Math 508 – Lie Algebras
(lecture notes)

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Contents

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1 Lecture 1 (January 4): Basic definitions and Examples
Scribe: Raymond Guo

We work over a field k. Often, we'll restrict to fields with characteristic that is not 2 (it is often safer to assume this is the case).

1.1 Definition (Lie Algebra)**.** Let $\mathfrak{g}$ be a vector space over k, where the vector spaces are supplied with a bilinear form $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, such that

1. $[x, x] = 0$.
2. The Jacobi identity: $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$.

A vector space $\mathfrak{g}$ endowed with such an operator is a Lie algebra.

By condition 1, we have $0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x] \implies [x, y] = -[y, x]$. If $\mathrm{char}(k) \neq 2$, the condition $[x, y] = -[y, x]$ is equivalent to condition 1 above.

Recall that a k-linear map $D : A \to A$ is a derivation if $D(ab) = D(a)b + aD(b)$. Then the Jacobi identity is equivalent to the condition that $[c, \cdot] : \mathfrak{g} \to \mathfrak{g}$ is a derivation in this sense, for any element $c \in \mathfrak{g}$.

1.2 Definition (Lie Algebra Homomorphism)**.** A map $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a Lie algebra homomorphism if

1. f is k-linear.
2. $f([a, b]) = [f(a), f(b)]$.

Monomorphisms, epimorphisms, and isomorphisms are defined in the usual manner (in particular, isomorphisms have an inverse that is also a map of Lie algebras).

1.3 Example. Let A be any associative algebra with 1. Define $[\cdot, \cdot] : A \times A \to A$ by $[a, b] = ab - ba$. This makes A into a Lie algebra. For any associative algebra A, we denote this lie algebra structure by A^{lie}. We often work in the case where $A = M_n(k)$ (the set of all $n \times n$ matrices over k). In this case, we write $A^{\mathrm{lie}} = \mathfrak{gl}_n(k)$.

Again pick V a vector space over k. Take the algebra $\mathrm{End}_k(V)$ and define the bracket $[\phi, \psi] = \phi \circ \psi - \psi \circ \phi$. The corresponding Lie algebra is denoted $\mathfrak{gl}(V)$, and is the same as the previous example except that we don't make a choice of basis.

1.4 Example. Let $(\mathbb{R}^3, \times)$ be the 3-dimensional Euclidean space with the bracket $[u, v] = u \times v$ (the cross product). This can directly be checked to be a Lie algebra.

1.5 Definition (Lie subalgebra)**.** Define a subset $\mathscr{H} \subset \mathfrak{g}$ to be a Lie subalgebra if it's a subspace closed under the bracket.

1.6 Example. Consider $\mathbb{H}$, the quaternions, as an associative algebra over the reals (with the standard basis $\{1,i,j,k\}$). Consider $\mathbb{H}^{\mathrm{lie}}$, as in Example 1.3. Taking the subspace spanned by i, j, and k gives a subalgebra that can be identified with $(\mathbb{R}^3, \times)$.

1.7 Definition (Lie Ideal)**.** $I \subset \mathfrak{g}$ is a Lie ideal if for all $x \in \mathfrak{g}$ and $a \in I$, $[x, a] \in I$. Claiming instead that $[a, x] \in I$ yields the same definition.

1.8 Definition (Center)**.** The center of a Lie algebra $\mathfrak{g}$ is $\{x \in \mathfrak{g} : \forall a,\ [x, a] = 0\}$.

1.9 Definition (Adjoint Homomorphism)**.** For $\mathfrak{g}$ a lie algebra, we define the adjoint homomorphism $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ by $\mathrm{ad}_x(y) = [x, y]$.

1.10 Exercise. Show that $\mathrm{ad}_{[x,y]}(z) = (\mathrm{ad}_x \circ \mathrm{ad}_y - \mathrm{ad}_y \circ \mathrm{ad}_x)(z)$. This shows that ad is a Lie algebra homomorphism.

1.11 Remark. Note that we have $\ker \mathrm{ad} = \mathbb{Z}(g)$, directly from the definitions.

1.12 Example. $\mathfrak{sl}_n(k) := \{x \in \mathfrak{gl}_n(k) : \mathrm{Trace}(x) = 0\}$. This is a Lie subalgebra because $\mathrm{Trace}([x, y]) = 0$ for any $x, y \in \mathfrak{gl}_n(k)$. In fact, it can be shown conversely that if $\mathrm{Trace}(x) = 0$, we can write $x = [y, z]$ for $y, z \in \mathfrak{g}$.

1.13 Exercise. Pick an element $S \in \mathfrak{gl}_n$. Let $\mathfrak{gl}_n^S := \{X \in \mathfrak{gl}_n : X^T S = -SX\}$.

1. Show that $\mathfrak{gl}_n^S$ is a lie subalgebra of $\mathfrak{gl}_n$.
2. Find $S \in \mathfrak{gl}_3(\mathbb{R})$ such that $(\mathbb{R}^3, \times) \cong \mathfrak{gl}_3^S(\mathbb{R})$.

1.14 Remark (Alternative description)**.** Let $B : V \times V \to k$ be a bilinear form. Let $\dim(V) = n$, with $\mathfrak{gl}_n = \mathrm{End}_k(V) = \mathfrak{gl}(V)$. Consider the Lie subalgebra $\{X \in \mathfrak{gl}(V) : B(X(v), w) = -B(v, X(w))\}$. This is the same as Exercise 1.13. Once we choose a basis $\{e_i\}_{i=1}^n$, there exists an $n \times n$ matrix S such that $B(v, w) = v^T S w$. Here $S_{i,j} = B(e_i, e_j)$.

2 Lecture 2 (January 6): More examples, Lie algebra of a Lie group

Scribe: Haoming Ning

2.1 Example (Symplectic Lie Algebras)**.** Let $n = 2l$ and let

$$S = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}.$$

Denote $\mathfrak{sp}_{2l} := \mathfrak{gl}_{2l}^S = \{X \in \mathfrak{gl}_{2l} : X^T S = -SX\}$. The lie algebra $\mathfrak{sp}_{2l}$ is called the **symplectic Lie algebra**.

As a subexample, take $n = 2$ and V a 2-dimensional vector space. The bilinear form $B : V \times V \to k$ where $B(v, w) = v^T S w = v_1 w_2 - v_2 w_1$ corresponds to the matrix S.

2.2 Exercise. Check that $x \in \mathfrak{sp}_{2n}$ if and only if x is of the form

$$x = \begin{pmatrix} a & b \\ c & -a^T \end{pmatrix},$$

where $a, b, c \in \mathfrak{gl}_l$, and b, c are symmetric.

2.3 Example (Orthogonal Matrices)**.** Let $S = I$, then $\mathfrak{gl}_n^S = \{X \in \mathfrak{gl}_n : X^T = -X\}$. This is called the orthogonal Lie algebra, and denoted $\mathfrak{so}_n = \mathfrak{gl}_n^S$.

2.4 Exercise. Consider two cases for the orthogonal Lie algebra $\mathfrak{so}_n$.

1. When $n = 2l + 1$ is odd, let

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{pmatrix},$$

   Check that $\mathfrak{so}_{2l+1} = \mathfrak{gl}_{2l+1}^S$.

2. When $n = 2l$ is even, let

$$S = \begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix},$$

   then one can check that $\mathfrak{so}_{2l} = \mathfrak{gl}_{2l}^S$.

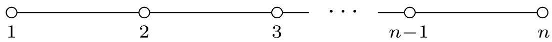
2.5 Remark. Exercise 2.4 shows that S is not uniquely determined by $\mathfrak{gl}^S$.

2.6 Notation. We use the following notation for families of simple Lie algebras over $\mathbf{C}$.

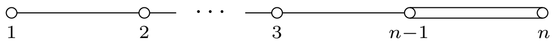
$$\begin{array}{rl} \mathfrak{sl}_{n+1} & A_n \\ \mathfrak{so}_{2l+1} & B_l \\ \mathfrak{sp}_{2l} & C_l \\ \mathfrak{so}_{2l} & D_l. \end{array}$$

There are other simple Lie algebras denoted E_6, E_7, E_8, F_4, G_2. These correspond to the Dynkin diagrams, drawn below.

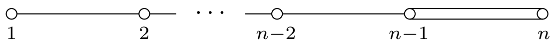
A_n $(n \geq 1)$:



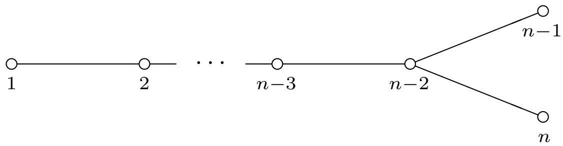
B_n $(n \geq 2)$:



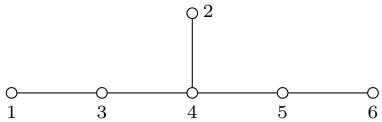
C_n $(n \geq 3)$:



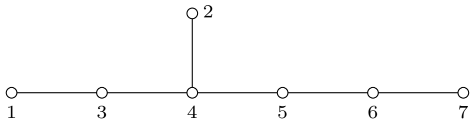
D_n $(n \geq 4)$:



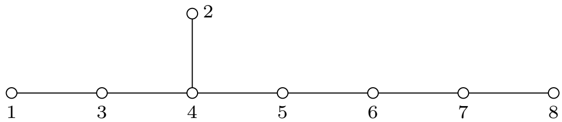
E_6:



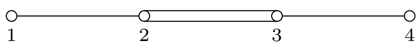
E_7:



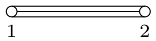
E_8:



F_4:



G_2:



2.7 Remark. Lie algebras are geometric objects. However, We will study Lie algebras in this course from the algebraic perspective.

2.8 Definition. Let A be an algebra over k. A map $D : A \to A$ is a derivation if D is k-linear and $D(ab) = D(a)b + aD(b)$. By definition, $\mathrm{Der}_k(A) \subseteq \mathrm{End}_k(A)$.

2.9 Exercise. Let $D_1, D_2 \in \mathrm{Der}_k(A)$, then $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1 \in \mathrm{Der}_k(A)$. Therefore, $\mathrm{Der}_k(A)$ is a Lie algebra.

2.10 Example. Let $A = k[x_1, \ldots, x_n]$, consider the formal differentiation map $\partial/\partial x_i : A \to A$.

Claim: Any $D \in \mathrm{Der}_k(A)$ is of the form $D = \sum_{i=1}^n f_i(x_1, \ldots, x_n)\partial/\partial x_i$ for $f_i \in k[x_1, \ldots, x_n]$. Therefore $\mathrm{Der}_k(A)$ is a free A-module of rank n. (Note that this is related to the concept of D-modules).

2.11 Remark (Geometric interpretation of derivations)**.** Let M be a smooth manifold, $A = C^\infty(M)$. Then $\mathrm{Der}\, C^\infty(M)$ is in one-to-one correspondence with smooth vector fields $\mathfrak{X}(M)$ on M, given by sending a vector field Y to the derivation $f \mapsto Yf$.

2.12 Example. Let G be a Lie group, $A = C^\infty(G)$ and consider $\mathrm{Der}(A)$. The multiplication map $G \times G \to G$ induces an action of G on A, given by $G \times A \to A$ by $(gf)(-) = f(g^{-1}-)$. We denote this $g \cdot f$.

2.13 Definition. Let G be a Lie group, $A = C^\infty(G)$, $D \in \mathrm{Der}\, A$. D is left invariant if $D(g \cdot f) = g \cdot D(f)$ for every $f \in C^\infty(G)$.

2.14 Exercise. If D_1, D_2 are left invariant, then $[D_1, D_2]$ is too. So that the left-invariant derivations form a Lie sub-algebra of $\mathrm{Der}(A)$.

2.15 Definition. For a Lie group G, we define $\mathrm{Lie}\, G$ to be the Lie algebra of left invariant derivations.

3 Lecture 3 (January 9): Representations of Lie Algebras
Scribe: Bashir abdel-Fattah

3.1 Example. If G is an algebraic group over k (that is, a group object in the category Var_k of varieties over k, or equivalently a variety with a group structure such that multiplication and inversion are morphisms of varieties) with the algebra of regular functions $\mathscr{O}(G) = k[G]$, then we can define

$$\operatorname{Lie} G := \text{ left-invariant derivations of } \mathscr{O}(G).$$

Then we can identify

$$\operatorname{Lie} G \cong T_e G = (\mathfrak{m}_e/\mathfrak{m}_e^2)^*,$$

where $\mathfrak{m}_e \lhd \mathscr{O}_e$ is the maximal ideal consisting of germs of regular functions vanishing at e.

3.2 Example. Given the exact sequence

$$\mathrm{SL}_n(\mathbb{R}) \hookrightarrow \mathrm{GL}_n(\mathbb{R}) \xrightarrow{\det} \mathbb{R}^\times,$$

by differentiating we induce the maps

$$\mathfrak{sl}_n(\mathbb{R}) = T_I\,\mathrm{SL}_n(\mathbb{R}) \hookrightarrow \mathfrak{gl}_n(\mathbb{R}) = T_I\,\mathrm{GL}_n(\mathbb{R}) \xrightarrow{d(\det)_I} T_1\,\mathbb{R}^\times \cong \mathbb{R}\,.$$

In fact, $d(\det)_I$ is just the trace operator, because if we take any $A \in \mathfrak{gl}_n(\mathbb{R})$ and consider the path $I + tA$ in $\mathrm{GL}_n(\mathbb{R})$ (for some sufficiently small interval $(-\epsilon, \epsilon)$) passing through I at time $t = 0$ with velocity A, we see that

$$\det(I + tA) = 1 + t \cdot \operatorname{Trace} A + O(t^2),$$

so by taking the derivative at time $t = 0$ we have that

$$d(\det)_I(A) = \frac{d}{dt}\Big|_{t=0} \det(I + tA) = \operatorname{Trace} A.$$

3.3 Exercise. Consider the unit quaternion group

$$\mathbb{S}^3 = \{q \in : \|q\| = 1\}$$

Then $\operatorname{Lie} \mathbb{S}^3 \cong (\mathbb{R}^3, \times)$.

3.4 Definition (Lie Algebra Representation)**.** Given a Lie algebra $\mathfrak{g}$ over k, a representation of $\mathfrak{g}$ is a vector space $V \in \mathrm{Vect}_k$ together with a Lie algebra homomorphism $\rho_V : \mathfrak{g} \to \mathfrak{gl}(V)$. Alternatively, this is equivalent to specifying an action $G \times V \to V$ that is k-bilinear and satisfies $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$ for all $x, y \in \mathfrak{g}$ and $v \in V$.

3.5 Definition (Morphism of Lie Algebra Representations)**.** A morphism in the category $\mathrm{Rep}_k\,\mathfrak{g}$ of representations of $\mathfrak{g}$ is a linear map $\varphi : V_1 \to V_2$ such that

$$\varphi(x \cdot v) = x \cdot \varphi(v)$$

for all $x \in \mathcal{G}$ and $v \in V_1$.

3.6 Example. If $\mathfrak{g} = \mathfrak{gl}(V)$, then the standard action of $\mathfrak{gl}(V)$ on V (that is, the map $\mathfrak{gl}(V) \times V \to V$ given by $(T, v) \mapsto T(v)$ for $T \in \mathfrak{gl}(V) = \mathrm{End}_{\mathrm{Vect}_k}(V)$ and $v \in V$) induces the ***standard representation***

$$\rho_{st} : \mathfrak{gl}(V) \xrightarrow{\mathrm{id}} \mathfrak{gl}(V).$$

3.7 Example. The ***adjoint representation*** of a Lie algebra $\mathfrak{g}$ is the map

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

sending $x \in \mathfrak{g}$ to the function $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$ given by $\mathrm{ad}_x(y) = [x, y]$. Alternatively, this representation is described by the action $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ given by $(x, y) \mapsto [x, y]$.

If G is a Lie group and $\mathfrak{g} = \mathrm{Lie}\, G$, then the action of G on itself by conjugation gives a map $\Psi : G \to \mathrm{Aut}(G)$ (that is, $\Psi(g)$ is the map $x \mapsto gxg^{-1}$ for any $g \in G$). Note that given any Lie group automorphism $\psi : G \to G$, taking the differential at the origin induces an automorphism $d\psi_e : T_eG \to T_eG$ and thus determines an automorphism $\mathfrak{g} \to \mathfrak{g}$ via the canonical identification $\mathfrak{g} \cong T_eG$. Then composing $G \xrightarrow{\Psi} \mathrm{Aut}(G) \to \mathrm{Aut}(\mathfrak{g}) = \mathrm{GL}(\mathfrak{g})$ gives us a group representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$. If G is a sub-Lie group of $\mathrm{GL}_n(\mathbb{R})$, then the elements of G and $\mathfrak{g}$ are all matrices, and the adjoint group representation is given by the action of G on $\mathfrak{g}$ by conjugation (that is, given $g \in G$, the map $\mathrm{Ad}(G) : \mathfrak{g} \to \mathfrak{g}$ is given by $X \mapsto gXg^{-1}$). Differentiating the map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ again gives us the adjoint Lie algebra representation

$$\mathrm{ad} = d(\mathrm{Ad})_e : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}).$$

3.8 Example. If $\mathscr{A} = C^\infty(M)$, then the action $\mathrm{Der}_k(\mathscr{A}) \times \mathscr{A} \to \mathscr{A}$ given by $(D, a) \mapsto Da$ determines an (almost always) infinite-dimensional representation of $\mathrm{Der}_k(\mathscr{A})$.

In $\mathrm{Rep}_k\, \mathfrak{g}$, we have the following operations:

1. Direct sums $V_1 \oplus V_2$ (with the action $x \cdot (v_1 \oplus v_2) = (xv_1) \oplus (xv_2)$ for $x \in \mathfrak{g}$, $v_1 \in V_1$, and $v_2 \in V_2$)
2. Subrepresentations $V' \subset V$ (where V' is a subspace of V that is closed under the action of $\mathfrak{g}$ on V).
3. Tensor products $V_1 \otimes V_2$ (with the action given on pure tensors by $x \cdot (v_1 \otimes v_2) = (xv_1) \otimes v_2 + v_1 \otimes (xv_2)$, and extended to mixed tensors by linearity).

3.9 Exercise. Prove that the tensor product construction above does indeed determine a representation of $\mathfrak{g}$.

$\mathrm{Rep}_k\, \mathfrak{g}$ is an abelian tensor/monoidal category. The identity object with respect to the tensor product is the trivial representation $\mathfrak{g} \times k \to k$ that is identically equal to zero.

3.10 Definition. A representation V is ***irreducible/simple*** if it doesn't have any nontrivial subrepresentations (that is, no nonzero proper subrepresentations).

3.11 Definition. A representation V is ***faithful*** if the map $\rho_V : \mathfrak{g} \to \mathfrak{gl}(V)$ is injective (equiv. if $x \cdot V = 0$ implies $x = 0$).

3.12 Definition. A representation V is ***indecomposable*** if there are not any nontrivial subrepresentations $V_1, V_2 \subset V$ such that $V = V_1 \oplus V_2$.

3.13 Example. Ways of constructing new representations from old ones:

1. The tensor power representation $T^n(V) = \bigotimes_1^n V = V^{\otimes n}$ (that is, inductively applying the binary tensor product of representations defined previously).

2. The symmetric group Σ_n on n elements acts on $V^{\otimes n}$ by permuting the factors (i.e., $\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)})$. This action commutes with the action of $\mathfrak{g}$ on $V^{\otimes n}$, thus we can define the symmetric power representation

$$\begin{aligned} S^n V &:= V^{\otimes n}/\Sigma_n \\ &:= V^{\otimes n}/\langle \sigma \cdot (v_1 \otimes \cdots \otimes v_n) - v_1 \otimes \cdots \otimes v_n : \sigma \in \Sigma_n, v_1 \otimes \cdots \otimes v_n \in V^{\otimes n}\rangle. \end{aligned}$$

   We denote the equivalence class of $v_1 \otimes \cdots \otimes v_n$ in $S^n V$ by $v_1 \cdot v_2 \cdot \cdots \cdot v_n$.

3. We define $\Gamma^n(V) := (V^{\otimes n})^{\Sigma_n} = \{t \in V^{\otimes n} : \sigma \cdot t = t \text{ for all } \sigma \in \Sigma_n\}$.

3.14 Exercise. If $\operatorname{char} k = 0$, then there is an isomorphism

$$\mathrm{sym} : S^n(V) \to \Gamma^n(V)$$

4 Lecture 4 (January 11): Examples of representations of $\mathfrak{gl}(V)$ *Scribe: Justin Bloom*

4.1 Notation. The following notations will be used throughout:

1. $S^n(V) = V^{\otimes n}/\langle \sigma(v_1 \otimes \cdots \otimes v_n) - v_1 \otimes \cdots \otimes v_n \rangle_\sigma \in \Sigma_n$, called 'coinvariants' of action by Σ_n, denoted also $(V^{\otimes n})_{\Sigma_n}$
2. $\Gamma^n(V) = (V^{\otimes n})^{\Sigma_n} \subset V^{\otimes n}$, the elements fixed by Σ_n, such as
$$v \otimes \cdots \otimes v,$$
and
$$v_1 \otimes v_2 + v_2 \otimes v_1$$
for $n = 2$, called 'invariants'.
3. $\bigwedge^n(V) = V^{\otimes n}/\langle \sigma(v_1 \otimes \cdots \otimes v_n) + (-1)^{|\sigma|} v_1 \otimes \cdots \otimes v_n \rangle_{\sigma \in \Sigma_n}$.

We also have representations built from these, each with a graded algebra structure concatenating tensors together:

1. $S^*(V) = \bigoplus_{n \geq 0} S^n(V)$
2. $\Gamma^*(V) = \bigoplus_{n \geq 0} \Gamma^n(V)$
3. $\bigwedge^*(V) = \bigoplus_{n \geq 0} \bigwedge^n(V)$.

The first two are infinite dimensional, but for the last, we have $\bigwedge^m(V) = 0 \quad \text{for } m > \dim(V)$ if $\operatorname{char} k \neq 2$.

4.2 Remark. Assume $\operatorname{char} k = 0$. We have a map

$$\begin{aligned} \mathrm{Sym} : S^n(V) &\to \Gamma^n(V) \\ v_1 \dots v_n &\mapsto \sum_{\sigma \in \Sigma_n} \sigma(v_1 \otimes v_n). \end{aligned}$$

4.3 Exercise. Show the following:

1. Sym respects action of $\mathfrak{gl}(V)$.
2. Sym is an isomorphism.

4.4 Remark. Assume $\operatorname{char} k = p > 0$, $n = p$. Then we have an exact sequence

$$V^{(1)} \to S^p(V) \xrightarrow{\mathrm{Sym}} \Gamma^p(V) \to V^{(1)},$$

where $V^{(1)}$ is V with a 'Frobenius twist'.

Notice $v^p \mapsto \sum_{\sigma \in \Sigma_p} \sigma(v^{\otimes p}) = 0$, so $V^{(1)} \hookrightarrow \langle v^p \rangle \subset S^p(V)$.

4.5 Example. Let $V = \bigoplus_{i=1}^n kx_i$, i.e. $\{x_1, \ldots, x_n\}$is a basis. Take $\mathscr{A} = k[x_1, \ldots, x_n] \cong S^*(V)$.

We have

$$S^d = k[x_1, \ldots, x_n]_{(d)},$$

the degree d monomials.

We have an action

$$\begin{aligned} \mathfrak{gl}_n &\hookrightarrow \mathrm{Der}_k(\mathscr{A}) \\ (a_{ij}) &\mapsto \sum_{1 \leq i,j \leq n} a_{ij} x_i \frac{\partial}{\partial x_j}. \end{aligned}$$

4.6 Exercise. Check that this defines the same action on $S^*(V)$ as the one in Example 3.13., i.e. the one induced by the action on the tensor representation $T^n(V)$ for each n.

Repesentations of $\mathfrak{sl}_2$:

Recall that

$$\mathfrak{sl}_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a + d = 0 \right\}.$$

So $\mathfrak{sl}_2$ is has dimension 3, with standard basis (check)

$$\begin{aligned} e &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ f &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ h &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \end{aligned}$$

and relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

4.7 Remark. An object $V \in \mathrm{Rep}\,\mathfrak{sl}_2$ is the data of

1. A vector space over k,
2. 3 linear operators satisfying the above bracket relations.

For a representation V, h is diagonalizable as an operator on V, since h is a semi-simple element of $\mathfrak{sl}_2$.

4.8 Example. V $= kx + ky = S^1(V)$, the standard representation of 2. Then

$$h : x \mapsto x; y \mapsto -y,$$

so $kx = V_1$ and $ky = V_{-1}$, the eigenspaces corresponding to values $1, -1$. We also have

$$e : x \mapsto 0; y \mapsto x,$$

$$f : x \mapsto y; y \mapsto x$$

We have

$$S^2(V) = k[x,y]_{(2)} = kx^2 + kxy + ky^2,$$

and an action by derivation:

$$\begin{aligned} h(x^2) &= 2xh(x) = 2x^2, \\ h(xy) &= yh(x) + xh(y) = 0, \\ h(y^2) &= 2yh(y) = -2y^2, \\ e(x^2) &= 0, \\ e(xy) &= x^2, \\ e(y^2) &= 2yx, \\ f(x^2) &= 2xy, \\ f(xy) &= y^2, \\ f(y^2) &= 0. \end{aligned}$$

We may continue on $S^3(V)$:

$$\begin{aligned} h(x^3) &= 3x^2h(x) = 3x^3, \\ h(x^2y) &= x^2h(y) + yh(x^2) = x^2y, \\ h(xy^2) &= xh(y^2) + y^2h(x) = -xy^2, \\ h(y^3) &= 3y^2h(y) = -3y^3. \\ &\vdots \end{aligned}$$

Notice on each of the $S^i(V)$ the monomials are a basis of eigenvectors of h.

4.9 Definition. For $\lambda \in k, V \in \operatorname{Rep}\mathfrak{sl}_2$, if λ is an eigenvalue of h acting on V, λ is called a *weight* of V. The eigenspace V_λ of the value λ is called the λ-weightspace of V.

4.10 Lemma. *If λ is weight of $V \in \operatorname{Rep}\mathfrak{sl}_2$, then*

$$\begin{aligned} e &: V_\lambda \to V_{\lambda+2}, \\ f &: V_\lambda \to V_{\lambda-2}, \\ h &: V_\lambda \to V_\lambda. \end{aligned}$$

In particular, if λ is the largest weight of V, $e(V_\lambda) = 0$, and if λ is the smallest weight of V, $f(V_\lambda) = 0$.

5 Lecture 5 (January 13): Classification of irreducible representations of $\mathfrak{sl}_2$

Scribe: William Dudarov

Let $V \in \operatorname{Rep} \mathfrak{sl}_2$. We first prove our lemma from last class.

5.1 Lemma. *If λ is a weight of V, then*

$$\begin{aligned} e &: V_\lambda \mapsto V_{\lambda+2} \\ f &: V_\lambda \mapsto V_{\lambda-2} \\ h &: V_\lambda \mapsto V_\lambda. \end{aligned}$$

Proof. Let $v \in V_\lambda$. Note that

$$\begin{aligned} h(ev) &= ehv + 2ev \\ &= e(\lambda v) + 2ev \\ &= (\lambda + 2)ev. \end{aligned}$$

We have the same calculation for f. ■

5.2 Proposition. *Let $k = \overline{k}$, and let V be a finite-dimensional irreducible representation of $\mathfrak{sl}_2$. Then*

1. $V \cong \bigoplus_{\lambda \in k} V_\lambda$,
2. $e, f : V_\lambda \to V_{\lambda \pm 2}$.

