Recall that the group $M$ of all rigid motions is generated by the following three “families” of elements:

1. Translations $t_{\vec{a}}$ ($t_{\vec{a}}\vec{v} = \vec{v} + \vec{a}$)
2. Rotations around the origin counterclockwise $\rho_{\phi}$.
   Rotation preserves the origin, and can be described by a rotation matrix
   \[ \rho_{\phi} = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \]
3. And just ONE reflection $r$ - reflection through (around, over, under or without respect to) the $x$-axis. The reflection $r$ also fixes the origin and correspond to the matrix $r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Any orientation-reversing rigid motion of the plane can be obtained by subsequent compositions of a reflection $r$ followed by a rotation followed by a translation; for orientation-preserving motion skip the reflection.

**Definition.** Let $s_1, \ldots, s_n$ be $n$ points on the plane. The center of gravity is the point whose coordinates are the arithmetic means of the coordinates of $s_i$:
\[
p = \frac{s_1 + \ldots + s_n}{n}
\]

**Exercise.** Show that rigid motions preserve centers of gravity.

*Hint:* Since the group of all rigid motions is generated by translations, rotations and a reflection, it suffices to do the exercise for those three. So, here is a reformulation:

**Exercise 1.** Show that the following rigid motions preserve centers of gravity:
1. Rotation $\rho_{\phi}$.
2. Translation $t_{\vec{a}}$.
3. Reflection through the $x$-axis $r$.

Recall that we denote by $M$ the group of all rigid motions of the plane. An (orthogonal) subgroup $\mathcal{O} < M$ is the subgroup of all motions which fix the origin. A subgroup $T < M$ is the subgroup of all translations of the plane.

**Definition.** A subgroup of $M$ is called *discrete* if it does not contain arbitrarily small rotations or translations.

**Exercise 2.** Show that a discrete subgroup of $\mathcal{O}$ is a finite group.

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