta + \cos\theta\sin\phi & \cos\phi\cos\theta - \sin\phi\sin\theta \end{pmatrix}$$

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[1] that is, commutes with addition and scalas multiplication of vectors

and

$$\rho_{\phi+\theta} = \begin{pmatrix} \cos(\phi+\theta) & -\sin(\phi+\theta) \\ \sin(\phi+\theta) & \cos(\phi+\theta) \end{pmatrix}$$

Since these two matrices are the same, we have proved the Laws of sines and cosines:

$$\cos(\phi+\theta) = \cos\phi\cos\theta - \sin\phi\sin\theta$$

$$\sin(\phi+\theta) = \cos\phi\sin\theta + \cos\theta\sin\phi$$

The last hour of Lecture III is taken by the Math Auction. The teams bid on groups of order 3,4,5,6,7,8,17.
Then we do subgroup lattice for D_4, time permitting.

Lecture IV.
Recall that we were trying to answer: "Which linear transformations are rigid motions"?

Definition 4.4. Let $A = \begin{bmatrix} \text{a} & \text{b} \\ \text{c} & \text{d} \end{bmatrix}$. Then the *transpose* of A, A^T is defined as

$$A^T = \begin{bmatrix} \text{a} & \text{c} \\ \text{b} & \text{d} \end{bmatrix}.$$

Question 3. Express $(AB)^T$ in terms of A^T and B^T.

Proposition 4.5. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *The following conditions are equivalent:*

(1) *The rows* (a, b) *and* (c, d) *are orthogonal vectors of length* 1.
(2) *The columns* (a, c) *and* (b, d) *are orthogonal vectors of length* 1.
(3) $a^2 + b^2 = c^2 + d^2 = 1$ *and* $ac + bd = 0$.
(4) $a^2 + c^2 = b^2 + d^2 = 1$ *and* $ab + cd = 0$.
(5) $AA^T = I$.

Definition 4.6. A matrix A satisfying the conditions of the Proposition is called an *orthogonal matrix.*

Lemma 4.7. *If* A *is orthogonal then* $\det A = \pm 1$.

Proof. $(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2$ or multiplicativity of the determinant. □

Example 4.8. Rotation matrices, reflection though the x-axis $\begin{bmatrix} 1 & 0 \\ 0 & \text{-}1 \end{bmatrix}$

Theorem 4.9. *A matrix* A *determines a rigid motion of the plane if and only if* A *is orthogonal.*

Proof. 1. Orthogonal matrices form a group under matrix multiplication. This is because $(AB)^T = B^T A^T$.
2. Any reflection is a composition of a rotation and the reflection through the x-axis. 3. If $\det A = 1$, then taking $a = \cos\phi$, we get $c, b = \pm\sin\phi$ and $d = \pm\cos\phi$ from the length one condition. If $d = -\cos\phi$ then b, c have the same sign and the determinant is negative. Contradiction. Hence, $d = \cos\phi$. Switching ϕ to $-\phi$ if necessary, we can have $b = -\sin\phi$ and $c = \sin\phi$. □

5. Finite groups of rigid motions

At this point we pause and discuss the long term goal - classifying the possible "groups of symmetries" for regular wallpaper patterns. Look at Escher prints. Decide if each one has finitely or infinitely many symmetries. Same question if we concetrate on symmetries given by orthogonal matrices (rotations/reflections, possibly around different centers/axes). This should motivate the following question: what are the FINITE subgroups of M?

Example 5.1. C_n, D_n.

Our nearest goal is to prove that these are the only finite subgroups of $\mathbb{M}$. And the goal of the next problem set is to prove the useful "Fixed Point theorem".

Problem set 6.

Homework problem: find an Escher print with the finite group of symmetries being D_n for some $n \geq 3$. Look at ghosts and fries - what are the finite groups of symmetry there?

Lecture V.

We start with the second problem of the Exercise set 6: "The Fixed Point Theorem":

Theorem 5.2. *Let G be a finite group of rigid motions of the plane (a finite subroup of $\mathbb{M}$. Prove that G has a fixed point (that is, there is a point P on the plane which is fixed by all elements of G).*

Theorem 5.3. *The only finite subgroups of M are cyclic or dihedral groups.*

Proof. Let G be a finite subgroup of $\mathbb{M}$. By the fixed point theorem, there is a point fixed by G. Assume this is the origin. Consider several cases.