Proof. We only need to prove 1. Let λ be an eigenvalue of h as an operator on V, i.e. let λ be a weight of V. Then $V_\lambda \neq 0$. Consider $\bigoplus_{\mu \in \{\lambda + 2\mathbb{Z}\}} V_\mu \subseteq V$. By lemma 5.1 above, this is an $\mathfrak{sl}_2$-invariant subspace in V. Since V is irreducible,

$$V = \bigoplus_{\mu \in \{\lambda + 2\mathbb{Z}\}} V_\mu.$$

■

What can λ be?

5.3 Example. Consider $V = k[x, y]_{(d)}$ with basis $\{x^d, x^{d-1}y, \ldots, y^d\}$ and dimension $d + 1$.

$$x^d \underset{e(1)}{\overset{f(d)}{\rightleftarrows}} x^{d-1}y \underset{e(2)}{\overset{f(d-1)}{\rightleftarrows}} \cdots \underset{e(d-1)}{\overset{f(2)}{\rightleftarrows}} xy^{d-1} \underset{e(d)}{\overset{f(1)}{\rightleftarrows}} y^d.$$

$$* \begin{cases} [h, e] &= 2e \\ [h, f] &= -2f \\ [e, f] &= h \end{cases}$$

$$\#\begin{cases} ev_i & = (d-i)v_{i+1} \\ fv_i & = iv_{i-1} \\ hv_i & = (2i-d)v_i \\ \mathcal{V}(d) & = \{v_0, \dots, v_d\} \end{cases}$$

$F = 1a$ (ex*)

$$\begin{aligned} v_d &= x^d \\ v_{d-1} &= x^{d-1}y \\ &\vdots \\ v_0 &= y^d \end{aligned}$$

5.4 Definition. $\mathcal{V}(d) \cong k[x,y]_{(d)}$ (as in #) is the **highest weight module of weight** d.

5.5 Proposition. *If k has characteristic 0, then $\mathcal{V}(d)$ is irreducible.*

Proof. Towards contradiction, suppose not. Then there is some $W \subsetneq \mathcal{V}(d)$ and weight vector $w \in W$ with $w = av_i$, $i = 0, \dots, d$. But, using the relations (#), we get the entire highest weight module $\mathcal{V}(d)$ of weight d by applying e and f. ∎

5.6 Remark. Let the characteristic of k be p. Let's consider $V(p)$.

$$x^p \xrightarrow{f(p)} x^{p-1}y \to \cdots \to y^p.$$

$$fx^p = 0$$

$$ex^p = 0,$$

and this implies that $kx^p \subseteq V(p)$ is a sub-representation and so is ky^p. This means that $V(p)$ is not irreducible, but it is indecomposable.

5.7 Theorem (Classification of irredudible representations of $\mathfrak{sl}_2$.)**.** *Let k be algebraically closed with characteristic 0.*

1. *Any finite-dimensional irreducible representation of $\mathfrak{sl}_2$ is isomorphic to $\mathcal{V}(d)$.*
2. $\dim(V_\lambda) \leq 1$ *for all* λ.

Proof. Let λ be a weigth of V. All of the weights of V sit in an arithmetic progression $\{\lambda + 2\mathbb{Z}\}$ by lemma 5.1 above. The dimension of V is finite, and there exist $\lambda_{\max}$ and $\lambda_{\min}$ weights, with $\lambda_{\max} - \lambda_{\min} = 2N$ for $N \in \mathbb{N}_0$. Let v^+ be the weight vector for $\lambda_{\max}$. Our claim then is that

$$V = \bigoplus_{i \geq 0} kf^i v.$$

We observe that $e(v) = 0$, $e : V_{\lambda_{\max}} \to V_{\lambda_{\max}+2}$. We now prove the claim - we need to show that $\bigoplus_{i\geq 0} kf^i v$ is invariant under h, f, e.

1. By construction, it's invariant under f.
2. $h(f^i v) = (\lambda_{\max} - 2i) f^i v$.
3. (Ex*) $e(f^i v) = i(\lambda_{\max} - (i-1)) f^{i-1} v$.

This finished the proof of the claim. Observe that we showed that $\dim(V_{\lambda_{\max}-2i} \leq 1$. What can $\lambda_{\max}$ be ? Let $\dim(V) = d+1$. Then we must have

$$f^{d+1} v = 0$$

(and $f^d v \neq$). But then $e f^{d+1} v = 0$, implying that $(d+1)(\lambda_{\max} - d) f^d v = 0$, so $\lambda_{\max} = d = 0$. We claim that from here, we get

$$V \cong V(d)$$

$$v^+ \leftarrow v_d \cong x^d$$

$$f^i v^+ \leftarrow \frac{i!}{d!} v_{d-i}.$$

■

Some homework problems:

5.8 Exercise. 1. Extend the proof of Theorem 5.7 to any k.

2. For char p, the complete list of the irreducible modules is $V(0), \ldots, V(p-1)$.

6 Lecture 6 (January 18): BGG resolution for $\mathfrak{sl}_2$, Weyl character formula
Scribe: Soham Ghosh

Last time:(Representations of $\mathfrak{sl}_2/\mathbb{C}$) Recall that isomorphism classes of irreducible finite dimensional representations of $\mathfrak{sl}_2\}$ are in one-one correspondence with the representations $V(d) \cong k[x,y]_{(d)}$, such that $\dim V(d) = d+1$, for all $d \in \mathbb{Z}$.

6.1 Exercise (Homework problems (contd.))**.** $\mathrm{ad}_{\mathfrak{sl}_2} : \mathfrak{sl}_2 \to \mathfrak{gl}(\mathfrak{sl}_2) \cong \mathfrak{gl}_3$ be the adjoint representation. Calculate $e, f, h \mapsto ?$ as matrices.

Verma modules and Weyl character formulas (for $\mathfrak{sl}_2$)

6.2 Notation. $\lambda \in k$; $M(\lambda)-$ highest weight module of weight λ (Verma module)

$M(\lambda) = \bigoplus_{i \geq 0} k f^i v^+$ along with $\mathfrak{sl}_2$-action:

1. $f(f^i v^+) = f^{i+1} v^+$
2. $hv^+ = \lambda v^+$, $h(f^i v^+) = (\lambda - 2i) f^i v^+$
3. $e(f^i v^+) = i(\lambda - i + 1) f^{i-1} v^+$ (in particular, $ev^+ = 0$)

$M(\lambda)$ is often seen as:

$$\begin{array}{c} v^+ \\ | \\ fv^+ \\ | \\ f^2 v^+ \\ | \\ \vdots \end{array}$$

6.3 Exercise. Show that $M(\lambda)$ is an $\mathfrak{sl}_2$-representation.

6.4 Proposition. *$M(\lambda)$ satisfies the universal property.*

1. $hv^+ = \lambda v^+$*;* $ev^+ = 0$*.*
2. *For all* $(W, w^+) \in \mathrm{Rep}\,\mathfrak{sl}_2$ *satisfying* (i)*, there exists unique* $\phi : M(\lambda) \to W$ *sending* $v^+ \mapsto w^+$*.*

6.5 Exercise. Prove Proposition 6.4.

Let $\lambda = d \in \mathbb{Z}$. By the universal property $M(d) \twoheadrightarrow V(d)$ mapping $v^+ \mapsto v^+$, and thus $f^i v^+ \mapsto f^i v^+$ for all $0 \leq i \leq d$. Since $V(d)$ is finite dimensional, the higher iterations $f^i v^+$ for $i \geq d+1$ map to 0 in $V(d)$.

$$\begin{tikzcd}
\ker(\lambda) \arrow[r, hook] & M(\lambda) \arrow[r, two heads] & V(d) \\
 & v^+ \arrow[d, dash] & v^+ \arrow[d, dash] \\
 & fv^+ \arrow[d, dash] & fv^+ \arrow[d, dash] \\
 & \vdots \arrow[d, dash] & \vdots \arrow[d, dash] \\
 & f^d v^+ \arrow[d, dash] & f^d v^+ \\
f^{d+1}v^+ \arrow[d, dash] & f^{d+1}v^+ \arrow[d, dash] & \\
\vdots & \vdots &
\end{tikzcd}$$

Thus, $\ker(\lambda) := \ker(M(d) \twoheadrightarrow V(d)) = \bigoplus_{i>d} kf^i v^+ = M(-d-2)$. So we have the following short exact sequence, known as the **BGG resolution** (Bernstein-Gelfand-Gelfand) of $V(d)$:

$$0 \to M(-d-2) \to M(d) \to V(d) \to 0$$

Weyl Character formula:

6.6 Definition. Let V be a $\mathfrak{sl}_2$ representation. Assume:

1. $V = \bigoplus_{\lambda \in k} V_\lambda$, where V_λ are the weight spaces.
2. $\dim V_\lambda < \infty$.

The character of the representation V is given by $\chi_V(t) := \sum_{\lambda \in k} \dim V_\lambda t^\lambda$.

Claim: $\chi_V(t)$ is an additive function $\chi_V(t) : \operatorname{Rep} \mathfrak{sl}_2 \to \mathbb{Z}[t^\lambda]$, i.e., for all short exact sequences

$$0 \to V_1 \to V_1 \to V_3 \to 0,$$

we have $\chi_{V_2} = \chi_{V_1} + \chi_{V_3}$.

Observation: $\chi_{M(d)}(t) = \sum_{i \geq 0} \dim V_{d-2i} t^{d-2i} = \sum_{i \geq 0} t^{d-2i} = \frac{t^d}{1-t^{-2}}$.

By additivity and BGG resolution, we obtain:

$$\chi_{V(d)}(t) = \chi_{M(d)}(t) - \chi_{M(-d-2)}(t) = \frac{t^d - t^{-d-2}}{1-t^{-2}} = \frac{t^{1+d} = t^{-1-d}}{t - t^{-1}}$$

.

7 Lecture 7 (January 20): Universal Enveloping Algebra and PBW basis *Scribe: Leo Mayer*

7.1 Definition. Let $\mathfrak{g}$ be a Lie algebra over k. The Universal Enveloping Algebra $\mathscr{U}(\mathfrak{g})$ is an associative unital algebra over k satisfying the following properties:

1. There exists a k-linear map $i : \mathfrak{g} \to \mathscr{U}(\mathfrak{g})$ such that
$$i([x, y]) = i(x)i(y) - i(x)i(y)$$
2. For any associative unital algebra A over k and k-linear map $j : \mathfrak{g} \to A$, satisfying (1), there exists a unique algebra homomorphism $\tilde{j} : \mathscr{U}(\mathfrak{g}) \to A$ making the following diagram commute:

$$\begin{tikzcd} \mathfrak{g} \arrow[rr, "j"] \arrow[dr, "i"'] & & A \\ & \mathscr{U}(\mathfrak{g}) \arrow[ur, dashed, "\tilde{j}"'] & \end{tikzcd}$$

7.2 Lemma. *If $\mathscr{U}(\mathfrak{g})$ exists, then it is unique up to unique isomorphism (commuting with the structure map).*

Proof. General nonsense. ∎

We now need to show that $\mathscr{U}(\mathfrak{g})$ indeed exists. Let
$$T(\mathfrak{g}) = \bigoplus_{n \geq 0} \mathfrak{g}^{\otimes n}$$
be the tensor algebra, where the multiplication map
$$\mathfrak{g}^{\otimes n} \otimes \mathfrak{g}^{\otimes m} \to \mathfrak{g}^{\otimes (n+m)}$$
is defined on simple tensors in the natural way and extended linearly. Let I be the 2-sided ideal of $T(\mathfrak{g})$ given by
$$I = \langle x \otimes y - y \otimes x - [x, y] \mid x, y \in \mathfrak{g} \rangle$$
We can finally define
$$\mathscr{U}(\mathfrak{g}) := T(\mathfrak{g})/I.$$

7.3 Lemma. *$\mathscr{U}(\mathfrak{g}) := T(\mathfrak{g})/I$ satisfies conditions for the Universal Enveloping Algebra.*

Proof. We define i as the composition
$$\mathfrak{g} \to T(\mathfrak{g}) \to T(\mathfrak{g})/I.$$
Then condition (1) of Definition 7.1 is satisfied by construction of I. Condition (2) of Definition 7.1 is left as an exercise. ∎

We have a functor Lie : $\underline{\mathrm{Alg}_{\mathrm{k}}} \to \underline{\mathrm{Lie}_{\mathrm{k}}}$ from the category of associative unital algebras over k to the category of Lie algebras over k. This construction gives a left adjoint to the functor Lie- $\mathscr{U}$: $\underline{\mathrm{Lie}_{\mathrm{k}}} \to \underline{\mathrm{Alg}_{\mathrm{k}}}$, i.e. we have for any Lie algebra $\mathfrak{g}$ and associative unital algebra A that

$$\mathrm{Hom}_{\underline{\mathrm{Lie}_{\mathrm{k}}}}(\mathfrak{g}, \mathrm{Lie}(A)) \cong \mathrm{Hom}_{\underline{\mathrm{Alg}_{\mathrm{k}}}}(\mathscr{U}(\mathfrak{g}), A).$$

Indeed, the property (2) of Definition 7.1 states that the map $j \mapsto \tilde{j}$ is an isomorphism between the Hom sets.

7.4 Remark. Note that we have an equivalence of additive categories $\mathrm{Rep}_k\, \mathfrak{g} \cong \mathscr{U}(\mathfrak{g})\mod$. Since the latter category is an abelian category, we conclude the former is as well.

7.5 Remark. The algebra $\mathscr{U}(\mathfrak{g})$ is actually a Hopf algebra under the maps

$$\nabla : \mathscr{U}(\mathfrak{g}) \to \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}) \text{ defined by } x \mapsto x \otimes 1 + 1 \oplus x$$

$$\epsilon : \mathscr{U}(\mathfrak{g}) \to k \text{ defined by } x \mapsto 0, \quad 1 \mapsto 1$$

$$S : \mathscr{U}(\mathfrak{g}) \to \mathscr{U}(\mathfrak{g}) \text{ defined by } x \mapsto -x$$

which are defined first for $x \in \mathfrak{g}$ and then extended multiplicitavely to all of $\mathscr{U}(\mathfrak{g})$.

7.6 Remark. The category of modules over a Hopf algebra is monoidal. We have already seen that $\mathrm{Rep}_k\, \mathfrak{g}$ is a monoidal category. It turns out the above equivalence of categories respects the monoidal structure.

We next turn to the Poincare-Birkhoff-Witt (PBW) theorem. Before doing so, we need to recall some definitions.

7.7 Definition. A *graded algebra* is an algebra A with a decomposition $A = \bigoplus_i A^i$ such that the multiplication map decomposes as

$$A^i \otimes A^j \to A^{i+j}$$

7.8 Example. The polynomial algebra $A = k[x_1, \ldots, x_n]$. Here $A^i = k[x_1, \ldots, x_n]_{(j)}$, the subspace of homogeneous degree i polynomials.

7.9 Definition. An associative unital algebra A is *filtered* if there exists a chain of subspaces

$$\ldots \subset A_i \subset A_{i+1} \subset \ldots$$

satisfying

$$A_i \cdot A_j \subseteq A_{i+j}$$

7.10 Example. $A = k[x_1, \ldots, x_n]$, and A_i is subspace of polynomials of degree at most i.

7.11 Definition. If A is filtered, the associated graded algebra $\mathrm{gr}^* A$ is defined by

$$\mathrm{gr}^i A = A_i / A_{i-1}$$

and

$$gr^* A = \bigoplus \mathrm{gr}^i A.$$

7.12 Exercise. Verify that we have

$$(A_i/A_{i-1}) \oplus (A_j/A_{j-1}) \to A_{i+j}/A_{i+j-1}$$

7.13 Theorem (Poincare-Birkhoff-Witt)**.** *The algebra* $\mathscr{U}(\mathfrak{g})$ *has a filtration such that* $\mathrm{gr}^* \mathscr{U}(\mathfrak{g}) \cong S^*(\mathfrak{g})$.

We first define the filtration on $\mathscr{U}(\mathfrak{g})$. Set

$$T_i(\mathfrak{g}) = k \oplus \mathfrak{g} \oplus \ldots \oplus \mathfrak{g}^{\otimes i},$$

and set

$$\mathscr{U}_i(\mathfrak{g}) = T_i(\mathfrak{g})/(T_i(\mathfrak{g}) \cap I),$$

where I was the two-sided ideal of $T(\mathfrak{g})$ which defined $\mathscr{U}(\mathfrak{g})$. Then the $\mathscr{U}_i(\mathfrak{g})$ will define a filtration on $\mathscr{U}(\mathfrak{g})$.

Explicitly, if $x_1, \ldots, x_n$ is a basis for g, then $\mathscr{U}_d(\mathfrak{g})$ will be generated by

$$\{x_{i_1} \cdot x_{i_1} \cdot \ldots \cdot x_{i_m} \mid m \leq d\}$$

7.14 Proposition. *Let* $x \in \mathscr{U}_p(\mathfrak{g})$ *and* $y \in \mathscr{U}_q(\mathfrak{g})$

1. $xy \in \mathscr{U}_{p+q}(\mathfrak{g})$
2. $xy - yx \in \mathscr{U}_{p+q-1}(\mathfrak{g})$
3. $\mathscr{U}_p(\mathfrak{g})$ *is generated as a vector space by*

$$\{x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n} \mid \alpha_1 + \ldots + \alpha_n \leq p\}.$$

Proof. 1. Clear.

2. We use induction on p. For the base case $p = 1$, we have $x \in \mathfrak{g}$ and $y = y_1 \ldots y_q$, where each $y_i \in \mathfrak{g}$. We have

$$xy = x(y_1 \ldots y_q) = (xy_1)y_2 \ldots y_q = (y_1 x)y_2 \ldots y_q + [x, y_1]y_2 \ldots y_q$$

since $xy - yx = [x, y]$ in $\mathscr{U}(\mathfrak{g})$. Note that $[x, y_1]y_2 \ldots y_q \in \mathscr{U}_q(\mathfrak{g})$ We can continue using the commutator relations to "push x right" and arrive at

$$xy = y_1 y_2 \ldots y_q x + r,$$

where $r \in \mathscr{U}_q(\mathfrak{g})$, and we conclude $xy - yx \in \mathscr{U}_q(\mathfrak{g})$.

The inductive step is similar, and left as an exercise.

3. By part (2), we can commute any of the products $x_{i_1} \ldots x_{i_m}$ mod $\mathscr{U}_{m-1}$.

■

8 Lecture 8 (January 23): Proof of the Poincare-Birkhoff-Witt theorem

Scribe: Jackson Morris

Last Time: We defined the Universal Enveloping Algebra $\mathscr{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ and placed a filtration on it:

$$\mathscr{U}_n(\mathfrak{g}) = T_n(\mathfrak{g})/T_n(\mathfrak{g}) \cap I,$$

where $I = \langle x \otimes y - y \otimes x - [x, y] | x, y \in \mathfrak{g} \rangle$ is the two sided ideal of the tensor algebra $T(\mathfrak{g})$ previously defined. Additionally, we proved Proposition 7.14.

Here are some immediate corollaries.

8.1 Corollary. 1. *$gr^* \mathscr{U}(\mathfrak{g})$ is commutative. (This follows directly from Proposition 7.14(2).)*

2. *There is a surjective linear map $\varphi : S^*(\mathfrak{g}) \to gr^*(\mathscr{U}(\mathfrak{g}))$. In particular, for a basis $\{x_1, \dots, x_n\}$ of $\mathfrak{g}$, let $\{z_1, \dots, z_n\}$ be the corresponding basis for $S^1(\mathfrak{g})$; then $\varphi(z_i) = x_i$.*

8.2 Theorem (Poincare-Birkhoff-Witt, Theorem 7.13)**.** *The map $\varphi : S^*(\mathfrak{g}) \to gr^* \mathscr{U}(\mathfrak{g})$ is an isomorphism. Alternatively, $\{x_{i_1}, \dots, x_{i_m} | m \in \mathbb{Z}_{\geq 0}, i_1 \leq \dots \leq i_m\}$ is a basis for $\mathscr{U}(\mathfrak{g})$.*

For example, let $\mathfrak{g} = \mathfrak{sl}_2$. Fix the basis $\{e, h, f\}$ with the relations we have discussed. The PBW theorem says that a basis for $\mathscr{U}(\mathfrak{sl}_2)$ is $\{e^a h^b f^c\}$.

1. $eh \cdot e = e(eh) + 2e^2 = e^2h + 2e^2$
2. $(ehf)(ehf) = (ehf)^2 = e^2h^2f^2 + 4e^2hf^2 + 4e^2f^2 - eh^3f$ (this answer was offered by Nelson and verified by Leo).

Proof. We already know that $\{x_{i_1}, \dots, x_{i_m} | m \in \mathbb{Z}_{\geq 0}, i_1 \leq \dots \leq i_m\}$ is a generating set. Now, we must show that the x_i are linearly independent. This is involved; the idea is that we will construct a linear map $f : U^*(\mathfrak{g}) \to S^*(\mathfrak{g})$ shich takes generators to generators such that

$$f(x_1^{\alpha_1} \cdots x_n^{\alpha_n}) = z_1^{\alpha_1} \cdots z_n^{\alpha_n}.$$

If we can do this, we're good!

Construction:

Define a map $\widetilde{f} : T(\mathfrak{g}) \to S^*(\mathfrak{g})$ such that

1. $\widetilde{f}(x_{i_1} \otimes \dots \otimes x_{i_m}) = z_{i_1} \cdots z_{i_m}$ for $i_1 \leq \dots \leq i_m$.
2. $\widetilde{f}(I) = 0$.

Notice that if (2) is satisfied, then we can pass to the quotient, i.e. get our desired map $f : \mathscr{U}(\mathfrak{g}) \to S^*(\mathfrak{g})$. Let's formalize this; we are defining (1) on the simple tensors. Now, if we extend this with the Lie brackets, then (2) will be satisfied. Inductively, we define $\widetilde{f}$ on $x_{j_1} \otimes \cdots \otimes x_{j_m}$ by induction on m and the number of transpositions in $(j_1 \ldots j_m)$. When $m = 0$, $\widetilde{f}(1) = 1$ and for $i_1 \leq \ldots \leq i_m$, we define

$$\widetilde{f}(x_{i_1} \otimes \cdots \otimes x_{i_m}) := z_{i_1} \cdots z_{i_m}.$$

Now, suppose we have $x_{j_1} \otimes \cdots \otimes x_{j_m} \in \mathfrak{g}^{\otimes m}$. Let $(j_t j_{t+1})$ be a transposition. Define

$$\begin{aligned}\widetilde{f}(x_{j_1} \otimes \cdots \otimes x_{j_m}) := \widetilde{f}(x_{j_1} \otimes \cdots \otimes x_{j_{t+1}} \otimes x_{j_t} \otimes \cdots \otimes x_{j_m}) \\ + \widetilde{f}(x_{j_1} \otimes \cdots \otimes [x_{j_t}, x_{j_{t+1}}] \otimes \cdots \otimes x_{j_m}).\end{aligned}$$

Notice that the bracket $[x_{j_t}, x_{j_{t+1}}] \in \mathfrak{g}^{\otimes m-1}$. We claim now that this is well-defined. The first case, when two transpositions are non overlapping is formal; the second case, where two transpositions are overlapping, reduces to the Jacobi Identity (check).

Now, we need to show that $\widetilde{f}(I) = 0$. So, we need to show that

$$\widetilde{f}(A(x_i \otimes x_j - x_j \otimes x_i - [x_i, x_j]B) = 0$$

for any simple tensors A and B; everything in I is a k-linear combination of these guys. But this is true from our definition and verification. So, we are done. ■

9 Lecture 9 (January 25): Nilpotent and solvable Lie algebras

Scribe: Ranjan Pradeep

Last Time: Why do we care about universal enveloping algebras?

1. Equivalence of categories, giving a category of representation of lie algebras
2. Abelian category for modules over universal enveloping algebra, allowing homological algebra
3. In order to quantize, you often study the enveloping algebra rather than the lie algebra

Consequences of the Poincare-Birkhoff-Witt Theorem

9.1 Corollary. *1. The universal map from the Lie algebra $\mathfrak{g}$ to $\mathscr{U}(\mathfrak{g})$ is a monomorphism, ie. $i : \mathfrak{g} \to \mathscr{U}(\mathfrak{g})$ is injective*

2. $\mathscr{U}$ is an additive functor

$$\mathscr{U}(\mathfrak{g}_1 \oplus \mathfrak{g}_2) = \mathscr{U}(\mathfrak{g}_1) \otimes \mathscr{U}(\mathfrak{g}_2)$$

3. $\mathscr{U}(\mathfrak{g})$ has no zero divisors

4. Because we have a "leading" term, one can often do induction in $\mathscr{U}(\mathfrak{g})$

9.2 Remark (Symmetrization Map)**.** There is a map s (in char $k = 0$) that takes a monomial to a symmetric element

$$s : S^*(\mathfrak{g}) \to \mathscr{U}(\mathfrak{g})$$

$$z_{i_1} z_{i_2} \dots z_{i_m} \to \frac{1}{n} \sum_{\sigma \in \Sigma_n} x_{\sigma(i_1)} x_{\sigma(i_2)} \dots x_{\sigma(i_m)}$$

This will be an isomorphism of vector spaces (in char 0). Summarizing:

$$T(\mathfrak{g})/\Sigma_n \cong S^*(\mathfrak{g}) \xrightarrow[s]{\sim} T(\mathfrak{g})^{\Sigma_n} \cong \mathscr{U}(\mathfrak{g})$$

9.3 Remark (On the center of $\mathscr{U}(\mathfrak{g})$)**.**

9.4 Example.

$$\mathfrak{g} = \mathfrak{sl}_2 \text{ with basis } (e, h, f)$$

9.5 Definition. (char $k \neq 2$) Casimir element: $c = ef + fe + \frac{1}{2}h^2$.

Claims:

1. $\mathrm{ad}_c = 0$ in $\mathrm{End}(\mathfrak{sl}_2)$ (equivalently, $\mathrm{ad}_e(c) = \mathrm{ad}_f(c) = \mathrm{ad}_h(c) = 0$).
2. Center is a polynomial algebra on c,

$$Z(\mathscr{U}(\mathfrak{g})) \cong k[c].$$

This preserves the action of $\mathfrak{g}$ so we can compute invariants (invariants are elements $V^g = \{v \in V, g \cdot v = 0, \ \forall g \in G\}$, ie. elements that commute with everything)

$$S^*(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} \mathscr{U}(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} Z(\mathscr{U}(\mathfrak{g}))$$

Nilpotent and Solvable Lie Algebras

9.6 Definition. 1. Let I, J be ideals in $\mathfrak{g}$. Then

$$[I, J] := \langle [x_i, y_i], x_i \in I, y_i \in J \rangle$$

is the ideal generated by all commutators

2. $[\mathfrak{g}, \mathfrak{g}]$ is the derived subalgebra of $\mathfrak{g}$

3. The derived series of $\mathfrak{g}$ is defined by

$$\mathfrak{g}^0 = \mathfrak{g}, \ \mathfrak{g}^{(i)} = [\mathfrak{g}^{(i-1)}, \mathfrak{g}^{(i-1)}].$$

9.7 Definition. $\mathfrak{g}$ is a simple lie algebra if it does not have proper nontrivial ideals. Note that this is by convention unlike groups, where a cyclic group of order p is simple, a one dimensional lie algebra is not simple.

9.8 Remark. If $\mathfrak{g}$ is simple, then $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ and the derived series terminates immediately. Since the center can not be the whole ring, $Z(\mathfrak{g}) = 0$

9.9 Example. $\mathfrak{sl}_2$ is simple unless char $k = 2$, In character 2, $Z(\mathfrak{sl}_2) \supset \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The identity matrix will be in the center because it has zero trace, and so the center is not zero.

