Math 509 – Homological Algebra
(lecture notes)

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Contents

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1 Lecture 1 (March 27): Basic definitions and Examples

Scribe: Haoming Ning

We use the following setup. Let R be a (commutative ring) and consider the abelian category of (left) R-modules.

1.1 Definition. A chain complex (of R-modules) is a exact sequence

$$\cdots \to C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \to \dots$$

such that $d_n \circ d_{n+1} = 0$. A cochain complex is a exact sequence

$$\cdots \leftarrow C^{n+1} \xleftarrow{d^{n+1}} C^n \xleftarrow{d^n} C^{n-1} \leftarrow \dots$$

such that $d^{n=1} \circ d^n = 0$.

Remark. We can turn a chain complex into a cochain complex and vice versa by defining $\tilde{C}^n := C_{-n}$.

1.2 Definition. A morphism $f_\bullet : B_\bullet \to C_\bullet$ consists of the data $f_n : B_n \to C_n$ such that each square commutes

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & C_{n+1} & \longrightarrow & C_n & \longrightarrow & C_{n-1} & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\dots & \longrightarrow & B_{n+1} & \longrightarrow & B_n & \longrightarrow & B_{n-1} & \longrightarrow & \dots
\end{array}$$

We denote $\mathbf{CH}_{\bullet,R}$ the category of chain complexes of R-modules.

As homework, show that $\mathbf{CH}_{\bullet,\mathbf{R}}$ (and $\mathbf{CH}^\bullet_{\mathbf{R}}$) is an abelian category. Reference: MacLane "Categories for the working mathematician".

1.3 Definition. Let $C_\bullet \in \mathbf{CH}_{\bullet,R}$. Define $Z_n = \ker d_n \subseteq C_n$, $B_n = \operatorname{im} d_{n+1} \subseteq C_n$. Note that $d^2 = 0$ implies that $B_n \subseteq Z_n$. We define $H_n(C_\bullet) = Z_n/B_n$, called the *n-th homology group* of $C_\bullet$. The *n-th cohomology group* of a cochain complex $H^n(C^\bullet)$ is define similarly.

1.4 Definition. A chain complex $C_\bullet$ is exact (acyclic) if $\ker d_n = \operatorname{im} d_{n+1}$, or equivalently $H_n(C_\bullet) = 0$, for every n.

Remark. As an exercise, show that H_n is a function from $\mathbf{CH}_{\bullet,R} \to \mathbf{Mod}_R$

Notation. We use $\mathbf{CH}_{\bullet,b}$ to denote the category of bounded complexes, and $\mathbf{CH}_{\bullet,\leq 0}, \mathbf{CH}_{\bullet,\geq 0}$ to denote the category of complexes bounded above and below, respectively.

1.5 Definition. A morphism $f_\bullet : B_\bullet \to C_\bullet$ is a quasi-isomorphism if $H_n(f_\bullet) : H_n(B_\bullet) \to H_n(C_\bullet)$ is an isomorphism for every n.

Remark. A chain complex $C_\bullet$ is acyclic if and only if the map $0_\bullet \to C_\bullet$ is a quasi-isomorphism.

Example (Motivational)**.** Let X be a topological space. Denote $\Delta^n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} : x_0 + \dots + x_n = 1\}$ the standard n-simplex. Put $S_n(X) = \mathbb{Z}[\mathrm{Hom}_{\mathrm{cont}}(\Delta^n, X)]$. Define first $\partial_i : \Delta^{n-1} \to \Delta^n$ by

$$(x_0, \dots, x_{n-1}) \mapsto (x_0, x_1, \dots, x_{i-1}, 0, x_i, \dots, x_{n-1}),$$

which by functoriality of Hom induces $\partial_n^i : \mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(\Delta^{n-1}, X)$. Now define the boundary map $d_n : S_n(X) \to S_{n-1}(X)$ by

$$d_n = \sum_{i=0}^{n} (-1)^i \partial_n^i.$$

It turns out that $d_{n-1} \circ d_n = 0$, so we have constructed a chain complex $S_\bullet(X)$. We define the *singular homology* of X to be $H_n(S_\bullet(X))$.

Lemma (Snake Lemma)**.** Suppose we have a commutative diagram in an abelian category with exact rows

$$\begin{tikzcd}
 & A' \arrow[r] \arrow[d, "f"] & B' \arrow[r] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C,
\end{tikzcd}$$

then there exists a long exact sequence

$$\ker f \to \ker g \to \ker h \xrightarrow{\delta} \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h.$$

Additionally, if we start with two short exact sequences in each row, namely

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r] \arrow[d, "f"] & B' \arrow[r] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0,
\end{tikzcd}$$

then we have the following long exact sequence

$$0 \to \ker f \to \ker g \to \ker h \xrightarrow{\delta} \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h \to 0.$$

Proof idea. Perform a diagram chase to build the connecting homomorphism δ, for proof see this Canvas page. ■

2 Lecture 2 (March 29): Long Exact Sequences in H_* and Chain Homotopies
Scribe: Bashir Abdel-Fattah

2.1 Proposition. Suppose that

$$0 \longrightarrow A_\bullet \longrightarrow B_\bullet \longrightarrow C_\bullet \longrightarrow 0$$

is a short exact sequences of chain complexes. Then there is a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A_\bullet) \arrow[r] & H_n(B_\bullet) \arrow[r] & H_n(C_\bullet) \arrow[dll, "\delta"] & \\
& H_{n-1}(A_\bullet) \arrow[r] & H_{n-1}(B_\bullet) \arrow[r] & H_{n-2}(C_\bullet) \arrow[r] & \cdots
\end{tikzcd}$$

Proof. For each n, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r] \arrow[d, "d_n^A"] & B_n \arrow[r] \arrow[d, "d_n^B"] & C_n \arrow[r] \arrow[d, "d_n^C"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r] & B_{n-1} \arrow[r] & C_{n-1} \arrow[r] & 0
\end{tikzcd}$$

which induces maps on the kernels and cokernels such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \ker(d_n^A) \arrow[r] \arrow[d] & \ker(d_n^B) \arrow[r] \arrow[d] & \ker(d_n^C) \arrow[d] & \\
0 \arrow[r] & A_n \arrow[r] \arrow[d, "d_n^A"] & B_n \arrow[r] \arrow[d, "d_n^B"] & C_n \arrow[r] \arrow[d, "d_n^C"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
& \operatorname{coker}(d_n^A) \arrow[r] & \operatorname{coker}(d_n^B) \arrow[r] & \operatorname{coker}(d_n^C) \arrow[r] & 0.
\end{tikzcd}$$

In particular, noting that $\ker(d_{n-1}^A) = Z_{n-1}(A_\bullet)$ and $\operatorname{coker}(d_n^A) = A_{n-1}/\operatorname{im}(d_n^A) = A_{n-1}/B_{n-1}(A_\bullet)$ (and similarly for $B_\bullet$ and $C_\bullet$), we have the commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & Z_{n-1}(A_\bullet) \arrow[r] \arrow[d] & Z_{n-1}(B_\bullet) \arrow[r] \arrow[d] & Z_{n-1}(C_\bullet) \arrow[d] & \\
0 \arrow[r] & A_{n-1} \arrow[r] \arrow[d, "d_{n-1}^A"] & B_{n-1} \arrow[r] \arrow[d, "d_{n-1}^B"] & C_{n-1} \arrow[r] \arrow[d, "d_{n-1}^C"] & 0 \\
0 \arrow[r] & A_{n-2} \arrow[r] & B_{n-2} \arrow[r] & C_{n-2} \arrow[r] & 0
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r] \arrow[d, "d_n^A"] & B_n \arrow[r] \arrow[d, "d_n^B"] & C_n \arrow[r] \arrow[d, "d_n^C"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r] \arrow[d] & B_{n-1} \arrow[r] \arrow[d] & C_{n-1} \arrow[r] \arrow[d] & 0 \\
& {}^{A_{n-1}}/_{B_{n-1}(A_\bullet)} \arrow[r] & {}^{B_{n-1}}/_{B_{n-1}(B_\bullet)} \arrow[r] & {}^{C_{n-1}}/_{B_{n-1}(C_\bullet)} \arrow[r] & 0.
\end{tikzcd}$$

Then, because the map $d_n^A : A_n \to A_{n-1}$ has kernel containing $\mathrm{im}(d_{n+1}^A) = B_n(A_\bullet)$ and image contained in $\ker(d_{n-1}^A) = Z_{n-1}(A_\bullet)$ and hence factors as

$$A_n \longrightarrow {}^{A_n}/_{B_n(A_\bullet)} \longrightarrow Z_{n-1}(A_\bullet) \longrightarrow A_{n-1}$$

(where we denote the middle map also by d_n^A by an abuse of notation), and similarly for $B_\bullet$ and $C_\bullet$, by combining with the induced maps on the kernels and cokernels above we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r] \arrow[d] & B_n \arrow[r] \arrow[d] & C_n \arrow[r] \arrow[d] & 0 \\
& {}^{A_n}/_{B_n(A_\bullet)} \arrow[r] \arrow[d, "d_n^A"] & {}^{B_n}/_{B_n(B_\bullet)} \arrow[r] \arrow[d, "d_n^B"] & {}^{C_n}/_{B_n(C_\bullet)} \arrow[r] \arrow[d, "d_n^C"] & 0 \\
0 \arrow[r] & Z_{n-1}(A_\bullet) \arrow[r] \arrow[d] & Z_{n-1}(B_\bullet) \arrow[r] \arrow[d] & Z_{n-1}(C_\bullet) \arrow[d] & \\
0 \arrow[r] & A_{n-1} \arrow[r] & B_{n-1} \arrow[r] & C_{n-1} \arrow[r] & 0
\end{tikzcd}$$

Also notice that the map $d_n^A : A_n/B_n(A_\bullet) \to Z_{n-1}(A_\bullet)$ has kernel $Z_n(A_\bullet)/B_n(A_\bullet) = H_n(A_\bullet)$, and because it's image is $B_{n-1}(A_\bullet)$ it also has cokernel $Z_{n-1}(A_\bullet)/B_{n-1}(A_\bullet) = H_{n-1}(A_\bullet)$. Then by applying the (most general statement of) the snake lemma to the middle two rows above, we have that there is a homomorphism $\delta : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$ such that the following diagram commutes

$$\begin{tikzcd}
& H_n(A_\bullet) \arrow[r] \arrow[d] & H_n(B_\bullet) \arrow[r] \arrow[d] & H_n(C_\bullet) \arrow[d] \arrow[dddll, rounded corners, "\delta"] & \\
& {}^{A_n}/_{B_n(A_\bullet)} \arrow[r] \arrow[d] & {}^{B_n}/_{B_n(B_\bullet)} \arrow[r] \arrow[d] & {}^{C_n}/_{B_n(C_\bullet)} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Z_{n-1}(A_\bullet) \arrow[r] \arrow[d] & Z_{n-1}(B_\bullet) \arrow[r] \arrow[d] & Z_{n-1}(C_\bullet) \arrow[d] & \\
& H_{n-1}(A_\bullet) \arrow[r] & H_{n-1}(B_\bullet) \arrow[r] & H_{n-1}(C_\bullet) &
\end{tikzcd}$$

as desired. ■

Chain Homotopies and Other Constructions

2.1 Definition.

1. Suppose we have two maps of chain complexes $f, g : C_\bullet \to D_\bullet$. Then f and g are ***homotopic*** (denoted $f \sim g$) if there exists a collection of maps $\{S_n : C_n \to D_{n+1}\}$ such that
$$f - g = dS + Sd.$$
2. If $f \sim 0$, then we say that f is ***null homotopic***.
3. $C_\bullet$ is ***contractible*** (null-homotopic) if $\mathrm{id}_{C_\bullet} \sim 0$.
4. $C_\bullet$ and $D_\bullet$ are ***homotopy equivalent*** (written $C_\bullet \sim D_\bullet$) if there exist chain maps $f_\bullet : C_\bullet \to D_\bullet$ and $g_\bullet : D_\bullet \to C_\bullet$ such that
$$f \circ g \sim \mathrm{id}_{D_\bullet} \qquad \text{and} \qquad g \circ f \sim \mathrm{id}_{C_\bullet}$$

3 Lecture 3 (March 31): Chain homotopy and homotopy category *Scribe: Eric Zhang*

3.1 Definition.

1. Suppose we have two maps of chain complexes $f, g : C_\bullet \to D_\bullet$. Then f and g are ***homotopic*** (denoted $f \sim g$) if there exists a collection of maps $\{s_n : C_n \to D_{n+1}\}$ such that
$$f - g = ds + sd.$$
2. If $f \sim 0$, then we say that f is ***null homotopic***.
3. $C_\bullet$ is ***contractible*** (null-homotopic) if $\mathrm{id}_{C_\bullet} \sim 0$.
4. $C_\bullet$ and $D_\bullet$ are ***homotopy equivalent*** (written $C_\bullet \sim D_\bullet$) if there exist chain maps $f_\bullet : C_\bullet \to D_\bullet$ and $g_\bullet : D_\bullet \to C_\bullet$ such that
$$f \circ g \sim \mathrm{id}_{D_\bullet} \qquad \text{and} \qquad g \circ f \sim \mathrm{id}_{C_\bullet}$$

Example. Let $f, g : X \to Y$ be morphisms of topological spaces and a homotopy $H : X \times [0,1] \to Y$ such that $H_{x\times 0} = f$ and $H_{x\times 1} = g$. Then exists a chain homotopy $S = S_*(X) \to S_*(Y)[1]$ such that $f_* \sim g_*$. This can be found in Hatcher's Algebraic Topology Theorem 2.10.

3.1 Proposition. Let $f, g : C_\bullet \to D_\bullet$ be chain maps. Then $f \sim g$ implies $f_* = g_* : H_*(C) \to H_*(D)$.

Proof. Recall H_n is an additive functor. So it suffices to show $f \sim 0$ implies $f_* = 0$ on $H_*(C)$. Consider the complexes

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r] \arrow[d, "f"] & C_n \arrow[r] \arrow[d, "f"] & C_{n-1} \arrow[r] \arrow[d, "f"] & \cdots \\ \cdots \arrow[r] & D_{n+1} \arrow[r] & D_n \arrow[r] & D_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

where $f = sd + ds$. Let $z \in Z_n(C)$ (so $d(z) = 0$). Then $f(z) = (sd + ds)(z) = ds(z)$ which is a boundary and hence is zero in $H_n(D)$. ■

3.2 Definition. The homotopy category $K(\mathbf{R\text{-}mod})$ has the following data:

1. Objects are chain complexes of R-modules.
2. Morphisms are chain maps up to homotopy equivalence.

3.2 Proposition. The homotopy category $K(\mathbf{R\text{-}mod})$ has the following properties.

1. It is an additive category.
2. It is not abelian.
3. It is triangulated, which implies in general it is not abelian.

4. The homology functor $H_n : \mathbf{CH}_{\bullet,\mathbf{R}} \to \mathbf{R\text{-}mod}$ factors through $K(\mathbf{R\text{-}mod})$.

$$\begin{tikzcd}
\mathbf{CH}_{\bullet,\mathbf{R}} \arrow[r, "H_n"] \arrow[dr] & \mathbf{R\text{-}mod} \\
& K(\mathbf{R\text{-}mod}) \arrow[u]
\end{tikzcd}$$

3.3 Definition. A chain complex $C.$ is split exact if it is an exact complex that splits. That is, for complexes

$$\begin{tikzcd}
\cdots \arrow[r, "d_{n+2}"] & C_{n+1} \arrow[r, "d_{n+1}"] \arrow[dr] & C_n \arrow[r, "d_n"] \arrow[dr] & C_{n-1} \arrow[r, "d_{n-1}"] & \cdots \\
& B_{n+1}(C) \arrow[u] & B_n(C) \arrow[u] & B_{n-1}(C) \arrow[u]
\end{tikzcd}$$

each sequence $0 \to B_n(C) \hookrightarrow C_n \twoheadrightarrow B_{n-1}(C) \to 0$ is exact split.

3.4 Remark. Split exact complex is null homotopic.

Proof. Consider the complexes

$$\begin{tikzcd}
\cdots \arrow[r, "d"] & B_{n+1} \oplus B_n \arrow[r, "d"] \arrow[d, "id"] & B_n \oplus B_{n-1} \arrow[r, "d"] \arrow[d, "id"] \arrow[dl, "s_n"] & B_{n-1} \oplus B_{n-2} \arrow[r, "d"] \arrow[d, "id"] \arrow[dl, "s_{n-1}"] & \cdots \\
\cdots \arrow[r, "d"] & B_{n+1} \oplus B_n \arrow[r, "d"] & B_n \oplus B_{n-1} \arrow[r, "d"] & B_{n-1} \oplus B_{n-2} \arrow[r, "d"] & \cdots
\end{tikzcd}$$

where $s_n : B_n \oplus B_{n-1} \to B_{n+1} \oplus B_n$ is the projection to B_n and the differentials $d_n : B_n \oplus B_{n-1} \to B_{n-1} \oplus B_{n-2}$ kills B_n and preserves B_{n-1}. ∎

3.5 Remark. A chain complex is split exact $\iff$ Null homotopic $\implies$ Acyclic/exact. But exactness does not imply split exact by the following counterexample.