Case 1 . All elements of G are rotations. Let θ be the minimal angle of rotation. Since the subgroup generated by ρ_ϕ is finite, the element ρ_ϕ has finite order n. That is $\rho_\phi^n = e$, and n is minimal such. Hence, $\phi = 2\pi/n$. Suppose G has a rotation that is not a power of ρ_ϕ. That is, by an angle α, $\alpha \neq m\phi$. Then there exists $m \in \mathbb{Z}$ such that $m\phi < \alpha < (m+1)\phi$. In that case $\psi = \alpha - m\phi < phi$ and $\rho_\psi = \rho_{\alpha - m\phi} = \rho_\alpha \circ ((\rho_\phi)^m)^{-1}$ but this contradicts the minimality of ϕ. We conclude that ϕ generates the entire group G, that is $G \simeq C_n$.

Case II . There is a reflection. WLOG, assume it is r, reflection through the x-axis. Let $H < G$ be the subgroup of ALL rotations of G. By Case I, $H \simeq C_n$ and is generated by some rotation ρ_ϕ. Consider the group generated by ρ_ϕ and r (which is isomorphic to D_n. We claim that it is the entire group G. Indeed, G cannot have any other rotations since they are ALL in H. Suppose G has a reflection r'. Then $rr' = \rho' \in H$. Hence, $r' = r\rho' \in \langle r, \rho_\phi \rangle$. Done.

□

6. 2-dimensional Crystallographic (wallpaper) groups

The group M has two important subgroups:

(1) An (orthogonal) subgroup $\mathbb{O} < M$ which consists of all motions that fix the origin (these are the ones given by the orthogonal matrices).
(2) A subgroup $T < M$ of all translations of the plane.

Question 4. Different name for T?
Answer: $T \simeq \mathbb{R}^2$ under vector addition.

$\mathbb{M} = \{t_{\vec{a}}\rho_\theta, t_{\vec{a}}\rho_\theta r\}$ (we already stated it is generated by rotations, reflection and translations. Adding relations, we can prove everything looks like this.

We have a group homomorphism

$$\phi : \mathbb{M} \to \mathbb{O}$$

which "drops" translations.

Definition 6.1. Let $f : H \to G$ be a group homomorphism. Then $\operatorname{Ker} f = \{h \in H \mid f(h) = e\}$

Exercise 1. $\operatorname{Ker} f$ is a (normal) subgroup of H, $\operatorname{Im} f$ is a subgroup of G.

Question 5. What is the kernel of $\phi : \mathbb{M} \to \mathbb{O}$?

Definition 6.2. A subgroup G of M is called *discrete* if it does not contain arbitrarily small rotations or translations.

Let G be a discrete subgroup of $\mathbb{M}$. We have

$$\phi \downarrow_G : G \to \mathbb{O}$$

(also, just drop translations from the symmetries in G. Or, identify elements which only differ by a translation).

Definition 6.3. Let G be a discrete group of motions of the plane. The translation subgroup of G, L_G, is the kernel of the map $\phi \downarrow_G : G \to \mathbb{O}$. This is a subgroup of all translations of G
The "point group of G, $\overline{G}$, it the image of $\phi \downarrow_G$. It is a subgroup of $\mathbb{O}$, that is, a group consisting of rotations and reflections around the origin.

Note that if G is discrete, then L_G and $\overline{G}$ are both discrete. Now we study some Escher prints and determine lattice subgroups and point groups. Also, discuss the homework problem which asked to find a print with the "finite group of symmetries" being non-cyclic.

Lecture VI: 17 Crystallographic groups.

Crystallography studies the arrangements of atoms in solids. Because these arrangements (crystals) are highly symmetric, the study of symmetries of space and plane was very relevant to crystallography; that's why the groups of symmetries of the "regular divisions of the plane" are called crystallographic groups.

Determine the point groups for the Escher prints the teams got last time. Also note that in each case the point group leaves the lattice of the print intact.

Lemma 6.4. *A discrete subgroup of M consisting of rotations around the origin is cyclic and is generated by some rotation ρ_θ.*

Proof. Draw a unit circle. For each $\rho_\phi \in \overline{G}$, mark all multiplies of ϕ on the unit circle. Now let ϵ be such that there are no angles less than ϵ in $\overline{G}$ (exists by the definition of a discrete group). Divide the circle into sectors of size $\leq \epsilon$. Each sector has no more than one point! Hence, there are finitely many points. In particular, there is a minimal angle of rotation, which will generate everything else. □

Proposition 6.5. *A discrete subgroup of* $\mathbb{O}$ *is a finite group.*

Proof. Exercise (just check the proof of the theorem from last lecture and do the same thing). □

Corollary 6.6. *Let* G *be a discrete group of rigid motions of the plane. The point group* $\overline{G}$ *of* G *is finite.*

Proof. The point group $\overline{G}$ is also discrete; and it fixes the origin. Hence, it is a discrete subgroup of $\mathbb{O}$. Now use the proposition. □

There are three possibilities for the translation subgroups of a discrete group of rigid motions of the plane: L is trivial (then G is a finite group of rigid motions), L is generated by just one translation (this leads to frieze patterns), and L is generated by two linearly independent translations. We shall prove this if we have time left today - most likely we won't, though.