9.10 Definition. A lie algebra is solvable if the derived series terminates in a finite number of steps

9.11 Definition. The descending (lower) central series of $\mathfrak{g}$ is given by:

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^1 = \mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}^i = [\mathfrak{g}, \mathfrak{g}^{(i-1)}]$$

9.12 Definition. $\mathfrak{g}$ is nilpotent if the descending central series terminates

9.13 Remark. Nilpotent implies solvable, but solvable does not imply nilpotent

9.14 Example. Take $\mathfrak{g} = \mathfrak{gl}_n$.

b_n- consisting of upper triangular matrices is known as a **Borel subalgebra**, and is solvable

u_n- consisting of strictly upper triangular matries (ie. with 0s on the main diagonal) is nilpotent, and is the **unipotent radical** of b_n

u_n is an ideal inside b_n, so one can quotient to get a short exact sequence

$$0 \to u_n \to b_n \to t_n \to 0 \text{ where,}$$

t_n- consisting of diagonal matrices, with every element off the diagonal being zero, is known as the **Cartan subalgebra**.

9.15 Remark. Is $\mathfrak{gl}_n$ simple? No, since $[\mathfrak{gl}_n, \mathfrak{gl}_n] = \mathfrak{sl}_n$, and since $\lambda \cdot I_n$ is central.

9.16 Definition. $\operatorname{Rad}\mathfrak{g}$ is the maximal, solvable ideal in $\mathfrak{g}$.

9.17 Exercise. Show that the radical $\operatorname{Rad}\mathfrak{g}$ of a lie algebra $\mathfrak{g}$ is well defined.

9.18 Definition. $\mathfrak{g}$ is semi-simple if and only if $\operatorname{Rad}\mathfrak{g} = 0$

9.19 Theorem (Weyl's complete reducibility theorem)**.** *In char $k = 0$, $\mathfrak{g}$ is semi-simple* $\iff$ $\operatorname{Rep}_k \mathfrak{g}$ *is semi-simple (i.e., every representation is completely reducible).*

9.20 Remark.

1. $\mathfrak{g}/\operatorname{Rad}\mathfrak{g}$ is semi-simple (this is well-defined since Rad is an ideal).
2. simple implies semi-simple.
3. In char $k = 0$, being semi-simple lie algebra is equivalent to being direct sum of simple lie algebras.

10 Lecture 10 (January 27): Engel's theorem *Scribe: Nelson Niu*

Today we discuss two analogous theorems on the common eigenvalues of Lie algebras—one for nilpotent Lie algebras, the other for solvable Lie algebras.

10.1 Definition. For $\mathfrak{g} \subseteq \mathfrak{gl}(V)$, we say that $x \in \mathfrak{g}$ is nilpotent if $x^n = 0$ in $\mathfrak{gl}(V)$ for some n.

10.2 Definition. We say that $x \in \mathfrak{g}$ is ad-nilpotent if it is nilpotent in the adjoint representation; that is, if $\mathrm{ad}_x \in \mathfrak{gl}(\mathfrak{g})$ is nilpotent.

10.3 Remark. If $\mathfrak{g}$ is nilpotent as a Lie algebra, then by definition ad_x is nilpotent for all $x \in \mathfrak{g}$ (i.e. every element of $\mathfrak{g}$ is ad-nilpotent).

10.4 Remark. The two definitions above are not equivalent. For example, if $\mathfrak{g}$ is an abelian Lie algebra, then $\mathrm{ad}_x = 0$ for every $x \in \mathfrak{g}$, so every $x \in \mathfrak{g}$ is ad-nilpotent. But certainly not every $x \in \mathfrak{g}$ must be nilpotent: take, for example, the abelian and thus nilpotent Lie algebra of diagonal matrices—certainly not all of its elements are nilpotent.

On the other hand, ad-nilpotence does relate to the nilpotence of the entire Lie algebra: Engel's theorem states that if every element of a Lie algebra is ad-nilpotent, then the Lie algebra must be nilpotent. This result will be a consequence of the following theorem.

10.5 Theorem. *Given $\mathfrak{g} \subseteq \mathfrak{gl}(V)$, if $\mathfrak{g}$ consists of nilpotent elements, then there exists a common eigenvector $v \in V$ such that $\mathfrak{g}\, v = 0$.*

The idea is that if $\mathfrak{g}$ consists of all nilpotent elements, you could make its elements strictly upper triangular by finding the Jordan form.

Also note that we don't need any additional assumptions on the base field k: it doesn't need to be algebraically closed, because the eigenvalue we're looking for is 0, and it doesn't need to have characteristic 0 either.

Proof. Induct on $n = \dim \mathfrak{g}$. When $n = 1$, we can write $\mathfrak{g} = kx$. Now just take an eigenvector for x: as x is nilpotent, the eigenvalue must be 0, and every other element of kx should have the same eigenvector and eigenvalue.

For the inductive step, assume the result holds when $\dim \mathfrak{g} < n$, and let H be a maximal proper Lie subalgebra of $\mathfrak{g}$. Then H acts on $\mathfrak{g} / \mathrm{H}$ via ad. By the inductive hypothesis, there is a common eigenvector $\bar{x} \in \mathfrak{g} / \mathrm{H}$ for which $\mathrm{H}\,\bar{x} = 0$; or equivalently, there exists $x \in \mathfrak{g}$ with $x \notin \mathrm{H}$ for which $[\mathrm{H}, x] \subseteq \mathrm{H}$. So if we let $\mathcal{N}_{\mathfrak{g}}(\mathrm{H})$ be the normalizer of H in $\mathfrak{g}$, consisting of all $x' \in \mathfrak{g}$ for which $[\mathrm{H}, x'] \subseteq \mathrm{H}$, we can write that $\mathrm{H} \subsetneq \mathcal{N}_{\mathfrak{g}}(\mathrm{H}) \subseteq \mathfrak{g}$. As $\mathcal{N}_{\mathfrak{g}}(\mathrm{H})$ is also a subalgebra of $\mathfrak{g}$ but H is maximal, it follows that $\mathcal{N}_{\mathfrak{g}}(\mathrm{H}) = \mathfrak{g}$. Hence $[\mathrm{H}, \mathfrak{g}] \subseteq \mathrm{H}$.

Now let W be the subspace of V consisting of all $w \in V$ for which $\mathrm{H}\, w = 0$. By the inductive hypothesis, $W \neq 0$. We claim that W is a $\mathfrak{g}$-stable subspace of V; that is, for all $w \in W$ and $x \in \mathfrak{g}$, we have $xw \in W$. Indeed, for all $h \in \mathrm{H}$, we have that $[h, x] \in \mathrm{H}$, so $[h, x]w = 0 = hw$ and thus

$$h(xw) = x(hw) + [h, x]w = x(0) + 0 = 0,$$

implying that $xw \in W$. So $\mathfrak{g}$ acts on W.

Finally, note that H must have codimension 1 in $\mathfrak{g}$: if not, we could always find some nonzero proper subalgebra $\mathfrak{g}' \subsetneq \mathfrak{g} / \mathrm{H}$ and lift it to a proper subalgebra $\tilde{\mathfrak{g}}' \subsetneq \mathfrak{g}$ with $\mathrm{H} \subsetneq \tilde{\mathfrak{g}}'$, contradicting the maximality of H. So there exists $z \in \mathfrak{g}$ with $z \notin \mathrm{H}$ for which $\mathfrak{g} = \mathrm{H} + kz$. We have that z acts nilpotently on W, so there exists an eigenvector $v \in W$ for z with eigenvalue 0. Then v is the common eigenvector we seek:

$$\mathfrak{g}\, v = (\mathrm{H} + kz)v = \mathrm{H}\, v + kzv = 0 + k0 = 0.$$

■

There is analogous theorem (and proof) for solvable algebras, but it is more involved, as not all eigenvalues will be 0 as in the nilpotent case.

As promised, Engel's theorem follows.

10.6 Theorem (Engel)**.** *Let $\mathfrak{g}$ be a Lie algebra. If every $x \in \mathfrak{g}$ is ad-nilpotent, then $\mathfrak{g}$ is nilpotent.*

Proof. Apply the previous theorem to the adjoint representation $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ (so take $V = \mathfrak{g}$). Then there exists a common eigenvector $z \in \mathfrak{g}$ such that $\mathrm{ad}_x(z) = 0$ for all $x \in \mathfrak{g}$. In other words, $z \in Z(\mathfrak{g})$, so $Z(\mathfrak{g}) \neq 0$.

Now induct on $\dim g$, modding out the center, building an ascending central series (i.e. with abelian quotients). Then $\mathfrak{g} / Z(\mathfrak{g})$ is nilpotent by induction, implying that $\mathfrak{g}$ is nilpotent via lifting. ■

Note that nilpotent Lie algebras are analogous to p-groups in having nontrivial centers.

10.7 Remark (on "flags")**.** Let $n = \dim V$ and fix $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ with all nilpotent elements. Then our earlier theorem implies the existence of a common eigenvector $v_n \in V$ for which $\mathfrak{g}\, v_n = 0$. Taking $V_n = kv_n \subseteq V$ and modding it out, we are left with a space of dimension $\dim(V/V_n) = n - 1$.

Now repeat this process on V/V_n: again we find a common eigenvector $\bar{v}_{n-1} \in V/V_n$ (the residue of some $v_{n-1} \in V$) for which $\mathfrak{g}\, \bar{v}_{n-1} = 0$ in V/V_n (and thus $\mathfrak{g}\, v_{n-1} \subseteq V_n$). Take $V_{n-1} = kv_n + kv_{n-1} \supset V_n$.

Iterating this process, we obtain a basis $\{v_1, \ldots, v_n\}$ for V and nested subspaces $V_n \subset \cdots \subset V_1 = V$ given by $V_i = kv_i + \cdots + kv_n$ satisfying $\mathfrak{g}\, V_i \subseteq V_{i+1}$. This basis induces an isomorphism $\mathfrak{gl}(V) \simeq \mathfrak{gl}_n$ that sends $\mathfrak{g}$ to $\mathfrak{u}_{-n}$, the strictly (zeroes along the diagonal) lower triangular matrices of size $n \times n$.

We call the nested sequence of subspaces $V_n \subset \cdots \subset V_1 = V$ a "flag"; here it satisfies $\mathfrak{g}\, V_i \subseteq V_{i+1}$.

10.8 Corollary. *Given a nilpotent Lie algebra $\mathfrak{g}$ and any ideal $\mathrm{H} \subseteq \mathfrak{g}$, we have $\mathrm{H} \cap Z(\mathfrak{g}) \neq 0$.*

As mentioned, there are analogous results for solvable Lie algebras, although now we must assume that k is algebraically closed (to have the necessary eigenvalues) and has characteristic 0.

10.9 Theorem. *If $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ is solvable (i.e. its derived series terminates), then there exists a common eigenvalue $v \in V$ for $\mathfrak{g}$: for all $x \in \mathfrak{g}$, there exists some $\lambda_x \in k$ for which $xv = \lambda_x v$.*

Proof. See [Hum73] II.4.1. ∎

10.10 Remark. If $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ is solvable, then there exists a flag $0 \subsetneq V_1 \subset V_2 \subset \cdots \subset V_n = V$ for which $\mathfrak{g}\, V_i \subseteq V_i$. So there exists a basis for which $\mathfrak{g}$ consists of the upper triangular matrices. (Compare this to the nilpotent case.) In a way, the upper triangular matrices are the "ultimate" solvable Lie algebras.

10.11 Exercise. The assumption that k has characteristic 0 is crucial in the solvable case. If instead k had characteristic 2, show that in $\mathfrak{gl}_2$ with

$$x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

we have that $[x, y] = x$, making $\mathrm{H} = kx + ky$ solvable. Is there a common eigenvalue?

Next time, we will cover Jordan decomposition.

11 Lecture 11 (January 27): Lie's theorem and Lie's lemma *Scribe: Eric Zhang*

We start with a proof of a corollary from last time:

11.1 Corollary (Corollary 10.8)**.** *Given a nilpotent Lie algebra* $\mathfrak{g}$ *and any ideal* $\mathrm{H} \subseteq \mathfrak{g}$*, we have* $\mathrm{H} \cap Z(\mathfrak{g}) \neq 0$.

Proof. Note that $\mathfrak{g}$ acts on the ideal H via the adjoint action. Since $\mathfrak{g}$ is nilpotent, there exists $v \in \mathrm{H}$ such that $\mathfrak{g}\, v = 0$ which implies $v \in Z(\mathfrak{g})$. ■

11.2 Exercise. Find all nilpotent, non-abelian, 3-dimensional lie algebras, up to isomorphism.

Recall Lie's theorem and note the assumption on the base field k.

11.3 Theorem (Lie's theorem)**.** *Let char* $k = 0$ *and* $\overline{k} = k$*. If* $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ *is solvable (i.e. its derived series terminates), then there exists a common eigenvector* $v \in V$ *for* $\mathfrak{g}$*: for all* $x \in \mathfrak{g}$*, there exists some* $\lambda_x \in k$ *for which* $xv = \lambda_x v$.

11.4 Corollary. *Let char* $k = 0$ *and* $\overline{k} = k$*. Any irreducible representation of a solvable lie algebra is* 1*-dimensional.*

Proof. Let $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ be an irreducible representation. By Lie's theorem, there exists a common eigenvector $v \in V$ which spans a 1-dimensional $\mathfrak{g}$-invariant subspace (v). Hence $V = (v)$ and is of 1-dimensional by irreducibility. ■

11.5 Corollary. *Let char* $k = 0$ *and* $\overline{k} = k$*. Then* $\mathfrak{g}$ *solvable implies* $[\mathfrak{g}, \mathfrak{g}]$ *nilpotent.*

Proof. (Exercise.) Consider the adjoint action restricted to $[\mathfrak{g}, \mathfrak{g}]$ and the induced short exact sequence $0 \to Z([\mathfrak{g}, \mathfrak{g}]) \to [\mathfrak{g}, \mathfrak{g}] \to \mathrm{ad}([\mathfrak{g}, \mathfrak{g}]) \to 0$. It suffices to show that $\mathrm{ad}([\mathfrak{g}, \mathfrak{g}])$ is nilpotent. Note $\mathrm{ad}([\mathfrak{g}, \mathfrak{g}]) = [\mathrm{ad}\,\mathfrak{g}, \mathrm{ad}\,\mathfrak{g}]$ as ad preserves lie bracket. Since $\mathfrak{g}$ is solvable, it follows from Lie's theorem (here we use the assumption on k) that $\mathrm{ad}\,\mathfrak{g} \subseteq b_n$. Thus $[\mathrm{ad}\,\mathfrak{g}, \mathrm{ad}\,\mathfrak{g}] \subseteq [b_n, b_n] = u_n$ and is nilpotent. ■

We then turn to a more general lemma.

11.6 Theorem (Lie's lemma)**.** *Let char* $k = 0$ *and* $\overline{k} = k$*. For* $\mathfrak{g} \subseteq \mathfrak{gl}(V)$ *and* $\mathrm{H} \subseteq \mathfrak{g}$ *any ideal, denote the restricted representation as* $V \downarrow_{\mathrm{H}}$*. Suppose* $\left(V \downarrow_{\mathrm{H}}\right)_\lambda$ *is a weight space. Then* $\left(V \downarrow_{\mathrm{H}}\right)_\lambda$ *is a* $\mathfrak{g}$*-invariant subspace of V.*

11.7 Remark. It can be shown that Lie's theorem follows from Lie's lemma.

11.8 Proposition. *Let char* $k = 0$ *and* $\overline{k} = k$*. Suppose* $\mathfrak{g}$ *is any lie algebra and* $V \in \mathrm{Rep}(\mathfrak{g})$ *is irreducible. Then*

1. *If* $x \in \mathrm{Rad}(\mathfrak{g})$*, then* x *acts by a particular scalar* λ *on V.*
2. *If* $x \in [\mathfrak{g}, \mathrm{Rad}(\mathfrak{g})]$*, then* x *acts by zero on V.*

Proof of part 1. If $\mathrm{Rad}(\mathfrak{g}) = 0$, then the statement is vacuously true. Assume $\mathrm{Rad}(\mathfrak{g}) \neq 0$. Recall $\mathrm{Rad}(\mathfrak{g})$ is solvable and consider its action on V. By Lie's theorem (here we need the assumption on k), there exists a common eigenvector $v \in V$ such that $xv = \lambda v$ for all $x \in \mathfrak{g}$. It follows that $\mathrm{span}(v)$ is a $\mathrm{Rad}(\mathfrak{g})$-invariant subspace. In particular, $\mathrm{span}(v) \subseteq (V \downarrow_{\mathrm{Rad}(\mathfrak{g})})_\lambda$ which implies the latter is a nontrivial weight space. Then by Lie's lemma, $(V \downarrow_{\mathrm{Rad}(\mathfrak{g})})_\lambda$ is $\mathfrak{g}$-invariant. By irreducibility of $V \in \mathrm{Rep}(\mathfrak{g})$, we may conclude $V = (V \downarrow_{\mathrm{Rad}(\mathfrak{g})})_\lambda$ and it follows that x acts by scalar on V. ■

Proof of part 2. (Exercise.) To see $[\mathfrak{g}, \mathrm{Rad}(\mathfrak{g})]$ acts by 0 on V, it suffices to show its generators acts by 0. Recall $[\mathfrak{g}, \mathrm{Rad}(\mathfrak{g})]$ is spanned by $[x, y]$ where $x \in \mathfrak{g}$ and $y \in \mathrm{Rad}(\mathfrak{g})$. Then $[x, y]$ acts on V as $[x, y]v = (xy)v - (yx)v = x(yv) - y(xv)$. Note $[\mathfrak{g}, \mathrm{Rad}(\mathfrak{g})] \subseteq \mathrm{Rad}(\mathfrak{g})$ and, by part 1 $yv = \lambda v, \forall y \in [\mathfrak{g}, \mathrm{Rad}(\mathfrak{g})]$ for a fixed λ. It follows that $[x, y]v = x(yv) - y(xv) = x(\lambda v) - \lambda(xv) = 0$. ■

11.9 Definition (reductive lie algebra)**.** A lie algebra $\mathfrak{g}$ is reductive if $\mathfrak{g}/Z(\mathfrak{g})$ is semisimple, or equivalently, if $\mathrm{Rad}(\mathfrak{g}) \subseteq Z(\mathfrak{g})$.

11.10 Example. $\mathfrak{gl}_n$ is not simple nor semisimple (since it has nontrivial center) but it is reductive. So are $\mathfrak{sl}_n$, $\mathfrak{sp}_{2m}$, SO_{2n}, and SO_{2n+1}.

11.11 Example. Simple or semisimple lie algebras are reductive because they have zero center or radical.

11.12 Remark. The following inclusion relations holds.

1. simple $\Longrightarrow$ semisimple $\Longrightarrow$ reductive
2. abelian $\Longrightarrow$ nilpotent $\Longrightarrow$ solvable

11.13 Definition (Generalized eigenspace)**.** Let $x \in \mathfrak{g}$. A generalized eigenspace is $V_{(\lambda)} = \{v \in V : (x - \lambda I)^n v = 0\}$.

11.14 Remark. If $x \in \mathfrak{g}$ is nilpotent, then $V_{(0)} = V$.

11.15 Proposition (Jordan canonical form)**.** *Let $\overline{k} = k$ and $x \in \mathfrak{gl}(V)$. Then there exists unique $(\lambda_1, \ldots, \lambda_s) \in k^s$ and $(n_1, \ldots, n_s) \in \mathcal{N}^s$ such that $V = \bigoplus_{i=1}^s V_{(\lambda_i)}$ where $dim_k V_{(\lambda_i)} = n_i$. If we choose basis for each $V_{(\lambda_i)}$, then x can be put into Jordan blocks.*

11.16 Definition. For $x \in \mathfrak{gl}(V)$, then x is semisimple if x is diagonalizable, or equivalently if the minimum polynomial of x has distinct roots.

12 Lecture 12 (February 1): Bilinear forms and reductive Lie algebras *Scribe: Raymond Guo*

Last time, we claimed that if V is a irreducible representation of $\mathfrak{g}$, then $[\mathfrak{g}, \mathrm{Rad}(\mathfrak{g})]$ acts by 0 on V. This was left as an exercise, but we go over the proof now. If $x \in [\mathfrak{g}, \mathrm{Rad}(\mathfrak{g})]$ then $x = [y, z], y \in \mathfrak{g}$ and $z \in \mathrm{Rad}(\mathfrak{g})$. By Proposition 11.8(1), x acts by a particular scalar λ on V, that is, $zv = \lambda v$ for all $v \in V$. Thus, we have,

$$[y, z]v = y(zv) - z(yv) = y(\lambda v) - \lambda(yv) = 0.$$

Note, for x in Jordan Canonical form and with

$$V_{(\lambda)} = \{v \in V \mid (x - \lambda I)^n v = 0\},$$

we have the invariant subspace decomposition

$$V = \bigoplus_{i=1}^{s} V_{(\lambda_i)}.$$

12.1 Definition (Rational Canonical Form)**.** Letting $L : V \to V$ be a linear map from a vector space to itself, we give V an $k[x]$ module structure by letting x act by L. With this, we can place the $k[x]$ module V in rational canonical form by writing

$$V \cong \bigoplus_{i=1}^{\ell} \frac{k[x]}{d_i(x)}$$

where each d_i is a polynomial and $d_i | d_{i+1}$. d_ℓ is the minimal polynomial.

12.2 Remark. x is semisimple if and only if the minimal polynomial of x has distinct roots.

12.3 Definition. Let $x \in \mathfrak{gl}(V)$. $x = x_s + x_n$ is a Jordan decomposition if x_s is semisimple, x_n is nilpotent, and $[x_s, x_n] = 0$.

12.4 Proposition. *For such a decomposition, there exist $p, q \in k[t]$ such that $x_s = p(x)$ and $x_n = q(x)$.*

Proof. Shown in [Hum73], uses the Chinese Remainder Theorem. ■

12.5 Proposition. *The Jordan decomposition exists.*

Proof. Write x in Jordan canonical form. The diagonal is the semisimple part and the strictly upper triangular entries are the nilpotent part. ■

12.6 Proposition. *The Jordan decomposition is unique for all matrices.*

Proof. Suppose $x_s + x_n = x'_s + x'_n$ with x_s and x'_s semisimple, x_n and x'_n nilpotent. Then $x_s - x'_s = x_n - x'_n$. We conclude that the LHS is semisimple (diagonalizable) and the RHS is nilpotent, but the only diagonalizable nilpotent matrix is 0. ■

12.7 Corollary. *Let $x, y \in \mathfrak{gl}(V)$, such that $[x, y] = 0$. Then*

1. *all generalized eigenspaces of V with respect to x are y-invariant.*
2. *$[y, x_s] = [y, x_n] = 0$.*

Proof. $x \in \mathfrak{g}$. $x = x_s + x_n \implies \mathrm{ad}_x = \mathrm{ad}_{x_s} + \mathrm{ad}_{x_n}$ is a Jordan decomposition for ad_x, from which the result follows. ∎

12.8 Remark. If the characteristic of k is 0, $\mathfrak{g}$ is solvable if and only if $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. This does not hold in characteristic p.

12.9 Example. Let $\mathrm{char}(k) = 2$. We have a 4-dimensional lie algebra with generators y_1, y_2, x_1, x_2. Let us have $[x_1, x_2] = 0$ and $[y_1, y_2] = y_1$. Let us also have $[y_1, x_1] = x_2$, $[y_1, x_2] = x_1$ $[y_2, x_1] = 0$, $[y_2, x_2] = x_2$ (to show that this is a lie algebra, we require that the characteristic of k is 2, in order to check the Jacobi identity. For example, $[[y_1, y_2], x_1] + [[y_2, x_1], y_1] + [[x_1, y_1], y_2] = x_2 + 0 + x_2 = 2x_2 = 0$) We claim this defines a lie algebra.

$L = \langle x_1, x_2 \rangle \subset \mathfrak{g}$ is an abelian ideal. $\mathfrak{g}/L = \langle y_1, y_2 \rangle$ is solvable. Thus $0 \subset L \subset \mathfrak{g}$ demonstrates that $\mathfrak{g}$ is solvable. We show that $[\mathfrak{g}, \mathfrak{g}]$ is not nilpotent. Note that $[\mathfrak{g}, \mathfrak{g}] = \langle y_1, x_1, x_2 \rangle = h$, $[h, h] = L$, and $[h, L] = L$, whereby we see $[\mathfrak{g}, \mathfrak{g}]$ is not nilpotent.

12.10 Definition (Invariant Bilinear Operator)**.** Let $\mathcal{B} : \mathfrak{g} \times \mathfrak{g} \to k$ be a bilinear form. B is said to be invariant if it satisfies $\mathcal{B}([x, y], z) = \mathcal{B}(x, [y, z])$ for all $x, y, z \in \mathfrak{g}$. (Humphreys calls this associative instead).

12.11 Example. Let $V \in \mathrm{Rep}_k\, \mathfrak{g}$ be a finite dimensional representation with $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$. $B_V : \mathfrak{g} \times \mathfrak{g} \to k$ defined by $B_V(x, y) = \mathrm{Trace}(\rho(x)\rho(y))$ is such a form.

13 Lecture 13 (February 3): Killing Form
Scribe: Goutham Seshadri

13.1 Proposition. *B_V as defined in Example 12.11 is bilinear, symmetric and invariant.*

Proof. Bilinearity is a consequence of linearity of the trace, and symmetry comes from the fact that $\mathrm{Trace}(ab) = \mathrm{Trace}(ba)$. To show that B_v is invariant, we need to show that $\mathcal{B}_V$ satisfies $\mathcal{B}_V([x, y], z) = \mathcal{B}_V(x, [y, z])$ for all $x, y, z \in \mathfrak{g}$. Directly computing, and using the fact that $\mathrm{Trace}(ab) = \mathrm{Trace}(ba)$ again, we see that

$$\begin{aligned}\mathcal{B}_V([x,y],z) - \mathcal{B}_V(x,[y,z]) &= \mathrm{Trace}(\rho([x,y])\rho(z)) - \mathrm{Trace}(\rho(x)\rho([z,y])) \\ = \mathrm{Trace}\left[(\rho(x)\rho(y) - \rho(y)\rho(x))\rho(z)\right] &- \mathrm{Trace}\left[\rho(x)(\rho(y)\rho(z) - \rho(z)\rho(y))\right] \\ &= \mathrm{Trace}(\rho(x)\rho(y)\rho(z) - \rho(y)\rho(x)\rho(z)) = 0.\end{aligned}$$

■

13.2 Lemma. *$\mathcal{B}_V$ is "additive"; i.e. that for any short exact sequence $0 \to V_1 \to V_2 \to V_3 \to 0$ in $\mathrm{Rep}\,\mathfrak{g}$, $B_{V_2} = B_{V_1} + B_{V_3}$.*

Proof. Homework. ■

13.3 Theorem. *If $\mathfrak{g}$ is a Lie algebra, $V \in \mathrm{Rep}_k \mathfrak{g}$, and B_V is non-degenerate then $\mathfrak{g}$ is reductive (i.e. $\mathfrak{g}/Z(\mathfrak{g})$ is semisimple).*

Proof. It suffices to show that $[\mathfrak{g}, \mathrm{Rad}\,\mathfrak{g}] = 0$, since this immediately implies that $\mathrm{Rad}\,\mathfrak{g} \subseteq Z(\mathfrak{g})$ and thus, $\mathfrak{g}/Z(\mathfrak{g})$ is semisimple.

Let W be an irreducible representation of $\mathfrak{g}$, then by Proposition 11.8, $[\mathfrak{g}, \mathrm{Rad}\,\mathfrak{g}]$ acts as 0 on W, so that $B_W(x, -) = 0$ for all $x \in [\mathfrak{g}, \mathrm{Rad}\,\mathfrak{g}]$. By Lemma 13.2 and induction on the composition series of V, we must have that $\mathcal{B}_V(x, -) = 0$ for all $x \in [\mathfrak{g}, \mathrm{Rad}\,\mathfrak{g}]$. But this means that $[\mathfrak{g}, \mathrm{Rad}\,\mathfrak{g}] \subseteq \ker \mathcal{B}_v = 0$, by non-degeneracy of B_V. ■

13.4 Remark. $\mathfrak{gl}_n$, $\mathfrak{sl}_n$, $\mathfrak{sp}_{2n}$ and the other classical Lie algebras can be shown to be reductive via this theorem by considering the standard representation.