Example. Consider the following chain complex of abelian groups.

$$\cdots \to \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \xrightarrow{\cdot 2} \mathbb{Z}/4 \to \cdots$$

Suppose there is a homotopy s such that $sd + ds = id$. Then $a = (sd + ds)(a) = s_{n-1}d(a) + ds_n(a) = 2s_n(a) + 2s_{n-1}(a)$. But not every $a \in \mathbb{Z}/4$ is divisible by 2.

3.6 Definition. We define some operations on complexes.

1. The shift of a complex C is a complex $C[j]_i = C_{i+j}$ with differentials $d^i_{C[j]} = (-1)^j d^i_C$.

2. A double complex is a collection $\{C_{p,q}\}$ with maps

$$\begin{tikzcd} & {} \arrow[d, "d^v"] & {} \arrow[d, "d^v"] & \\ \cdots \arrow[r] & C_{p,q} \arrow[r, "d^h"] \arrow[d, "d^v"] & C_{p-1,q} \arrow[r] \arrow[d, "d^v"] & \cdots \\ \cdots \arrow[r] & C_{p,q-1} \arrow[r, "d^h"] \arrow[d, "d^v"] & C_{p-1,q-1} \arrow[r] \arrow[d, "d^v"] & \cdots \\ & {} & {} & \end{tikzcd}$$

such that $d^h d^h = 0$, $d^v d^v = 0$, and $d^h d^v + d^v d^h = 0$.

3. Let $\{C_{p,q}\}$ be a double complex. Then total complex is $\mathrm{Tot}(C_{\bullet,\bullet}) = \bigoplus_{p+q=n} C_{p,q}$ with differentials $d : \bigoplus_{p+q=n} C_{p,q} \to \bigoplus_{i+j=n-1} C_{i,j}$ where $d = d^v + d^h$. Note it is indeed a differential since $d^2 = (d^v + d^h)^2 = d^h d^h + d^v d^h + d^h d^v + d^v d^v = 0$.

4. The mapping cone for a chain map $f : C_\bullet \to D_\bullet$ is the total complex of the following double complex

$$\begin{tikzcd} {} \arrow[d, "-d_C"] & {} \arrow[d, "d_D"] \\ C_{n+1} \arrow[r, "-f"] \arrow[d, "-d_C"] & D_{n+1} \arrow[d, "d_D"] \\ C_n \arrow[r, "-f"] \arrow[d, "-d_C"] & D_n \arrow[d, "d_D"] \\ C_{n-1} \arrow[r, "-f"] \arrow[d, "-d_C"] & D_{n-1} \arrow[d, "d_D"] \\ {} & {} \end{tikzcd}$$

In particular, $\mathrm{cone}_n(f) = C_{n-1} \oplus D_n$ and $\mathrm{cone}_{n-1}(f) = C_{n-2} \oplus D_{n-1}$ where the differential is $\begin{pmatrix} -d_C & -f \\ 0 & d_D \end{pmatrix}$.

3.3 Proposition. The mapping cone $\mathrm{cone}(f)$ has the following properties.

1. $D_\bullet \hookrightarrow \mathrm{cone}(f)$.
2. $\mathrm{cone}(f) \twoheadrightarrow C[-1]$.
3. $C_\bullet \xrightarrow{f} D_\bullet \hookrightarrow \mathrm{cone}(f) \twoheadrightarrow C[-1]$ which is called a triangle.

4 Lecture 4 (April 3): Abstract nonsense
Scribe: Soham Ghosh

4.1 Proposition. A map $f : C_\bullet \to D_\bullet$ of chain complexes is a quasi-isomorphism if and only if $\text{cone}_\bullet(f)$ is exact.

Proof. Apply long exact sequence in homology to the short exact sequence

$$D_\bullet \to \text{cone}_\bullet(f) \to C_\bullet[-1],$$

i.e., this is a short exact sequence of chain complexes, which implies that for all n, we have $D_n \to C_{n-1} \oplus D_n \to C_{n-1}$ [Warning: this is not split exact because the differential of $\text{cone}_\bullet(f)$ is not diagonal]. ∎

4.1 Exercise. Finish the above proof.

Exactness

4.2 Definition. Consider a covariant functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ between Abelian categories.

1. $\mathcal{F}$ is **left exact** if for all short exact sequences $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, we have $0 \to \mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(C)$ is exact in $\mathcal{B}$.
2. $\mathcal{F}$ is **right exact** if for all short exact sequences $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, we have $\mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(C) \to 0$ is exact in $\mathcal{B}$.
3. $\mathcal{F}$ is **exact** if it takes short exact sequences to short exact sequences, i.e., it is both left and right exact.

4.3 Remark. For left exactness of $\mathcal{F}$, it suffices to start with left exact sequences $0 \to A \to B \to B \to C$ in $\mathcal{A}$ and require them to go to left exact sequences in $\mathcal{B}$. Analogous statement holds for right-exactness.

4.4 Remark. Exactness of contravariant functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ is determined by exactness of the covariant functor $\mathcal{F}^{op} : \mathcal{A}^{op} \to \mathcal{B}$ by above.

4.5 Example (also Exercise to check).
1. $\text{Hom}_{\mathcal{A}}(M, -) : \mathcal{A} \to \text{Ab}$ is left exact for all $M \in \mathcal{A}$.
2. $\text{Hom}_{\mathcal{A}}(-, M) : \mathcal{A} \to \text{Ab}$ is contravariant and left exact for all $M \in \mathcal{A}$.
3. Let R be a commutative ring and $M \in R-\text{Mod}$. Then $M \otimes_R - : R-\text{Mod} \to R-\text{Mod}$ and $- \otimes_R M : R-\text{Mod} \to R-\text{Mod}$ are right exact.

4.6 Lemma. *Let $A, B, C \in \mathcal{A}$, then $0 \to A \to B \to C$ is exact if and only if for all $M \in \mathcal{A}$, the sequence $0 \to \text{Hom}(M, A) \to \text{Hom}(M, B) \to \text{Hom}(M, C)$ is exact.*

4.7 Exercise. Prove the above lemma.

4.8 Lemma. *$f : A \xrightarrow{\sim} B$ is an isomorphism in abelian category $\mathcal{A}$ if and only if for all $M \in \mathcal{A}$, the induced map $\text{Hom}(f) : \text{Hom}(M, A) \to \text{Hom}(M, B)$ is isomorphism in* Ab.

Yoneda lemma and functor categories

Functor category $\mathrm{Fun}(\mathcal{A}^{op}, \mathrm{Ab}) := \{\mathcal{F} : \mathcal{A} \to \mathrm{Ab} \text{ contravariant}\}$, with morphisms being natural transformations. This category is also denoted by $\mathrm{Ab}^{\mathcal{A}^{op}}$ or $2^{\mathcal{A}}$. Thus, we have a functor,

$$\mathcal{A} \xrightarrow{h} \mathrm{Fun}(\mathcal{A}^{op}, \mathrm{Ab}),$$

mapping $A \in \mathcal{A}$ to the functor $h_A := \mathrm{Hom}_{\mathcal{A}}(-, A)$. The following can be checked:

1. h is an exact fully faithful embedding.
2. $\mathrm{Hom}_{\mathcal{A}}(A, B) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Fun}}(h_A, h_B)$.

4.2 Proposition (Yoneda lemma)**.** Let $\mathcal{F} : \mathcal{A} \to \mathrm{Ab}$ be a contravariant functor. Then we have

$$\mathrm{Hom}_{\mathrm{Fun}}(h_A, \mathcal{F}) = \mathcal{F}(A).$$

4.9 Exercise. Prove the Yoneda lemma.

Note that when $\mathcal{F} = h_B$ in Yoneda lemma, we get the above version as $h_B(A) = \mathrm{Hom}_{\mathcal{A}}(A, B)$.

4.10 Theorem (Freyd-Mitchell)**.** *For a small Abelian category, there exists ring R, such that the functor $\mathcal{F} : \mathcal{A} \to R-\mathrm{Mod}$ is fully-faithful (i.e., $\mathcal{A}$ is a full subcategory of R-*Mod*).*

Derived functors

Let $\mathcal{A}$ be an abelian category.

4.11 Definition. $P \in \mathcal{A}$ is a projective object if any of the following equivalent conditions hold:

1. all surjections $A \twoheadrightarrow B$ and maps $P \to B$ lift to maps $P \to A$.

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d] \\ A \arrow[r, two heads] & B \end{tikzcd}$$

2. $\mathrm{Hom}(P, -)$ is exact.
3. In the category $R - \mathrm{Mod}$, projectives are direct summands of free modules.

Fact: Projective objects in $\mathrm{Ch}_\bullet(R - \mathrm{Mod})$ are split exact sequences of projective modules.

5 Lecture 5 (April 5): Derived functors
Scribe: Ansel Goh

5.1 Definition. A projective resolution of $M \in \mathcal{A}$ is a complex

$$P_i \to P_{i-1} \to \cdots \to P_0$$

with P_i projective for each i such that the complex

$$\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\epsilon} M \to 0$$

is exact. Equivalently, we can replace the exactness condition with the condition that for the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r] \arrow[d, "\epsilon"] & 0 \\
 & & 0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd},$$

ϵ is a quasi-isomorphism of chain complexes.

5.2 Definition. $\mathcal{A}$ has "enough" projectives if for all $M \in \mathcal{A}$, there exists $P \in \mathcal{A}$ projective such that $P \twoheadrightarrow M$.

5.3 Remark. if $\mathcal{A}$ has enough projectives, then any object in $\mathcal{A}$ has a projective resolution. Indeed, we have the diagram

$$\begin{tikzcd}
\ker(\epsilon) = M_0 \arrow[r, hook] & P_0 \arrow[r, two heads, "\epsilon"'] & M \\
\ker(d_0) = M_1 \arrow[r, hook] & P_1 \arrow[r, two heads] \arrow[u] & M_0
\end{tikzcd}$$

from which we can form the sequence

$$\begin{tikzcd}
\arrow[r] & P_2 \arrow[rr, dashed] \arrow[dr, two heads] & & P_1 \arrow[rr, dashed] \arrow[dr, two heads] & & P_0 \arrow[r] & M \\
 & & M_1 \arrow[ur, hook] & & M_0 \arrow[ur, hook] & &
\end{tikzcd}$$

Additionally, for each i, $M_i = \Omega^{i+1}M$ which is called the syzygy.

5.4 Definition. Given $M \in \mathcal{A}$, a chain complex $Q_\bullet$ with a map $\epsilon\colon Q_\bullet \to M$ where $\epsilon\colon Q_0 \to M$ is a resolution of M if the chain complex

$$\cdots \to Q_2 \to Q_1 \to Q_0 \xrightarrow{\epsilon} M \to 0$$

is exact.

5.5 Theorem. *Let $M, N \in \mathcal{A}$ and $f\colon M \to N$. Also, let*

$$P_\bullet \xrightarrow{\epsilon} M \to 0$$

be a projective resolution of M and

$$Q_\bullet \to N \xrightarrow{\epsilon'} 0$$

be any resolution of of N. Then, there exists a unique chain map $F\colon P_\bullet \to Q_\bullet$ such that the diagram

$$\begin{tikzcd} \arrow[r] & P_1 \arrow[r] \arrow[d, "F_1"] & P_0 \arrow[r, "\epsilon"] \arrow[d, "F_0"] & M \arrow[d, "f"] \\ \arrow[r] & Q_1 \arrow[r] & Q_2 \arrow[r, "\epsilon'"] & N \end{tikzcd}$$

commutes.

5.6 Corollary. *Given $M \in \mathcal{A}$, any two projective resolutions of M are homotopy equivalent.*

5.7 Definition. I is injective if either of the following equivalent definitions hold:

1. For any maps $M \hookrightarrow N$ and $M \to I$, there exists a map $N \to I$ such that the diagram

$$\begin{tikzcd} & I \\ N \arrow[ur, dashed] & M \arrow[l, hook'] \arrow[u] \end{tikzcd}$$

   commutes.

2. $\mathrm{Hom}(-, I)$ is an exact functor.

5.8 Definition. $\mathcal{A}$ has enough injectives if for all $M \in \mathcal{A}$, there exists $I \in \mathcal{A}$ injective such that M embeds into I.

5.9 Definition. An injective resolution of $M \in \mathcal{A}$ is a complex

$$I_0 \to I_1 \to I_2 \to \cdots$$

with I_i injective for each i such that the complex

$$0 \xrightarrow{\epsilon} M \to I_1 \to I_2 \to I_3 \to \cdots$$

is exact.

5.10 Remark. Let $\mathcal{A} = \textbf{R-mod}$. Then,

1. $\mathcal{A}$ has enough projectives (free modules).

2. $\mathcal{A}$ has enough injectives.

An example of a category with enough injectives but not enough projectives is $\mathrm{Rep}\, GL_n$. In fact, this category has no projective modules. It has enough injectives because $\mathrm{Rep}\, GL_n$ is equivalent to $k[GL_n]$-comod.

5.11 Definition. Let $\mathcal{A}$ have enough projectives and let $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ be a right exact functor. For $M \in \mathcal{A}$, let

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

be a projective resolution of M. Then, we define

$$(L_i\mathcal{F})(M) := H_i(\mathcal{F}(P_\bullet)).$$

5.12 Proposition. *$L_i\mathcal{F}$ has the following properties:*

1. *$L_i\mathcal{F}$ is independent of choice of $P_\bullet \to M$*
2. *$L_i\mathcal{F}$ are additive functors*
3. *$L_0\mathcal{F} = \mathcal{F}$*
4. *Let $M = P$ be projective. Then, for $i > 0$,*
$$(L_i\mathcal{F})(P) = 0$$
We say that projectives are "$\mathcal{F}$ acyclic".
5. *Given a short exact sequence*
$$0 \to M_1 \to M_2 \to M_3 \to 0,$$
there exists a long exact sequence
$$\cdots \to (L_1\mathcal{F})(M_3) \to \mathcal{F}(M_1) \to \mathcal{F}(M_2) \to \mathcal{F}(M_3) \to 0.$$

5.13 Remark. Weibel calls $\{L_i\mathcal{F}\}_{i\geq 0}$ a "δ"-functor

5.14 Definition. Let $\mathcal{A}$ have enough injectives and let $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor. For $M \in \mathcal{A}$, let
$$0 \to M \to I_0 \to I_1 \to I_2 \to \cdots$$
be an injective resolution of M. Then, we define
$$(R^i\mathcal{F})(M) := H_i(\mathcal{F}(I^\bullet)).$$

5.15 Proposition. *$R^i\mathcal{F}$ has the following properties:*

1. *$R^i\mathcal{F}$ is independent of choice of $I^\bullet \to M$*
2. *$R^i\mathcal{F}$ are additive functors*
3. *$R^0\mathcal{F} = \mathcal{F}$*
4. *Let $M = I$ be injective. Then, for $i > 0$,*
$$(R^i\mathcal{F})(I) = 0$$
We say that injectives are "$\mathcal{F}$ acyclic".
5. *Given a short exact sequence*
$$0 \to M_1 \to M_2 \to M_3 \to 0,$$
there exists a long exact sequence
$$0 \to \mathcal{F}(M_1) \to \mathcal{F}(M_2) \to \mathcal{F}(M_3) \to R^1\mathcal{F}(M_1) \to \cdots$$

5.16 Example. Let R be a commutative ring and consider the functor $- \otimes_R N \colon \textbf{R-mod} \to \textbf{R-mod}$. Then, define

$$\mathrm{Tor}_i^R(M, N) := L_i(- \otimes_R N)(M).$$

Note that $M \otimes_R N$ is symmetric so

$$\mathrm{Tor}_i^R(M, N) = L_i(M \otimes_R -)(N).$$

Weibel calls this "balanced tor."