Definition 6.7. A discrete group of rigid motions of the plane is called a 2-dimensional **crystallographic group** if the subgroup L of G is a lattice, i.e., L is generated by two linearly independent vectors $\vec{a}$, $\vec{b}$.

There is one-to-one correspondence:

$$\text{Wallpaper patterns} \longleftrightarrow \text{Crystallographic groups}$$

Theorem 6.8. *Let* G *be a (2-dimensional) crystallographic group. Then its point group* $\overline{G}$ *preserves its translation subgroup* L_G*.*

So, the point group is a subgroup of symmetries of a two dimensional lattice. We also now that since it is a finite subgroup of $\mathbb{O}$, it must be C_n or D_n (by the theorem we proved last time). We now figure out the possibilities for n.

Theorem 6.9. *(Crystallographic restriction) Let* $H < \mathbb{O}$ *be a finite subgroup of the group of symmetries of a lattice* L*. Then*
(a) Every rotation in H *has order* $1, 2, 3, 4,$ *or* 6.
(b) H *is one of the groups* C_n *or* D_n *for* $n = 1, 2, 3, 4,$ *or* 6.

If G is a crystallographic group, and L is the lattice of G, then $\overline{G} = G/L$ is the point group. The group $\overline{G}$ consists only of rotations and reflections, and is a finite subgroup of $\mathbb{O}$. It also carries the lattice L to itself. Hence, we have the following important corollary:

Corollary 6.10. *Let* G *be a 2-dimensional crystallographic group; that is,* G *is a group of symmetries of a wallpaper pattern. Then the choice for the point group of* G *(the group which "encodes" all rotations, reflections and glide reflections) is very limited: it is one of the eight (only!!) groups from the list in Theorem 6.9(b).*

This theorem is the content of the first problem set.

Theorem 6.11. *There are 17 non-isomorphic crystallographic groups (equivalently, 17 regular divisions of the plane with different groups of symmetries).*

The proof that there are actually 17 different crystallographic (wallpaper pattern) groups is more complicated - we have all the tools but just not the time to complete it. In the next exercise set we shall determine the point groups of all 17 wallpaper patterns.

Lecture VII: Penrose tilings.

We look at all 17 wallpaper groups - geometric representation, description, notation (crystallographic and orbifold). Handouts. Explanation of the crystallographic and orbifold notation.

Crystallographic notation.
Excerpt from [Sch]: The crystallographic notation consists of four symbols which identify the conventionally chosen "cell," the highest order of rotation, and other fundamental symmetries. Usually a "primitive cell" (a lattice unit) is chosen with centers of highest order of rotation at the vertices. In two cases a "centered cell" is chosen so that reflection axes will be normal to one or both sides of the cell. The "x-axis" of the cell is the left edge of the cell (the vector directed downward). The interpretation of the full international symbol (read left to right) is as follows: (1) letter p or c denotes primitive or centered cell; (2) integer n denotes highest order of rotation; (3) symbol denotes a symmetry axis normal to the x-axis: m (mirror) indicates a reflection axis, g indicates no reflection, but a glide-reflection axis, l indicates no symmetry axis; (4) symbol denotes a symmetry axis at angle alpha to x-axis, with alpha dependent on n, the highest order of rotation: alpha = 180 degree for n=1 or 2, alpha = 45 degree for n = 4, alpha = 60 degree for n = 3 or 6; the symbol m, g, l are interpreted as in (3). No symbol in the third and fourth position indicate that the group contains no reflections or glide-reflections... [The four alpha-numeric symbols can be shortened without loss of identification and the shortened form is most popular.]

Orbifold notation.
From wikipedia article: Orbifold notation for wallpaper groups, introduced by John Horton Conway ([Conway]), is based not on crystallography, but on topology. We fold the infinite periodic tiling of the plane into its essence, an orbifold, then describe that with a few symbols.