13.5 Definition (The Killing Form)**.** $K_{\mathfrak{g}} := \mathrm{Trace}(\mathrm{ad}\,x\,\mathrm{ad}\,y)$

13.6 Example. We can compute the matrix of $K_{\mathfrak{sl}_2}$ with respect to e, h, f by starting with the ad matrices:

$$\mathrm{ad}\,e = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}\,h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \mathrm{ad}\,f = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{pmatrix}$$

We end up with the following matrix which is non-degenerate unless $\mathrm{char}(k) = 2$:

$$K_{\mathfrak{sl}_2} = \begin{pmatrix} 0 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 0 \end{pmatrix}$$

13.7 Lemma. *If B is an invariant, symmetric, bilinear form on $\mathfrak{g}$ and $I \subset \mathfrak{g}$ is an ideal, then $I^\perp := \{x \in \mathfrak{g} \mid B(I, x) = 0\}$ is an ideal.*

Proof. Given any $x \in I^\perp$ and $y \in \mathfrak{g}$, we must show that $[y, x] \in I^\perp$. But for any $z \in I$, invariance of $\mathcal{B}$ gives that $\mathcal{B}(z, [y, x]) = \mathcal{B}([z, y], x) = 0$ since $[z, y] \in I$ and $x \in I^\perp$. ■

13.8 Theorem (Cartan's Criterion)**.** *If $\operatorname{char}(k) = 0$ and $k = \bar{k}$, then $\mathfrak{g}$ is a solvable Lie algebra if and only if $B_V(x, y) = 0$ for all $x \in \mathfrak{g}$, $y \in [\mathfrak{g}, \mathfrak{g}]$.*

Proof. See [Hum73] Section 4.3. ■

13.9 Theorem (Lie's Theorem)**.** *$\mathfrak{g}$ is semisimple if and only if its associated Killing form, $K_{\mathfrak{g}}$ is non-degenerate.*

Proof. ($\Longleftarrow$) Suppose $K_{\mathfrak{g}}$ is non-degenerate. Then by Theorem 13.3, $\mathfrak{g}$ is reductive. But for all $x \in Z(\mathfrak{g})$, $K_{\mathfrak{g}}(x, -) = 0$ so that $Z(\mathfrak{g}) \subseteq \ker K_{\mathfrak{g}} = 0$. We conclude that $\mathfrak{g}$ is semisimple.

($\Longrightarrow$) Suppose $I = \ker K_{\mathfrak{g}} \neq 0$. Then I is an ideal by the invariance of $K_{\mathfrak{g}}$. Moreover $K_I = (K_{\mathfrak{g}}) \downarrow_I$ (Homework), so that $K_I = 0$. But by Cartan's criterion (Theorem 13.8), I must be solvable, so that $\mathfrak{g}$ cannot be semisimple. ■

14 Lecture 14 (February 6): Categorical properties of representations and homological algebra *Scribe: Haoming Ning*

14.1 Lemma. *Let B be a symmetric bilinear form on V, let $U \subseteq V$ such that $B\downarrow_U$ is non-degenerate, then $V = U \oplus U^\perp$.*

Sketch of proof. Pick a basis $e_1, \ldots, e_m$ for U, complete it to a complementary bases $e_{m+1}, \ldots, e_n$ for $U^\perp$. Define

$$Q = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}, \quad P = \begin{pmatrix} I & -S \\ 0 & I \end{pmatrix}$$

where $S = A^{-1}B$. Then

$$P^T Q P = \begin{pmatrix} A & 0 \\ 0 & * \end{pmatrix}.$$

■

14.2 Exercise. Check the detail in the proof of the above lemma.

14.3 Lemma. *Let $\mathfrak{g}$ be a semi-simple Lie algebra, $I \subseteq \mathfrak{g}$ an ideal. Then the killing form $K_\mathfrak{g}\downarrow_I$ is non-degenerate. (Recall that when* $\operatorname{char}(k) = 0$*, then $\mathfrak{g}$ is semi-simple if and only if $K_\mathfrak{g}$ is non degenerate.)*

Proof. By homework $K_\mathfrak{g}\downarrow_I = K_I$. Suppose that K_I is degenerate. Let $\mathrm{H} = I \cap I^\perp \subseteq \mathfrak{g}$, then H is an ideal in $\mathfrak{g}$. We have $K_\mathrm{H} = K_\mathfrak{g}\downarrow_\mathrm{H} = 0$, so that $K_\mathfrak{g}(a,b) = 0$ for every $a, b \in I \cap I^\perp$. By Cartan's criterion, H is solvable. But then $\mathfrak{g}$ has a solvable ideal H, contradicting the semi-simple hypothesis. ■

14.4 Theorem. *Let $\mathfrak{g}$ be semi-simple. Then $\mathfrak{g} = \bigoplus_{i=1}^s \mathfrak{g}_i$ where $\mathfrak{g}_i$ are simple.*

Proof. We induct on the dimension of $\mathfrak{g}$. If $\mathfrak{g}$ is not simple, there exists an ideal $I \subseteq \mathfrak{g}$. By Lemma 14.3, $K_\mathfrak{g}\downarrow_I \neq 0$ (so that I is semi-simple). By Lemma 14.1, $\mathfrak{g} = I \oplus I^\perp$. Apply induction to $I, I^\perp$. ■

14.5 Theorem. *If $\mathfrak{g}$ is semi-simple, then* $\operatorname{Der}\mathfrak{g} = \operatorname{ad}\mathfrak{g} := \{\operatorname{ad} x \mid x \in \mathfrak{g}\}$. *That is, all derivations are inner.*

Note that this can be viewed as an additive analogue of the Noether-Skolem theorem for central simple algebras.

Weyl complete reducibility theorem

Recall the following facts and operations on $\operatorname{Rep}_k \mathfrak{g}$:

- $\operatorname{Rep}_k \mathfrak{g}$ is an abelian category.
- Tensor operation $\otimes_R$ exists on $\operatorname{Rep}_k \mathfrak{g}$.

- Inner $\mathrm{Hom}_k(V, W)$, where $\mathfrak{g}$ acts on $\varphi : V \to W$ by $x \cdot \varphi(v) = \varphi(v) - \varphi(xv)$ for every $x \in \mathfrak{g}$. (Note that in Hopf algebras: $x \in H$, $(S \otimes 1) \cdot \Delta(x)$ we have $x \mapsto \sum S(x') \otimes x''$). Note also that $\mathrm{Hom}_{\mathfrak{g}}(V, W) = \mathrm{Hom}_k(V, W)^{\mathfrak{g}}$.
- Duals $V^\sharp = \mathrm{Hom}_k(V, k)$. If $\dim V < \infty$, then $\mathrm{Hom}_K(V, W) \simeq V^\sharp \otimes_k W$.
- Jordan-Holder theorem holds in this category.
- Schur's lemma holds. If V, W are irreducible representations of $\mathfrak{g}$, then $\mathrm{Hom}_{\mathfrak{g}}(V, W) = 0$ for $V \not\simeq W$, and $\mathrm{End}_{\mathfrak{g}}(V) = k = \mathrm{End}_k(V)^{\mathfrak{g}}$.

Recall also the following facts on homological algebra:

To show that $\mathrm{Rep}_k\, \mathfrak{g}$ (category of finite dimension representations) is semi-simple (every object is a direct sum of simple ones), it is equivalent to show that $\mathrm{Ext}^1_{\mathfrak{g}}(V, W) = 0$ for every V, W, which implies that every short exact sequence $0 \to W \to U \to V \to 0$ splits.

14.6 Fact. The functor $V \mapsto V^{\mathfrak{g}} = \mathrm{Hom}_{\mathfrak{g}}(k, V)$ is left exact, we denote this functor simply by $V^{\mathfrak{g}}$. Therefore we may define its right derived functor $\mathrm{Ext}^i_{\mathfrak{g}}(k, v) = R^i V^{\mathfrak{g}}$. We define in general $\mathrm{Ext}^i_{\mathfrak{g}}(V, W) = R^i\, \mathrm{Hom}_{\mathfrak{g}}(V, W)$.

14.7 Fact. Ext has the long exact sequence. Any short exact sequence $0 \to V' \to V \to V'' \to 0$ gives

$$0 \to \mathrm{Hom}_{\mathfrak{g}}(W, V') \to \mathrm{Hom}_{\mathfrak{g}}(W, V) \to \mathrm{Hom}_{\mathfrak{g}}(W, V'') \to \mathrm{Ext}^1_{\mathfrak{g}}(W, V) \to \dots.$$

14.8 Fact. We have $\mathrm{Ext}^i_{\mathfrak{g}}(V, W) \simeq \mathrm{Ext}^i_{\mathfrak{g}}(k, V^\sharp \otimes_k W)$. Define $H^*(\mathfrak{g}, V) := \mathrm{Ext}^*_{\mathfrak{g}}(k, V)$.

Now let $\mathfrak{g}$ be a semi-simple Lie algebra.

14.9 Definition (Casimir element)**.** Let B be a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$. Let x_i be a basis of $\mathfrak{g}$. Choose x^i to be the dual basis with respect to B, so that $B(x_i, x^j) = \delta_{ij}$. Define $c_B = \sum x_i x^i \in \mathscr{U}(g)$. In the case that $B = K$ is the Killing form on a semisimple Lie algebra $\mathfrak{g}$ (so that K is nondegenerate), then we call $c_K = c$ the Casimir element of $\mathfrak{g}$.

15 Lecture 15 (February 8): Casimir Element
Scribe: Bashir Abdel-Fattah

15.1 Proposition. *If $\mathfrak{g}$ is a semisimple Lie algebra and B is a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$, then*

(1) c_B is independent of the choice of basis for $\mathfrak{g}$.

(2) $c_B \in Z(\mathscr{U}(\mathfrak{g}))$.

(3) If V is a representation of $\mathfrak{g}$ (with respect to the action $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$) and $B = B_V$, then define

$$c_\rho := \sum \rho(x_i)\rho(x^i) \in \mathfrak{gl}(V)$$

(note that the Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ extends uniquely to a k-algebra homomorphism $\mathscr{U}(\mathfrak{g}) \to \mathfrak{gl}(V)$ by the universal property of the universal enveloping algebra, and that c_ρ is the image of $c_B \in \mathscr{U}(\mathfrak{g})$ under this map). Then $\mathrm{Trace}(c_\rho) = \dim \mathfrak{g}$.

Proof. (1) Calculate (see [Hum73] section 6.2).

(2) Consider the map

$$\mathrm{End}_k(\mathfrak{g}) \cong \mathfrak{g}^\# \otimes \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g} \xrightarrow{M} \mathscr{U}(\mathfrak{g})$$

(noting that B determines a canonical isomorphism $\mathfrak{g}^\# \cong \mathfrak{g}$ by mapping $x \in \mathfrak{g}$ to $B(x, -) \in \mathfrak{g}^\#$, hence there is a canonical isomorphism $\mathfrak{g}^\# \otimes \mathfrak{g} \cong \mathfrak{g} \otimes \mathfrak{g}$). Then c_B is the image of $\mathrm{id}_\mathfrak{g} \in \mathrm{End}_k(\mathfrak{g})$ in $\mathscr{U}(\mathfrak{g})$ under the above composition of morphisms of representations. Thus the fact that $\mathrm{id}_\mathfrak{g}$ is ad-invariant (i.e., $\mathrm{id}_\mathfrak{g} \in \mathrm{End}_k(\mathfrak{g})^\mathfrak{g}$), it follows that c_B is also ad-invariant and hence $c_B \in Z(\mathscr{U}(\mathfrak{g}))$.

(3) Given the basis $\{x_1, \dots, x_n\}$ of $\mathfrak{g}$ and the corresponding dual basis $\{x^1, \dots, x^n\}$, we have that $B_V(x_i, x^j) = \mathrm{Trace}(\rho(x_i)\rho(x^j)) = \delta_i^j$ by definition. Then we can calculate

$$\mathrm{Trace}\, c_\rho = \mathrm{Trace}(\sum_{i=1}^n \rho(x_i)\rho(x^i)) = \sum_{i=1}^n \mathrm{Trace}(\rho(x_i)\rho(x^i)) = \sum_{i=1}^n 1 = n = \dim \mathfrak{g}$$

∎

15.2 Theorem. *If $\mathfrak{g}$ is a semisimple Lie algebra and V is an irreducible representation of $\mathfrak{g}$, then there exists an element $c_V \in Z(\mathscr{U}(\mathfrak{g}))$ which acts on V as a scalar $\lambda \in k$. If V is not the trivial representation $V = k$, then $\lambda \neq 0$.*

Proof. Take $B = B_V$, and let $I = \ker B$ and $J = I^\perp$. Then J is semisimple, $\mathfrak{g} = I \perp J$, and $B|_J$ is nondegenerate. Let

$$c = c_V := c_{(B|_J)} \in Z(\mathscr{U}(J)) \subseteq \mathscr{U}(J) \subseteq \mathscr{U}(\mathfrak{g}).$$

We already know that c_V commutes with every element of J by the fact that it's in $Z(\mathscr{U}(J))$, and it also commutes with every element of I because c_V is a sum of products of elements in J, all of which commute with I by the fact that

$$[I, J] \subseteq I \cap J = \{0\},$$

so $c = c_V \in \mathscr{U}(\mathfrak{g})$. This also means that the map $V \to V, v \mapsto cv$ is in fact an endomorphism of representations (since $c(xv) = x\,cv$ for all $x \in \mathfrak{g}$), and $\mathrm{End}_{\mathfrak{g}}(V) \cong k$ by Schur's lemma, so c_V must act by a scalar on V.

15.3 Exercise. If V is an irreducible representation of a semisimple Lie algebra $\mathfrak{g}$, then $B_V \equiv 0$ if and only if V is the trivial representation.

By the exercise, if V is nontrivial then $J = (\ker B)^{\perp} \neq 0$, so

$$\mathrm{Trace}\, c_V = \dim J \neq 0$$

and c_V must in fact act by a nontrivial scalar. ■

15.4 Theorem. *Suppose* $\mathrm{char}(k) = 0$, $\overline{k} = k$, *and* $\mathfrak{g}$ *is a semisimple Lie algebra over* k. *Then the category* $\mathrm{Rep}_k\, \mathfrak{g}$ *is semisimple (meaning that every finite-dimensional representation of* $\mathfrak{g}$ *is completely reducible).*

Proof. This is equivalent to showing that $\mathrm{Ext}^1_{\mathfrak{g}}(V, W) = 0$ for all $V, W \in \mathrm{Rep}\,\mathfrak{g}$. We proceed via the following steps

- Step 1: Show that $\mathrm{Ext}^1_{\mathfrak{g}}(k, V) = 0$ for every irreducible representation V
- Step 3/2: Show that $\mathrm{Ext}^1_{\mathfrak{g}}(k, V) = 0$ for any arbitrary representation V by induction on the length of the composition series of V and by using the long exact sequence for Ext^1.
- Step 2: Conclude that $\mathrm{Ext}^1_{\mathfrak{g}}(V, W) \cong \mathrm{Ext}^1_{\mathfrak{g}}(k, V^{\#} \otimes W) = 0$ for every $V, W \in \mathrm{Rep}\,\mathfrak{g}$.

In order to prove step 1, suppose that V is a nontrivial representation $V \ncong k$, and suppose for the sake of contradiction that $\mathrm{Ext}^1_{\mathfrak{g}}(k, V) \neq 0$. Then there exists a short exact sequence of representations

$$0 \longrightarrow V \longrightarrow W \longrightarrow k \longrightarrow 0.$$

Let $\{v_1, \ldots, v_n\}$ be a basis for $V \subset W$, and complete it to a basis $\{v_1, \ldots, v_n, \widetilde{w}\}$ of W, so that $W \cong V \oplus k\widetilde{w}$ as a vector space. Let $c = c_V$ be a central element that acts on V by a nonzero scalar $\lambda \in k$. Then the action of c on W has the matrix expression

$$\left[\begin{array}{cccc|c} \lambda & 0 & \cdots & 0 & * \\ 0 & \lambda & \cdots & 0 & * \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & \lambda & * \\ \hline 0 & 0 & \cdots & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} \lambda I_n & * \\ \hline \mathbf{0} & 0 \end{array}\right]$$

with respect to the basis $\{v_1, \ldots, v_n, \widetilde{w}\}$. Proof to be continued...

■

16 Lecture 16 (February 10): Weyl complete reducibility theorem
Scribe: Justin Bloom

We continue with the proof of Theorem 15.4 from last time.

Proof. (of Theorem 15.4, *continued*) By the matrix representation, c has a 0-eigenvector, say w. We claim now kw is $\mathfrak{g}$-invariant and hence $W = V \oplus kw$ is a splitting of the short exact sequence of representations.

Let $x \in \mathfrak{g}$. Then $c \cdot xw = xcw = 0$ since $c \in \mathscr{U}(\mathfrak{g})$ is central, and hence xw is also a 0-eigenvector for c. But the eigenspace of 0 for c is kw, so $xw \in kw$, proving our claim.

Consider the case where $V \cong k$ the trivial irreducible representation, with a short exact sequence

$$k \hookrightarrow W \twoheadrightarrow k.$$

If $k \hookrightarrow W$ is the embedding to the $\mathfrak{g}$-subspace kv, we have $W = kv \oplus kw$ as vector spaces, where the image of w generates k in $W \twoheadrightarrow k$. If $x \in \mathfrak{g}$ is any element, we have the matrix representation of x is

$$\begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix}$$

since kv is $\mathfrak{g}$-invariant. Since $\mathfrak{g}$ is semisimple and $\operatorname{char} k = 0$, we may consider an arbitrary simple component of $s \subset \mathfrak{g}$ acting on W by restriction. Since s is simple, $[s, s] = s$, and hence the matrix representation for any $x \in s$ acting on W must have $* = 0$. Then this is true also of any $x \in \mathfrak{g}$ so $\mathfrak{g}$ acts trivially on W, and the short exact sequence splits. Hence $\operatorname{Ext}^1_{\mathfrak{g}}(k, k) = 0$, and we conclude $\operatorname{Ext}^1_{\mathfrak{g}}(k, V) = 0$ for any finite dimensional representation V. ∎

16.1 Example. Let $\mathfrak{g} = \mathfrak{sl}_2$, and consider the Casimir element

$$c_{\mathcal{B}} = \sum x_i x^i \in Z(\mathscr{U})$$

for a basis $\{x_i\}$ of $\mathfrak{g}$ and $\{x^i\}$ dual w.r.t $\mathcal{B}$.

Let $V = k^2$ the standard representation for $\mathfrak{g} = \langle e, f, h \rangle$, i.e. matrices

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let $\mathcal{B} = \mathcal{B}_V$, i.e. $\mathcal{B}(x, y) = \operatorname{Trace}(\rho_V(x)\rho_V(y))$. Find a dual basis for $\mathcal{B}$, and compute $c_{\mathcal{B}}$:

First, we compute some traces, identifying e, f, h with their matrices:

$$\begin{array}{ll} \mathrm{Trace}(eh) = 0, & \mathrm{Trace}(e^2) = 0 \\ \mathrm{Trace}(fh) = 0, & \mathrm{Trace}(h^2) = 2 \\ \mathrm{Trace}(ef) = 1, & \mathrm{Trace}(f^2) = 0. \end{array}$$

Then we have dual elements

$$e^\perp = f, \quad h^\perp = \frac{1}{2}h, \quad f^\perp = e,$$

so our Casimir element is

$$c_{\mathcal{B}} = ef + \frac{h^2}{2} + fe = 2ef + \frac{h^2}{2} - h.$$

Now consider the adjoint representation: the killing form is

$$\begin{pmatrix} & & 4 \\ & 8 & \\ 4 & & \end{pmatrix}.$$

The dual basis is then

$$e^\perp = \frac{f}{4}, \quad h^\perp = \frac{h}{8}, \quad f^\perp = \frac{e}{4}$$

and

$$c = \frac{ef}{2} + \frac{h^2}{8} - \frac{h}{4}.$$

Note c and $c_{\mathcal{B}}$ generate the same linear subspace.

(Abstract) Jordan decompositions

Let $\mathfrak{g}$ be semisimple Lie algebra. Jordan decomposition:

$$\operatorname{ad} x = (\operatorname{ad} x)_s + (\operatorname{ad} x)_n$$

for derivations $(\operatorname{ad} x)_s, (\operatorname{ad} x)_n$. Since all derivations are inner, there exists $\widetilde{x}_s, \widetilde{x}_n$ such that $(\operatorname{ad} x)_s = \operatorname{ad} \widetilde{x}_s$ and $(\operatorname{ad} x)_n = \operatorname{ad} \widetilde{x}_n$. We also have $\widetilde{x}_s, \widetilde{x}_n$ gives a Jordan decomposition

$$x = \widetilde{x}_s + \widetilde{x}_n$$

called the abstract Jordan decomposition.

For any representation V, $\rho_V : \mathfrak{g} \to \mathfrak{gl}(V)$, we have $\rho(\widetilde{x}_s) = \rho(x)_s$ and $\rho(\widetilde{x}_n) = \rho(x)_n$.

16.2 Proposition. *If $\mathfrak{g}' \subset \mathfrak{g}$ is any Lie subalgebra and $x \in \mathfrak{g}'$, then x_s, x_n are both in $\mathfrak{g}'$.*

17 Lecture 17 (February 13): Root decompositions and Root spaces
Scribe: William Dudarov

Root decompositions

Let $k = \overline{k}$ be of characteristic 0, and let $\mathfrak{g}$ be a semisimple Lie algebra over k.

17.1 Definition. A subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$ is **toral** if it consists entirely of semisimple elements.

17.2 Example. For $\mathfrak{sl}_n$, we have the example of a subalgebra of diagonal matrices of the form $\operatorname{diag}(a_1, \dots, a_n)$, such that $a_1 + \cdots + a_n = 0$.

17.3 Proposition. *Toral subalgebras are abelian.*

The proof is in [Hum73], Section 8.1.

The classification of semisimple Lie algebras rests on the choice of maximal toral subalgebra $\mathfrak{h}$.

17.4 Remark. Making such a choice of $\mathfrak{h}$ a maximal toral subalgebra, $\mathfrak{h}$ is abelian by Proposition 17.3, and so $\operatorname{ad}_{\mathfrak{g}} \mathfrak{h}$ is simultaneously diagonalizable.

17.5 Remark. $\mathfrak{h} = C_{\mathfrak{g}}(\mathfrak{h})$, i.e. $\mathfrak{h}$ is self-normalizing ($[\mathfrak{h}, x] = 0 \implies x \in \mathfrak{h}$).

17.6 Definition. A **Cartan subalgebra (CSA)** of $\mathfrak{g}$ is a nilpotent self-normalizing Lie sub-algebra.

An observation: note that any maximal toral subalgebra is a Cartan subalgebra, and any Cartan subalgebra is abelian.

17.7 Definition. The **rank** of $\mathfrak{g}$ is the dimension of $\mathfrak{h}$.

17.8 Remark. All CSAs are "conjugate" - that is, in a finite-dimensional Lie algebra over a field of characteristic 0, all CSAs are isomorphic, and conjugate under automorphisms.

Root spaces

Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Cartan subalgebra. Then $\mathfrak{h}$ is simultaneously diagonalizable, which is equivalent to

$$\mathfrak{g} = \bigoplus_{\alpha} \mathfrak{g}_\alpha,$$

where the $\mathfrak{g}_\alpha$s are the eigenspaces of $\mathfrak{g}$ with respect to the action of $\mathfrak{h}$. That is, if we let $\mathfrak{h}^* = \operatorname{Hom}_k(\mathfrak{h}, k)$, and let α run over the elements of $\mathfrak{h}^*$, we get the above direct sum for $\mathfrak{g}$ where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathfrak{h}\}.$$

Note that $\mathfrak{g}_0 = \mathfrak{h}$. Denote by Φ the $\alpha \in \mathfrak{h}^*$ such that $\alpha \neq 0$.

17.9 Definition. Fix $\mathfrak{h} \subseteq \mathfrak{g}$, and let $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$.

1. $\alpha \in \Phi$ is called a **root** for $\mathfrak{g}$.
2. Φ is called a **root system** for $\mathfrak{g}$.
3. The eigenspace $\mathfrak{g}_\alpha$ is called a **root space**.

17.10 Example. Let $\mathfrak{g} = \mathfrak{sl}_2$ be generated by

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \ h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We have that $\mathfrak{g} = kh \oplus ke \oplus kf$. Our Cartan subalgebra is the one generated by h, $\mathfrak{h} = kh$. We have

1. $[h, h] = 0$,
2. $[h, e] = 2e$,
3. $[h, f] = -2f$.

What are our roots α? We have $\alpha(h) = 2, -\alpha(h) = -2$.

The root system Φ can be represented as the following.

Root system for $\mathfrak{sl}_2$

17.11 Example. Let $\mathfrak{g} = \mathfrak{sl}_3$. We have $\mathfrak{h}$ generated by

$$h_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } h_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \text{ with } h_3 = h_1 + h_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

We have

$$e_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ e_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ e_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$f_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ f_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \ f_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

We have the decomposition

$$\mathfrak{sl}_3 = \mathfrak{h} \oplus \bigoplus_{i=1}^{3} ke_i \oplus \bigoplus_{i=1}^{3} kf_i.$$

18 Lecture 18 (February 15): Killing form and $\mathfrak{sl}_2$-triples

Scribe: Soham Ghosh

We continue with Example 17.11 we saw in last class.

18.1 Example. $\mathfrak{g} = \mathfrak{sl}_3$. Let

$$h_1 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & \end{pmatrix} \quad h_2 = \begin{pmatrix} & & \\ & 1 & \\ & & -1 \end{pmatrix} \quad h_3 = h_1 + h_2$$

$$e_1 = \begin{pmatrix} & 1 & \\ & & \\ & & \end{pmatrix} \quad e_2 = \begin{pmatrix} & & 1 \\ & & \\ & & \end{pmatrix} \quad e_3 = \begin{pmatrix} & & \\ & & 1 \\ & & \end{pmatrix}$$

$$f_1 = \begin{pmatrix} & & \\ 1 & & \\ & & \end{pmatrix} \quad f_2 = \begin{pmatrix} & & \\ & & \\ 1 & & \end{pmatrix} \quad f_3 = \begin{pmatrix} & & \\ & & \\ & 1 & \end{pmatrix}$$

Recall the decomposition $\mathfrak{sl}_3 = \mathfrak{h} \oplus \bigoplus_{i=1}^3 ke_i \oplus \bigoplus_{i=1}^3 kf_i$.

Let $\alpha_1, \alpha_2 \in \mathfrak{h}^\star$ and let $\alpha_3 = \alpha_1 + \alpha_2$. Let K be the Killing form on $\mathfrak{g}$, and henceforth we shall write $\langle x, y\rangle$ for $K(x,y)$. $K|_{\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate. We have isomorphism $\mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^\star$.