Proof. See Weibel section 2.7 ■

6 Lecture 6 (April 7): Derived functors *Scribe: Ting Gong*

6.1 Example. Let $\mathcal{A}$ be abelian category with enough projectives. Then we consider $\mathrm{Hom}(-,M) : \mathcal{A}^{op} \to \mathbf{Ab}$ which is a left exact functor. We define

$$\mathrm{Ext}^i_R(-,M) = R^i(\mathrm{Hom}_{\mathcal{A}}(-,M)).$$

In practice, let $P_\bullet \to N$ be a projective resolution, then applying the $\mathrm{Hom}(-,M)$ to $\to P_3 \to P_2 \to P_1 \to P_0$, we have

$$\ldots \leftarrow \mathrm{Hom}(P_1,M) \leftarrow \mathrm{Hom}(P_0,M)$$

Then take the cohomology, we have

$$\mathrm{Ext}^i_R(N,M) = H^i(\mathrm{Hom}(P_\bullet,M)).$$

Alternatively, assume $\mathcal{A}$ has enough injectives, consider $A \to I^\bullet$ be an injective resolution, then we have

$$\mathrm{Hom}(N,I^0) \to \mathrm{Hom}(N,I^1) \to \ldots$$

and we can define

$$\mathrm{Ext}^i_R(N,M) = H^i(\mathrm{Hom}(N,I^\bullet)).$$

Adjoint Functors and Injective Modules

6.2 Definition. Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ be two functors between categories $\mathcal{A}, \mathcal{B}$, then we call $(\mathcal{F},\mathcal{G})$ an adjoint pair, $\mathcal{F}$ left adjoint to $\mathcal{G}$, $\mathcal{G}$ right adjoint to $\mathcal{F}$ if

$$\mathrm{Hom}_{\mathcal{B}}(\mathcal{F}(A),B) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(A,\mathcal{G}(B))$$

for $A \in \mathcal{A}, B \in \mathcal{B}$.

6.3 Example. (1) Let $\mathcal{A} = \underline{\mathrm{Lie}}_{\mathrm{k}}$ and $\mathcal{B} = \underline{\mathrm{Alg}}_{\mathrm{k}}$, where we have $\mathrm{Lie} : \underline{\mathrm{Alg}}_{\mathrm{k}} \to \underline{\mathrm{Lie}}_{\mathrm{k}}$ by defining a Lie bracket, then we have the universal envelopping algebra functor $U : \underline{\mathrm{Lie}}_{\mathrm{k}} \to \underline{\mathrm{Alg}}_{\mathrm{k}}$ being the left adjoint.

(2) Let R be a commutative ring, the forgetful functor $\mathrm{For} : R\text{-}\mathbf{Mod} \to \mathbf{Ab}$ is an exact functor, and it has right adjoint $\mathrm{Hom}_{\mathbf{Ab}}(R,-) : \mathbf{Ab} \to R\text{-}\mathbf{Mod}$ with

$$\mathrm{Hom}_{\mathbf{Ab}}(M,B) \xrightarrow{\sim} \mathrm{Hom}_R(M,\mathrm{Hom}_R(R,B))$$

Moreover, it has left adjoint $R \otimes_{\mathbb{Z}} - : \mathbf{Ab} \to R\text{-}\mathbf{Mod}$, with

$$\mathrm{Hom}_{\mathbf{Ab}}(M,B) \xrightarrow{\sim} \mathrm{Hom}_R(R \otimes_{\mathbb{Z}} A, M)$$

(3) Let R,S be rings, A an R-module, B a (R,S)-bimodule, and C an S-module. Then we have

$$\mathrm{Hom}_S(A \otimes B, C) \xrightarrow{\sim} \mathrm{Hom}_R(A,\mathrm{Hom}_S(B,C))$$

6.4 Remark. Consider $\mathbb{Z} \xrightarrow{\varphi} R$, then it induces a map R-**Mod**$\xrightarrow{\varphi^*}$ $\mathbb{Z}$-**Mod**. Generally, assume $S \xrightarrow{\varphi} R$, it induces a map R-**Mod**$\xrightarrow{\varphi^*}$ S-**Mod**.

6.5 Proposition. *(1) Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$, assume $\mathcal{F}$ has a left adjoint $\mathcal{G}$ which is exact, then $\mathcal{F}$ takes injectives to injectives.*

(2) If $\mathcal{F}$ has a right adjoint functor $\mathcal{G}$ which is exact, then $\mathcal{F}$ takes projectives to projectives.

Proof. We prove (1), then (2) should follow similarly. Let $I \in \mathcal{A}$ be injective, we want to show that $\mathcal{F}(I)$ is injective; that is, we want to show $\operatorname{Hom}_{\mathcal{B}}(-, \mathcal{F}(I))$ is exact.

Let $0 \to B_1 \to B_2 \to B_3 \to 0$ be a short exact sequence. Apply $\operatorname{Hom}_{\mathcal{B}}(-, \mathcal{F}(I))$, we have the below commutative diagram, there the vertical maps being isomorphisms from the adjunction, and the exactness of the bottom row comes from the exactness of $\mathcal{G}$:

$$\begin{tikzcd}
& \operatorname{Hom}_{\mathcal{B}}(B_1, \mathcal{F}(I)) \arrow[d, "\cong"] & \operatorname{Hom}_{\mathcal{B}}(B_2, \mathcal{F}(I)) \arrow[l] \arrow[d, "\cong"] & \operatorname{Hom}_{\mathcal{B}}(B_1, \mathcal{F}(I)) \arrow[l] \arrow[d, "\cong"] & 0 \arrow[l] \\
0 & \operatorname{Hom}_{\mathcal{A}}(\mathcal{G}(B_1), I) \arrow[l] & \operatorname{Hom}_{\mathcal{A}}(\mathcal{G}(B_2), I) \arrow[l] & \operatorname{Hom}_{\mathcal{A}}(\mathcal{G}(B_3), I) \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

Hence the top sequence is exact on the left when adding a term 0 as well. ■

6.6 Proposition. *Let $(\mathcal{F}, \mathcal{G})$ be an adjoint pair of functors. Then $\mathcal{F}$ is right exact, and $\mathcal{G}$ is left exact.*

6.7 Proposition (Baer's Criterion)**.** *A R-module M is injective if and only if for all $\mathfrak{a} \subset R$ ideals, the module homomorphism from $\mathfrak{a} \to M$ can be extended to a homomorphism $R \to M$.*

7 Lecture 7 (April 12): Injectives in R-mod *Scribe: Raymond Guo*

7.1 Theorem. *(Baer's Criterion)*

Let R be a commutative ring. If an R-module I satisfies the property that for every ideal J of R, any map $J \to I$ can be extended to a map $R \to I$, then I is an injective R-module.

7.2 Corollary. *In $\mathbb{Z}$-mod, the injective modules are the divisible groups. (exercise)*

7.3 Corollary. *$\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are both injective $\mathbb{Z}$-modules.*

7.4 Definition. $M \in \mathbb{Z}$-mod. $M^\vee := \operatorname{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ is called the Pontryagin dual.

7.5 Lemma. *There exists an injective natural map from M into $M^{\vee\vee}$. We call M reflexive if it's an isomorphism.*

7.6 Lemma. *If P is projective, $P^\vee$ is injective.*

Proof. We need to show $\operatorname{Hom}_{\mathbb{Z}}(-, P^\vee)$ is exact.

We note that

$$\operatorname{Hom}_{\mathbb{Z}}(M, \operatorname{Hom}_{\mathbb{Z}}(P, \mathbb{Q}/\mathbb{Z})) \cong \operatorname{Hom}_{\mathbb{Z}}(M \otimes_{\mathbb{Z}} P, \mathbb{Q}/\mathbb{Z})$$

by the Hom-Tensor adjunction. Tensoring with a projective module and Hom into an injective module are both exact, so the RHS is a composition of two exact functors, which is exact. Thus the LHS is exact, which is the desired result. ■

7.7 Theorem. *$\mathbb{Z}$-mod has enough injectives.*

Proof. Let M be a $\mathbb{Z}$-module. There's a free module F with a surjective map into $M^\vee$. If we dualize, we have a map $M \hookrightarrow M^{\vee\vee} \hookrightarrow F^\vee$ where injectivity of the second map, given surjectivity of the map before taken the dual, is an exercise. The module $F^\vee$ is injective by the lemma. ■

7.8 Theorem. *Let R be arbitrary, not necessarily commutative. $R-$mod has enough injectives.*

Proof. There's a forgetful functor $U : R$-mod $\to \mathbb{Z}$-mod.

$(U, \operatorname{Hom}_{\mathbb{Z}}(R, -))$ is an adjoint pair. Since U is exact, $\operatorname{Hom}_{\mathbb{Z}}(R, -)$ takes injectives to injectives. This yields that if T is a divisible abelian group, $\operatorname{Hom}_{\mathbb{Z}}(R, T)$ is an injective R-module.

Let M be an R-module.

There's an embedding $M \hookrightarrow T$ of abelian groups where T is a divisible (injective) abelian group (which isn't necessarily a map of R-modules) by the previous theorem.

We then write the adjunction

$$\mathrm{Hom}_{\mathbb{Z}}(M, T) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(R, T))$$

Let $f : M \to T$ be the map of $\mathbb{Z}$ modules discussed above. It gets identified with the map $f' : M \to \mathrm{Hom}_{\mathbb{Z}}(R, T)$ in R-mod.

To show that f' is injective, we must write the adjunction explicitly. The isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(M, T) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(R, T))$$

takes $\phi : M \to T$ to $\phi' : M \to (\mathrm{Hom}_{\mathbb{Z}}(R, T))$ defined by $\phi'(m)(r) = \phi(rm)$.

If ϕ is injective and if $\phi'(m) = \phi'(m')$, then $\phi'(m)(1) = \phi'(m')(1)$, which is to say that $\phi(m) = \phi(m')$. Since ϕ is injective, $m = m'$, so we have that ϕ' is injective. This completes the proof. ■

We compute the following for Tor and Ext:

7.1 Proposition. $\mathrm{Tor}_i^{\mathbb{Z}}(M, N) = \mathrm{Ext}_i^{\mathbb{Z}}(M, N) = 0$ for $i > 1$.

Proof. We work first with Tor. Taking a projective resolution of M, we have a projective resultion of the form $F_1 \hookrightarrow F_0 \to M$ because submodules of free $\mathbb{Z}$-modules are free. Since this resolution only has two modules, after tensoring with N and taking homology, $\mathrm{Tor}_i^{\mathbb{Z}}(M, N) = 0$ for $i > 1$. ■

The same can be said for Ext, taking a projective resolution of M and applying contravariant Hom.

7.2 Proposition. We compute $\mathrm{Tor}_1^{\mathbb{Z}}(M, N)$.

Proof. Let us first do this for $M = \mathbb{Z}/n$. We resolve $\mathbb{Z}/n$ by $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n$, where the maps are multiplication by n and the quotient. After tensoring with N (and removing M from the front), the resulting chain complex takes the form

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \otimes_{\mathbb{Z}} N \arrow[r, "\times n \otimes \mathrm{id}"] \arrow[d, "="'] & \mathbb{Z} \otimes_{\mathbb{Z}} N \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & N \arrow[r, "\times n"] & N \arrow[r] & 0
\end{tikzcd}$$

Identifying the top row with the bottom, it's clear that the homology of this chain complex in the first degree is $\{x \in N : nx = 0\}$, so $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, N)$ is this group, called the n-torsion of N.

Since $\mathbb{Z}$ is free, it has a projective resolution $\mathbb{Z} \to \mathbb{Z}$ with one nonzero module, so $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, N) = 0$. ■

8 Lecture 8 (April 12): Ext and Tor
Scribe: Nathan Louie

8.1 Proposition. Tor_i^R *commutes with filtered colimits. That is,*

$$\mathrm{Tor}_i^R(\varinjlim M_j, N) = \varinjlim \mathrm{Tor}_i^R(M_j, N).$$

8.2 Remark. Tensor product functor $\bigotimes_R$ commutes with $\bigoplus$, $\varinjlim$, and coker and hom functor Hom_R commutes with $\prod$, $\varprojlim$, and ker.

8.3 Definition. A R-module M is flat if any of the following equivalent conditions hold:

1. $M \otimes_R -$ is exact,
2. $\mathrm{Tor}_i^R(M, N) = 0$ for all $i > 0$ and for any R-module N,
3. $\mathrm{Tor}_1^R(M, N) = 0$ for any R-module N,

8.4 Exercise. Prove the three equivalences for flatness.

8.5 Remark. Flat modules are "Tor-acyclic" (i.e. makes Tor vanish), so we can calculate Tor using flat resolutions.

8.6 Proposition. *In $\mathbb{Z}$-**Mod**, a module is flat if and only if it is torsion-free.*

Proof. Assume M is a flat $\mathbb{Z}$-module. Then $0 = \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n, M) = {}_nM$, the n-torsion of M, for any n by calculation on 4/10. Hence, there is no n-torsion for any n, so M is torsion-free.

Assume M is torsion free. Let N be an R-module. We can write $N = \varinjlim N_i$, where N_i is finitely generated. By structure theorem, $N_i = \mathbb{Z}^{n_i} \bigoplus \oplus_{j=1}^{l_i} \mathbb{Z}/n_{i,j}$. In particular, Tor commutes with direct sums, so $\mathrm{Tor}_1^{\mathbb{Z}}(N_i, M) = 0$ because M is torsion free. Since Tor is symmetric and commutes with limits, we can write

$$\mathrm{Tor}_1^{\mathbb{Z}}(N, M) = \mathrm{Tor}_1^{\mathbb{Z}}(\varinjlim N_i, M) = \varinjlim \mathrm{Tor}_1^{\mathbb{Z}}(N_i, M) = 0.$$

■

8.7 Remark. For general R-**Mod**, free $\subset$ projective $\subset$ flat $\subset$ torsion-free. For $\mathbb{Z}$-**Mod**, free = projective $\subsetneq$ flat = torsion-free. For example, $\mathbb{Q}$ is torsion-free, but not free as a $\mathbb{Z}$-module.

Local Properties

8.8 Proposition. *Let R be commutative, and M be an R-module. Then the following are equivalent:*

1. $\mathrm{M} \cong 0$,
2. *For all prime ideals $\mathfrak{p} \subset R$, $M_{\mathfrak{p}} \cong 0$.*

3. *For all maximal ideals $\mathfrak{m} \subset R$, $M_{\mathfrak{m}} \cong 0$.*

8.9 Proposition. *Let F be a flat R-algebra. Then for an R-module M, define the F-module $M_F := F \otimes_R M$, the extension of scalars. There is a natural isomorphism*

$$F \otimes_R \operatorname{Tor}_i^R(M, N) \xrightarrow{\sim} \operatorname{Tor}_i^F(M_F, N_F).$$

8.10 Proposition. *(Localization for* Tor*) Let R be commutative and M, N be R-modules. Then the following are equivalent:*

1. $\operatorname{Tor}_i^R(M, N) = 0$,
2. *For all prime ideals $\mathfrak{p} \subset R$, $\operatorname{Tor}_i^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) = 0$,*
3. *For all maximal ideals $\mathfrak{m} \subset R$, $\operatorname{Tor}_i^{R_{\mathfrak{m}}}(M_{\mathfrak{m}}, N_{\mathfrak{m}}) = 0$,*

8.11 Corollary. *Vanishing for* Tor *and, in turn, being flat are local properties.*

8.12 Remark. For local ring R, a R-module is free iff projective iff flat. By Quillen-Suslin Theorem, a $k[x_1, ..., x_n]$-module is free iff projective. There are local properties also related to Ext.

Ext and Extension

8.13 Definition. An extension of length n is an exact sequence $0 \to M_0 \to ... \to M_{n+1} \to 0$. In this case, we say that we extend M_{n+1} by M_0.

8.14 Remark. A short exact sequence is an extension of length 1.

For the rest of the section, we illustrate how to construct the Yoneda product. For $n \in \mathbb{N}$, let us define

$$\mathcal{E}\mathrm{xt}_R^n(M, N) := \frac{\{0 \to N \to E_1 \to ... \to E_n \to M \to 0 \mid \text{extension of length } n\}}{\sim}$$

where two extensions of length n are equivalent if there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, equal] & E_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_n \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r] & E_1' \arrow[r] & \cdots \arrow[r] & E_n' \arrow[r] & M \arrow[r] & 0
\end{tikzcd}.$$

We can (and later will) introduce an addition structure on short exact sequences, which will give us an abelian group structure on $\mathcal{E}\mathrm{xt}_R^1$ (and on $\mathcal{E}\mathrm{xt}_R^n$). Then, we will also show that there exists a bijection $\operatorname{Ext}_R^n(M, N) \xrightarrow{\sim} \mathcal{E}\mathrm{xt}_R^n(M, N)$, which respects the group structure. This will allow us to treat elements of $\operatorname{Ext}_R^n(M, N)$ as extensions from M to N.

Given an extension of M by N, and of L by by M, we can produce an extension of L by N by concatenation, thereby defining a pairing

$$\operatorname{Ext}_R^n(M, N) \times \operatorname{Ext}_R^m(L, M) \to \operatorname{Ext}_R^{n+m}(L, N)$$

Explicitly, we have:

$$(N \to E_1 \to \cdots \to E_n \to M) \times (M \to F_1 \to \cdots \to F_m \to L) \\ \mapsto N \to E_1 \to \cdots \to E_n \to F_1 \to \cdots \to F_m \to L.$$

where the middle map $E_n \to F_1$ is the composition $E_n \to M \to F_1$ of the tail of the first extensions and the head of the second.