A digit, n, indicates a center of n-fold rotation corresponding to a cone point on the orbifold. By the crystallographic restriction theorem, n must be 2, 3, 4, or 6. An asterisk, *, indicates a mirror symmetry corresponding to a boundary of the orbifold. It interacts with the digits as follows: 1.Digits before * denote centers of pure rotation (cyclic). 2.Digits after * denote centers of rotation with mirrors through them, corresponding to "corners" on the boundary of the orbifold (dihedral). A cross, x, occurs when a glide reflection is present and indicates a crosscap on the orbifold. Pure mirrors combine with lattice translation to produce glides, but those are already accounted for so we do not notate them. The "no symmetry" symbol, o, stands alone, and indicates we have only lattice translations with no other symmetry. The orbifold with this symbol is a torus; in general the symbol o denotes a handle on the orbifold. Consider the group denoted in crystallographic notation by cmm; in Conway's notation, this will be 2*22. The 2 before the * says we have a 2-fold rotation centre with no mirror through it. The * itself says we have a mirror. The first 2 after the * says we have a 2-fold rotation centre on a mirror. The final 2 says we have an independent second 2-fold rotation centre on a mirror, one that is not a duplicate of the first one under symmetries.

The group denoted by pgg will be 22x. We have two pure 2-fold rotation centers, and a glide reflection axis. Contrast this with pmg, Conway 22*, where crystallographic notation mentions a glide, but one that is implicit in the other symmetries of the orbifold.

To finish our study of regular divisions of the plane and Escher's drawings, we'll have a team competition classifying Escher's drawings into one of the 17 types. The teams have 2 minutes per drawing which will be displayed on the screen. After two minutes, the answer should be written on a piece of paper and displayed. Correct answer gets 50 TAC bucks times the highest order of the rotational symmetry in the print; or times two if there is no rotational symmetry but a reflection symmetry.

7. Penrose Tilings (or "What about $2\pi/5$?")

Periodic tilings (tesselations) = tilings with translational symmetries. The entire tiling can be recovered by translating a finite piece of it.

Aperiodic tilings = tilings WITHOUT translational symmetries (cannot be reconstructed from a finite piece).
We can easily turn a periodic tiling into an aperiodic one. E.g., take a standard "brick" example of the cm wallpaper group and slice bricks in half in an irregular manner.

Definition 7.1. Aperiodic set of tiles is a finite set of tiles which can tile a plane ONLY aperiodically.

The phrase "aperiodic tiling" usually refers to a tiling by an aperiodic set if tiles.

Some history: in 1961 Hao Wang, a Harvard professor specializing in logic, formulated the following conjecture based on his studies of the Hilbert 18th problem (about groups of symmetries of a Euclidean n-space): is it true that if a set of dominoes with colored edges can tile the plane aperiodically then it can tile the plane periodically (that is, no aperiodic domino tiles). In fact, he actually used this fact in devising an algorithm which would allow to decide whether a given set of colored dominoes can tile a plane (he was a logician after all).

1966 P. Berger: example of **20426** aperiodic domino tiles.

1996 Karel Culik found a set of 13.
We look at the picture on wikipedia under "Wang's tiles".

Roger Penrose was studying the question of what is the minimal set of aperiodic tilings. He is actually a math physicist from Oxford, both he and his dad are responsible for the "Penrose staircase". In 1970 he suggested just 2 shapes (plus matching rules) which made a set of aperiodic tiles. These are the famous kite and dart; the tilings you obtain from them do not have translational symmetry but have a 5-fold rotational symmetry.

We now study "kites" and "darts". Look at the picture from Wolfram math. $\phi = \frac{1+\sqrt{5}}{2}$; the angles in the triangles making up kite and dart are all related to $\pi/5$. Note that the matching rules disallow forming a rombus which would have had the angles $2\pi/5$ and $3\pi/5$. Also, one can construct kite and dart with a compass and a ruler.

Our first exercise today is to make
a) a bigger kite from kites and darts
b) a bow-tie.

Now let's look at the patterns. And the next puzzle is to construct a batman (you can look at it first). The first team gets 100 TAC bucks. 5 minutes maximum.

Techniques for constructing tilings: deflation and inflation. Watch a movie. Some fun facts about Penrose tilings:

(1) Any finite piece of any Penrose tiling is present in any other Penrose tiling. So if you are inside a Penrose tiling, you can never figure out which one that is.
(2) The ratio of darts to kites "approaches" ϕ, the golden ratio (that is, if you take a big enough piece of the tiling, this is what the ratio should roughly be).
(3) Applications: study of quasi-crystals

Last activity - making a Penrose tile competition.

References

[Conway] J. Conway, *The Orbifold Notation for Surface Groups*. In: M. W. Liebeck and J. Saxl (eds.), Groups, Combinatorics and Geometry, Proceedings of the L.M.S. Durham Symposium, 1992

[Sch] Doris Schattschneider, *Visions of Symmetry*, Freeman, 1990