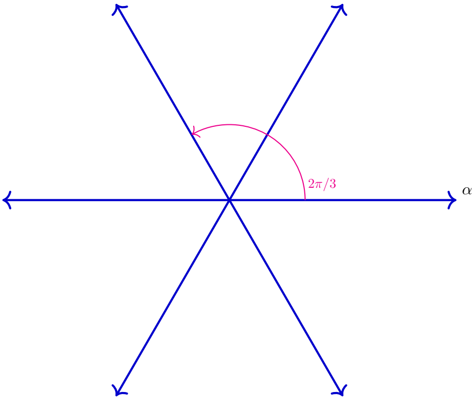
$$\begin{aligned} (\operatorname{ad} h_1)(e_1) = [h_1, e_1] = 2e_1 \qquad & (\operatorname{ad} h_1)(e_2) = [h_1, e_2] = -e_2 \\ (\operatorname{ad} h_1)(e_3) = (\operatorname{ad} h_1)([e_1, e_2]) = [h_1, [e_1, e_2]] &= [[h_1, e_1], e_2] + [e_1, [h, e_2]] \\ & [2e_1, e_2] + [e_1, -e_2] = [e_1, e_2] = e_3 \end{aligned}$$

So we have, in matrix form:

$$\operatorname{ad} h_1 = \begin{pmatrix} 2 &&&&&&& \\ & -1 &&&&&& \\ && 1 &&&&& \\ &&& 0 &&&& \\ &&&& 0 &&& \\ &&&&& -2 && \\ &&&&&& 1 & \\ &&&&&&& -1 \end{pmatrix} \quad \operatorname{ad} h_2 = \begin{pmatrix} -1 &&&&&&& \\ & 2 &&&&&& \\ && 1 &&&&& \\ &&& 0 &&&& \\ &&&& 0 &&& \\ &&&&& 1 && \\ &&&&&& -2 & \\ &&&&&&& -1 \end{pmatrix}$$

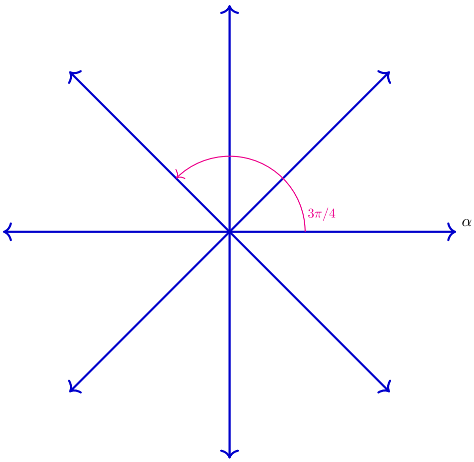
We have $\langle h_i, h_j\rangle = \operatorname{Trace}(\operatorname{ad} h_i, \operatorname{ad} h_j)$. Note that $\langle h_1, h_1\rangle = 12 = \langle h_2, h_2\rangle$ and $\langle h_1, h_2\rangle = -6$. Thus, $\alpha_1(h_1) = 2$, $\alpha_1(h_2) = -1$.

We claim that $\alpha_1 = h_{\alpha_1}^\star$, i.e., $\alpha_1(h) = \langle h_{\alpha_1}, h\rangle$ for all h. Note that $h_{\alpha_1} = \frac{h_1}{6}$ and $h_{\alpha_2} = \frac{h_2}{6}$. To find the angle ϕ between h_1 and h_2, we see that $\cos\phi = \frac{\langle h_1, h_2\rangle}{\|h_1\|\|h_2\|} = -1/2$, i.e., $\phi = 2\pi/3$.

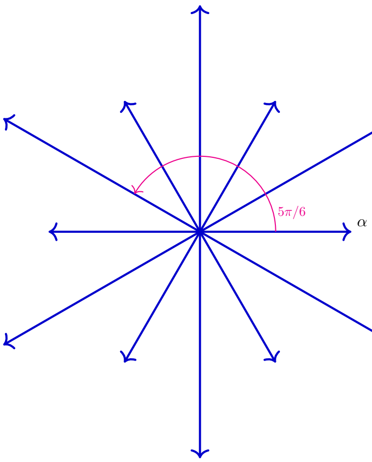


Type A_2 rank 2 Root system for $\mathfrak{sl}_3$

The following are the root system diagrams of type B_2 rank 2 and type G_2 root systems:



Type B_2 rank 2 Root system



Type G_2 Root system

18.2 Lemma. *Let $\mathfrak{g}$ be a semisimple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra, and K be a Killing form on $\mathfrak{g}$. Then:*

1. $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$ *is Root space decomposition of* $\mathfrak{g}$.
2. α, $\beta \in \Phi$, *then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$. *In particular if* $\alpha + \beta \notin \Phi$, *and* $\alpha \neq -\beta$, *then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$.
3. $\alpha + \beta \neq 0$ *implies* $K(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$.
4. *For all* $\alpha \in \Phi$, $K|_{\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}}$ *is non-degenerate.*

Proof. For (2), note that if $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$, then $[h, [x, y]] = [[h, x], y] + [x, [h, y]] = \alpha(h)([x, y]) + \beta(h)([x, y]) = (\alpha + \beta)(h)([x, y])$, which implies that $[x, y] \in \mathfrak{g}_{\alpha+\beta}$.

For (3), let $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_\beta$ and suppose $K(x,y) \neq 0$. Then

$$\alpha(h)K(x,y) = K([h,x],y) = -K(x,[h,y]) = -\beta(h)K(x,y),$$

which implies $\alpha = -\beta$.

For (4) note that K is non-degenerate, and by (3), we have $K(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ if $\alpha + \beta \neq 0$, and $K(\mathfrak{g}_\alpha, h) = 0$ implying $K|_{\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}}$ is non-degenerate. ■

Upshot: Given a semisimple Lie algebra $\mathfrak{g}$, we get a root space decomposition with root space Φ, by choosing a Cartan subalgebra $\mathfrak{h}$. We get a decomposition of $\mathfrak{g}$ into simple lie algebra $\mathfrak{g} = \bigoplus \mathfrak{g}_i$, ($\mathfrak{g}_i$ simple), such that $\mathfrak{h} = \bigoplus \mathfrak{h}_i$, where $\mathfrak{h}_i$ is a Cartan subalgebra in $\mathfrak{g}_i$, which yields a root system Φ_i for $\mathfrak{g}_i$, such that $\Phi = \bigsqcup \Phi_i$.

$\mathfrak{sl}_2$-triples

For all roots $\alpha \in \Phi$, there exists a triple $\langle e, \mathfrak{h}_\alpha, f\rangle \subset \mathfrak{g}$ such that $\mathfrak{h}_\alpha \cong \mathfrak{sl}_2$, $\alpha(h_\alpha) = 2$, $e \in \mathfrak{g}_\alpha$, and $f \in \mathfrak{g}_{-\alpha}$.

Note that we have an isomorphism $\mathfrak{h} \xrightarrow{\sim} \mathfrak{h}^\star$ via the Killing form K given by $\alpha \mapsto \mathfrak{h}_\alpha$, the dual of α, defined by $\alpha(h) = \langle \mathfrak{h}_\alpha, h\rangle$ for all $h \in \mathfrak{h}$. Let $h_\alpha = \frac{2H_\alpha}{\langle\alpha,\alpha\rangle}$ (have to show $\langle \alpha, \alpha\rangle \neq 0$ for this). We can define $\langle \alpha, \beta\rangle := \langle \mathfrak{h}_\alpha, \mathfrak{h}_\beta\rangle$, whereby we have $\alpha(\mathfrak{h}_\alpha) = \langle \mathfrak{h}_\alpha, \mathfrak{h}_\beta\rangle = \beta(\mathfrak{h}_\alpha)$.

$\mathfrak{sl}_2$**:** $\langle h\rangle := \mathfrak{h}$, where $h = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$. Recall decomposition $\mathfrak{sl}_2 = kh \oplus ke \oplus kf$.

Define $\alpha : \mathfrak{h} \to k$ by $\alpha(h) = 2$. Then $\mathfrak{h}_\alpha = \frac{h}{4}$ and $\langle h, h\rangle = 8$. Also we see then $\alpha(h) = \langle \mathfrak{h}_\alpha, h\rangle = \langle \frac{h}{4}, h\rangle = 2$.

$$h_\alpha = \frac{2\mathfrak{h}_\alpha}{\langle \alpha, \alpha\rangle} = \frac{2\mathfrak{h}_\alpha}{\langle \mathfrak{h}_\alpha, \mathfrak{h}_\alpha\rangle} = \frac{2h/4}{\langle h/4, h/4\rangle} = \frac{h/2}{8/16} = h$$

19 Lecture 19 (February 17): Blitz through semisimple Lie algebras

Scribe: Leo Mayer

19.1 Proposition (Properties of Φ). *The following are the properties of root systems:*

1. Φ *spans* $\mathrm{H}^\star$.
2. $\dim \mathfrak{g}_\alpha = 1$ *for all* $\alpha \in \Phi$.
3. *For all* $\alpha, \beta \in \Phi$ *we have* $S_\alpha(\beta) \in \Phi$. *In particular, if* $\alpha \in \Phi$, *then* $-\alpha \in \Phi$.
4. *No other multiple of* $\alpha \in \Phi$ *is a root.*
5. *If* $\alpha, \beta \in \Phi$, *the subspace* $\bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_{\beta+n\alpha}$ *is a representation of* $_2$ *(in particular, we there are* $r \le q \in \mathbb{Z}$ *such that* $\beta + r\alpha, \beta + (r+1)\alpha, \ldots, \beta + q\alpha$ *are all roots).*
6. $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.
7. $\beta(\mathfrak{h}_\alpha) = \frac{2\langle \alpha,\beta\rangle}{\langle \alpha,\alpha\rangle} \in \mathbb{Z}$.

Observation: Since $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \subset k$, we may consider $\mathrm{H}_\mathbb{Q}$, the $\mathbb{Q}$ vector space spanned by $\{\mathfrak{h}_\alpha | \alpha \in \Phi\}$. We may also consider $\mathrm{H}_\mathbb{R} := \mathrm{H}_\mathbb{Q} \otimes \mathbb{R}$. The following properties hold:

1. Let $\{\alpha_1, \ldots, \alpha_s\}$ be a basis for H^*. Then each α can be written as $\alpha = \sum c_i \alpha_i$, where $c_i \in \mathbb{Q}$.
2. For all α, β, $(\alpha, \beta) \in \mathbb{Q}$.
3. $K_{\mathrm{H}^*_\mathbb{Q} \times \mathrm{H}^*_\mathbb{Q}}$ is positive definite.

Summary: $E = \mathbb{Q}\Phi$ and $E_\mathbb{R} = E \otimes_\mathbb{Q} \mathbb{R}$, K defines an inner product $(-,-)$ on $E_\mathbb{R}$. The following conditions $(*)$ hold

1. $0 \notin \Phi$, $|\Phi| < \infty$, Φ spans E.
2. If $\alpha \in \Phi$, then $-\alpha \in \Phi$, and no other scalar multiple of α is in Φ.
3. If $\alpha, \beta \in \Phi$, then $S_\alpha(\beta) \in \Phi$.
4. If $\alpha, \beta \in \Phi$, then $\langle \beta, \alpha \rangle \in \mathbb{Z}$, where $\langle \beta, \alpha \rangle := \frac{2(\beta,\alpha)}{\alpha,\alpha}$

Abstract Root Systems

Let E be a Euclidean vector space, i.e. a real vector space with an inner product denoted $(-,-)$. If $\mu \in E$, define the reflection S_μ by $S_\mu(\lambda) = \lambda - \langle \lambda, \mu \rangle \mu$. The *perpendicular hyperplane* to μ is $P_\mu = \{\lambda \in E | \langle \lambda, \mu \rangle = 0\}$.

19.2 Definition. An *abstract root system* in E is a subset $\Phi \subset E$ satisfying the four conditions $(*)$ above.

Simple Roots

Choose a vector $t \in E$ not normal to any root in Φ. We then get a decomposition $\Phi = \Phi^+ \coprod \Phi^-$, where $\Phi^+ := \{\alpha \in \Phi \mid (\alpha, t) > 0\}$, and similarly for $\Phi-$. Such a decomposition is called a *polarization*.

19.3 Definition. A root $\alpha \in \Phi^+$ is *simple* if $\alpha \neq \beta + \gamma$ for all $\beta, \gamma \in \Phi^+$.

Some facts:

1. If $\alpha \in \Phi^+$, then α is a sum of simple roots.
2. If α, β are simple, then $(\alpha, \beta) \leq 0$
3. The simple roots in Φ^+ are linearly independent.

19.4 Definition. Let $\Delta = \{\alpha_1, \ldots, \alpha_s\}$ be the set of simple roots in Φ^+. The *Cartan matrix* is the matrix (a_{ij}), where $a_{ij} = \langle \alpha_i, \alpha_j \rangle$. The *Dynkin diagram* is the graph where:

1. The vertices are simple roots.
2. The number of edges between α_i and α_j is $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$.
3. If $\|\alpha_i\| > \|\alpha_j\|$, there is an arrow pointing from α_i to α_j.

20 Lecture 20 (February 24): Abstract root systems and Weyl group
Scribe: Jackson Morris

The idea here is that to each semi-simple Lie algebra, we can assign a root system Φ. Remember that an **abstract root system** is a euclidean vector space E such that: Φ spans E; $|\Phi| < \infty$; $0 \notin \Phi$; for every $\alpha \in \Phi$, we have $-\alpha \in \Phi$ and that no other scalar multiplies of α are; $\langle \beta, \alpha \rangle = 2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$, where parantheses denote the killing form; rkΦ=dim E.

Fact: If Φ is irreducible, then there are at most 2 different root lengths, short roots and long roots.

For rank 2, we have that $A_1 \times A_1$ and A_2 have only one root length, while B_2 and G_2 have two root lengths.

21 Lecture 21 (February 27)
Scribe: Nelson Niu

The type A root system

Consider the A_{n-1} type root system Φ (corresponding to the simple Lie algebra $\mathfrak{sl}_n$). The reflections generating its Weyl group W can be thought of as transpositions, making every element of W a permutation; so $W \cong S_n$. More precisely, if we identify its underlying $(n-1)$-dimensional Euclidean space with

$$E = \frac{\bigoplus_{i=1}^{n} \mathbb{R}\, e_i}{\mathbb{R}(e_1 + \cdots + e_n)},$$

where each e_i acts on the diagonal matrix $E_{jj} \in \mathfrak{sl}_n$ (with a 1 in the j^{th} row, j^{th} column and zeroes everywhere else) by $e_i(E_{jj}) = \delta_{ij}$, then each simple root α_i in $\Delta = \{\alpha_i\}_{i=1}^{n-1}$ can be identified with $e_i - e_{i+1} = (0, \ldots, 0, 1, -1, 0, \ldots, 0)$, with a 1 in the i^{th} entry and a -1 in the $(i+1)^{\text{th}}$ entry.

These $n-1$ simple roots determine the $n(n-1)/2$ positive roots $e_i - e_j$ with $i < j$, each of which can be written as a sum of consecutive simple roots:

$$e_i - e_j = (e_i - e_{i+1}) + \cdots + (e_{j-1} - e_j) = \alpha_i + \cdots + \alpha_{j-1}.$$

In terms of the A_{n-1} Dynkin diagram, consisting of the $n-1$ simple roots in a line with a single edge connecting each pair of consecutive roots, the positive roots are given by adding simple roots along connected edges. One can visualize this as a triangle of positive roots written above the line of simple roots in the Dynkin diagram.

Then the Weyl group $W \cong S_n$ acts by permutations on the entries of these roots. Simple reflections, corresponding to simple roots, are *transpositions*: the reflection S_{α_i} swaps the i^{th} and $(i+1)^{\text{th}}$ components of the vector, so it corresponds to the transposition $(i, i+1)$. These transpositions generate the rest of the permutations in W.

We can also think of W as acting on the Weyl chambers, the cones given by subdividing the space E with the hyperplanes determined by the roots. One of these chambers is the ***fundamental Weyl chamber***: its scalar products with simple roots are always positive. In the A_{n-1} case, since the scalar product of a vector with simple root $\alpha_i = e_i - e_{i+1}$ is just its i^{th} entry minus its $(i+1)^{\text{th}}$ entry, the fundamental Weyl chamber consists of all vectors $(a_1, \ldots, a_n) \in E$ with $a_1 > \cdots > a_n$.

In the A_{n-1} case, you can get from any root to any other root via elements of W; that is, for all $\alpha \in \Phi$, its orbit $W(\alpha)$ is equal to all of Φ. This is true in general if every root has the same length; i.e. if the root system is ***simply-laced***.

21.1 Definition. A root system is ***simply-laced*** if there are no multi-edges in its Dynkin diagram; equivalently, its Cartan matrix has only 0 and -1 entries outside the main diagonal.

A root system is simply-laced if and only if all of its roots are the same length. The irreducible simply-laced root systems are exactly the type A, type D, and type E root systems; unsurprisingly, we call these the ***ADE type*** root systems.

21.2 Proposition. *There are at most two root lengths in any irreducible root system.*

Proof. By inspection of the three irreducible rank 2 root systems. ■

If Φ is an irreducible root system that is not simply-laced, it has exactly two lengths of roots, short roots and long roots. Elements of the Weyl group cannot change root lengths. But the Weyl group orbit of a short root is the set of all short roots; the Weyl group orbit of a long root is the set of all long roots.

We think of the positive roots as "larger" than the negative roots; motivated by this, we can define an ordering on a root system as follows.

21.3 Definition. Fix a root system Φ with simple roots $\Delta \subset \Phi$. Given $\alpha, \beta \in \Phi$, we write $\alpha \succeq \beta$ if $\alpha - \beta \in \mathbb{Z}_{\geq 0}\Delta$. (Note that $\alpha - \beta$ need not be in Φ^+.)

21.4 Definition. Fix a root system Φ with simple roots $\Delta = \{\alpha_i\}_{i=1}^s \subset \Phi$ inducing positive roots $\Phi^+ \subset \Phi$. Given $\alpha \in \Phi^+$, we can write $\alpha = \sum_i c_i \alpha_i$, and we define the ***height*** of α to be $\mathfrak{h}eight\alpha = \sum_i c_i$. We call the root with maximal height the ***longest*** or ***maximal root***.

The maximal root is always a long root (if there are two different lengths of roots), and the root space corresponding to the maximal root commutes with the entire Lie algebra: it is in the algebra's center. It is often convenient to begin at the maximal root when performing induction and go down by height.

Serre relations

Root systems help us classify (semi)simple Lie algebras.

21.5 Definition. An ***isometry*** $\varphi\colon (\Phi, \Delta) \to (\Phi', \Delta')$ of root systems with fixed simple roots is a linear map $\varphi\colon E \to E'$ of their underlying Euclidean spaces that preserves lengths (so $(\alpha, \beta)_E = (\varphi(\alpha), \varphi(\beta))_{E'}$) and sends Φ to Φ' and Δ to Δ'.

21.6 Theorem. *Let $\mathfrak{g}$ and $\mathfrak{g}'$ be semisimple Lie algebras over an algebraically closed field of characteristic* 0. *Then*

1. *$\mathfrak{g}$ is simple if and only if its root system is irreducible.*
2. *$\mathfrak{g} \cong \mathfrak{g}'$ as Lie algebras if and only if there is an isometry between their root systems.*

So the classification of simple Lie algebras comes down to the classification of irreducible root systems, or equivalently Dynkin diagrams. Indeed, every Dynkin diagram is the Dynkin diagram for some irreduicble Lie algebra. This can be proven directly by explicitly constructing a simple Lie algebra of each type, or it can be proven more generally with a rule for converting Dynkin diagrams to the generators and relations of the corresponding Lie algebra and proving that the resulting Lie algebra is simple.

Let $\mathfrak{g}$ be a semisimple Lie algebra, Φ be its corresponding root system, (a_{ij}) be its Cartan matrix given by

$$a_{ij} = \langle \alpha_i, \alpha_j \rangle = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$$

so that

$$S_{\alpha_i}(\beta) = \beta - \langle \beta, \alpha_i \rangle \alpha_i,$$

$\Delta = \{\alpha_1, \dots, \alpha_s\} \subset \Phi$ be a fixed set of simple roots, Φ^+ be the positive roots, and Φ^- be the negative roots. Then each positive root $\alpha \in \Phi^+$ gives rise to an $\mathfrak{sl}_2$ triple in $\mathfrak{g}$: $\langle e_\alpha, h_\alpha, f_\alpha \rangle$.

In particular,

$$h_\alpha = \frac{2\mathfrak{h}_\alpha}{(\alpha, \alpha)},$$

where $\mathfrak{h}_\alpha$ is dual to α, so that $\alpha(h_\alpha) = 2$. Then we choose $e_\alpha \in \mathfrak{g}_\alpha$ and $f_\alpha \in \mathfrak{g}_{-\alpha}$ so that

$$(e_\alpha, f_\alpha) = \frac{2}{(\alpha, \alpha)}.$$

Then $[h_\alpha, e_\alpha] = 2e_\alpha$, $[h_\alpha, f_\alpha] = -2f_\alpha$, and $[e_\alpha, f_\alpha] = h_\alpha$, making $\langle e_\alpha, h_\alpha, f_\alpha \rangle$ an $\mathfrak{sl}_2$ triple.

We have the following decomposition of $\mathfrak{g}$, modeled after the strict lower triangular, diagonal, and strict upper triangular decomposition of $_n$.

21.7 Theorem (Triangular decomposition)**.** *We have*

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+,$$

where

$$\mathfrak{n}_- = \sum_{\alpha \in \Phi^-} \mathfrak{g}_\alpha \quad \textit{and} \quad \mathfrak{n}_+ = \sum_{\alpha \in \Phi^+} \mathfrak{g}_\alpha \,.$$

*Here $\mathfrak{n}_- \oplus \mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, known as the **negative Borel subalgebra**, as is $\mathfrak{h} \oplus \mathfrak{n}_+$, known as the **positive Borel subalgebra**.*

We have that

- $\{e_{\alpha_i} \mid \alpha_i \in \Delta\}$ generates $\mathfrak{n}_+$,
- $\{f_{\alpha_i} \mid \alpha_i \in \Delta\}$ generates $\mathfrak{n}_-$, and
- $\{h_{\alpha_i} \mid \alpha_i \in \Delta\}$ generates $\mathfrak{h}$.

So together, $\{e_{\alpha_i}, h_{\alpha_i}, f_{\alpha_i} \mid \alpha_i \in \Delta\}$ generates $\mathfrak{g}$ subject to the ***Serre relations***, which we will give next class.

22 Lecture 22 (March 1): Serre relations *Scribe: Eric Zhang*

Let $\mathfrak{g}$ be semisimple over $k = \overline{k}$ with char$k = 0$. Let Φ be a root system and Φ^+ be positive roots. Each positive root $\alpha \in \Phi^+$ gives rise to an $\mathfrak{sl}_2$ triple in $\mathfrak{g}$: $\langle e_\alpha, h_\alpha, f_\alpha \rangle$. Denote the simple roots as $\Delta = \{\alpha_1, \ldots, \alpha_s\}$. The Cartan matrix is denoted as (a_{ij}) where $a_{ij} = \langle \alpha_i, \alpha_j \rangle$. Then $\mathfrak{g} = \mathfrak{n}_- \oplus \mathrm{H} \oplus \mathfrak{n}_+$ where $\mathfrak{n}_- = \oplus \, \mathfrak{g}_{-\alpha}$ and $\mathfrak{n}_+ = \oplus \, \mathfrak{g}_\alpha$ for $\alpha \in \Phi^+$.

22.1 Theorem. *e_i, h_i, and f_i have the following relations:*

S1. $[h_i, h_j] = 0$

S2. $[h_i, e_j] = a_{ij} e_j$ *and* $[h_i, f_j] = -a_{ij} f_j$

S3. $[e_i, f_j] = \delta_{ij} h_i$

S4. $(\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0$

S5. $(\operatorname{ad} f_i)^{1-a_{ij}} f_j = 0$

Proof. S1 holds because H is abelian. To see S2 holds, note $[h_i, e_j] = \alpha_i(h_i) e_j = \alpha_j(\frac{2H_{\alpha_i}}{(\alpha_i,\alpha_i)}) e^j = \frac{2}{(\alpha_i,\alpha_i)} \alpha_j(H_{\alpha_i}) e_j = \frac{2}{(\alpha_i,\alpha_i)} (\alpha_j, \alpha_i) e^j = \langle \alpha_j, \alpha_i \rangle e^j$. Similar argument applies to $[h_i, f_j]$. Relation S3 comes from $\mathfrak{sl}_2$. Relation S4 and S5 are usually called Serre's relations. Note $[e_{\alpha_i}, e_{\alpha_i}] \in \mathfrak{g}_{\alpha_i+\alpha_j}$ and $[e_{\alpha_i}, [e_{\alpha_i}, e_{\alpha_i}]] \in \mathfrak{g}_{2\alpha_i+\alpha_j}$. So $(\operatorname{ad} e_i)^r e_j \in \mathfrak{g}_{\alpha_j + r\alpha_i}$. Then $\{\beta + r\alpha\}_{r \geq 0}$ is a $\alpha-$string of roots through β which translates at the length of this root string. In particular, $r = -\langle \beta, \alpha \rangle$. Then this reduces to verifying the vaildity of the relations on all rank 2 root sytstems. ∎

22.2 Theorem (Serre's relation)**.** *Let Φ be a root system and $\Delta = \{\alpha_1, \ldots, \alpha_s\}$ be simple roots for a choice of polarization. Let $\mathfrak{g}$ is a complex lie algebra generated by $\{e_i, h_i, f_i\}_{1 \leq i \leq s}$ subjected to relations S1 to S5. Then $\mathfrak{g}$ is semisimple with root system Φ.*

23 Lecture 23 (March 3):Representation of Simple Lie Algebras *Scribe: Ranjan Pradeep*

We have $\mathfrak{g}$ (simple, complex lie algebra), Φ, $\Phi^{\pm}$, $\Delta = \{\alpha_1, \dots, \alpha_n\}$ (simple positive roots), $\Phi^+ = \{\beta_1, \dots \beta_n\}$, $R = \mathbb{Z}_\Phi \subset E \subset \mathfrak{h}^*$ (lattice),

- Weight lattice:
$$\Lambda = \{\lambda \in \mathfrak{h}^* | \langle \lambda, \alpha \rangle \in \mathbb{Z}, \alpha \in \Phi\}$$

- Dominant integral weights:
$$\Lambda^+ = \{\lambda \in \mathfrak{h}^* | \langle \lambda, \alpha_i \rangle \geq 0\}$$

- Basis of fundamental dominant integral weights:
$$\{\overline{\omega_1} \dots \overline{\omega_n}\} | \lambda = \sum \langle \lambda, \alpha_i \rangle \omega_i$$
$$\langle \overline{\omega_i}, \alpha_i \rangle = \delta_{ij}$$

This geometry is responsible for representation theory. There is, in general, a correspondence between dominant integral weights (Λ^+) and irreducible representations of a lie algebra. Recall, as categories we have $\mathrm{Rep}_k\, \mathfrak{g} = \mathscr{U}(\mathfrak{g}) -$ mod .

In the decomposition,
$$\mathfrak{g} = h^- \oplus \mathfrak{h} \oplus n^+$$

The first term is generated by negative weights E_{-B_i} and the last term is generated by positive weights E_{B_i}. Let $V \in \mathrm{Rep}_k\, \mathfrak{g}$. We have weight space decomposition:

$$\oplus_{\lambda \in \mathfrak{h}^*} V_\lambda, \text{ where } V_\lambda = \{v \in V | hv = \lambda(h)v\}$$

1. decomposition in eigenspaces: $\dim V < \infty \implies V = \oplus_{\lambda \in \mathfrak{h}^*} V_\lambda$
2. $E_\alpha V_\lambda \subset V_{\lambda+\alpha}, \alpha \neq 0$

   23.1 Remark. If $\dim V = \infty$ some of this doesn't work. BBG introduced the Category O, which includes the conditions necessary to do weight theory

   23.2 Definition. $V \in \mathrm{Rep}_k\, \mathfrak{g}$ is a highest weight module of weight λ if $\exists v^+ \in V$ such that

   (a) $hv^+ = \lambda(h)v^+$

   (b) $n^+v^+ = 0$

   (c) $V = gv^+$

   Construction of universal highest weight module M_λ, Verma module: Take the universal embedding algebra, kill the positive part and everything necessary from Cartan

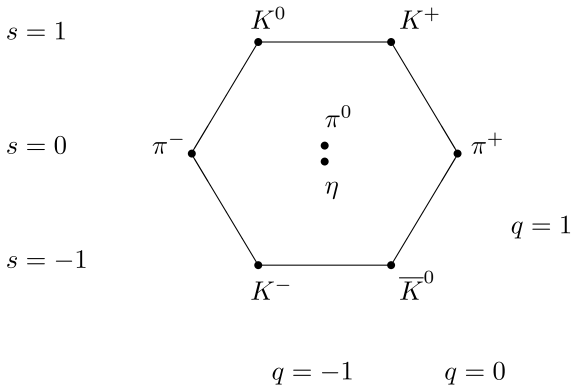
$$U(\mathfrak{g}) / \langle U(n^+), h - \lambda(h).1 | h \in \mathfrak{h} \rangle$$

3. **Claim**: For all V of highest weight λ, any highest weight module V is an image of M_λ
$$M_\lambda \to V$$

4. For all λ, M_λ has a unique max submodule and a unique simple quotient
$$\text{max sub} \to M_\lambda \to V(\lambda)$$
This is a beginning of the BGG resolution, helps prove the Weyl character formula in general

5. Let V be a module of highest weight λ. $\dim V_\lambda = 1$, $\dim V_\mu < \infty$ for all weight μ of V such that $\mu \subset \{\lambda - \sum c_i \alpha_i | c_i \in \mathbb{Z}_{\geq 0}\}$

6. $\dim V(\lambda) < \infty \iff \lambda \in \Lambda^+$



24 Presentation Notes

24.1 Root systems of Type A_n
Presenter: Jackson Morris

Here, we will provide an explicit construction to prove the existence of complex simple Lie algebras of type A_n. We work over a field k of characteristic different from 2.

Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$, recalling that

$$\mathfrak{sl}_{n+1} = \{X \in \mathfrak{gl}_{n+1} : \mathrm{tr}X = 0\}$$

. Lie algebras of this sort are called **special linear Lie algebras**. The simplest Lie algebra, it is the one we have become the most familiar with throughout the course. We take for the Cartan subalgebra the subalgebra $\mathfrak{h}$ of diagonal matrices in $\mathfrak{g}$. Let $E_{i,j}$ denote the matrix with a 1 in position (i, j) and 0 elsewhere. This subalgebra has the basis $\{E_{i,i} - E_{i+1,i+1} : 1 \leq i \leq n\}$. Now, it is clear that $\mathfrak{h}$ is a toral subalgebra since each matrix here is diagonal. To see maximality, suppose that $\mathfrak{m}$ is another toral subalgebra containing $\mathfrak{h}$. Since $\mathfrak{m}$ is toral, it is abelian. Then, for any $X \in \mathfrak{h}$ and $Y \in \mathfrak{m}$, $XY = YX$. But for any entry b_{ij} in the matrix Y, this says that $a_j b_{ij} = b_{ij} a_i$, where a_i is the i-the entry of X along the main diagonal. Since our choice of X was arbitrary, though, it must be that $Y \in \mathfrak{h}$, showing maximality.

The other root spaces ofv$\mathfrak{g}$ are given by the basis $\{E_{ij} : i \neq j\}$. Observe the following brackets:

$$[E_{ij}, E_{ji}] = E_{ii} - E_{jj} \in \mathfrak{h}$$
$$[[E_{ij,E_{ji}}], E_{ij}] = 2E_{ij} \neq 0$$

This implies that the a basis of Φ is

$$\{E_{ij} : i \neq j\}$$

24.2 Root systems of Type B_n and C_n
Presenter: William Dudarov

In this note, we provide explicit constructions to prove the existence of complex simple Lie algebras of types B_n and C_n, covering two of the four types of classical Lie algebras.

That is, we take a family of Lie algebras existing "in nature," and show it is of type B_n, and another family, showing it is of type C_n.

We work with a field k not of characteristic 2, and follow the book by Erdmann & Wildon.

Type C_n:

We start off with the case of type C_n since it is slightly simpler.

Let $\mathfrak{g} = \mathfrak{gl}_{2n}^S$, where S is the matrix

$$S = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

Recall that this means that

$$\mathfrak{g} = \{X \in \mathfrak{gl}_{2n} : X^T S = -SX\}.$$

Lie algebras of this sort are called **symplectic Lie algebras**, often denoted $\mathfrak{sp}_{2n}$. This terminology comes from symplectic geometry, the study of symplectic manifolds - a special kind of smooth manifold arising in classical mechanics.

At any point of a symplectic manifold, the tangent space is a vector space equipped with a non-degenerate skew-symmetric bilinear form whose group of structure-preserving transformations is the Lie group $Sp(2n, k)$, with corresponding Lie algebra $\mathfrak{sp}_{2n}$.

We show that these Lie algebras $\mathfrak{sp}_{2n}$ are of type C_n.

We can also describe $\mathfrak{g}$ the following way:

$$\mathfrak{g} = \left\{ \begin{bmatrix} m & p \\ q & -m^T \end{bmatrix} : p = p^T, q = q^T \right\}.$$

This is because if X is in $\mathfrak{gl}_{2n}$ with $X^T S = -SX$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T S = S \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

which yields

$$\begin{bmatrix} -c^T & a^T \\ -d^T & b^T \end{bmatrix} = \begin{bmatrix} -c & -d \\ a & b \end{bmatrix}.$$

We take for our Cartan subalgebra the subalgebra $\mathfrak{h}$ of diagonal matrices in $\mathfrak{g}$.

Let $H \in \mathfrak{h}$ have diagonal entries denoted $x_1, \ldots, x_n, -x_n, \ldots, -x_n$. Then

$$H = \sum_{i=1}^{n} x_i (e_{i,i} - e_{i+n,i+n}),$$

where $e_{i,j}$ is the matrix with a 1 in the (i, j)th position and zeroes everywhere else.

This is in fact a maximal toral subalgebra since all of the elements are semi-simple/diagonalizable (in fact, already diagonal!), and if $\mathfrak{h} \subset \mathfrak{m}$ a toral subalgebra, since $\mathfrak{m}$ is toral and thus abelian, then for $X \in \mathfrak{h}, Y \in \mathfrak{m}$, we have $XY = YX$. We show that $Y \in \mathfrak{h}$, i.e. Y is diagonal. Note $x_j y_{i,j} = y_{i,j} x_i$, so that $y_{i,j} = 0$ when $i \neq j$ so that Y is diagonal and $\mathfrak{h}$ is maximal, as desired.

What are the root spaces of $\mathfrak{g}$?

They are given by the basis

$$\begin{aligned}
m_{i,j} &= e_{j,i} - e_{n+j,n+i} \text{ for } 1 \leq i \neq j \leq n \\
p_{i,j} &= e_{i,n+j} + e_{j,n+i} \text{ for } 1 \leq i < j \leq n \\
p_{i,i} &= e_{i,n+i} \text{ for } 1 \leq i \leq n \\
q_{j,i} &= p_{i,j}^T = e_{n+j,i} + e_{n+i,j} \text{ for } 1 \leq i < j \leq n \\
q_{i,i} &= e_{n+i,i} \text{ for } 1 \leq i \leq n.
\end{aligned}$$

Checking the bracket with h, we have

$$\begin{aligned}
[H, m_{i,j}] &= (x_i - x_j) m_{i,j} \\
[H, p_{i,j}] &= (x_i + x_j) p_{i,j} \\
[H, q_{i,j}] &= -(x_i + x_j) q_{j,i}.
\end{aligned}$$

Let $\alpha_i \in \mathfrak{h}^*$ be such that

$$\alpha_i(H) = x_i.$$

By the above, we have roots

$$\begin{aligned}
&\alpha_i - \alpha_j \\
&\alpha_i + \alpha_j \\
-&(\alpha_i + \alpha_j) \\
&2\alpha_i \\
-&2\alpha_i.
\end{aligned}$$

We claim that

$$\{\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \ldots, \alpha_{n-1} - \alpha_n, 2\alpha_n\}$$

is a basis for Φ.

Proof. This is in fact the case since

$$\begin{aligned}
\alpha_i - \alpha_j &= (\alpha_i - \alpha_{i+1}) + (\alpha_{i+1} - \alpha_{i+2}) + \cdots + (\alpha_{j-1} - \alpha_j) \\
\alpha_i + \alpha_j &= (\alpha_i - \alpha_{i+1}) + (\alpha_{i+1} - \alpha_{i+2}) + \cdots + (\alpha_{j-1} - \alpha_j) + 2((\alpha_j - \alpha_{j-1}) + \cdots + (\alpha_{n-1} - \\
2\alpha_i &= 2((\alpha_j - \alpha_{j-1}) + \cdots + (\alpha_{n-1} - \alpha_n)) + 2\alpha_n.
\end{aligned}$$

■

We show that $\mathfrak{g}$ is semi-simple by showing that the Killing form is non-degenerate. For $H, H' \in \mathfrak{h}$, we have

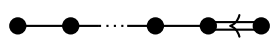
$$\begin{aligned}
K(H, H') &= \sum_{\alpha \in \Phi} \alpha(H)\alpha(H') \\
&= 2\sum_{i<j}(x_i - x_j)(x_i' - x_j') + 2\sum_{i<j}(x_i + x_j)(x_i' + x_j') + 2\sum_{i=1}^{n} 4x_i x_i' \\
&= (4n+1)\sum_{i=1}^{n} x_i x_i' \\
&= (2n+2)\,\mathrm{Trace}(HH').
\end{aligned}$$

This is non-degenerate since $K(H, H) = 0$ if and only if $H = 0$.

The Cartan matrix is given by

$$\begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -2 & 2 \end{bmatrix}.$$

The Dynkin diagram is given by

C_n 

which is connected, so that $\mathfrak{g}$ is simple.

The Weyl group of type C_n is isomorphic to $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$, where the factors of $\mathbb{Z}/2\mathbb{Z}$ switch the signs of the basis vectors.

Type B_n:

Let $\mathfrak{g} = \mathfrak{gl}^S_{2n+1}(\mathbb{C})$ for $n \geq 1$, where

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{bmatrix}.$$

Recall once again that this means that

$$\mathfrak{gl}^S_{2n+1} = \{X \in \mathfrak{gl}_{2n+1} : X^T S = -SX\}.$$

Lie algebras of this sort are called **odd-dimensional orthogonal Lie algebras**, often denoted $\mathfrak{so}_{2n+1}$. They are the Lie algebras of odd-dimensional orthogonal groups, and orthogonal groups are groups of isometries on Euclidean spaces.

We show that these Lie algebras $\mathfrak{so}_{2n+1}$ are of type B_n.

We can also describe $\mathfrak{g}$ the following way:

$$\mathfrak{g} = \left\{ \begin{bmatrix} 0 & c^T & -b^T \\ b & m & p \\ -c & q & -m^T \end{bmatrix} : p = -p^T, q = -q^T \right\}.$$

As above, we take $\mathfrak{h}$ the subalgebra of diagonal matrices to be our Cartan subalgebra, for the same reasons as in the case of C_n above. We take $H \in \mathfrak{h}$ as above, with diagonal entries $0, x_1, \ldots, x_n, -x_1, \ldots, x_n$.

Here, instead, the root spaces are spanned by the matrices where non-zero entries occur only on the blocks labelled b and c.

As above, we define $m_{i,j}, p_{i,j}$, and $q_{i,j}$ in the same way, with the same bracket with H, except that we define

$$b_i = e_{i,0} - e_{0,n+1}$$
$$c_i = e_{0,i} - e_{n+i,0},$$

where we calculate that

$$[H, b_i] = x_i b_i$$
$$[H, c_i] = -x_i c_i.$$

Let $\alpha_i(H) = x_i$ as above.

We have roots

$$\begin{aligned} \alpha_i &\text{ corresponding to } b_i \\ -\alpha_i &\text{ corresponding to } c_i \\ \alpha_i - \alpha_j &\text{ corresponding to } m_{i,j} (i \neq j) \\ \alpha_i + \alpha_j &\text{ corresponding to } p_{i,j} (i < j) \\ -(\alpha_i + \alpha_j) &\text{ corresponding to } q_{j,i} (i < j). \end{aligned}$$

We have a basis for our root system given by

$$\{\alpha_i - \alpha_{i+1} : 1 \leq i \leq n\} \cup \{\alpha_n\}.$$

We show that $\mathfrak{g}$ is semi-simple by since the Killing form, using the same kind of calculations as above, is given by $K(H, H') = (2n-1)\operatorname{Trace}(HH')$ where $H, H' \in \mathfrak{h}$.

This is non-degenerate since $K(H, H) = 0$ if and only if $H = 0$, as desired. The Cartan matrix is given by

$$\begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & -1 \\ & & & & & -1 & 2 \end{bmatrix}.$$

The Dynkin diagram is given by

B_n 

which is connected, so that $\mathfrak{g}$ is simple.

The Weyl group of type B_n is, just like C_n, isomorphic to $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$, where the factors of $\mathbb{Z}/2\mathbb{Z}$ switch the signs of the basis vectors.

24.3 Root systems of Type D_n

Presenter: Ranjan Pradeep

In this presentation, we give an explicit construction to prove existence of the complex simple Lie algebras of D_n. The idea is to take a Lie algebra existing "in nature" (ie. as vector spaces of linear transformations, [Hum73]) and show it has a given type.

What is D_n:

$$D_n := \mathfrak{so}(2n, \mathbb{C})$$

is the Lie algebra of the special orthogonal group in $2n$ variables, $\mathrm{SO}(2n)$. It consists of complex orthogonal $n \times n$ matrices, those that satisfy $x + x^{'} = 0$, where $x^{'}$ is the transposition of x with respect to the anti-diagonal.

$$\mathfrak{g} = \{\mathfrak{gl}(2n, \mathbb{C}) | x + x^{'} = 0\}$$

An alternate description is given by

$$\mathfrak{g} = \mathfrak{gl}_S(2n, \mathbb{C}) = \{x \in \mathfrak{gl}(2n, \mathbb{C}) | Sx + x^t S = 0\}$$

where

$$S = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$$

This is isomorphic to the presentation of D_n as skew-symmetric matrices, by $x \mapsto xS$, but this presentation is more convenient for finding a Cartan subalgebra, as we will see.

Proof. Viewing S as a permutation matrix, we find that Sx sends row i to row $n - i$. Transposing and reapplying S, we get $S(Sx)^t = x'$ and $SSx = x$. So,

$$x^t S + Sx = 0$$

$$(Sx)^t + Sx = 0$$

$$S(Sx)^t + x = 0$$

And,

$$x + x' = 0$$

. ∎

Writing the elements of $\mathfrak{g}$ as block matrices (ie. say $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is a $2n \times 2n$ matrix in $\mathfrak{g}$ where each block is an $n \times n$ matrix) , we calculate to find that

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} : B^t = -B, C^t = -C \right\}$$

24.3.1 Low dimensional special cases:

The low dimensional cases are special. $\mathfrak{so}(2)$ is one-dimensional, abelian, and not simple. It consists of matrices $\begin{pmatrix} 0 & -z \\ z & 0 \end{pmatrix}$. $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, and so is semi-simple but not simple ($D_2 \cong A_1 \oplus A_1$). $\mathfrak{so}(6) \cong \mathfrak{sl}(4)$, so D_3 occurs as A_3. So while it's possible to define D_n for $n \geq 1$, the discussion proceeds with the typical restriction that $n \geq 4$.

Cartan Subalgebra:

We show that the Cartan subalgebra is diagonal matrices in $\mathfrak{g}$.

$$\mathfrak{h} = \begin{pmatrix} a_1 & & & & & \\ & \cdots & & & & \\ & & a_n & & & \\ & & & -a_1 & & \\ & & & & \cdots & \\ & & & & & -a_n \end{pmatrix}$$

This is the set of all diagonal matrices in $\mathfrak{g}$.

Proof. It's immediately clear every matrix in $\mathfrak{h}$ as described lies in $\mathfrak{g}$, but we must show that any diagonal matrix in $\mathfrak{g}$ is contained in $\mathfrak{h}$. Say $x = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in \mathfrak{g}$, Then,

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}^t \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} + \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}^t = 0$$

So,

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} + \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} = \begin{pmatrix} A+B & 0 \\ 0 & A+B \end{pmatrix} = 0$$

And, $x = \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix}$ is an element of $\mathfrak{h}$ as desired. ■

It is the maximal toral subalgebra of $\mathfrak{g}$

Proof. It's clear that $\mathfrak{h}$ is a toral subalgebra, since every element is a diagonal matrix, and so semi-simple. Let $\mathfrak{m}$ be a toral subalgebra that contains $\mathfrak{h}$ and let $a \in \mathfrak{h}$, $b \in \mathfrak{m}$. Since $\mathfrak{m}$ is a toral algebra, it is abelian, and so $ab = ba$. Considering an element b_{ij} we have $a_j b_{ij} = b_{ij} a_i$. Since the choice of a was arbitrary, this forces $b_{ij} = 0$ when $i \neq j$, and we find that $b \in \mathfrak{m}$. So, $\mathfrak{h}$ is maximal. ■

With the description $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$, the Cartan subalgebra consists of block-diagonal matrices

$$\mathfrak{h} = \left\{ \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \cdots & \\ & & & A_n \end{pmatrix} \right\}, A_i = \begin{bmatrix} 0 & h_i \\ -h_i & 0 \end{bmatrix}$$

Root Spaces:

Next we find the root spaces for $\mathfrak{h}$. Since $\dim(\mathfrak{g}) = n(2n-1)$ and $\dim(\mathfrak{h}) = n$ there are $2n^2 - 2n$ roots.

E_{ij} is the matrix with a 1 at the i,j element and zeros elsewhere. Let $h \in \mathfrak{h}$ be an element with diagonal entries $a_1, \ldots a_n, -a_1 \ldots a_n$,

$$h = \sum_{i=1}^{n} a_i(E_{ii} - E_{i+n,i+n})$$

Consider the subspace of $\mathfrak{g}$ spanned by matrices whose only elements are at positions labeled b and c. This has a subspace $b_i = E_{i,0} - E_{0,i+n}$ and $c_i = E_{0,i} - E_{i+n,0}$ for $1 \le i \le n$. Calculation gives that $[h, b_i] = a_i b_i$ and $[h, c_i] = -a_i c_i$

This suggests the following choice of basis:

$$m_{ij} = E_{ij} - E_{j+n,i+n} \text{ for } i \ne j$$

$$p_{ij} = E_{i,j+n} - E_{j,i+n} \text{ for } i < j$$
$$q_{ji} = p_{ij}^t = E_{i+n,j} - E_{j+n,i} \text{ for } i < j$$

Then calculation works out such that the obvious basis elements are simultaneously eigenvectors for the action of $\mathfrak{h}$. This is determined by calculating:

$$[h, m_{ij}] = (a_i - a_j)E_{ij} - (-a_j + a_i)E_{j+n,i+n} = (a_i - a_j)(E_{ij} - E_{j+n,i+n}) = (a_i - a_j)m_{ij}$$

With similar calculations showing that

$$[h, p_{ij}] = (a_i + a_j)p_{ij}$$
$$[h, q_{ji}] = -(a_i + a_j)q_{ji}$$

In summary we have the following root subspaces:

root: $e_i - e_j$, eigenvector: m_{ij}

root: $e_i + e_j$, eigenvector: p_{ij}

root: $-(e_i + e_j)$, eigenvector: q_{ji}

The root system is

$$\Phi = \{-e_i - e_j, -e_i + e_j, e_i - e_j, e_i + e_j : i < j\}$$

There are $4\binom{n}{2}$ roots, as expected.

Basis:

A base for our root system is given by

$$\Delta = \{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n\} \cup \{e_{n-1} + e_n\}$$

For the sake of simpler notation, $\alpha_i = e_i - e_{i+1}$ and $\alpha_n = e_{n-1} + e_n$, so our set of simple roots is just $\{\alpha_1, \ldots, \alpha_n\}$ Going through the set of roots from the last subsection, we can see that if $\gamma \in \Phi$, then either γ or $-\gamma$ appears as a non-negative linear combination of elements of Δ with integer coefficients. See that

$$e_i - e_j = \sum_{k=i}^{j-1} \alpha_k$$

$$e_i + e_j = \sum_{k=i}^{n-2} \alpha_k + \sum_{k=j}^{n-1} \alpha_k + \alpha_n$$

Since Δ has n (dim $\mathfrak{h}$) elements, as expected, so Δ is a base for our root system.

We have a root space decomposition,

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{a \in \Phi} \mathfrak{g}_a$$

Killing Form:

If the killing form is non-degenerate, then $\mathfrak{g}$ is semi-simple,

Let $h \in \mathfrak{h}$ be an element with entries $a_1, \ldots, a_n, -a_i, \ldots, -a_n$ and h' be another matrix with elements $a'_1, \ldots, a'_n, -a'_i, \ldots, -a'_n$

Then,

$$\begin{aligned}
K(h, h') &= \sum_{\alpha \in \Phi} \alpha(h)\alpha(h') \\
&= 2\sum_{i<j} (a_i - a_j)(a'_i - a'_j) + 2\sum_{i<j} (a_i + a_j)(a'_i + a'_j) \\
&= 4\sum_{i<j} (a_i a'_i + a_j a'_j) \\
&= 4(n-1)\sum_{i=1}^{n} a_i a'_i \\
&= 4(n-1)\,\mathrm{Trace}(hh')
\end{aligned}$$

We have that K is nondegenerate because $\sum_{i=1}^{n} a_i a'_i$ is the usual inner product, so $K(h, h) = 0$ only if $h = 0$

Cartan Matrix:

For $n > j = i + 1 > 0$ we have

$$\langle a_i, a_j \rangle = \langle E_i - E_{i+1}, E_j - E_{j+1} \rangle = -1$$

If $n > j > i + 1 > 0$,

$$\langle a_i, a_j \rangle = 0$$

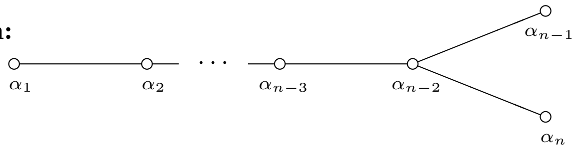
The branching comes from

$$\langle \alpha_{n-2}, \alpha_{n-1} \rangle = -1$$
$$\langle \alpha_{n-1}, \alpha_n \rangle = 0$$

The Cartan matrix of type D_n is

$$\begin{pmatrix}
2 & -1 & 0 & & & & & & & \\
-1 & 2 & -1 & & & & & & & \\
 & -1 & 2 & -1 & & & & & & \\
 & & \cdots & \cdots & \cdots & & & & & \\
 & & & \cdots & \cdots & \cdots & & & & \\
 & & & & \cdots & \cdots & \cdots & & & \\
 & & & & & -1 & 2 & -1 & & \\
 & & & & & & -1 & 2 & -1 & -1 \\
 & & & & & & & -1 & 2 & 0 \\
 & & & & & & & -1 & 0 & 2
\end{pmatrix}$$

Dynkin Diagram:



Since all roots have the same length, D_n is simply laced.

Weyl Group:

The Weyl group of type D_n is isomorphic to the semidirect product of the symmetric group S_n and the group $\mathbb{Z}_2$.

24.4 Exceptional Lie Algebras and the Freudenthal Magic Square

Presenter: Justin Bloom

Let $F = \mathbb{C}$ be the complex numbers.

24.1 Definition. We take F-algebras $K, Q, \mathbb{O}$ to be $F \times F$, the quaternion, and the octonion algebras of dimension 2, 4, 8 respectively.

These, together with the trivial algebra F are called *composition algebras*, and each composition algebra C is equipped with an involution π_C, which we denote $\pi_C(x) = \bar{x}$ when context is clear.

The involution π_K is the matrix $\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$ with the basis defining the algebra $F \times F$. With respect to the basis $(1,1),(1,-1)$, the involution is the matrix $\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, which is closer to how the other composition algebras are defined.

The involution π_Q with respect to the familiar basis $1, i, j, k$ is

$$\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

(these are algebras over $\mathbb{C}$, it's more convenient here to call $\omega \in \mathbb{C}$ instead of i, the quaternion element. Notice $\omega - i \in Q$ is a zero divisor)

The involution $\pi_{\mathbb{O}}$ is also diagonal, with a 1 followed by seven -1s.

24.2 Definition. For any composition algebra C, we define the *Freudenthal algebra* associated to C by

$$\mathcal{H}(C) = \{X \in M_3(C) \mid \bar{X}^t = X\}$$

where $\bar{X} = (\bar{x}_{ij})$ for $X = (x_{ij})$. We endow this space with the multiplication

$$M \bullet N = \frac{1}{2}(MN + NM).$$

Ordinary matrix multiplication is not necessarily associative, because $C = \mathbb{O}$ is not an associative composition algebra. It's helpful to see

$$\mathcal{H}(C) = \left\{ \begin{pmatrix} \xi_1 & x_1 & x_2 \\ \bar{x}_1 & \xi_2 & x_3 \\ \bar{x}_2 & \bar{x}_3 & \xi_3 \end{pmatrix} \;\middle|\; \xi_i \in F, x_i \in C \right\}.$$

Now we define the algebras:

$$J_1 = \mathcal{H}(F), \quad J_2 = \mathcal{H}(K), \quad J_4 = \mathcal{H}(Q), \quad J_8 = \mathcal{H}(\mathbb{O})$$

so that $\dim J_n = 3(n+1)$.

The Freudenthal algebras $(J_i, \bullet)$ are examples of *Jordan algebras*

24.3 Definition. A given composition or Jordan algebra B has an involution π_B. We may define the *trace* of an element $x \in B$ by $\mathrm{Tr}_B(x) = x + \pi(x)$. We denote by B^0 the trace free elements of B, i.e.

$$B^0 = \{x \mid \bar{x} = -x\}.$$

For a composition algebra C, we define a bilinear product $*$ on C^0 by

$$a * b = ab - \frac{1}{2}\operatorname{Tr}_C(ab),$$

and similarly on J^0 by

$$x * y = xy - \frac{1}{3}\operatorname{Tr}_J(xy).$$

For any algebra B, denote left and right multiplication by $b \in B$ with maps $\ell_b, r_b \in \operatorname{End}_F(B)$.

It can be checked that for our composition algebras C, for any $a, b \in C$ the map

$$\partial_{a,b} = [\ell_a, \ell_b] + [\ell_a, r_b] + [r_a, r_b]$$

is a derivation of $\operatorname{Der}(C, C)$, where $[\ ,\ ]$ is taken in $\operatorname{End}_F(C) = \mathfrak{gl}(C)$

24.4 Definition. Given a composition algebra C, and a Jordan algebra J, we may define a Lie algebra structure $\mathfrak{L}(C, J)$ on the vector space

$$\mathfrak{L}(C, J) = \operatorname{Der}(C, C) \oplus (C^0 \otimes_F J^0) \oplus \operatorname{Der}(J, J).$$

To define $[\ ,\ ]$, we quantify

$$\forall \quad D \in \operatorname{Der}(C, C), \quad D' \in \operatorname{Der}(J, J), \quad a, b, \in C^0, \quad x, y \in J^0 :$$

(1) $[\ ,\ ]$ is the usual bracket on $\operatorname{Der}(C, C)$ and $\operatorname{Der}(J, J)$, and $[D, D'] = 0$,

(2) $[a \otimes x, D + D'] = D(a) \otimes x + a \otimes D'(x)$,

(3) $[a \otimes x, b \otimes y] = \frac{1}{12}\operatorname{Tr}(xy)\partial_{a,b} + (a * b) \otimes (x * y) + \frac{1}{2}\operatorname{Tr}(ab)[r_x, r_y]$, where it can be checked $[r_x, r_y] \in \operatorname{Der}(J, J)$ for each Jordan algebra.

24.5 Remark. Denote $J_0 = F \times F \times F$. Denote the algebra $F \times \cdots \times F$ by $F^{\times n}$, so $J_0 = F^{\times 3}$. Denote $|V| = \dim V$ for vector spaces.