8.15 Definition. Given above, we define

$$\mathrm{Ext}_R^*(M, M) := \bigoplus_{n=0}^{\infty} \mathrm{Ext}_R^n(M, M),$$

which has a structure of a graded ring via *Yoneda product* defined as above for $M = N = L$.

9 Lecture 9 (April 12): Ring structure on $\mathcal{E}\mathbf{xt}^*_R(M,N)$ *Scribe: Jackson Morris*

9.1 Equality of the two Ext's

$\mathcal{E}\mathrm{xt}^1_R(M,N) = \{0 \to N \to E \to M \to 0\}/\sim$, where we identify two short exact sequences if there exists a diagram of the following kind:

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] \arrow[d, "="] & E \arrow[r] \arrow[d, "\cong"] & M \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & N \arrow[r] & E' \arrow[r] & M \arrow[r] & 0 \end{tikzcd}.$$

This set has a group structure: Given two short exact sequences

$$0 \to N \to E \to M \to 0$$

and

$$0 \to N \to E' \to M \to 0$$

in $\mathcal{E}\mathrm{xt}^1_R(M,N)$, we can get a new short exact sequence by taking direct sums:

$$0 \to N \oplus N \to E \oplus E' \to M \oplus M \to 0.$$

This is no longer an element of $\mathcal{E}\mathrm{xt}^1_R(M,N)$. However, we can modify this slightly to get what we want.

9.1 Remark. Given two maps $f, G : A \to B$ say of abelian groups, we can do the following procedure:

$$A \xrightarrow{\Delta} A \oplus A \xrightarrow{f \oplus g} B \oplus B \xrightarrow{\mu} B$$

$$a \mapsto (a, -a) \mapsto (f(a), g(-a)) \mapsto f(a) + g(-a)$$

We can apply functoriality of hom to get a natural sequence

$$\mathrm{Hom}(A^{\oplus 2}, B^{\oplus 2}) \xrightarrow{\Delta^*} \mathrm{Hom}(A, B^{\oplus 2}) \xrightarrow{\mu_*} \mathrm{Hom}(A, B)$$

Then, we can pass to $\mathcal{E}\mathrm{xt}$:

$$\mathcal{E}\mathrm{xt}^1(A^{\oplus 2}, B^{\oplus 2}) \xrightarrow{\Delta^*} \mathcal{E}\mathrm{xt}^1(A, B^{\oplus 2}) \xrightarrow{\mu_*} \mathcal{E}\mathrm{xt}^1(A, B)$$

This is the motivation for how we define the addition for the Yoneda $\mathcal{E}\mathrm{xt}$.

9.2 Definition. Take two short exact sequences $\xi, \zeta \in \mathcal{E}\mathrm{xt}^1_R(M,N)$, and form the direct sum short exact sequence as above. We can form a module $\tilde{E}$ by pulling back the right square of the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & N \oplus N \arrow[r] \arrow[d, "="] & \tilde{E} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & M \arrow[r] \arrow[d, "\Delta"] & 0 \\ 0 \arrow[r] & N \oplus N \arrow[r] & E \oplus E' \arrow[r] & M \oplus M \arrow[r] & 0 \end{tikzcd}$$

This is a map of short exact sequences (Exercise: show that we get a map $N \oplus N \to \tilde{E}$ that makes this true), and we can get a new map of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & N \oplus N \arrow[r] \arrow[d, "\mu"'] & \tilde{E} \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "="'] & 0 \\
0 \arrow[r] & N \arrow[r] & \tilde{\tilde{E}} \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

by pushing out to get the middle term on the bottom row (Exercise: show that we get a map $\tilde{\tilde{E}} \to M$ that makes this true). This bottom short exact sequence is the sum $\xi + \zeta = \mu_* \Delta^*(\xi \oplus \zeta)$.

9.3 Theorem. *There is a group isomorphism map* $\mathcal{E}xt^1_R(M, N) \to \operatorname{Ext}^1_R(M, N)$.

9.4 Remark. This is actually true for all n and not just $n = 1$, but we will not prove this.

Proof. Sketch. First, let's produce a map $\mathcal{E}\text{xt} \to \text{Ext}$. Take a projective resolution $P_\bullet \to \epsilon \to 0$. Applying $\hom(-, N)$ gives us

$$\ldots \longleftarrow \operatorname{Hom}(P_2, N) \xleftarrow{d_1^*} \operatorname{Hom}(P_1, N) \xleftarrow{d_0^*} \operatorname{Hom}(P_0, N)$$

To calculate $Ext^1(M, N)$, we calculate cohomology at $\hom(P_1, N)$, i.e. the elements in the kernel of d_1^*. Well, an element

$$\xi : 0 \to N \to E \to M \to 0$$

is a resolution of M. By the fundamental theorem of projective resolutions, there is a commutative diagram

$$\begin{tikzcd}
\ldots \arrow[r] & P_2 \arrow[r, "d_1"] \arrow[d] & P_1 \arrow[r, "d_0"] \arrow[d, "f"'] & P_0 \arrow[r, "\epsilon"] \arrow[d] & M \arrow[r] \arrow[d, "="] & 0 \\
\ldots \arrow[r] & 0 \arrow[r] & N \arrow[r] & E \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

We define our map by sending ξ to this f. Notice that by construction $d_1^*(f) = f \circ d_1 = 0$, so indeed $f \in \operatorname{Ext}^1(M, N)$. There are still the issues of this being well-defined, but we will skip them for the moment.

For an inverse $\text{Ext} \to \mathcal{E}\text{xt}$, take some $f \in \operatorname{Ext}^1(M, N)$ and a projective resolution $P_\bullet \to M \to 0$. Since f represents a map $P_1 \to N$ in the kernel of d_1, we can sort of work in reverse from above: by arranging the projective resolution in to the top row and placing the map $f : P_1 \to N$ in the same location, we can form a module E by pushing out:

$$\begin{tikzcd}
\ldots \arrow[r] & P_2 \arrow[r, "d_1"] & P_1 \arrow[r, "d_0"] \arrow[d, "f"'] & P_0 \arrow[r, "\epsilon"] \arrow[d] & M \arrow[r] \arrow[d, "="] & 0 \\
 & & N \arrow[r] & E \arrow[r] & M &
\end{tikzcd}$$

More precisely, $E = \underline{N \oplus P_0}/(f(x), d_0(x))$. The map we get from $N \hookrightarrow E$ is injective. Suppose that $\overline{(n,0)} = 0$; then $n = f(x)$ $0 = d_0(x)$, hence $x \in P_1$ such that $x = d_1(y)$. Then $f(d_1(y)) = d_1^* f(y) = 0 = n$. It is an exercise to show that the cokernel of this map is always M. This then gives us a short exact sequence $0 \to N \to E \to M \to 0$, hence an element of $\mathcal{E}\mathrm{xt}(M,N)$. We again omit a proof of the bijection being well-defined and a group homomorphism. ■

9.2 Yoneda Product

Assuming the previous theorem, we can now define a product on Ext:

$$\mathrm{Ext}_R^n(M,N) \times \mathrm{Ext}_R^m(L,M) \to \mathrm{Ext}_R^{n+m}(L,N),$$

where, given $(N \to E_1 \to \ldots \to E_n \to M) \in \mathrm{Ext}_R^n(M,N)$ and $(M \to F_1 \to \ldots \to F_m \to L) \in \mathrm{Ext}_R^m(L,M)$, we form a new exact sequence by concatenating:

$$(N \to E_1 \to \ldots \to E_n \to M = M \to F_1 \to \ldots \to F_m \to L) \in \mathrm{Ext}_R^{n+m}(L,N)$$

Notice that if we fix a module M in both components, this defines a ring structure on $\mathrm{Ext}_R^*(M,M) = \bigoplus \mathrm{Ext}_R^n(M,M)$.

9.3 Augmentation

An algebra R over a field k is called *augmented* if there is a k-linear map $R \xrightarrow{\varepsilon} k$, called the augmentation, such that

$$k \hookrightarrow R \xrightarrow{\varepsilon} k = id_k$$

9.5 Example. If G is any group, then the group algebra kG is augmented, where $\varepsilon(g) = 1$ for any $g \in G$. If $\mathfrak{g}$ is any Lie algebra, then the Universal enveloping algebra $U(\mathfrak{g})$ is augmented, where $\varepsilon(x) = 0$ for any $x \in \mathfrak{g}$.

9.6 Remark. In particular, there are Hopf algebras, which are always augmented.

If R is augmented, then we can view k as a module over R. Returning to the algebra $\mathrm{Ext}_R^*(M,M)$, and letting $M = k$, the Yoneda product tells us that we have a map

$$\mathrm{Ext}_R^*(k,k) \times \mathrm{Ext}_R^*(M,k) \to \mathrm{Ext}_R^*(M,k),$$

which allows us to view $\mathrm{Ext}_R^*(M,k)$ as a module over $\mathrm{Ext}_R^*(k,k)$!

9.7 Theorem. $\mathrm{Ext}_{kG}^*(k,k) = H^*(G;k)$ *is a graded commutative algebra*

9.8 Theorem. *If G is a finite group, then the group cohomology algebra is a finitely generated k-algebra (or, equivalently, Noetherian).*

10 Lecture 10 (April 17): Ext examples, Künneth formula and universal coefficient theorem *Scribe: Nelson Niu*

10.1 Ext examples

By definition $\mathrm{Ext}^0_R(A,B) = \mathrm{Hom}_R(A,B)$; and every $\mathbb{Z}$-module has a projective resolution of length 1, so $\mathrm{Ext}^i_{\mathbb{Z}}(A,B) = 0$ for $i > 1$. Hence the only interesting Ext groups over $\mathbb{Z}$ are when $i = 1$. Here are some sample computations of $\mathrm{Ext}^1_{\mathbb{Z}}(A,B)$.

10.1 Example. We compute $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p,\mathbb{Z}/p)$. The standard projective resolution of $\mathbb{Z}/p$ is

$$0 \to \mathbb{Z} \xrightarrow{p} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/p \to 0;$$

applying $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Z}/p)$ to the resolution yields

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}/p) \simeq \mathbb{Z}/p \xrightarrow{p=0} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}/p) \simeq \mathbb{Z}/p \to 0,$$

so $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p,\mathbb{Z}/p) \simeq \mathbb{Z}/p$. It follows that there are p isomorphism classes of length-1 extensions of the form

$$0 \to Z/p \hookrightarrow E \twoheadrightarrow \mathbb{Z}/p \to 0$$

for some abelian group E, corresponding to the p elements of $\mathbb{Z}/p$; Homework Problem 10 asks you to find them.

10.2 Example. While Tor commutes with colimits in either factor, meaning that computing Tor on abelian groups reduces to computing them for finitely generated abelian groups, Ext does not commute with colimits in the first factor. The obstacle is that the hom functor turns colimits in the first factor a limit, and the limit functor is not exact; indeed it has a nontrivial right derived functor $\lim^1$. So we must take care when computing $\mathrm{Ext}^1_{\mathbb{Z}}(A,B)$ for an arbitrary, possibly not finitely generated abelian group A. (On the other hand, computing $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n,A)$ is much the same as computing a Tor group, but with the arrows reversed; indeed there is a duality with Tor. See Homework Problem 11.)

Consider $\mathrm{Ext}^1_{\mathbb{Z}}(A,\mathbb{Z})$. Note that $\mathbb{Z}$ is *not* injective, as it is not divisible, so this Ext group is not necessarily trivial. We take an injective resolution for $\mathbb{Z}$ by embedding it in $\mathbb{Q}$, which is injective; the quotient of any divisible group is also divisible, so indeed every $\mathbb{Z}$-module has an injective resolution of length at most 1; for $\mathbb{Z}$, it is

$$0 \to \mathbb{Z} \hookrightarrow \mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z} \to 0.$$

Applying $\mathrm{Hom}_{\mathbb{Z}}(A,-)$ to the resolution yields

$$\mathrm{Hom}_{\mathbb{Z}}(A,\mathbb{Q}) \to \mathrm{Hom}_{\mathbb{Z}}(A,\mathbb{Q}/\mathbb{Z}) = A^\vee \to 0;$$

recall that $A^\vee = \mathrm{Hom}_{\mathbb{Z}}(A,\mathbb{Q}/\mathbb{Z})$ is the Pontryagin dual.

In the special case where A is torsion, $\mathrm{Hom}_{\mathbb{Z}}(A,\mathbb{Q}) = 0$, so $\mathrm{Ext}^1_{\mathbb{Z}}(A,\mathbb{Z}) \simeq A^\vee$. For instance, let $A = \mathbb{Z}/p^\infty = \varinjlim \mathbb{Z}/p^n$, the colimit (i.e. direct limit) of the sequence

$$\mathbb{Z}/p \to \cdots \to \mathbb{Z}/p^n \xrightarrow{p} \mathbb{Z}/p^{n+1} \xrightarrow{p} \mathbb{Z}/p^{n+2} \to \cdots$$

We have that A is torsion (every element has order p^n for some finite n), so

$$\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \simeq A^\vee = \mathrm{Hom}_{\mathbb{Z}}(\varinjlim \mathbb{Z}/p^n, \mathbb{Q}/\mathbb{Z}).$$

If we pull out the colimit from the first factor, it becomes a limit, yielding

$$\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \simeq \varprojlim \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^n, \mathbb{Q}/\mathbb{Z}).$$

To compute $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^n, \mathbb{Q}/\mathbb{Z})$, we note that each map is uniquely determined by where 1 is sent in $\mathbb{Q}/\mathbb{Z}$, and since p^n annihilates 1 in $\mathbb{Z}/p^n$ it must also annihilate the image of 1 in $\mathbb{Q}/\mathbb{Z}$. The p^n-torsion elements of $\mathbb{Q}/\mathbb{Z}$ are precisely the residues of $0, 1/p^n, \ldots, (p^n-1)/p^n$, so the p^n-torsion subgroup of $\mathbb{Q}/\mathbb{Z}$ is isomorphic to $\mathbb{Z}/p^n$. Hence $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p^n, \mathbb{Q}/\mathbb{Z}) \simeq \mathbb{Z}/p^n$, making

$$\mathrm{Ext}^1_{\mathbb{Z}}(A, \mathbb{Z}) \simeq \varprojlim \mathbb{Z}/p^n,$$

i.e. the limit of the sequence

$$\cdots \to \mathbb{Z}/p^{n+1} \xrightarrow{\text{mod } p^n} \mathbb{Z}/p^n \to \cdots \to \mathbb{Z}/p,$$

which is the group of p-adic integers $\mathbb{Z}_p^\wedge$.

10.2 Künneth formula and universal coefficient theorem

The motivating questions come from topology; let X and Y be topological spaces.

1. Given $H_\bullet(X)$ and $H_\bullet(Y)$, how can we compute $H_\bullet(X \times Y)$? The answer to this question will lead to the *Künneth formula.*
2. Given $H_\bullet(X)$, how can we compute $H_*(X; M)$ for some other abelian group M of coefficients, where we tensor the singular complex of X with M before computing homology? The answer to this question will lead to the *universal coefficient theorem.*

We can answer the second question in the special case where M is a flat $\mathbb{Z}$-module: $H_n(X; M) = H_n(X) \otimes_{\mathbb{Z}} M$. But what about in general?

The Eilenberg-Zilber theorem gives singular chain complex of $X \times Y$ in terms of those of X and Y; from there, the questions become purely algebraic ones. We begin by giving the construction of this chain complex, called the *tensor product of chain complexes.* We will work in the category of R-modules for a commutative ring R (commutativity can be dropped, but that would require us to be careful about left versus right modules and such).

10.3 Definition. Let $P_\bullet, Q_\bullet$ be chain complexes in the category of R-modules. Define $P_\bullet \otimes_R Q_\bullet$ to be the double complex $(P_\bullet \otimes_R Q_\bullet)_{i,j} = P_i \otimes_R Q_j$ with differentials

$$\begin{tikzcd}
P_{i-1} \otimes Q_j \arrow[d, "(-1)^{i-1}(1\otimes d^Q)"'] & P_i \otimes Q_j \arrow[l, "d^P \otimes 1"'] \arrow[d, "(-1)^i(1\otimes d^Q)"] \\
P_{i-1} \otimes Q_{j-1} & P_i \otimes Q_{j-1}. \arrow[l, "d^P \otimes 1"']
\end{tikzcd}$$

Then the ***tensor product*** of $P_\bullet$ and $Q_\bullet$ is the total complex of this double complex: $\mathrm{Tot}_\bullet(P_\bullet \otimes_R Q_\bullet)$, so that

$$\mathrm{Tot}_n(P_\bullet \otimes_R Q_\bullet) = \bigoplus_{i+j=n} P_i \otimes Q_j$$

with differentials as above. (We will often write this complex as simply $P_\bullet \otimes_R Q_\bullet$, leaving it to context to determine whether we are referring to the double complex or its total complex.)