(1) $\mathfrak{L}(F, J) = \operatorname{Der}(J, J)$ for each Jordan algebra J, as $\operatorname{Der}(F, F) = 0$ and $F^0 = 0$. Similarly $\mathfrak{L}(C, F) = \operatorname{Der}(C, C)$.

(2) $\operatorname{Der}(F^{\times n}, F^{\times n}) = 0$ by directly computing de for $e^2 = e$.

(3) Assume tables 1 and 2 are accurate. Then we may deduce

   (a) $L(\mathbb{O}, J_0) = D_4$, and has dimension 28.

   (b) $\operatorname{Der}(\mathbb{O}, \mathbb{O}) = G_2$, and $\operatorname{Der}(J_8, J_8) = F_4$.

   (c) The trace of J_0 is the sum of entries, so its trace-free subspace is of codimension 1. In fact $\operatorname{codim} B^0 = 1$ for each algebra $B = C, J$ in the arguments for $\mathfrak{L}$.

   (d) $|D_4| = 28 = |G_2| + (8-1)(3-1) + 0 = |G_2| + 14$ so $|G_2| = 14$.

   (e) $\operatorname{Der}(J_1, J_1) = A_1$ and $F_4 = \mathfrak{L}(\mathbb{O}, J_1)$, so

$$|F_4| = |G_2| + (8-1)(6-1) + |A_1| = 14 + 35 + 3 = 52.$$

Table 1: $\mathfrak{L}(A, J)$ as A ranges through composition algebra, and J ranges through Jordan algebras.
The rightmost 4 columns are known as *Freudenthal's magic square.*

	F	J_0	J_1	J_2	J_4	J_8
F	0	0	A_1	A_2	C_3	F_4
K	0	$F \oplus F$	A_2	$A_2 \oplus A_2$	A_5	E_6
Q	A_1	$A_1 \oplus A_1 \oplus A_1$	C_3	A_5	D_6	E_7
$\mathbb{O}$	G_2	D_4	F_4	E_6	E_7	E_8

(f) $|E_6| = 0 + (2-1)(27-1) + 52 = 78$.

(g) $|E_7| = 3 + (4-1)(27-1) + 52 = 133$.

(h) $|E_8| = 14 + (8-1)(27-1) + 52 = 248$.

24.5 Root systems of Type F_4
Presenter: Leo Mayer

Let $\mathbb{O}$ denote the Octionian algebra. Recall that this is an 8-dimensional algebra over $\mathbb{R}$ which is unital, but neither commutative nor associative. There is also a linear involution $\mathbb{O} \to \mathbb{O}$, written as $a \mapsto \overline{a}$, which satisfies the following properties:

1. $\overline{ab} = \overline{b}\overline{a}$.
2. $a = \overline{a}$ if and only if $a \in \mathbb{R}$.
3. The bilinear form $n(a, b) := \frac{1}{2}(a\overline{b} + b\overline{a})$ is nondegenerate and symmetric.
4. The quadratic form $n(a) := n(a, a) = a\overline{a}$ is multiplicative, i.e. $n(ab) = n(a)n(b)$.

24.6 Definition. Let $\mathcal{H}_3(\mathbb{O})$ be the set of 3×3 matrices in $\mathbb{O}$ satisfying $M = \overline{M}^t$. Give $\mathcal{H}_3(\mathbb{O})$ the structure of a commutative, non-associative algebra over k with the operation $M * N := \frac{1}{2}(MN + NM)$.

24.7 Notation. We can see that

$$\mathcal{H}_3(\mathbb{O}) = \left\{ \begin{pmatrix} \lambda_1 & a & b \\ \overline{a} & \lambda_2 & c \\ \overline{b} & \overline{c} & \lambda_3 \end{pmatrix} \mid \lambda_i \in \mathbb{R}, a, b, c \in \mathbb{O} \right\}.$$

Table 2: The same table, with all dimensions known a priori:

	1	3	6	9	15	27
1	0	0	3	8	21	F_4
2	0	2	8	16	24	E_6
4	3	9	21	24	66	E_7
8	G_2	28	F_4	E_6	E_7	E_8

For $i = 1, 2, 3$ let e_i denote the matrix with a 1 in the ith diagonal and 0s elsewhere. We also define

$$\mathbb{O}_{12} = \left\{ a_{12} := \begin{pmatrix} 0 & a & 0 \\ \overline{a} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid a \in \mathbb{O} \right\},$$

$$\mathbb{O}_{13} = \left\{ b_{13} := \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ \overline{b} & 0 & 0 \end{pmatrix} \mid b \in \mathbb{O} \right\},$$

$$\mathbb{O}_{23} = \left\{ c_{23} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & \overline{c} & 0 \end{pmatrix} \mid c \in \mathbb{O} \right\},$$

so that $\mathcal{H}_3(\mathbb{O}) = \mathbb{R}\, e_1 \oplus \mathbb{R}\, e_2 \oplus \mathbb{R}\, e_3 \oplus \mathbb{O}_{12} \oplus \mathbb{O}_{13} \oplus \mathbb{O}_{23}$.

This decomposition is very well-behaved with respect to the operation $*$. In particular,

24.8 Lemma. *For $M \in \mathcal{H}_3(\mathbb{O})$, we have*

1. *$M \in \mathbb{O}_{12} \oplus \mathbb{O}_{13}$ if and only if $M = 2e_1 * M$,*
2. *$M \in \mathbb{O}_{12} \oplus \mathbb{O}_{23}$ if and only if $M = 2e_2 * M$,*
3. *$M \in \mathbb{O}_{13} \oplus \mathbb{O}_{23}$ if and only if $M = 2e_3 * M$.*

Proof. This follows from an immediate computation. For example,

$$\begin{pmatrix} \lambda_1 & a & b \\ \overline{a} & \lambda_2 & c \\ \overline{b} & \overline{c} & \lambda_3 \end{pmatrix} = 2e_1 * \begin{pmatrix} \lambda_1 & a & b \\ \overline{a} & \lambda_2 & c \\ \overline{b} & \overline{c} & \lambda_3 \end{pmatrix} = \begin{pmatrix} 2\lambda_1 & a & b \\ \overline{a} & 0 & 0 \\ \overline{b} & 0 & 0 \end{pmatrix}$$

holds if and only if each λ_i and c are all 0. ■

We are finally ready to define our main object of study.

24.9 Definition. F_4 is the Lie algebra of derivations of $\mathcal{H}_3(\mathbb{O})$.

24.10 Notation. We write $D_0 := \{D \in F_4 \mid De_i = 0, i = 1,2,3\}$.

Then D_0 is a Lie subalgebra of F_4, and $\mathcal{H}_3(\mathbb{O})$ is a representation of D_0.

24.11 Lemma. *For $1 \leq i < j \leq 3$, we have $D_0\mathbb{O}_{ij} \subseteq \mathbb{O}_{ij}$.*

Proof. By the first lemma, $M \in \mathbb{O}_{ij}$ if and only if $2e_i * M = M = 2e_j * M$. Applying $D \in D_0$ to this equation gives $2e_i * (DM) = DM = 2e_j * (DM)$, and we conclude $DM \in \mathbb{O}_{ij}$ as well. ∎

Since $\mathbb{O}_{ij} \cong \mathbb{O}$ as a vector space, we obtain three induced representations $\rho_{ij} : D_0 \to \mathfrak{gl}(\mathbb{O})$ by restricting the action of D_0 to the invariant subspace $\mathbb{O}_{ij}$. Concretely, we associate to $D \in D_0$ the map $D_{ij} : \mathbb{O} \to \mathbb{O}$ defined by $(D_{ij}a)_{ij} = Da_{ij}$.

24.12 Lemma. *Each D_{ij} is skew-symmetric with respect to the norm on $\mathbb{O}$. The three representations ρ_{ij} are irreducible, inequivalent, and induce isomorphisms $D_0 \cong D_4$, where D_4 is the Lie algebra of all skew endomorphisms of $\mathbb{O}$.*

Proof. For the first claim we need to show that for $a, b \in \mathbb{O}$ we have $n(D_{ij}a, b) = -n(a, D_{ij}b)$. A computation shows that $a_{ij} * b_{ij} = n(a,b)(e_i + e_j)$, and so

$$0 = D(n(a,b)(e_i + e_j)) = Da_{ij} * b_{ij} + a_{ij} * Db_{ij} = n(D_{ij}a, b) + n(a, D_{ij}b).$$

The remaining claims require more machinery than we have time to develop here, and so we instead direct the curious reader to [Jac71]. ∎

We now turn to define three more related subspaces of F_4.

24.13 Definition. For $N \in \mathcal{H}_3(\mathbb{O})$, let R_N be the linear endomorphism $M \mapsto M * N$.

24.14 Definition. For $1 \leq i < j \leq 3$, let J_{ij} be the collection of endomorphisms of the form $[R_{e_i} R_{a_{ij}}]$, where $a \in \mathbb{O}$.

Quick computations show that each $D \in J_{ij}$ is a derivation, and so each J_{ij} is a subspace (although not a subalgebra) of F_4. We can do some example computations:

$$[R_{e_1}, R_{a_{12}}]e_1 = (e_1 * e_1) * a_{12} - (e_1 * a_{12}) * e_1 = \frac{1}{2}a_{12} - \frac{1}{4}a_{12} = \frac{1}{4}a_{12}$$

$$[R_{e_1}, R_{a_{12}}]e_2 = (e_2 * e_1) * a_{12} - (e_2 * a_{12}) * e_1 = 0 - \frac{1}{4}a_{12} = -\frac{1}{4}a_{12}$$

$$[R_{e_1}, R_{a_{12}}]e_3 = (e_3 * e_1) * a_{12} - (e_3 * a_{12}) * e_1 = 0$$

Similarly, we see that

$$[R_{e_1}, R_{b_{13}}] : e_1 \mapsto \frac{1}{4}b_{13}, \quad e_2 \mapsto 0, \quad e_3 \mapsto -\frac{1}{4}b_{12},$$

$$[R_{e_2}, R_{c_{23}}] : e_1 \mapsto 0, \quad e_2 \mapsto \frac{1}{4}c_{23}, \quad e_3 \mapsto -\frac{1}{4}c_{23}.$$

24.15 Proposition. *$F_4 = D_0 \oplus J_{12} \oplus J_{13} \oplus J_{23}$ as vector spaces.*

Proof. The above calculations show that any two of the listed subspaces have trivial intersection, so we need only show that all four span F_4.

Let $D \in F_4$ be arbitrary. Applying D to the relation $e_i * e_i = e_i$ gives $2e_i * (De_i) = De_i$. Then Lemma 24.8 implies we can write

$$De_1 = a_{12} - b_{13}, \quad De_2 = c_{23} - d_{12} \quad De_3 = e_{13} - f_{23}$$

for some $a, b, c, d, e, f \in \mathbb{O}$. For $i \neq j$, applying D to the relation $e_i * e_j = 0$ gives $De_i * e_j = -e_i * De_j$, and so

$$a = d, \quad b = e, \quad c = f.$$

Now let $D' = 4[R_{e_1} R_{a_{12}}] - 4[R_{e_1} R_{b_{13}}] + 4[R_{e_2} R_{c_{23}}]$. The above calculations show that $D'e_i = De_i$ for each i, and so $D - D' \in D_0$. Since $D' \in J_{12} \oplus J_{13} \oplus J_{23}$, this completes the proof. ■

24.16 Proposition. *Let* $\mathrm{H} \subset D_0$ *be a Cartan subalgebra of* D_0*. Then* H *is also a Cartan subalgebra of* F_4*.*

Proof. A similar computation as in Lemma 24.11 shows that D_{ij} is an invariant subspace for the adjoint representation of D_0 in F_4, and moreover the induced representation of D_0 on $D_{ij} \cong \mathbb{O}$ is equal to ρ_{ij}. Since D_{ij} is an irreducible representation of D_0, the induced representation of H decomposes as a direct sum of 1 dimensional weight spaces. Thus, F_4, viewed as the adjoint representation of H in F_4, decomposes as a direct sum of 1-dimensional weight spaces, and we conclude that H is a Cartan subalgebra. ■

We next turn to describing the root system of F_4. Since $D_0 \cong D_4$ is a subalgebra containing H, there is a basis $\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$ of H for which the roots in D_0 are $\{\pm\epsilon_i \pm \epsilon_j\}$. Using facts from the representation theory of D_4 which we will not develop here, the 24 additional roots from D_{12}, D_{13}, D_{23} are $\{\pm\epsilon_i\}$ and $\{\frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)\}$.

One choice of simple roots is $\alpha_1 = \epsilon_2 - \epsilon_3, \alpha_2 = \epsilon_3 - \epsilon_4, \alpha_3 = \epsilon_4$, and $\alpha_4 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)$. With respect to this basis, the Cartan matrix is

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

24.6 Classification of Coxeter Graphs
Presenter: Raymond Guo

Method 1:

24.17 Remark. Let (Φ, Δ) be a root system and base. For $\alpha \in \Delta$, let $\alpha' = \frac{\alpha}{|\alpha|}$, and let $\Delta' = \{\alpha' : \alpha \in \Delta\}$. We see that for $\alpha' \neq \beta' \in \Delta'$,

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = \langle \alpha', \beta' \rangle \langle \beta', \alpha' \rangle = 2(\alpha, \beta) \cdot 2(\beta, \alpha) = 4(\alpha, \beta)^2$$

Note also $(\alpha, \beta) \leq 0 \implies (\alpha', \beta') \leq 0$, and that Δ' consists of linearly independent unit vectors.

24.18 Definition. Admissible set, associated Coxeter graphs.

Let E be a Euclidean space. Define $U \subset E$ to be an admissible set if $U = \{e_1, e_2, .., e_n\}$ consists of linearly independent unit vectors, $(e_i, e_j) \leq 0$ for $i \neq j$, and $4(e_i, e_j)^2 \in \{0, 1, 2, 3\}$. Let the Coxeter graph induced by an admissible set have vertices $\{e_1, e_2, ..., e_n\}$ and have $4(e_i, e_j)^2$ edges between e_i and e_j. When (Φ, Δ) is a root system and base, the above remark shows that Δ' is admissible and the Coxeter graph on (Φ, Δ) is the same as the Coxeter graph induced by Δ'. **We classify the connected Coxeter graphs of admissible sets.**

For the remainder of this argument, let U be an admissible set, Γ be its coxeter graph. Assume Γ is connected.

24.19 Proposition. *1) Let $U' \subset U$, with U admissible. U' is admissible, and its Coxeter graph is the full subgraph of Γ induced by U'.*

Proof. Entirely obvious from the definitions. ■

24.20 Proposition. *2) The number of pairs of vertices in U' connected by any edges is less than n.*

Proof. Let $\epsilon = \sum_{i=1}^{n} \epsilon_i$. Then

$$0 < (\epsilon, \epsilon) = \sum_{i=1}^{n} (\epsilon_i, \epsilon_i) + 2 \sum_{i<j} (\epsilon_i, \epsilon_j) = n + 2 \sum_{i<j} (\epsilon_i, \epsilon_j)$$

That is, $-n < \sum_{i<j} 2(\epsilon_i, \epsilon_j)$. If ϵ_i and ϵ_j are connected, $2(\epsilon_i, \epsilon_j) \in \{-1, -\sqrt{2}, -\sqrt{3}\}$, so $2(\epsilon_i, \epsilon_j) \leq -1$. The above inequality shows that at most n such pairs exist.

■

24.21 Proposition. *3) Γ is acyclic.*

Proof. Consider replacing every set of multiple edges in Γ with a single edge. 2) yields that the resulting graph is a connected graph with less than n vertices, so it's a tree. Thus this "reduced" graph is acyclic, so Γ is acyclic. ■

24.22 Proposition. *4) Each vertex has degree at most 3.*

Proof. Let $\epsilon \in \Gamma$ be arbitrary, and let $\eta_1, \eta_2, ..., \eta_k$ be all of the adjacent vertices. Since ϵ and the η_i's are linearly independent, there's a unit vector η_0 in the span of $\{\epsilon, \eta_1, \eta_2, ..., \eta_k\}$ orthogonal to all η_i (Gram-Schmidt). $(\epsilon, \eta_0) \neq 0$ because ϵ isn't in the span of the $\eta_i's$. Then $\epsilon = \sum_{i=0}^{n} (\epsilon, \eta_i)\eta_i$ (standard identity for orthonormal bases) so

$$1 = (\epsilon, \epsilon) = \sum_{i=0}^{k} (\epsilon, \eta_i)^2$$

Since $(\epsilon, \eta_0)^2 > 0$, we must have $\sum_{i=1}^k (\epsilon, \eta_i)^2 < 1$, so $\sum_{i=1}^k 4(\epsilon, \eta_i)^2 < 4$. This is exactly the statement that the degree of ϵ is less than 4.

■

24.23 Proposition. *5) If Γ has a triple edge, it must be G_2.*

Proof. Noting that we're assuming that Γ is connected, this is obvious from 4. ■

24.24 Proposition. *6) Let $\{\epsilon_1, \epsilon_2, ..., \epsilon_k\}$ induce a full subgraph of Γ that is a simple path (a path where adjacent nodes are connected by a single edge), where specifically ϵ_i and ϵ_{i+1} are adjacent for each i. Let $\epsilon = \sum_{i=1}^k \epsilon_i$. Then $\Gamma' = U \backslash \{\epsilon_1, \epsilon_2, ..., \epsilon_k\} \cup \{\epsilon\}$ is admissible, with Coxeter graph formed by contracting the path to the one vertex ϵ.*

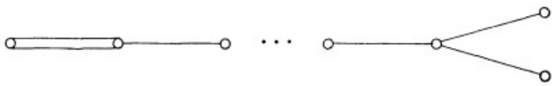
Proof. Linear independence is obvious. By hypothesis, for $i < j$, $2(\epsilon_i, \epsilon_{i+1}) = -\delta_j^{i+1}$, so

$$(\epsilon, \epsilon) = \sum_{i=1}^{k} (\epsilon_i, \epsilon_i) + \sum_{i<j} 2(\epsilon_i, \epsilon_j) = k - (k-1) = 1$$

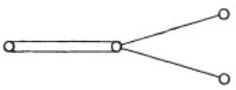
and thus ϵ is a unit vector. Let $\eta \in U \backslash \{\epsilon_i\}_{i=1}^k$. η is connected to at most one ϵ_i because the graph must be acyclic, so $4(\eta, \epsilon)^2 = 4(\eta, \epsilon_i)^2 \in \{0, 1, 2, 3\}$. This also shows that the new graph is formed by contracting the path to the single vertex, noting that the number of edges from η to ϵ is the same as the number of edges from η to ϵ_i (and η has no other edges to other ϵ_j's) ■

24.25 Proposition. *7) Let a vertex that connects to three other distinct vertices be called a node. Γ has at most one instance of either a double edge or a node.*

Proof. Assume for contradiction that Γ has say, both a node and a double edge. They're connected by some path, so Γ has a subgraph of the form:

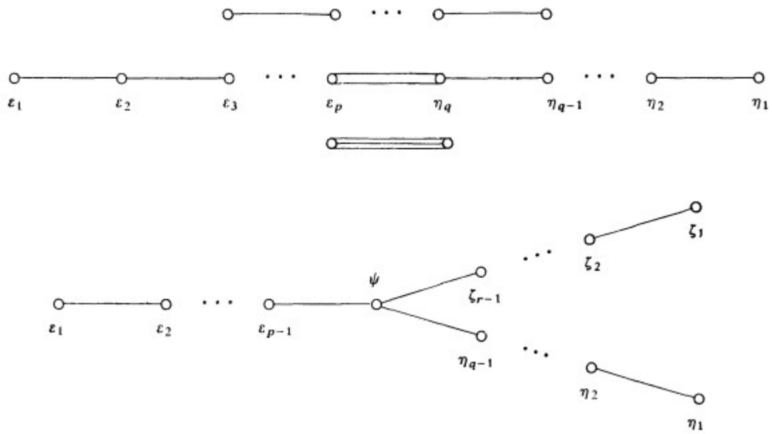


1) yields that this subgraph is itself a Coxeter Graph of an admissible set. 6) yields that we can contract the path in the middle of this graph, yielding another Coxeter Graph of an admissible set:



This contradicts 4), so Γ cannot have both a node and a double edge. Assuming that Γ has two nodes or two double edges yields similar contradictions. ■

24.26 Proposition. *8)* Γ *takes one of the following forms:*



In future arguments, we will refer to these as graphs of Type 1,2,3, and 4. In our arguments regarding Types 2 and 4, we will use the names ϵ_i*,* η_i*,* ζ_i*, and* ψ *to refer to the vectors associated with the vertices labeled in these diagrams.*

Proof. 7) yields that Γ either has one node, one double edge, or neither. The graphs of Type 2 are the only graphs with one double edge and no nodes. The graphs of Type 4 are the only graphs with one node and no double edges. We've noted above that the graph of Type 3 is the only graph with a triple edge. The only remaining case is a graph with no nodes and no multiple edges, which must be a simple path (Type 1). ■

24.27 Proposition. *9) If* Γ *is Type 2, it's either* F_4 *or* $B_n = C_n$*.*

Proof. Let $\epsilon = \sum_{i=1}^{p} i\epsilon_i$ and let $\eta = \sum_{i=1}^{q} i\eta_i$. Again for $i < j$, $2(\epsilon_i, \epsilon_j) = -\delta_{i+1}^{j}$, so

$$(\epsilon, \epsilon) = \sum_{i=1}^{p} i^2 - \sum_{i<j} 2ij(\epsilon_i, \epsilon_j) = \sum_{i=1}^{p} i^2 - \sum_{i=1}^{p-1} i(i+1) = \frac{p(p+1)}{2}$$

Similarly $(\eta, \eta) = \frac{q(q+1)}{2}$. $(\epsilon, \eta)^2 = (q\epsilon_q, p\epsilon_p)^2 = \frac{2p^2q^2}{4} = \frac{p^2q^2}{2}$. Cauchy-Schwarz (noting that they're not colinear) yields

$$(\epsilon, \eta)^2 < (\epsilon, \epsilon)(\eta, \eta)$$

By above computation, $\frac{p^2q^2}{2} < \frac{p(p+1)q(q+1)}{4}$, which yields $(p-1)(q-1) < 2$ after algebraic manipluation. Then either $p = q = 2$ (Γ is of the form F_2) or $p = 1$ and q takes any value (Γ is of the form $B_n = C_n$) or $q = 1$ and p takes any value (also $B_n = C_n$).

■

24.28 Proposition. *10) If* Γ *is type 4, it's either* D_n *or* E_n *with* $n = 6, 7, 8$*.*

Proof. Let $\epsilon = \sum_{i=1}^{p-1} i\epsilon_i$, $\eta = \sum_{i=1}^{q-1} i\eta_i$, and $\zeta = \sum_{i=1}^{r-1} i\zeta_i$. By the same argument as in 9, $(\epsilon, \epsilon) = \frac{p(p-1)}{2}$, $(\eta, \eta) = \frac{q(q-1)}{2}$, and $(\zeta, \zeta) = \frac{r(r-1)}{2}$. Let $\theta_1, \theta_2, \theta_3$ be the angles between ψ and each of ϵ, η, and ζ and respectively. An argument similar to the proof of 4) yields $\sum_{i=1}^{3} \cos^2(\theta_i) < 1$.

We note $(\epsilon, \psi)^2 = ((p-1)\epsilon_{p-1}, \psi)^2 = \frac{(p-1)^2}{4}$. Having shown that $(\epsilon, \epsilon) = \frac{p(p-1)}{2}$ and noting that $(\psi, \psi) = 1$, we now compute $\cos^2(\theta_1) = \frac{(\epsilon,\psi)^2}{(\epsilon,\epsilon)(\psi,\psi)} = \frac{1}{2}(1 - \frac{1}{p})$. Same for η and ζ. The above equality yields

$$\frac{1}{2}(1 - \frac{1}{p}) + \frac{1}{2}(1 - \frac{1}{q}) + \frac{1}{2}(1 - \frac{1}{r}) < 1$$

which gives, after simple manipluation,

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$$

Note that we assume $p, q, r \geq 2$ (otherwise ψ isn't actually a node, so Γ isn't actually type 4). WLOG let $\frac{1}{p} \leq \frac{1}{q} \leq \frac{1}{r}$. Then $\frac{3}{2} \geq \frac{3}{r} \geq \frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$ (since $r \geq 2$) so $r = 2$. This leaves $\frac{1}{p} + \frac{1}{q} > \frac{1}{2}$. If $q = 2$, p can be anything. If $q = 3$, $\frac{1}{p} > \frac{1}{6}$ so $p \in \{3, 4, 5\}$ (p cannot be 2 because $p \geq q$).

Thus we can have the triples $(p, q, r) = (p, 2, 2)$ (Γ is of the form D_n), or $(p, q, r) = (3, 3, 2), (4, 3, 2), (5, 3, 2)$ (Γ is of the form E_6, E_7, E_8). ■

The results of 8,9,10 complete our classification. If Γ is of Type 1, it's of the form A_n. If it's Type 2, it's of the form $B_n = C_n$ or F_4 by 9). If it's Type 3, it's of the form G_2. If it's Type 4, it's of the form D_n or E_6, E_7, E_8 by 10).

Setup For Method 2:

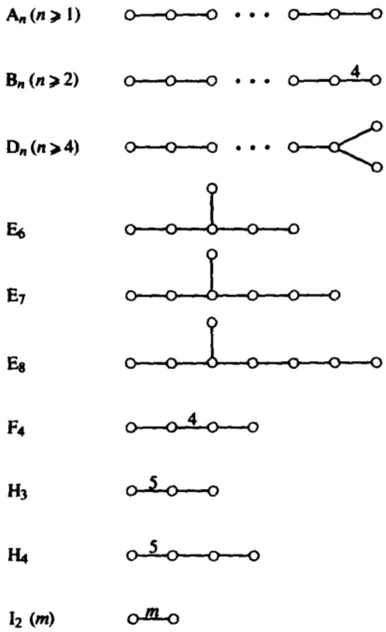
In the argument given in Reflection Groups and Coxeter Groups, we no longer require root systems to satisfy $\langle \alpha, \beta \rangle \in \mathbb{Z}$. Bases still exist. We define the Weyl group in the same way.

Additionally, letting s_α be the reflection across α, we define $m(\alpha, \beta)$ to be the order of $s_\alpha s_\beta$ in the Weyl group. An appeal to the dihedral group yields that for α, β in a base, $4(\alpha, \beta)^2 = -\cos(\pi/m(\alpha, \beta))$. Explicit computation shows that $4(\alpha, \beta)^2 = 0, 1, 2, 3$ corresponds to $m(\alpha, \beta) = 2, 3, 4, 6$.

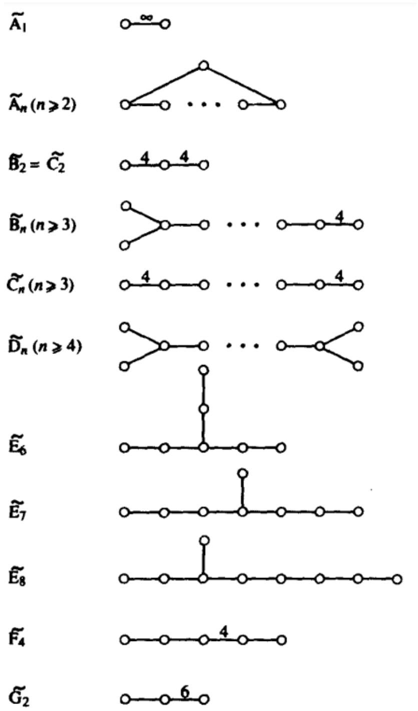
We redefine Coxeter graphs as well. A Coxeter graph for a root system will still have vertices as elements in the base Δ. Now, for $\alpha \neq \beta \in \Delta$, there is no edge from α to β if $m(\alpha, \beta) = 2$ ($(\alpha, \beta) = 0$), there is an unlabeled edge if $m(\alpha, \beta) = 3$ ($(\alpha, \beta) = 1$), and there is an edge labeled by $m(\alpha, \beta)$ otherwise.