Below, we let $P_\bullet$ and $Q_\bullet$ be chain complexes of R-modules, where $P_\bullet$ has differentials $d_n^P : P_{n+1} \to P_n$ as well as $B_n = d_n^P(P_\bullet)$ and $Z_n = \ker(d_{n-1}^P)$.

10.4 Theorem (Künneth formula)**.** *Assume P_n, B_n are flat R-modules for all n. Then there exists a natural short exact sequence*

$$0 \to \bigoplus_{i+j=n} H_i(P_\bullet) \otimes H_j(Q_\bullet) \hookrightarrow H_n(P_\bullet \otimes Q_\bullet) \twoheadrightarrow \bigoplus_{i+j+1=n} \mathrm{Tor}_i^R(H_i(P_\bullet), H_j(Q_\bullet)) \to 0.$$

Note that the universal coefficient theorem will simply be a special case of the Künneth formula when $Q_\bullet$ is concentrated in degree 0 (Weibel proves only this special case). We sketch the start of the proof here; this can also be proven using spectral sequences.

Start of Proof. First we show that if P_n and B_{n-1} are flat, then so is Z_n. Consider the short exact sequence

$$0 \to Z_n \hookrightarrow P_n \twoheadrightarrow B_{n-1} \to 0.$$

Applying the long exact sequence for Tor yields

$$\cdots \to \mathrm{Tor}_2(B_{n-1}, A) \to \mathrm{Tor}_1(Z_n, A) \to \mathrm{Tor}_1(P_n, A) \to \mathrm{Tor}_1(B_{n-1}, A) \to Z_n \otimes A \to \cdots,$$

where flatness implies $\mathrm{Tor}_i(P_n, A) \simeq \mathrm{Tor}_i(B_{n-1}, A) \simeq 0$ for all i and thus $\mathrm{Tor}_i(Z_n, A) \simeq 0$ for all i as well, implying that Z_n is also flat.

(As an aside, note that it would not be enough to assume that P_n and Z_n are flat; together these do not necessarily imply that B_{n-1} is flat, as the above argument fails. Indeed, an easy counterexample is $\mathbb{Z} \hookrightarrow \mathbb{Z} \twoheadrightarrow \mathbb{Z}/n$, where $\mathbb{Z}$ is flat but $\mathbb{Z}/n$ is not.)

Then

$$0 \to Z_\bullet \to P_\bullet \to B_\bullet[-1] \to 0$$

is a short exact sequence of complexes of flat modules, so we can tensor it with $Q_\bullet$ to obtain another short exact sequence of complexes:

$$0 \to Z_\bullet \otimes Q_\bullet \to P_\bullet \otimes Q_\bullet \to B_\bullet \otimes Q_\bullet[-1] \to 0.$$

Taking homology yields the long exact sequence

$$\cdots \to H_{n+1}(B_\bullet \otimes Q_\bullet[-1]) \simeq H_n(B_\bullet \otimes Q_\bullet) \xrightarrow{\partial_n} H_n(Z_\bullet \otimes Q_\bullet) \to H_n(P_\bullet \otimes Q_\bullet) \to \cdots$$

Since every module in $B_\bullet$ and $Z_\bullet$ is flat and every differential is trivial, we can pull each of them out: $H_n(B_\bullet \otimes Q_\bullet) \simeq (B_\bullet \otimes H_\bullet(Q))_n$ and $H_n(Z_\bullet \otimes Q_\bullet) \simeq (Z_\bullet \otimes H_\bullet(Q))_n$. So our long exact sequence in homology becomes

$$\cdots \to (B_\bullet \otimes H_\bullet(Q))_n \xrightarrow{\partial_n} (Z_\bullet \otimes H_\bullet(Q))_n \to H_n(P_\bullet \otimes Q_\bullet) \to \cdots$$

We leave the rest of the proof for next time. ■

11 Lecture 11 (April 19): Künneth formula and group cohomology

Scribe: Gavin Pettigrew

11.1 Künneth Formula Continued

Proof of Künneth formula cont'd. We left off with a long exact sequence

$$\cdots \to (B_\bullet \otimes H_\bullet(Q_\bullet))_n \xrightarrow{\partial_n} (Z_\bullet \otimes H_\bullet(Q_\bullet))_n \to H_n(P_\bullet \otimes Q_\bullet) \to H_{n-1}(B_\bullet \otimes Q_\bullet) \xrightarrow{\partial_{n-1}} \cdots .$$

By the definition of the total complex, this sequence can be rewritten as

$$\cdots \to \bigoplus_{i+j=n} B_i \otimes H_j(Q_\bullet) \xrightarrow{\partial_n} \bigoplus_{i+j=n} Z_i \otimes H_j(Q_\bullet) \to H_n(P_\bullet \otimes Q_\bullet) \to \bigoplus_{i+j=n-1} B_i \otimes H_j(Q_\bullet) \xrightarrow{\partial_{n-1}} \cdots ,$$

and one can check that $\partial_n = \bigoplus_{i+j=n} \partial_{ij}$, where ∂_{ij} is the tensor product of the inclusion $e_i : B_i \hookrightarrow Z_i$ with the identity map $1 : H_j(Q_\bullet) \to H_j(Q_\bullet)$. Focusing on the nth degree, we see that there is a short exact sequence

$$0 \to \operatorname{coker} \partial_n \to H_n(P_\bullet \otimes Q_\bullet) \to \ker \partial_{n-1} \to 0, \tag{1}$$

which turns out to be exactly what we want. To see this, note that

$$0 \to B_i \xrightarrow{e_i} Z_i \to H_i(P_\bullet) \to 0$$

is a flat resolution, so tensoring with $H_j(P_\bullet)$ results in a complex

$$0 \to B_i \otimes H_j(Q_\bullet) \xrightarrow{\partial_{ij}} Z_i \otimes H_j(Q_\bullet) \to H_i(P_\bullet) \otimes H_j(Q_\bullet) \to 0,$$

from which we derive $\ker \partial_{ij} \cong \mathrm{Tor}_1(H_i(P_\bullet), H_j(Q_\bullet))$. Since $(-) \otimes H_j(Q_\bullet)$ is right exact, we also have $\operatorname{coker} \partial_{ij} \cong H_i(P_\bullet) \otimes H_j(Q_\bullet)$. It follows that

$$\ker \partial_{n-1} = \bigoplus_{i+j=n-1} \ker \partial_{ij} \cong \bigoplus_{i+j=n-1} \mathrm{Tor}_1(H_i(P_\bullet), H_j(Q_\bullet)) \quad \text{and}$$
$$\operatorname{coker} \partial_n \cong \bigoplus_{i+j=n} \operatorname{coker} \partial_{ij} = \bigoplus_{i+j=n} H_i(P_\bullet) \otimes H_j(Q_\bullet),$$

so (1) becomes

$$0 \to \bigoplus_{i+j=n} H_i(P_\bullet) \otimes H_j(Q_\bullet) \to H_n(P_\bullet \otimes Q_\bullet) \to \bigoplus_{i+j=n-1} \mathrm{Tor}_1(H_i(P_\bullet), H_j(Q_\bullet)) \to 0.$$

■

11.1 Corollary (Universal Coefficient Theorem)**.** *Let M be an R-module, $P_\bullet$ a chain complex, and assume B_n and P_n are flat for each n. There is a short exact sequence*

$$0 \to H_n(P_\bullet) \otimes M \to H_n(P_\bullet \otimes M) \to \mathrm{Tor}_1^R(H_{n-1}(P_\bullet), M) \to 0.$$

11.2 Corollary. *If $R = \mathbb{Z}$, $P_\bullet \in \mathbb{Z}$, and $M \in \mathbb{Z}-$mod, the short exact sequence from the universal coefficient theorem splits noncanonically. In other words, there is an unnatural isomorphism $H_n(P_\bullet, M) \cong H_n(P_\bullet) \otimes M \oplus \mathrm{Tor}_1(H_{n-1}(P_\bullet), M)$.*

11.3 Example. Let X and Y be topological spaces. The Eilenberg-Zilber theorem says that $H_n(X \times Y) \cong H_n(S_\bullet(X) \otimes S_\bullet(Y))$, where $S_\bullet(X)$ and $S_\bullet(Y))$ denote the singular chain complexes of X and Y respectively. By the Künneth formula, there is a short exact sequence

$$0 \to \bigoplus_{i+j=n} H_i(X) \otimes H_j(Y) \to H_n(X \times Y) \to \bigoplus_{i+j=n-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_i(X), H_j(Y)) \to 0.$$

11.4 Definition. Given a chain complex $P_\bullet$ and a cochain complex $Q^\bullet$, we can construct a double complex whose squares look like

$$\begin{array}{ccc} \mathrm{Hom}(P_i, Q^{j+1}) & \xrightarrow{d_{P*}} & \mathrm{Hom}(P_{i+1}, Q^{j+1}) \\ {\scriptstyle (-1)^i d_Q^*}\big\uparrow & & {\scriptstyle (-1)^{i+1} d_Q^*}\big\uparrow \\ \mathrm{Hom}(P_i, Q^j) & \xrightarrow{d_{P*}} & \mathrm{Hom}(P_{i+1}, Q^j). \end{array} .$$

For the next theorem, let $\mathrm{Hom}(P_\bullet, Q^\bullet)$ denote the total complex of this double complex.

11.5 Theorem (Künneth formula for cohomology)**.** *Suppose $P_\bullet$ is a chain complex and $Q^\bullet$ is a cochain complex of R-modules. If P_n and $d(P_n)$ are projective for each n, then there is a short exact sequence*

$$0 \to \prod_{i+j=n-1} \mathrm{Ext}_R^1(H_i(P_\bullet), H^j(Q^\bullet)) \to H^n(\mathrm{Hom}(P_\bullet, Q^\bullet))$$
$$\to \prod_{i+j=n} \mathrm{Hom}(H_i(P_\bullet), H^j(Q^\bullet)) \to 0.$$

11.2 Group Cohomology

Assume G is a group and R is a commutative ring. We will be working in the abelian category $\mathrm{Rep}_R\, G \cong RG-$mod, which has enough injectives and projectives.

11.6 Definition. Let M be a representation of G over R. We define the *invariance* of M to be the R-module

$$M^G := \{m \in M \mid gm = m\}$$

and the *coinvariance* of M to be

$$M_G = M/\langle m - gm \mid m \in M, g \in G\rangle.$$

The operations $\{-\}^G$ and $\{-\}_G$ define left and right exact functors from $\mathrm{Rep}_R\, G$ to $R-$mod, respectively (ex*).

11.7 Definition. We define the *nth cohomology of G with coefficients in M* by

$$H^n(G, M) := R^n\{M\}^G.$$

Similarly, the *nth homology of G with coefficients in M* is defined by

$$H_n(G, M) = L_n\{M\}_G.$$

12 Lecture 12 (April 22): Group Cohomology *Scribe: Joseph Rogge*

Recall: the category $\operatorname{Rep}_R G$ is equivalent to the category RG-mod. Both are tensor categories: given $M, N \in \operatorname{Rep}_R G$, the product $M \otimes_R N \in \operatorname{Rep}_R G$ via $g(m \otimes n) := (gm) \otimes n$. Similarly, $\operatorname{Hom}_R(M, N) \in \operatorname{Rep}_R G$ via $g \cdot \varphi := g\varphi(g^{-1}-)$ for $\varphi : M \to N$. This is referred to as "internal Hom" since it lands back inside $\operatorname{Rep}_R G$.

Claim: $\operatorname{Hom}_G(M, N) = \operatorname{Hom}_R(M, N)^G$.

Proof. Fix $f : M \to N$ in $\operatorname{Hom}_G(M, N)$, i.e. $f(gm) = gf(m)$ for all $g \in G, m \in M$. Then $g \cdot f(m) = g^{-1} f(gm) = f(m)$, so f is G-invariant. □ ■

Similarly, $M \otimes_{RG} N = (M \otimes_R N)_G$. This yields

$$R \otimes_{RG} M = (R \otimes_R M)_G \cong M_G.$$

The left derived functor $L_i(R \otimes_{RG} -) = L_i\{-\}_G$, namely group homology is equal to $\operatorname{Tor}_i^{RG}(R, -)$.

Similarly, $\operatorname{Hom}_G(R, -) = \operatorname{Hom}(R, -)^G$ from which we obtain $R^i\{-\}^G = \operatorname{Ext}^i_G(R, -)$, from which we see that group cohomology is equal to $\operatorname{Ext}^i_G(R, -)$. Note, we supress R from the group ring notation, but Ext is computed over the ring RG, not the group G.

How does one actually compute group (co)homology? For cohomology, we can produce an injective resolution of M: $M \to I^0 \to I^1 \to \dots$, apply $\{-\}^G$, then take cohomology of this chain complex. This is the same data as $\operatorname{Ext}^i_G(R, M)$, so we could just as well resolve R instead. Let $\dots \to P_1 \to P_0 \to R$ be a projective resolution of RG-modules. To compute $\operatorname{Ext}^i_G(R, M)$, we apply Hom to this resolution: $\dots \leftarrow \operatorname{Hom}(P_1, M) \leftarrow \operatorname{Hom}(P_0, M) \leftarrow 0$. With this approach, we don't need to produce a new resolution, which makes computation easier in practice.

12.1 Examples

12.1 Example. $R = \mathbb{Z}, G = C_n = \langle \sigma | \sigma^n = 1 \rangle$. $RC_n = \mathbb{Z}\langle \sigma \rangle = \mathbb{Z}[\sigma]/\langle \sigma^n - 1 \rangle$. We can resolve $\mathbb{Z}C_n$ as follows: define $N := \sum_{j=0}^{n-1} \sigma^j$. Then we have the infinite repeating resolution

$$\dots \xrightarrow{(\sigma-1)} \mathbb{Z}\langle\sigma\rangle \xrightarrow{N} \mathbb{Z}\langle\sigma\rangle \xrightarrow{(\sigma-1)} \mathbb{Z}\langle\sigma\rangle \xrightarrow{N} \mathbb{Z}\langle\sigma\rangle \xrightarrow{(\sigma-1)} \mathbb{Z}\langle\sigma\rangle \xrightarrow{\varepsilon} \mathbb{Z}.$$

This is a periodic resolution.

$\operatorname{Hom}_{C_n}(-, M) = \operatorname{Hom}_{\mathbb{Z}C_n}(-, M)$, and $\operatorname{Hom}_{\mathbb{Z}C_n}(\mathbb{Z}C_n, M) \cong M$. Thus we have the complex

$$\dots \xleftarrow{(\sigma-1)} M \xleftarrow{N} M \xleftarrow{(\sigma-1)} M \xleftarrow{N} M \xleftarrow{(\sigma-1)} M \leftarrow 0.$$

Upshot:

$$H^0(G,M) = \ker\{\sigma - 1\} = \{m : \sigma m = m\} = M^G.$$
$$H^1(G,M) = \ker\{N\}/\operatorname{im}\{\sigma - 1\} = H^{\text{odd}}.$$
$$H^2(G,M) = \ker\{\sigma - 1\}/\operatorname{im}\{N\} = M^G/(\sigma^{n-1} + \sigma^{n-2} + \ldots + 1)M = H^{\text{even}}.$$

Exercise: compute H_i.
Solution: H^{odd} and H^{even} swap in homology.

12.2 Example. Computing cohomology with $n = p, M = \mathbb{Z}$. Our cochain complex ends up being

$$\ldots \xleftarrow{\sigma-1} \mathbb{Z} \xleftarrow{\sigma^{p-1}+\ldots+1} \mathbb{Z} \xleftarrow{\sigma-1} \mathbb{Z}.$$

Because $\mathbb{Z}$ is a trivial $\mathbb{Z}C_p$-module, $\sigma - 1 = 0$ and $\sigma^{p-1} + \ldots + 1 = p$, so the cochain complex is

$$\ldots \xleftarrow{p} \mathbb{Z} \xleftarrow{0} \mathbb{Z} \xleftarrow{p} \mathbb{Z} \xleftarrow{0} \mathbb{Z}.$$

Taking cohomology yields

$$H^0(C_p, \mathbb{Z}) = \mathbb{Z}.$$
$$H^1(C_p, \mathbb{Z}) = 0.$$
$$H^2(C_p, \mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}.$$

As in the first example, the even and odd positive homology is swapped with cohomology.

12.3 Example. $R = k, n = p$, and k is a field of characteristic p. The field k is a trivial C_p-module, and having characteristic p means $\sigma^p = 1$, so $\sigma - 1 = 0, \sigma^{p-1} + \ldots + 1 = p = 0$, so the cochain complex has all zero maps:

$$\ldots \xleftarrow{0} k \xleftarrow{0} k \xleftarrow{0} k.$$

Thus cohomology (and homology) is identically k.

12.4 Example. $R = \mathbb{C}, n$ arbitrary. Then $\operatorname{Ext}^i_{\mathbb{C}C_n}(\mathbb{C}, M) = \{$ extensions $0 \to M \to E \to \mathbb{C} \to 0\}$. The ring $\mathbb{C}C_n$ is semi-simple so any extension splits, namely the only extension is $0 \to M \to M \oplus \mathbb{C} \to \mathbb{C} \to 0$, so $\operatorname{Ext}^i_{\mathbb{C}C_n}(\mathbb{C}, M) = 0 \forall i > 0$. Therefore $H^0(\mathbb{C}C_n, M) = M^{C_n}$ and all higher cohomology vanishes.

Construction: The Ext groups $\operatorname{Ext}^*_{C_p}(k,k)$ can be decomposed as $\bigoplus_{i=0}^{\infty} \operatorname{Ext}^i_{C_p}(k,k) \cong \bigoplus_{i=0}^{\infty} k$, which can be given a graded ring structure isomorphic to $k[x]$.

Similarly, if $p = 2$, we can decompose $H^*(C_p, k)$ as $\bigoplus_{i=0}^{\infty} H^i(C_p, k) \cong \bigoplus_{i=0}^{\infty} k$, which again has a graded ring structure isomorphic to $k[x]$.

Now let p be an odd prime. The exterior algebra over k of some y is $\bigwedge^*(y) = k[x]/y^2$. Then $H^*(C_p, k)$ can be given a ring structure isomorphic to $k[x] \otimes \bigwedge^*(y)$ with grading defined by x having degree 2 and y having degree 1. This ring is graded

commutative, i.e. $st = (-1)^{|s|\cdot|t|}ts$, with $|s|, |t|$ denoting the degree of s and t respectively. In particular $xy = (-1)^2 yx = yx$.
Each graded piece is a one dimensional k vector space, where the degree $2i$ part is spanned by x^i and the degree $2i+1$ graded piece is spanned by $x^i y$.

13 Lecture 13 (April 24): Group Cohomology, Hopf Algebras
Scribe: William Dudarov

13.1 More Group Cohomology

13.1 Example. Let $C_\infty = \langle \sigma \rangle$, and consider the group ring $\mathbb{Z}\langle \sigma \rangle \cong \mathbb{Z}[x, x^{-1}]$. We compute the cohomology of C_∞ with $\mathbb{Z}$ coefficients.

$$H^n(C_\infty, \mathbb{Z}) = \mathrm{Ext}^n_{C_\infty}(\mathbb{Z}, \mathbb{Z}).$$

Take a projective resolution

$$0 \to \mathbb{Z}\langle \sigma \rangle \xrightarrow{(\sigma-1)} \mathbb{Z}\langle \sigma \rangle \xrightarrow{\sigma \to 1} \mathbb{Z},$$

which is finite.

Note

$$H^0(C_\infty, M) = M^\sigma$$
$$H^1(C_\infty, M) = M_\sigma,$$

and if we had wanted homology, we'd have

$$H_0(C_\infty, M) = M_\sigma$$
$$H_1(C_\infty, M) = M^\sigma,$$

switched around.

We also have

$$H^i(C_\infty, M) = 0, \quad \text{for } i > 1.$$

13.2 Observation. For k a field, the same calculation yields

$$H^1(G, kG) = Ext^1(k, kG) = k \neq 0.$$

$G = C_\infty$, kG is not injective

When G is finite, kG is always injective (and projective since it is a free module).

13.2 Hopf Algebras

A lyric sidenote about finite-dimensional Hopf algebras...

13.3 Definition. Let k be a field. A is called a **Hopf algebra** if we have the following three operations

$$\begin{aligned} &\Delta : A \to A \otimes A, && \text{comultiplication/coproduct,} \\ &\varepsilon : A \to k, && \text{counit,} \\ &S : A \to A, && \text{coinverse,} \end{aligned}$$

such that we have the following compatibility conditions.

Coassociativity:

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A \otimes A \arrow[d, "\Delta\otimes 1"] \\ A \otimes A \arrow[r, "1\otimes\Delta"'] & A \otimes A \otimes A \end{tikzcd}$$

.

The following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[dr] & A \otimes A \arrow[d, "Id\overline{\otimes}\varepsilon"] \\ & A. \end{tikzcd}$$

The following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\varepsilon"'] & A \otimes A \arrow[d, "Id\overline{\otimes}S"] \\ k \arrow[r] & A \end{tikzcd}$$

The coinverse is sometimes called an antipode, because it is not dual to anything, but rather attempts to turn this structure into a group.

One can also define **cocommutative Hopf algebras**, given by

$$\begin{aligned} A \otimes A &\xrightarrow{\tau} A \otimes A \\ a \otimes b &\mapsto b \otimes a. \end{aligned}$$

A Hopf algebra is **cocommutative** if $\Delta \circ \tau = \Delta$.

13.4 Example. The group algebra

$$kG$$

is a Hopf algebra.

We have

$$\begin{aligned} \Delta : kG &\to kG \otimes k \\ g &\mapsto g \otimes g. \end{aligned}$$

For all $a \in A$, if $\Delta(a) = a \otimes a$, then a is called "group-like."

The augmentation map/counit is given by

$$\begin{aligned} \varepsilon : kG &\to k \\ g &\mapsto 1 \end{aligned}$$

$$I = ker(\varepsilon) = \langle (g-1) \rangle$$

is called the **augmentation ideal**.

Finally, the antipode is given by

$$\begin{aligned} S : kG &\to kG \\ g &\mapsto g^{-1}. \end{aligned}$$

Claim: kG is a cocommutative Hopf algebra.

13.5 Example. Take $\mathfrak{g}$ a Lie algebra, and we define its universal enveloping algebra $U(\mathfrak{g})$, given by $k + \mathfrak{g} + \mathfrak{g}^{\otimes 2} + \dots$

We describe the Hopf structure on $U(\mathfrak{g})$.

The coproduct is given by

$$\begin{aligned} \Delta : \mathfrak{g} &\to U(\mathfrak{g}) \otimes U(\mathfrak{g}) \\ x &\mapsto x \otimes 1 + 1 \otimes x \end{aligned}$$

(such x is called primitive).

The counit is given by

$$\begin{aligned} \varepsilon : U(\mathfrak{g}) &\to k \\ x &\mapsto 0. \end{aligned}$$

The coinverse is given by

$$\begin{aligned} S : U(\mathfrak{g}) &\to U(\mathfrak{g}) \\ \mathfrak{g} &\to U(\mathfrak{g}) \\ x &\mapsto -x. \end{aligned}$$

Claim: $U(\mathfrak{g})$ is a cocommutative Hopf algebra, primitively generated.

There was a feeling in the air for a long time that all Hopf algebras were like this, but in the 1960s, Earl Taft came up with the **Taft algebra**,

$$\begin{aligned} g &\to g \otimes g \\ x &\to x \otimes 1 + g \otimes x. \end{aligned}$$

This is how you get quantum groups, Hopf algebras given by generators and relations, quantized Serre relations.

There are a lot of non-cocommutative Hopf algebras, even in characteristic 0, a complete classificiation is still wide open.

13.6 Definition. Let R be a finite-dimensional k-algebra. We say that R is **Frobenius** if there exists a bilinear associative nondegenerate form

$$\eta : R \otimes R \to k$$
$$\eta(ab \otimes c) = \eta(a \otimes bc).$$

13.7 Observation. If we have R a Frobenius algebr,a then η induces an isomorphism

$$R^* \cong R$$

an isomorphism of R-modules.

Let $a \in R$, then $a \iff \eta(a, -) : R \to k$.

13.8 Definition. R is moreover **symmetric** if the form is symmetric.

13.9 Theorem. *1. Let A be a finite-dimensional Hopf algebra over k. Then it is Frobenius.*

2. If G is a finite group. Then kG is not only Frobenius by the above, it is also symmetric.

EXERCISE: define the form on kG.

What does this all have to do with homological algebra?

13.10 Proposition. *If R is a Frobenius algebra, then in the category of R-modules, projectives and injectives and the same.*

13.11 Definition. A category is called Frobenius if projective and injective modules are the same.

What does this mean? In part,

$$H^i(G, kG) = 0, \quad i > 0.$$

Here, $R = kG$.

13.12 Corollary. *Suppose R is Frobenius. Suppose M is an R-module. Assume M is not projective. Then*

1. *Any projective resolution of M is infinite.*
2. *If $Ext^i(N, M) = 0$ for any $i > n$, for all N, then M is projective (a restatement of the claim above).*

All of this happens for finite groups.

14 Lecture 14 (May 8): More on Frobenius Algebras, Group Cohomology, and Products *Scribe: Haoming Ning*

From last time: Frobenius algebras (categories).

14.1 Definition. A finite-dimensional k-algebra R is *Frobenius* if there exists a bilinear associative nondegenerate form $\nu : R \otimes R \to k$.

A category $\mathcal{C}$ is Frobenius if projective objects are the same as injective objects.

14.2 Remark. If R is a Frobenius algebra, then the category of R-modules is Frobenius. (R is "self-injective").

14.3 Example. Let k be a field, G be a group such that $\operatorname{char} k \mid |G|$, then kG is Frobenius. In fact, we have a bilinear, associative, nondegenerate, and symmetric form $\nu : kG \otimes_k kG \to k$, defined by

$$\nu(a \otimes b) = \sum_{g \in G} a_g b_g^{-1} \cdot g$$

where $a = \sum_g a_g \cdot g$ and $b = \sum_g b_g \cdot g$.

14.4 Example. Consider the Lie algebra $\mathfrak{g} = \mathfrak{gl}_n$ over a field of characteristic p. We have an internal p-th power map, called the restriction map, $[p] : \mathfrak{g} \to \mathfrak{g}$, defined by $A^{[p]} \mapsto A^p$. (This structure exists on more general Lie algebras.) We construct

$$\mathcal{U}^{[p]}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})/\langle x^p - x^{[p]} | x \in \mathfrak{g}\rangle,$$

called the restricted enveloping algebra. This is a finite dimensional (primitively generated) Hopf algebra.

14.5 Exercise. Check that the restricted enveloping algebra $\mathcal{U}^{[p]}(\mathfrak{gl}_n)$ for $\mathfrak{gl}_n$ is Frobenius, by constructing the form using Poincaré-Birkhoff-Witt.

14.6 Exercise. As a further exercise, do this computation for $U^{[p]}(\mathfrak{sl}_2)$.

14.1 Restriction and Corestriction

14.7 Construction. Given $f : G \to G'$, we wish to define natural maps on homology $H_*(G, M) \to H_*(G', M)$. First, given a G'-module M, we can equip M with a G-action $g \cdot m = f(g)m$ to view it as a G-module (this is sometimes referred to as the pullback of M). Now, recall that

$$H_*(G, M) = \operatorname{Tor}_*^G(\mathbb{Z}, M) = H_*((P_G \otimes_{\mathbb{Z}} M)_G),$$

and similarly

$$H_*(G', M) = H_*((P'_G \otimes_{\mathbb{Z}} M)_{G'}).$$

Suppose we have a projective resolution $P_{G'} \to \mathbb{Z}$ of G'-modules. Pull-back is by definition an exact functor, so we obtain a (not necessarily projective) resolution

$P_G \to \mathbb{Z}$. By the fundamental theorem of homological algebra, we can lift the identity map $\mathbb{Z} \to \mathbb{Z}$ to obtain

$$\begin{tikzcd} P_G \arrow[r] \arrow[d, dotted, "F"] & \mathbb{Z} \arrow[d, equal] \\ P_{G'} \arrow[r] & \mathbb{Z} \end{tikzcd}$$

Tensoring with M and taking coinvariance, we have

$$\begin{tikzcd} (P_G \otimes_{\mathbb{Z}} M)_G \arrow[r, "F\otimes\mathrm{id}"] \arrow[dr, dotted] & (P_{G'} \otimes_{\mathbb{Z}} M)_G \arrow[d] \\ & (P_{G'} \otimes_{\mathbb{Z}} M)_{G'} \end{tikzcd}$$

This composition induces our desired map on H_*.

$$H_*(G, M) = H_*((P_G \otimes_{\mathbb{Z}} M)_G) \to H_*((P'_G \otimes_{\mathbb{Z}} M)_{G'}) = H_*(G', M).$$

14.8 Construction. As before, given $f : G \to G'$, we will construct a similar natural map on cohomology. Recall that

$$H^*(G, M) = H^*(\mathrm{Hom}_{\mathbb{Z}}(P_G, M)^G)$$

and

$$H^*(G', M) = H^*(\mathrm{Hom}_{\mathbb{Z}}(P_{G'}, M)^{G'}).$$

The same construction gives $F : P_G \to P_{G'}$, which induces

$$\mathrm{Hom}_{\mathbb{Z}}(P_{G'}, M) \to \mathrm{Hom}_{\mathbb{Z}}(P_G, M).$$

Taking G-invariance and pre-composing, we have

$$\begin{tikzcd} (\mathrm{Hom}_{\mathbb{Z}}(P_{G'}, M))^{G'} \arrow[d] \arrow[dr, dotted] & \\ (\mathrm{Hom}_{\mathbb{Z}}(P_{G'}, M))^G \arrow[r] & (\mathrm{Hom}_{\mathbb{Z}}(P_G, M))^G \end{tikzcd}$$

which in turn induces our desired map on cohomology.

14.2 Cup Products in Cohomology

We wish to equip $H^*(G, k) = \bigoplus_{i=0}^{\infty} H^i(G, k)$ with a product map, and the key is to utilize the diagonal map $\Delta : G \to G \times G$ and the cohomology of cross product.

Given groups G, H, we can express $H^*(G \times H, \mathbb{Z})$ in terms of $H^*(G, \mathbb{Z})$ and $H^*(H, \mathbb{Z})$. Take projective resolutions $P_G \to \mathbb{Z}$, $P_H \to \mathbb{Z}$, then we have a projective resolution (in $\mathrm{Rep}(G \times H)$)

$$P_G \otimes_{\mathbb{Z}} P_H \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{Z},$$

noting also that $\mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}H = \mathbb{Z}(G \times H)$. This computation to be continues next time.

15 Lecture 15(May 10): Cross product and Cup product on cohomology *Scribe: Eric Zhang*

15.1 cross product continued

We want to compute $H^*(G \times H, \mathbb{Z})$ using $H^*(G,\mathbb{Z})$ and $H^*(H,\mathbb{Z})$. To do this, we take $P_{\bullet,G} \to \mathbb{Z}$ and $Q_{\bullet,H} \to \mathbb{Z}$ be projective resolutions in Rep_G and Rep_H. Consider $\mathrm{Tot}(P_{\bullet,G} \otimes P_{\bullet,H})$.

15.1 Lemma. $\mathrm{Tot}(P_{\bullet,G} \otimes P_{\bullet,H}) \to \mathbb{Z}$ *is a projective resolution in* $\mathrm{Rep}_{G\times H}$.

15.2 Lemma. *In the above setting,*

1. *There is a well-defined map*

$$\mathrm{Hom}_G(P_i,\mathbb{Z}) \otimes \mathrm{Hom}_H(Q_j,\mathbb{Z}) \xrightarrow{f} \mathrm{Hom}_{G\times H}(P_{i,G} \otimes Q_{j,H},\mathbb{Z})$$

via $f(u \otimes v)(x \otimes y) = u(x)v(y)$ *for* $u : P_i \to \mathbb{Z}$ *and* $v : Q_j \to \mathbb{Z}$.

2. *If* G, H *are finite groups and* $P_\bullet, Q_\bullet$ *are finitely generated, then* f *induces an isomorphism of complexes*

$$\mathrm{Tot}(\mathrm{Hom}_G(P_{i,\mathbb{Z}}) \otimes \mathrm{Hom}_H(Q_j,\mathbb{Z})) \cong \mathrm{Hom}_{G\times H}(P_{\bullet,G} \otimes Q_{\bullet,H},\mathbb{Z}).$$

Note the RHS can be computed by Kunneth formula. We consider the following

$$\bigoplus_{i+j=n} H^i(G,\mathbb{Z}) \otimes H^j(H,\mathbb{Z}) \xrightarrow{\times} H^n(G\times H,\mathbb{Z}) \to \bigoplus_{p+q=n} \mathrm{Tor}^{\mathbb{Z}}(H^p(G,\mathbb{Z}), H^q(H,\mathbb{Z}))$$

where the first map is the cross product in cohomology. In particular Tor vanishes over k, and we have an isomorphism

$$\bigoplus_{i+j=n} H^i(G,k) \otimes H^j(H,k) \cong H^n(G\times H,k).$$

15.3 Remark. We note a few properties of cross product.

1. Construction does not depend on choice of projective resolution.
2. Cross product is associative.
3. Let $M \in \mathrm{Rep}_R\, G, N \in \mathrm{Rep}_R\, H$ for a commutative ring R, we retain a map

$$\bigoplus_{i+j=n} H^i(G,M) \otimes H^j(H,N) \to H^n(G\times H, M \otimes_R N).$$

So cohomology has algebra structure for R-algebras.