24.7 Classification of Coxeter Graphs
Presenter: Bashir Abdel-Fattah

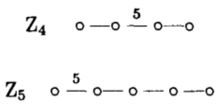
We call a Coxeter graph positive definite if its corresponding matrix is positive definite, and by convention we will say that it is positive semi-definite if its corresponding matrix is positive semi-definite but not positive definite. We also say that a Coxeter graph is of positive type if it is either positive definite or positive semi-definite. Some examples of positive definite Coxeter graphs include



In order to check that the above graphs have positive definite matrices, it suffices to check that the principal minors (the determinants of the square submatrices formed by taking the first k rows and first k columns of the original $n \times n$ matrix for some $1 \leq k \leq n$) are all strictly positive, which can be checked inductively for A_n, B_n, and D_n, and directly otherwise. In addition to the positive definite Coxeter graphs above, we also have the following positive semidefinite Coxeter graphs:



(where the number of nodes is the subscript plus one). In order to see that these are all positive semidefinite, we can note that these are all given by adding a single vertex to one of the corresponding positive definite graphs, so all of the proper principal minors of the corresponding matrix have positive determinant, and we just need to check that the matrix itself has zero determinant. This can be done by direct computation. It's also useful to note that the following graphs aren't of positive type, again by direct computation:



Next, we want to show that the list of examples of positive definite Coxeter graphs that Raymond talked about in fact enumerates *all* of the positive definite Coxeter graphs, which we accomplish by showing that any such graph cannot include any of the above non-positive definite graphs as a subgraph. By a subgraph of a Coxeter graph Γ, we mean a graph Γ' that can be obtained from Γ by eliminating some of its vertices and their adjacent edges and/or decreasing the weight labels of some of its edges. However, in order to do this we first need a technical lemma.

24.29 Lemma. *We say that an $n \times n$ matrix $A = (a_{ij})$ is indecomposable if there is no partition of the index set $\{1, \dots, n\}$ in nonempty subsets I, J such that $a_{ij} = 0$ for all $i \in I$ and $j \in J$.*

Suppose A is an indecomposable symmetric positive semidefinite matrix such that

$a_{ij} \leq 0$ for all $i \neq j$, and that $x = [x_1, \ldots, x_n]^T$ is a nontrivial vector such that $x^T Ax = 0$. Then $x_i \neq 0$ for all $i = 1, \ldots, n$.

Proof. Let $z = [z_1, \ldots, z_n]^T$ be defined by $z_i = |x_i|$. Then, using that A is positive semidefinite and $a_{ij} \leq 0$ for all $i \neq j$, we have that

$$\begin{aligned}
0 \leq z^t Az &= \sum_{i,j=1}^n a_{ij} z_i z_j = \sum_{i,j=1}^n a_{ij} |x_i||x_j| \\
&= \sum_{i=1}^n a_{ij} |x_i|^2 + \sum_{i \neq j} a_{ij} |x_i x_j| \leq \sum_{i=1}^n a_{ij} x_i^2 + \sum_{i \neq j} a_{ij} x_i x_j \\
&= \sum_{i,j=1}^n a_{ij} x_i x_j = x^t Ax = 0,
\end{aligned}$$

forcing equality throughout. Then note that the fact that $z^T Az = 0$ in fact implies that $Az = 0$; recalling from linear algebra that every symmetric positive semidefinite matrix is orthogonally diagonalizable, take P to be an orthogonal matrix such that

$$P^T AP = D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

(where $\lambda_i \geq 0$ for all i by the fact thay A is positive semidefinite). Letting $y = [y_1, \ldots, y_n]^T$ such that $z = Py$, then we have that

$$0 = z^t Az = (Py)^T A(Py) = y^T (P^T AP)y = y^T Dy = \sum_{i=1}^n d_i y_i^2.$$

Because each of the terms above is nonnegative and they collectively sum to zero, then each term must itself be zero, which implies that $d_i y_i = 0$ for all i and hence $Dy = 0$. Then

$$Az = A(Py) = P(P^T AP)y = P(Dy) = 0$$

as claimed. Now let $J \subset \{1, \ldots, n\}$ denote the (nonempty) set of indices j such that $z_j \neq 0$, and let I be its complement. The fact that $Az = 0$ means that

$$\sum_{j=1}^n a_{ij} z_j = \sum_{j \in J} a_{ij} z_j = 0$$

for all $i \in I$ in particular. Because $a_{ij} \leq 0$ for all $i \in I$ and $j \in J$ (noting that we must have that $i \neq j$) and that $z_j = |x_i| > 0$ for all $j \in J$, then every term of the above sum is nonpositive and we must have that they are all equal to zero, and thus a_{ij} must be equal to zero for all $i \in I$ and $j \in J$. However, if

$$I = \Big\{ i \in \{1, \ldots, n\} : z_i = 0 \Big\}$$

is also nonempty, this contradicts the assumption that A is indecomposable, thus we must have that $z_i = |x_i| \neq 0$ for all i and hence $x_i \neq 0$ for all i as desired. ∎

Using this lemma, we can prove the following result about Coxeter graphs:

24.30 Theorem. *Suppose Γ is a connected Coxeter graph of positive type. Then every (proper) subgraph of Γ is positive definite.*

Proof. Let n be the number of vertices of Γ, and let Γ' be a proper subgraph of Γ with $k \leq n$ vertices. By relabelling the vertices of Γ, we can suppose without loss of generality that the vertices of Γ' are exactly the first k vertices of Γ. Let A' and A be the matrices associated with Γ' and Γ, respectively, where A' is a $k \times k$ matrix and A is an $n \times n$ matrix. Note that the fact that Γ is connected means that A is an indecomposable matrix (since $a_{ij} = -\cos(\pi/m_{ij}) = 0$ if and only if $m_{ij} = 2$ if and only if there is no edge between the vertices i and j, so the existence of a partition I, J of the index set $\{1, \ldots, n\}$ such that $a_{ij} = 0$ for all $i \in I$ and $j \in J$ is equivalent to the existence of a partition I, J of the set of vertices of Γ such that there are no edges connecting a vertex in I to a vertex in J). Also, the fact that $m'_{ij} \leq m_{ij}$ for all $i, j \in \{1, \ldots, k\}$ means that

$$a'_{ij} = -\cos(\pi/m'_{ij}) \geq -\cos(\pi/m_{ij}) = a_{ij}$$

for all $i, j \in \{1, \ldots, k\}$. Then suppose for the sake of contradiction that A' is not a positive definite matrix, meaning that there exists some nonzero vector $x = [x_1, \ldots, x_k]^T$ such that $x^T A' x \leq 0$. Consider the vector

$$y = [|x_1|, \ldots, |x_k|, 0, \ldots, 0] \in \mathbb{R}^n;$$

Using that A is positive semidefinite, we can calculate

$$\begin{aligned}
0 \leq y^T A y &= \sum_{i,j=1}^{n} a_{ij} y_i y_j = \sum_{i,j=1}^{k} a_{ij} |x_i||x_j| \leq \sum_{i,j=1}^{k} a'_{ij} |x_i||x_j| \\
&= \sum_{i=1}^{k} a'_{ij} |x_i|^2 + \sum_{i \neq j} a'_{ij} |x_i x_j| \leq \sum_{i=1}^{k} a'_{ij} x_i^2 + \sum_{i \neq j} a'_{ij} x_i x_j \\
&= \sum_{i,j=1}^{k} a'_{ij} x_i x_j = x^T A' x \leq 0
\end{aligned}$$

(where we have used above that $a'_{ij} \leq 0$ for all $i \neq j$). Therefore we must have that $y^T A y = 0$, and because A is an indecomposable positive semidefinite matrix, the previous lemma tells us that we must have that all of the components of y are nonzero, meaning that $k = n$. Furthemore, the fact that

$$0 = \sum_{i,j=1}^{k} a'_{ij} |x_i||x_j| - \sum_{i,j=1}^{k} a_{ij} |x_i||x_j| = \sum_{i,j=1}^{k} (a'_{ij} - a_{ij}) |x_i||x_j|$$

and all of the terms above are non-negative, we must have $(a'_{ij} - a_{ij})|x_i||x_j| = 0$ for all $i, j \in \{1, \ldots, k\} = \{1, \ldots, n\}$, which in turn implies that $a'_{ij} = a_{ij}$ for all i and j by the fact that $|x_i|, |x_j| > 0$. However, we have contradicted the fact that Γ' is a proper subgraph of Γ, thus it must instead be the case that Γ' is a positive definite subgraph as desired. ■

Finally, we can proceed to the main result:

24.31 Theorem. *Every connected Coxeter graph of positive type must be one of the positive definite or positive semidefinite graphs listed previously.*

Proof. Suppose for the sake of contradiction that Γ is a connected Coxeter graph of positive type that is not among those enumerated, and that Γ has n vertices and maximum edge weight $m \in \mathcal{N} \cup \{\infty\}$. The previous theorem tells us that Γ cannot have any subgraphs that aren't positive definite, so we can rule out certain structures for Γ as follows:

1. Because all of the Coxeter graphs with fewer than two vertices were previously enumerated (A_1, $I_2(m)$, and $\widetilde{A}_1$), we must have that $n \geq 3$.

2. We must have that $m < \infty$, because otherwise Γ would have $\widetilde{A}_1$ as a proper subgraph, contradicting the fact that $\widetilde{A}_1$ isn't positive definite.

3. Γ cannot contain any cycles, or else it would contain $\widetilde{A}_n$ $(n \geq 2)$ as a subgraph. That is, Γ must be a tree.

Now suppose for a moment that $m = 3$. Then

4. Γ must have at least one branch node, by the assumption that it is distinct from A_n.

5. If Γ contained two or more branch points, then by connecting them via a path of edges (using connectedness) we would have that Γ contains a copy of $\widetilde{D}_n$ for $n > 4$, which is a contradiction.

6. Furthermore, Γ cannot contain $\widetilde{D}_4$, so its branch node has exactly three incident edges. Suppose the three branches of the tree have $a \leq b \leq c$ vertices, respectively (not counting the vertex at the center).

7. Because $\widetilde{E}_6$ is not a subgraph of Γ, we must have $a = 1$.

8. Because $\widetilde{E}_7$ is not a subgraph, we must have $b \leq 2$.

9. Because $\Gamma \neq D_n$, we cannot have $b = 1$, so we must have $b = 2$.

10. Because $\widetilde{E}_8$ is not a subgraph, we must have $c \leq 4$.

11. Recalling that $c \geq b = 2$, the only options are $c = 2, 3, 4$. In these cases, we would have $\Gamma = E_6, E_7, E_8$, respectively, contradicting the fact that Γ is not one of the previously listed graphs. Thus the case where $m = 3$ cannot occur.

Now we must have that $m \geq 4$.

12. If Γ has more than one edge with weight ≥ 4, then by connecting them via any path we would have $\widetilde{C}_n$ as a subgraph, a contradiction.

13. If Γ had a branch point, then by taking any path connecting it to the edge of weight ≥ 4 we would have a copy of $\widetilde{B}_n$, again a contradiction.

Now suppose that we have $m = 4$.

14. By the fact that $\Gamma \neq B_n$, the unique edge of weight 4 must be on the interior of the chain (rather than one of the two extremal edges).

15. Because Γ cannot contain $\widetilde{F}_4$, we must have that $n = 4$.

16. Then $\Gamma = F_4$, which is a contradiction. Thus we must have that $m \geq 5$.

Now we are in the case of $m \geq 5$.

17. Since Γ cannot contain $\widetilde{G}_2$, we must have that $m = 5$.

18. Γ also cannot contain Z_4, the unique edge of weight 5 must be one of the two extremal edges of the chain.

19. Because Γ doesn't contain Z_5, we must have $n \leq 4$.

20. Now we must have that Γ is either H_3 or H_4, contradicting the assumption that Γ was not one of the listed graphs.

Since every possibility has resulted in a contradiction, we conclude that the list of Coxeter graphs of positive type that we gave previously was in fact exhaustive. ■

24.8 Root system of Type G_2

Presenter: Nelson Niu

We construct the Lie algebra $\mathfrak{g}$ corresponding to the G_2 root system Φ as follows.

The Dynkin diagram for G_2 consists of two edges, corresponding to the two simple roots α and β, with a triple edge between them, corresponding to the fact that $\langle\alpha,\beta\rangle\langle\beta,\alpha\rangle = 3$. If we let α be the long simple root and β be the short simple root, then $\langle\alpha,\beta\rangle = -3$ and $\langle\beta,\alpha\rangle = -1$, making its Cartan matrix

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

From this we deduce that $\alpha, \alpha+\beta, \alpha+2\beta, \alpha+3\beta$, and β are all positive roots, that α and β have a length ratio of $\sqrt{3}$, and that the angle between α and β is $\cos^{-1}(-\sqrt{3}/2) = 5\pi/6$. Drawing out the roots, we deduce by inspection that $2\alpha+3\beta$ is the only other positive root (and the highest weight root), so we have 12 roots total: six short roots (the positive ones are $\beta, \alpha+\beta$, and $\alpha+2\beta$) and six long roots (the positive ones are $\alpha, \alpha+3\beta$, and $2\alpha+3\beta$). Each is a (scaled) copy of the A_2 root system.

Identifying the underlying Euclidean space of the root system with

$$E = \frac{\mathbb{R}\,\varepsilon_1 \oplus \mathbb{R}\,\varepsilon_2 \oplus \mathbb{R}\,\varepsilon_3}{\mathbb{R}(\varepsilon_1+\varepsilon_2+\varepsilon_3)},$$

we can identify β with $(1,0,-1)$ and $\alpha+\beta = (-1,1,0)$, making $\alpha+2\beta = (0,1,-1)$, as with the A_2 root system. Then $\alpha = (-2,1,1)$, so $\alpha+3\beta = (1,1,-2)$ and $2\alpha+3\beta = (-1,2,-1)$. Overall, the short roots are permutations of $(-1,0,1)$, while the long roots are permutations of $(-2,1,1)$ and $(-1,-1,2)$.

To construct $\mathfrak{g}$, it suffices to construct a Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\gamma\in\Phi} \mathfrak{g}_\gamma$ with Cartan subalgebra $\mathfrak{h}$ and each $\mathfrak{g}_\gamma = \mathbb{R}\,e_\gamma$ satisfying $[h, e_\gamma] = \gamma(h)e_\gamma$ for all $h \in \mathfrak{h}$, under some identification of E with $\mathfrak{h}^*$. Here $\dim\mathfrak{h} = 2$ and $\dim\mathfrak{g} = 2 + |\Phi| = 14$.

We will take $\mathfrak{g}$ to be a Lie subalgebra of $\mathfrak{so}_7$. Recall that $\mathfrak{so}_7$ consists of 7×7 matrices of the form

$$\begin{pmatrix} 0 & c^T & -b^T \\ b & M & Q \\ -c & P & -M^T \end{pmatrix},$$

where $b, c \in \mathbb{C}^3$ and M, P, and Q are 3×3 matrices. The subalgebra $\mathfrak{g}$ will consist of matrices of this form where $\mathrm{Trace}(M) = 0$ (so $M \in \mathfrak{sl}_3$),

$$P = \begin{pmatrix} 0 & w & -v \\ -w & 0 & u \\ v & -u & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix},$$

$b = \sqrt{2}(u,v,w)$, and $c = \sqrt{2}(x,y,z)$ for $u,v,w,x,y,z \in \mathbb{C}$. This is indeed a subspace of $\mathfrak{so}_7$ of dimension 14; to verify that it is a Lie subalgebra, we will eventually need to check that it is closed under the bracket. We can do this via casework on the basis elements we define below.

Let $\mathfrak{h} \subseteq \mathfrak{g}$ be the diagonal matrices in $\mathfrak{g}$ (so M, and thus $-M^T$, are diagonal); it is certainly an abelian Lie algebra, and we have $\dim\mathfrak{h} = 2$. Note that $\mathfrak{h}$ is spanned by the matrices

$$\begin{aligned} h_1 &= E_{22} - E_{33} - E_{55} + E_{66}, \\ h_2 &= E_{33} - E_{44} - E_{66} + E_{77}, \quad \text{and} \\ h_3 &= -h_1 - h_2 = -E_{22} + E_{44} + E_{55} - E_{77}. \end{aligned}$$

Then $\mathfrak{h}$ is dual to E with each h_ℓ dual to ε_ℓ.

For the following computations, we use the fact that given a diagonal matrix H and indices $i \neq j$, we have $[H, E_{ij}] = HE_{ij} - E_{ij}H = (H_{ii} - H_{jj})E_{ij}$. Then for $i, j \in \{2,3,4\}$, note that the six possible $E_{ij} - E_{j+3,i+3}$ matrices are linearly independent elements of $\mathfrak{g}$ with no diagonal entries, and for diagonal $h \in \mathfrak{h}$ we have

$$\begin{aligned} [h, E_{ij} - E_{j+3,i+3}] &= (h_{ii} - h_{jj})E_{ij} + (h_{i+3,i+3} - h_{j+3,j+3})E_{j+3,i+3} \\ &= (h_{ii} - h_{jj})(E_{ij} - E_{j+3,i+3}), \end{aligned}$$

as we always have $h_{ii} = h_{i+3,i+3}$ and $h_{jj} = h_{j+3,j+3}$ for $h \in \mathfrak{h}$. It follows, for instance, that

$$\begin{aligned}
&[h_1, E_{32} - E_{56}] = ((h_1)_{33} - (h_1)_{22})(E_{32} - E_{56}) = -2(E_{32} - E_{56}),\\
&[h_2, E_{32} - E_{56}] = ((h_2)_{33} - (h_2)_{22})(E_{32} - E_{56}) = E_{32} - E_{56}, \quad \text{and}\\
&[h_3, E_{32} - E_{56}] = -[h_1, E_{32} - E_{56}] - [h_2, E_{32} - E_{56}] = E_{32} - E_{56},
\end{aligned}$$

so since $\alpha = (-2, 1, 1)$, we have $[h, E_{32} - E_{56}] = \alpha(h)(E_{32} - E_{56})$. So we can set $e_\alpha = E_{32} - E_{56}$. Analogously, we have for the long roots that

$$\begin{aligned}
e_\alpha &= E_{32} - E_{56}, & e_{-\alpha} &= E_{23} - E_{65},\\
e_{\alpha+3\beta} &= E_{24} - E_{75}, & e_{-\alpha-3\beta} &= E_{42} - E_{57},\\
e_{2\alpha+3\beta} &= E_{34} - E_{76}, & e_{-2\alpha-3\beta} &= E_{43} - E_{67}.
\end{aligned}$$

Note that together, the h_i's and the e_γ's when γ is long span the subalgebra of $\mathfrak{g}$ consisting of matrices where b, c, P, and Q are all zero, a subalgebra isomorphic to $\mathfrak{sl}_3$ (and thus closed under the bracket).

We also have that setting $e_\beta = \sqrt{2}(E_{15} - E_{21}) + (E_{73} - E_{64}) \in \mathfrak{g}$ (i.e. setting $u = -1$ and all the other variables to 0) implies for all $h \in \mathfrak{h}$ that

$$[h, e_\beta] = \sqrt{2}((h_{11} - h_{55})E_{15} - (h_{22} - h_{11})E_{21}) + (h_{77} - h_{33})E_{73} - (h_{66} - h_{44})E_{64} = h_{22}e_\beta,$$

since $h_{11} = 0, h_{55} = -h_{22}, h_{66} = -h_{33}, h_{77} = -h_{44}$, and $h_{22} + h_{33} + h_{44} = 0$. So $[h_1, e_\beta] = 1, [h_2, e_\beta] = 0$, and $[h_3, e_\beta] = -1$, correctly yielding $[h, e_\beta] = \beta(h)e_\beta$ for all $h \in \mathfrak{h}$, as $\beta = (1, 0, -1)$. Analogously, we have for the short roots that

$$\begin{aligned}
e_\beta &= \sqrt{2}(E_{15} - E_{21}) + (E_{73} - E_{64}) = -e_{-\beta}^T,\\
e_{\alpha+\beta} &= \sqrt{2}(E_{13} - E_{61}) + (E_{27} - E_{45}) = -e_{-\alpha-\beta}^T,\\
e_{\alpha+2\beta} &= \sqrt{2}(E_{14} - E_{71}) + (E_{35} - E_{26}) = -e_{-\alpha-2\beta}^T.
\end{aligned}$$

We can verify that these are linearly independent elements of $\mathfrak{g}$ that, together with the h_i's and the e_γ's for long γ's, span $\mathfrak{g}$. This completes our verification that Φ is the root system induced by $\mathfrak{g}$.

25 Homework Problems

1 Homework problem. Find all nilpotent, nonabelian, 3-dim lie algebra, up to isomorphism.

Proof. Let $\mathfrak{g}$ be a nonabelian, nilpotent, 3-dim lie algebra and consider $[\mathfrak{g},\mathfrak{g}]$. Then $\dim[\mathfrak{g},\mathfrak{g}] \neq 0$ since otherwise $\mathfrak{g} \cong \mathfrak{g}/[\mathfrak{g},\mathfrak{g}]$ is abelian. Note the number of nonvanishing pairs $[x,y]$ must be no less than $\dim[\mathfrak{g},\mathfrak{g}]$. Elements in $Z(\mathfrak{g})$ induce vanishing pairs. So we have $\binom{\dim\mathfrak{g} - \dim Z(\mathfrak{g})}{2} \geq \dim[\mathfrak{g},\mathfrak{g}]$. Since $\mathfrak{g}$ is nilpotent, $\dim Z(\mathfrak{g}) \geq 1$. It follows that $\dim[\mathfrak{g},\mathfrak{g}] = \dim Z(\mathfrak{g}) = 1$. But $[\mathfrak{g},\mathfrak{g}] \cap Z(\mathfrak{g}) \neq 0$ for nilpotent $\mathfrak{g}$, so we may write $[\mathfrak{g},\mathfrak{g}] = Z(\mathfrak{g}) = \text{span }(z)$ for some $z \in Z(\mathfrak{g})$. Extend $\{z\}$ to a k-linear basis $\{x,y,z\}$ of $\mathfrak{g}$. Note that $[x,y]$ is the only nonvanishing bracket and hence $[x,y]$ spans $[\mathfrak{g},\mathfrak{g}] = Z(\mathfrak{g})$. We may assume $[x,y] = z$.

So, if exists, then $\mathfrak{g}$ must be a lie algebra generated by $\{x,y,z\}$ such that $[x,y] = z$ and $z \in Z(\mathfrak{g})$. And there is at most one of such lie algebra up to isomorphism because the above relations indeed define a lie bracket Note the Jacobi identity holds for the generators as $[x,[y,z]] + [y,[x,z]] + [z,[x,y]] = 0 + 0 + 0 = 0$.

Note that $x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, and $z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ with commutator bracket generate a lie subalgebra of $\mathfrak{gl}_n$ which is nonabelian, nilpotent and of dimension 3. This example shows the existence. ■

2 Homework problem. Let char $k = 0$ and $\overline{k} = k$. Then $\mathfrak{g}$ solvable implies $[\mathfrak{g},\mathfrak{g}]$ nilpotent.

Proof. Consider the adjoint action restricted to $[\mathfrak{g},\mathfrak{g}]$ and the induced short exact sequence $0 \to Z([\mathfrak{g},\mathfrak{g}]) \to [\mathfrak{g},\mathfrak{g}] \to \text{ad}([\mathfrak{g},\mathfrak{g}]) \to 0$. It suffices to show that $\text{ad}([\mathfrak{g},\mathfrak{g}])$ is nilpotent. Note $\text{ad}([\mathfrak{g},\mathfrak{g}]) = [\text{ad}\,\mathfrak{g}, \text{ad}\,\mathfrak{g}]$ as ad preserves lie bracket. Since $\mathfrak{g}$ is solvable, it follows from Lie's theorem (here we use the assumption on k) that $\text{ad}\,\mathfrak{g} \subseteq b_n$. Thus $[\text{ad}\,\mathfrak{g}, \text{ad}\,\mathfrak{g}] \subseteq [b_n, b_n] = u_n$ and is nilpotent. ■

3 Homework problem. Let V be a representation of a Lie algebra $\mathfrak{g}$, let $V_1 \subset V$ be a $\mathfrak{g}$-invariant subspace, and consider the corresponding short exact sequence

$$0 \longrightarrow V_1 \longrightarrow V \longrightarrow V_2 \longrightarrow 0$$

Let $B_V : \mathfrak{g} \times \mathfrak{g} \to k$ be the bilinear form defined by the formula $B_V(x,y) = tr(\rho_V(x)\rho_V(y))$ where $\rho_V : \mathfrak{g} \to gl(V)$ is the representation of $\mathfrak{g}$ on V; similarly for B_{V_1}, B_{V_2}. Show that

$$B_V = B_{V_1} + B_{V_2}$$

4 Homework problem. Let $I \subset \mathfrak{g}$ be an ideal in a Lie algebra $\mathfrak{g}$. Show that the restriction of the Killing form for $\mathfrak{g}$ to I coincides with the Killing form on I:

$$(K_{\mathfrak{g}})\downarrow_I = K_I.$$

Proof. $I \subset \mathfrak{g}$ is an ideal of $\mathfrak{g}$. Let $x \in I \subset \mathfrak{g}$. Then $[x, z] \in I$ for all $z \in \mathfrak{g}$. Choose a basis $\mathcal{B}_I$ of I as a vector space, and extend it to a basis $\mathcal{B}_\mathfrak{g}$ of $\mathfrak{g}$. Let the matrix of $\mathrm{ad}_I(x)$ with respect to $\mathcal{B}_I$ be $[\mathrm{ad}_I(x)]$. Then with respect to basis $\mathcal{B}_\mathfrak{g}$ of $\mathfrak{g}$, the matrix of $\mathrm{ad}_\mathfrak{g}(x)$ is given by the block matrix:

$$[\mathrm{ad}_\mathfrak{g}(x)] = \begin{pmatrix} [\mathrm{ad}_I(x)] & \star \\ 0 & 0 \end{pmatrix}$$

Note here the bottom right block is 0 precisely due to the fact that I is an ideal, and thus $\mathrm{ad}_\mathfrak{g}(x)(\mathfrak{g}) \subseteq I$. Thus, for $x, y \in I$, the matrix of $\mathrm{ad}_\mathfrak{g}(x)\,\mathrm{ad}_\mathfrak{g}(y)$ with respect to the basis $\mathcal{B}_\mathfrak{g}$ is the block matrix:

$$[\mathrm{ad}_\mathfrak{g}(x)\,\mathrm{ad}_\mathfrak{g}(y)] = \begin{pmatrix} [\mathrm{ad}_I(x)\,\mathrm{ad}_I(y)] & \star \\ 0 & 0 \end{pmatrix},$$

where $[\mathrm{ad}_I(x)\,\mathrm{ad}_I(y)] = [\mathrm{ad}_I(x)][\mathrm{ad}_I(y)]$ is the matrix of $\mathrm{ad}_I(x)\,\mathrm{ad}_I(y)$ with respect to the basis $\mathcal{B}_I$. Thus, clearly from the form of the matrix it follows that

$$(K_\mathfrak{g}) \downarrow_I (x, y) = \mathrm{Trace}(\mathrm{ad}_\mathfrak{g}(x)\,\mathrm{ad}_\mathfrak{g}(y)) = \mathrm{Trace}(\mathrm{ad}_I(x)\,\mathrm{ad}_I(y)) = K_I(x, y),\ \forall x, y \in I$$

. ■

5 Homework problem. Let x, y be two semisimple elements in gl_n.

1. Suppose $[x, y] = 0$. Show that $x + y$ is semisimple.
2. Give a counterexample to the semisimplicity of $x+y$ when they don't commute

6 Homework problem. Let $\mathfrak{g}$ be a simple Lie algebra. Show that an invariant bilinear symmetric form on $\mathfrak{g}$ is unique up to a scalar.

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