15.2 cup product

15.4 Definition. Recall the diagonal map $\Delta : \mathbb{Z}G \to \mathbb{Z}G \times \mathbb{Z}G$. We define the cup product $\smile: H^*(G,\mathbb{Z}) \otimes H^*(G,\mathbb{Z}) \to H^*(G,\mathbb{Z})$ to be the composition

$$H^*(G,\mathbb{Z}) \otimes H^*(G,\mathbb{Z}) \xrightarrow{\times} H^*(G\times G,\mathbb{Z}) \xrightarrow{\Delta^*} H^*(G,\mathbb{Z})$$

via $u \smile v = \Delta^*(u \times v)$ for $u \in H^i(G,\mathbb{Z})$ and $v \in H^j(G,\mathbb{Z})$.

15.5 Remark. We note a few properties of cup product.

1. Cup product is natural in G. That is, for $f : G \to G'$, there is $f^* : \mathrm{Hom}(G', \mathbb{Z}) \to \mathrm{Hom}(G, \mathbb{Z})$ such that $f^*(u \otimes v) = f^*(u) \smile g^*(v)$
2. Cup product is associative.
3. There is a unit $1 \in H^0(G, \mathbb{Z}) = \mathrm{Hom}_G(\mathbb{Z}, \mathbb{Z})$ such that $1 \smile u = u \smile 1 = u$.
4. $H^*(G, \mathbb{Z})$ is a graded commutative algebra.

15.6 Theorem. *Let A be finite dimensional, flat, hopf algebra over a commutative, noetherian ring R. Then*

1. $H^*(A, R) = \bigoplus_{i=0}^{\infty} H^i(A, R)$ *is a unital graded commutative algebra over R.*
2. *If A is cocommutative, then $H^(A, R)$ is finitely generated (and Noetherian).*

15.7 Remark. Part 1 is due to Hochschild. For finite group, part 2 is due to Vantov and Evans around 1961. For cocommutative hopf algebra over field k, part 2 is due to Friedlander and Suslin in 1995. For cocommutative hopf algebra flat over commutative, noetherian ring, part 2 is due to Van der Kaller in 2022. Etingof and Ostrik conjectured in 2004 the statement is true for any finite tensor category.

15.8 Remark. The associativity of cup product is by diagonal approximation due to Alexander and Whitney. Let $\Delta : G \to G \times G$ and the induced map $\tilde{\Delta} : P_G \to P_G \otimes P_G$ given by the fundamental lemma. Then the diagram

$$\begin{tikzcd}
P_G \arrow[r, "\tilde{\Delta}"] \arrow[d, "\tilde{\Delta}"] & P_G \otimes P_G \arrow[d, "\tilde{\Delta} \otimes 1"] \\
P_G \otimes P_G \arrow[r, "1 \otimes \tilde{\Delta}"] & P_G \otimes P_G \otimes P_G
\end{tikzcd}$$

can't commute but does commute up to homotopy. The explicit construction uses bar resolution.

15.9 Remark. The graded commutativity of $H^*(A, R)$ can be shown by Eckmann-Hilton argument. That is, for X a set with two binary operations $\circ$ and $*$ such that

1. $(a * b) \circ (c * d) = (a \circ c) * (b \circ d)$,
2. $\circ$ and $*$ have the same identity e.

Then $\circ = *$ and is commutative and associative. One may show the cup product is the Yoneda product on Hochschild cohomology, which is graded commutative.

15.10 Remark. The Yoneda product does not depend on the hopf strucutre.

16 Lecture 16 (May 12): The Bar Resolution *Scribe: Bashir Abdel-Fattah*

Let G be a group, and consider the free abelian group

$$\mathbb{Z}G^n := \underbrace{\mathbb{Z}G \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z}G}_{n \text{ times}} \cong \mathbb{Z}(\underbrace{G \times \cdots \times G}_{n \text{ times}}),$$

which is also a free $\mathbb{Z}G$-module of rank $|G|^{n-1}$. Then G acts on each factor $\mathbb{Z}G$ and hence on $\mathbb{Z}G^n$ by left multiplication:

$$g \cdot (x_1 \otimes \cdots \otimes x_n) = (gx_1) \otimes \cdots \otimes (gx_n).$$

Let $(g_1, \ldots, g_n)$ denote the element $g_1 \otimes \cdots \otimes g_n \in \mathbb{Z}G^n$. Consider the resolution

$$\cdots \longrightarrow \underset{(\text{degree } n)}{\mathbb{Z}G^{n+1}} \longrightarrow \cdots \longrightarrow \underset{(\text{degree } 1)}{\mathbb{Z}G^2} \longrightarrow \underset{(\text{degree } 0)}{\mathbb{Z}G} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

where $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ is the augmentation map $\sum n_i g_i \mapsto \sum n_i$, and the differentials $d_n : \mathbb{Z}G^{n+1} \to \mathbb{Z}G^n$ are given by

$$d\big((g_0, \ldots, g_n)\big) = \sum_{i=0}^{n} (-1)^i (g_0, \ldots, \widehat{g}_i, \ldots, g_n) = \sum_{i=0}^{n} (-1)^i (g_0, \ldots, g_{i-1}, g_{i+1}, \ldots, g_n)$$

and extending by linearity. Then this gives a free resolution of $\mathbb{Z}$ in $\mathbb{Z}G$-mod, where exactness follows from the fact that the resolution is null-homotopic. In particular, we can define a chain homotopy

$$\begin{tikzcd}
 & \cdots \arrow[r] & \mathbb{Z}G^3 \arrow[r] \arrow[dl, "S_2"'] & \mathbb{Z}G^2 \arrow[r] \arrow[dl, "S_1"'] & \mathbb{Z}G \arrow[r] \arrow[dl, "S_0"'] & \mathbb{Z} \\
\cdots \arrow[r] & \mathbb{Z}G^4 \arrow[r] & \mathbb{Z}G^3 \arrow[r] & \mathbb{Z}G^2 \arrow[r] & \mathbb{Z}G \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

such that $Sd + dS = \mathrm{id}$ by

$$S_n\big((g_0, \ldots, g_n)\big) = (e, g_0, \ldots, g_n)$$

and extending by linearity. Indeed, we can calculate that

$$\begin{aligned}
(Sd + dS)(g_0, \ldots, g_n) &= S\Big(\sum_{i=0}^{n} (-1)^i (g_0, \ldots, \widehat{g}_i, \ldots, g_n)\Big) + d(e, g_0, \ldots, g_n) \\
&= \sum_{i=0}^{n} (-1)^i (e, g_0, \ldots, \widehat{g}_i, \ldots, g_n) \\
&\qquad + (g_0, \ldots, g_n) - \sum_{i=0}^{n} (-1)^i (e, g_0, \ldots, \widehat{g}_i, \ldots, g_n) \\
&= (g_0, \ldots, g_n) = \mathrm{id}(g_0, \ldots, g_n).
\end{aligned}$$

We can give a basis of the module $B_n = \mathbb{Z}G^{n+1}$ over $\mathbb{Z}G$ as follows; first, we define

$$[g_1 | g_2 | \cdots | g_n] := (e, g_1, g_1 g_2, \ldots, g_1 g_2 \cdots g_n) \in B_n.$$

Then the set

$$\{[g_1|\cdots|g_n] : g_1, \ldots, g_n \in G\}$$

gives a basis for B_n. The complex (B_n, d_n) is called the ***Bar resolution*** (of $\mathbb{Z}$ as a $\mathbb{Z}G$-module). Note that

$$\begin{aligned}
d\big([g_1|\cdots|g_n]\big) &= d\big((e, g_1, g_1g_2, \ldots, g_1\cdots g_n)\big) \\
&= (g_1, g_1g_2, \ldots, g_1\cdots g_n) - (e, g_1g_2, g_1g_2g_3, \ldots) \\
&\qquad + (e, g_1, g_1g_2g_3, \ldots) - \cdots + (-1)^n(e, g_1, \ldots, g_1\cdots g_{n-1}) \\
&= g_1 \cdot (e, g_2, g_2g_3, \ldots, g_2\cdots g_n) - [g_1g_2|g_3|\cdots|g_n] \\
&\qquad + [g_1|g_2g_3|g_4|\cdots|g_n] - \cdots + (-1)^n[g_1|\cdots|g_{n-1}] \\
&= g_1 \cdot [g_2|\cdots|g_n] + \sum_{i=1}^{n-1}(-1)^i[g_1|\cdots|g_ig_{i+1}|\cdots|g_n] + (-1)^n[g_1|\cdots|g_{n-1}].
\end{aligned}$$

Define a subcomplex $D_\bullet \subset B_\bullet$ by taking $D_n \subset B_n$ to be submodule generated by the elements $[g_1|\cdots|g_n]$ such that $g_i = e$ for at least one index i. Then the differential $d_n : B_n \to B_{n-1}$ restricts to a map $d_n : D_n \to D_{n-1}$.

Next, define the ***normalized Bar complex*** by $\overline{B}_\bullet = B_\bullet/D_\bullet$, which still gives a free resolution of $\mathbb{Z}$ because S maps D_n into D_{n+1} and hence descends to a homotopy $\mathrm{id}_{\overline{B}_\bullet} \sim 0$.

17 The Hochschild Complex

Let $\cdots \to B_2 \to B_1 \to B_0 \to \mathbb{Z}$ be the Bar resolution of a group G, and take $M \in \mathrm{Rep}_{\mathbb{Z}} G = \mathbb{Z}G\text{-}\mathrm{mod}$. We want to calculate $H^i(G, M)$, so we compute

$$\begin{aligned}
\mathrm{Hom}_G(B_n, M) = \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G^{n+1}, M) &= \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}G^n, M) \\
&= \{\text{functions } f : G^n \to M \text{ (extended linearly)}\}.
\end{aligned}$$

Together with the differentials induced by the differentials in $B_\bullet$, this defines the ***Hochschild complex*** of M, denoted $C^\bullet(G, M)$. The differentials $d^n : C^n(G, M) \to C^{n+1}(G, M)$ are given explictly by taking any $f \in \mathrm{Hom}_{\mathrm{Set}}(G^n, M)$ and defining $d^n f \in \mathrm{Hom}_{\mathrm{Set}}(G^{n+1}, M)$ by

$$\begin{aligned}
(d^n f)(g_1, \ldots, g_{n+1}) = g_1\, f(g_2, \ldots, g_{n+1})& \\
&+ \sum_{i=1}^{n-1}(-1)^i f(g_1, \ldots, g_{i-1}, g_ig_{i+1}, g_{i+2}, \ldots, g_n) \\
&+ (-1)^n f(g_1, \ldots, g_n)
\end{aligned}$$

18 Presentation 2 (May 15): Hochschild (co)homology *Scribe: Soham Ghosh*

18.1 Simplicial objects

Let Δ be the category whose objects are the finite ordered sets $[n] = \{0 < 1 < \cdots < n\}$ for integers $n \geq 0$, and whose morphisms are nondecreasing monotone functions. If $\mathcal{A}$ is any category, a simplicial object A in $\mathcal{A}$ is a contravariant functor from Δ to $\mathcal{A}$, that is, $A : \Delta^{\mathrm{op}} \to \mathcal{A}$. For simplicity, we write A_n for $A([n])$. Similarly, a cosimplicial object C in $\mathcal{A}$ is a covariant functor $C : \Delta \to \mathcal{A}$, and we write A^n for $A([n])$. A morphism of simplicial objects is a natural transformation, and the category $\mathcal{SA}$ of all simplicial objects in $\mathcal{A}$ is just the functor category $\mathcal{A}^{\Delta^{\mathrm{op}}}$.

We want to give a more combinational description of simplicial (and cosimplicial) objects, and for this we need to study the simplicial category Δ directly. It is useful to introduce the face maps ε_i and degeneracy maps η_i. For each n and $i = 0, \cdots, n$ the map $\varepsilon_i : [n-1] \to [n]$ is the unique injective map in Δ whose image misses i and the map $\eta_i : [n+1] \to [n]$ is the unique surjective map in Δ with two elements mapping to i. Combinationally, this means that

$$\varepsilon_i(j) = \left\{ \begin{array}{ll} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{array} \right\}, \quad \eta_i(j) = \left\{ \begin{array}{ll} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{array} \right\}$$

These maps satisfy the following identities:

$$\begin{aligned} \varepsilon_j \varepsilon_i &= \varepsilon_i \varepsilon_{j-1} \quad \text{if } i < j \\ \eta_j \eta_i &= \eta_i \eta_{j+1} \quad \text{if } i \leq j \\ \eta_j \varepsilon_i &= \begin{cases} \varepsilon_i \eta_{j-1} & \text{if } i < j \\ \text{identity} & \text{if } i = j \text{ or } i = j+1 \\ \varepsilon_{i-1} \eta_j & \text{if } i > j+1 \end{cases} \end{aligned}$$

18.1 Proposition. *To give a simplicial object A in $\mathcal{A}$, it is necessary and sufficient to give a sequence of objects $A_0, A_1, \cdots$ together with face operators $\partial_i : A_n \to A_{n-1}$ and degeneracy operators $\sigma_i : A_n \to A_{n+1} (i = 0, 1, \cdots, n)$, which satisfy the following "simplicial" identities*

$$\begin{aligned} \partial_i \partial_j &= \partial_{j-1} \partial_i \quad \textit{if } i < j \\ \sigma_i \sigma_j &= \sigma_{j+1} \sigma_i \quad \textit{if } i \leq j \\ \partial_i \sigma_j &= \begin{cases} \sigma_{j-1} \partial_i & \textit{if } i < j \\ \textit{identity} & \textit{if } i = j \textit{ or } i = j+1 \\ \sigma_j \partial_{i-1} & \textit{if } i > j+1. \end{cases} \end{aligned}$$

Under this correspondence $\partial_i = A(\varepsilon_i)$ and $\sigma_i = A(\eta_i)$.

18.2 Remark. The dual of the above proposition holds, i.e., to give a cosimplicial object, with ∂_i and σ_i replaced by ∂^i and σ^i respectively.

Let A be a simplicial (or semi-simplicial) object in an abelian category $\mathcal{A}$. The associated, or unnormalized, chain complex $C(A)$ has $C_n = A_n$, and its boundary morphism $d : C_n \to C_{n-1}$ is the alternating sum of the face operators $\partial_i : C_n \to C_{n-1}$:

$$d = \partial_0 - \partial_1 + \cdots + (-1)^n \partial_n$$

The (semi-) simplicial identities for $\partial_i \partial_j$ imply that $d^2 = 0$.

18.2 Hochschild Homology and Cohomology of Algebras

We fix a field k. For legibility, we write $\otimes$ for $\otimes_k$ and $R^{\otimes n}$ for the n-fold tensor product $R \otimes \cdots \otimes R$. Let R be a k-algebra and M an R-R bimodule. We obtain a simplicial k-module $M \otimes R^{\otimes *}$ with $[n] \mapsto M \otimes R^{\otimes n}$ $\left(M \otimes R^{\otimes 0} = M\right)$ by declaring

$$\partial_i (m \otimes r_1 \otimes \cdots \otimes r_n) = \begin{cases} m r_1 \otimes r_2 \otimes \cdots \otimes r_n & \text{if } i = 0 \\ m \otimes r_1 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_n & \text{if } 0 < i < n \\ r_n m \otimes r_1 \otimes \cdots \otimes r_{n-1} & \text{if } i = n \end{cases}$$
$$\sigma_i (m \otimes r_1 \otimes \cdots \otimes r_n) = m \otimes \cdots \otimes r_i \otimes 1 \otimes r_{i+1} \otimes \cdots \otimes r_n,$$

where $m \in M$ and the r_i are elements of R. These formulas are k-multilinear, so the ∂_i and σ_i are well-defined homomorphisms, and the simplicial identities are readily verified. (Check this!) The Hochschild homology $HH_*(R, M)$ of R with coefficients in M is defined to be the k-modules

$$HH_n(R, M) = H_n C\left(M \otimes R^{\otimes *}\right).$$

Here $C\left(M \otimes R^{\otimes *}\right)$ is the associated chain complex with $d = \sum (-1)^i \partial_i$:

$$0 \longleftarrow M \xleftarrow{\partial_0 - \partial_1} M \otimes R \xleftarrow{d} M \otimes R \otimes R \xleftarrow{d} \cdots .$$

For example, the image of $\partial_0 - \partial_1$ is the k-submodule $[M, R]$ of M that is generated by all terms $mr - rm (m \in M, r \in R)$. Hence $HH_0(R, M) \cong M/[M, R]$.

Similarly, we obtain a cosimplicial k-module with $[n] \mapsto \mathrm{Hom}_k\left(R^{\otimes n}, M\right) = \{k$-multilinear maps $f : R^n \to M\}$ $\left(\mathrm{Hom}\left(R^{\otimes 0}, M\right) = M\right)$ by declaring

$$\left(\partial^i f\right)(r_0, \cdots, r_n) = \begin{cases} r_0 f(r_1, \ldots, r_n) & \text{if } i = 0 \\ f(r_0, \ldots, r_{i-1} r_i, \ldots) & \text{if } 0 < i < n \\ f(r_0, \ldots, r_{n-1}) r_n & \text{if } i = n \end{cases}$$
$$\left(\sigma^i f\right)(r_1, \cdots, r_{n-1}) = f(r_1, \ldots, r_i, 1, r_{i+1}, \ldots, r_n).$$

The Hochschild cohomology $HH^*(R, M)$ of R with coefficients in M is defined to be the k-modules

$$HH^n(R, M) = H^n C\left(\mathrm{Hom}_k\left(R^{\otimes *}, M\right)\right).$$

Here $C\,\mathrm{Hom}_k(R^*, M)$ is the associated cochain complex

$$0 \longrightarrow M \xrightarrow{\partial^0 - \partial'} \mathrm{Hom}_k(R, M) \xrightarrow{d} \mathrm{Hom}_k(R \otimes R, M) \xrightarrow{d} \cdots$$

For example, it follows immediately that

$$HH^0(R,M) = \{m \in M : rm = mr \quad \text{for all } r \in R\}$$

The above definitions, originally given by Hochschild in 1945, have the advantage of being completely natural in R and M. To put them in a homological framework, it is necessary to consider the enveloping algebra $R^e = R \otimes_k R^{\mathrm{op}}$ of R, where R^{op} is the "opposite ring"- with the same underlying abelian group structure as R, but multiplication in R^{op} is the opposite of that in R (the product $r \cdot s$ in R^{op} is the same as the product sr in R). The main feature of R^{op} is this: A right R-module M is the same thing as a left R^{op} -module via the product $r \cdot m = mr$ because associativity requires that

$$(r \cdot s) \cdot m = (sr) \cdot m = m(sr) = (ms)r = r \cdot (ms) = r \cdot (s \cdot m).$$

Similarly a left R-module N is the same thing as a right R^{op}-module via $n \cdot r = rn$. Consequently, the main feature of R^e is that an $R - R$ bimodule M is the same thing as a left R^e-module via the product $(r \otimes s) \cdot m = rms$, or as a right R^e-module via the product $m \cdot (r \otimes s) = smr$. (Check this!) This gives a slick way to consider the category $R - \mathrm{mod} - R$ of $R - R$ bimodules as the category of left R^e-modules or as the category of right R^e-modules. In particular, the canonical $R - R$ bimodule structure on R makes R into both a left and right R^e-module.

18.3 Proposition. *$HH_*(R,M) \cong \mathrm{Tor}_*^{R^e}(M,R)$ and $HH^*(R,M) \cong \mathrm{Ext}^*_{R^e}(R,M)$ (R is a k-algebra).*

Proof. The bar resolution of R (which is itself an R-bimodule) as an R-bimodule is:

$$\beta(R,R) = \cdots \xrightarrow{b'} R \otimes R \otimes R \otimes R \xrightarrow{b'} R \otimes R \otimes R \xrightarrow{b'} R \otimes R$$

here we write $\otimes$ for $\otimes_k$, $\beta(R,R)_n = R^{\otimes n+2}$ and $\beta(R,R)_0 = R \otimes R$,

$$b'_n\,(a_o \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1},$$

We can write the isormorphism $\beta(R,R)_n := R \otimes R^{\otimes n} \otimes R \simeq (R \otimes R^{op}) \otimes R^{\otimes n} = R^e \otimes R^{\otimes n}$ of $R - R$-bimodules for each $n \geq 0$; by R^e-flatness of such resolution and by definition of Tor we arrive at (Flat resolution lemma: If F is a flat resolution of R, then $\mathrm{Tor}(M,R) = H(M \otimes F)$)

$$\mathrm{Tor}^{R^e}(M,R) = H\left(M \otimes_{R^e} \beta(R,R)\right)$$

The isomorphism $\rho : M \otimes_{R^e} \beta_n(R,R) \to M \otimes R^n$, with $\rho\,(m, r_0, r_1, \ldots, r_n, r_{n+1}) := (r_{n+1} m r_0, r_1, \ldots, r_n)$ for all $n \geq 0$ (note that we used the fact that M is an $R - R$-bimodule) gives the identification $M \otimes_{R^e} \beta(R,R) = \mathcal{C}(M \otimes R^\star)$ (the chain complex of Hochschild homology).

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The following theorem tells how Hochschild (co)homology is affected on changing the algebra R.

18.4 Theorem. *(Change of rings) Let R be a k-algebra and M an $R-R$ bimodule.*

1. *(Product) If R' is another k-algebra and M' an $R'-R'$ bimodule, then*

$$HH_*\left(R \times R', M \times M'\right) \cong HH_*(R, M) \oplus HH_*\left(R', M'\right)$$
$$HH^*\left(R \times R', M \times M'\right) \cong HH^*(R, M) \oplus HH^*\left(R', M'\right).$$

2. *(Flat base change) If R is a commutative k-algebra and $R \to T$ is a ring map such that T is flat as a (left and right) R-module, then*

$$HH_*\left(T, T \otimes_R M \otimes_R T\right) \cong T \otimes_R HH_*(R, M)$$

3. *(Localization) If S is a central multiplicative set in R, then*

$$HH_*\left(S^{-1}R, S^{-1}R\right) \cong HH_*\left(R, S^{-1}R\right) \cong S^{-1}HH_*(R, R)$$

We can form the graded k-modules $HH_\star(R,M) = \bigoplus_{n\geq 0} HH_n(R,M)$ and $HH^\star(R,M) = \bigoplus_{n\geq 0} HH^n(R,M)$.In the case of cohomology, the graded k-module has an algebra structure, which we will see now.

18.3 Products in Hochschild cohomology

The associated graded k-module of the simplicial k-module $\mathrm{Hom}_k(R^{\otimes\star}, M)$ given by $C^*(R,M) = \bigoplus_{n\geq 0} \mathrm{Hom}_k\left(R^{\otimes n}, M\right)$ is the k-module of Hochschild cochains on R with coefficients in M. In what follows, we will take $M = R$. In the special case of $M = R$ with the standard $R-R$ bimodule structure, we will denote $HH^\star(R) := HH^\star(R,R)$. This can be made into an associative (graded commutative) k-algebra with the cup product.

18.5 Definition. Let $f \in \mathrm{Hom}_k\left(R^{\otimes m}, R\right)$ and $g \in \mathrm{Hom}_k\left(R^{\otimes n}, R\right)$. The cup product $f \smile g$ is the element of $\mathrm{Hom}_k\left(R^{\otimes(m+n)}, R\right)$ defined by

$$(f \smile g)\left(a_1 \otimes \cdots \otimes a_{m+n}\right) = (-1)^{mn} f\left(a_1 \otimes \cdots \otimes a_m\right) g\left(a_{m+1} \otimes \cdots \otimes a_{m+n}\right)$$

for all $a_1, \ldots, a_{m+n} \in R$. If $m = 0$, we interpret this formula to be

$$(f \smile g)\left(a_1 \otimes \cdots \otimes a_n\right) = f(1) g\left(a_1 \otimes \cdots \otimes a_n\right),$$

and similarly if $n = 0$.

This product induces an associative graded commutative product on the Hochschild cohomology, which is the cup product: $\smile: \mathrm{HH}^m(R) \times \mathrm{HH}^n(R) \to \mathrm{HH}^{m+n}(R)$. Consequently Proposition‘18.3 can be upgraded to the k-algebra isomorphism

$$HH^\star(R) \cong \mathrm{Ext}^\star_{R^e}(R, R)$$

18.3.1 Finite dimensional Hopf algebras

Let A be a finite dimensional k-Hopf algebra, and let M, M', N, N' be A-modules. There is a cup product for each $i, j \geq 0$,

$$\smile: \mathrm{Ext}^i_A\left(M, M'\right) \times \mathrm{Ext}^j_A\left(N, N'\right) \longrightarrow \mathrm{Ext}^{i+j}_A\left(M \otimes N, M' \otimes N'\right).$$

If $M = N = M' = N = k$, the trivial A-module via the counit $\epsilon : A \to k$, and identify $k \otimes k$ with k, then the cup product makes the graded k-module $H^\star(A,k) = \mathrm{Ext}^\star_A(k,k) := \bigoplus_{n\geq 0} \mathrm{Ext}^{\mathrm{n}}_{\mathrm{A}}(\mathrm{k},\mathrm{k})$ into an associative commutative k-algebra. This is the Hopf algebra cohomology ring of A over k.

Let A^{ad} be A with the left A-module structure given by the action $a.b = \sum a_1 b S(a_2)$ where S is the antipode of A and $\Delta(a) = \sum a_1 \otimes a_2$ in sweedler notation. The graded k-module $H^\star(A, A^{ad}) = \mathrm{Ext}^\star_A(k, A^{ad})$ has a commutative associated graded k-algebra structure given by cup product defined above with $M = N = k$ and $M' = N' = A^{ad}$ along with the induced map $A^{ad} \otimes A^{ad} \to A^{ad}$ from product of A. The following theorem due to Ginzburg and Kumar , relates this with Hochschild cohomology.

18.6 Theorem. *Let A be a Hopf algebra over k with bijective antipode S. There is an isomorphism of k-algebras*

$$\mathrm{HH}^*(A) \cong \mathrm{H}^*\left(A, A^{ad}\right)$$

18.4 Morita Invariance

18.7 Definition. Two rings R and S are said to be Morita equivalent if there is an $R - S$ bimodule P and an $S - R$ bimodule Q such that $P \otimes_S Q \cong R$ as $R - R$ bimodules and $Q \otimes_R P \cong S$ as $S - S$ bimodules. It follows that the functors $\otimes_R P : \mathrm{mod} - R \to \mathrm{mod} - S$ and $\otimes_S Q : \mathrm{mod} - S \to \mathrm{mod} - R$ are inverse equivalences, because for every right R-module M we have $(M \otimes_R P) \otimes S$ $Q \cong M \otimes_R (P \otimes_S Q) \cong M$ and similarly for right S-modules.

18.8 Remark (Facts)**.** A few facts about Morita equivalence

1. Morita equivalence is an equivalence relation.
2. If R and S are Morita equivalent, so are R^{op} and S^{op}.
3. If R and S are Morita equivalent, then the bimodule categories $R - \mathrm{mod} - R$ and $S - \mathrm{mod} - S$ are equivalent (via $Q \otimes_R - \otimes_R P$).

18.9 Proposition. *The matrix rings $M_m(R)$ are Morita equivalent to R.*

18.10 Corollary. *The isomorphism $R - \mathrm{mod} - R \to M_m(R) - \mathrm{mod} - M_m(R)$ associates to an $R - R$ bimodule M the $M_m(R) - M_m(R)$ bimodule $M_m(M)$ of all $m \times m$ matrices with entries in M.*

18.11 Lemma. *If P and Q define a Morita equivalence between R and S, then P is a finitely generated projective left R-module. P is also a finitely generated projective right S-module.*

18.12 Lemma. *If L is a left R-module and Q is a projective right R-module then $H_i(R, L \otimes Q) = 0$ for $i \neq 0$ and $H_0(R, L \otimes Q) \cong Q \otimes_R L$.*

18.13 Theorem. *(R. K. Dennis) Hochschild homology is Morita invariant. That is, if R and S are Morita equivalent rings and M is an $R - R$ bimodule, then*

$$H_*(R, M) \cong H_*\left(S, Q \otimes_R M \otimes_R P\right)$$

19 Spectral Sequences *Presenter: Jackson Morris*

Link

20 Homework Problems

1 Homework problem. Show that $\mathbf{CH}_{\bullet,\mathbf{R}}$ (and $\mathbf{CH}^{\bullet}_{\mathbf{R}}$) is an abelian category. Reference: MacLane "Categories for the working mathematician".

Proof. (Raymond Guo)

Claim 1) $\mathbf{CH}_{\bullet,\mathbf{R}}$ is preadditive

Proof. Let $A_\bullet, B_\bullet \in \mathbf{CH}_{\bullet,\mathbf{R}}$ be a pair of objects. Give $\mathbf{CH}_{\bullet,\mathbf{R}}(A_\bullet, B_\bullet)$ by defining the sum of $f, g \in \mathbf{CH}_{\bullet,\mathbf{R}}(A_\bullet, B_\bullet)$ pointwise at each module. Under this definition of the sum, the fact that $f+g$ is a chain map follows by linearity of the differentials in both chains: For $a \in A_i$, letting d be the differential,

$$d(f+g)(a) = d(f(a)+g(a)) = df(a)+dg(a) = f(d(a))+g(d(a)) = (f+g)(d)(a)$$

where the third equality holds because f and g are both chain maps. Thus $(f+g)d = d(f+g)$, so $f+g$ is a chain map.

Associativity and commutativity can be checked at each individual module, and follow easily from the associativity and commutativity of the addition at each module. Similarly, it can easily be checked at each module that $z : A_\bullet \to B_\bullet$ which is the zero map at each module is an identity under this addition, and that $-f$ can be defined at each module by $-f(a) = -(f(a))$ (which again is easily seen to be a chain map by linearity of the differentials). Thus this defines a valid abelian group structure on $\mathbf{CH}_{\bullet,\mathbf{R}}(A_\bullet, B_\bullet)$. We also see that for $f, f' : A_\bullet \to B_\bullet$ and $g, g' : B_\bullet \to C_\bullet$ that

$$(g+g')\circ(f+f')(a) = (g+g')(f(a)+f'(a)) = g(f(a))+g(f'(a))+g'(f(a))+g'(f'(a))$$

so composition is bilinear.

■

We then show the requirements for an abelian category in sequence.

Claim 2) $\mathbf{CH}_{\bullet,\mathbf{R}}$ has a null object

Proof. Let N be the chain complex that is 0 in every degree. Clearly it has a unique map to each chain complex (the zero map in each degree) and admits a unique map from each chain complex (the zero map in each degree). This makes it a null object by definition. ■

Claim 3) $\mathbf{CH}_{\bullet,\mathbf{R}}$ has binary biproducts

Proof. Let $A_\bullet, B_\bullet$ be chain complexes with differentials d_A and d_B respectively. Define a new chain complex $C_\bullet$ by defining the modules by $C_i = A_i \oplus B_i$ and the differentials by $d_A \oplus d_B : A_i \oplus B_i \to A_{i-1} \oplus B_{i-1}$. Since $(d_A \oplus d_B) \circ (d_A \oplus d_B) = (d_A \circ d_A) \oplus (d_B \circ d_B) = 0 \oplus 0 = 0$, this defines a valid chain complex.

Let $i_A : A_\bullet \to C_\bullet$ be inclusion in the A coordinate at each module and $p_A : C_\bullet \to A_\bullet$ be projection to the A coordinate at each module. Define i_B and p_B

analogously. Then i_A is a chain map because for $a \in A_i$, $d_C i_A(a) = d_A \oplus d_B(a, 0) = (d_A a, 0) = i_A d_A(a)$. Also p_A is a chain map because for $(a, b) \in C_i = A_i \oplus B_i$., $p_A d_C(a, b) = p_A(d_A a, d_B b) = d_A a = d_A(p_A(a, b))$. Entirely symmetric arguments show that i_B and p_B are chain maps.

We see that $p_A \circ i_A = \mathrm{id}_A$ at each module since for $a \in A_i$, $p_A(i_A(a)) = p_A(a, 0) = a$, and identically that $p_B \circ i_B = \mathrm{id}_B$. We also note that for $a \in A_i$, $b \in B_i$,

$$(i_A \circ p_A + i_B \circ p_B)(a, b) = i_A(p_A(a, b)) + i_B(p_B(a, b)) = i_A(a) + i_B(b) = (a, 0) + (0, b) = (a, b)$$

Thus $i_A \circ p_A + i_B \circ p_B = \mathrm{id}_C$. This completes all the checks that C is a binary biproduct of A and B.

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Claim 4) Let $f : A_\bullet \to B_\bullet$ be a chain map. Let $K_\bullet$ be the subcomplex of $A_\bullet$ which is the kernel of $f : A_i \to B_i$ at each homological degree. Let $i : K_\bullet \to A_\bullet$ be the inclusion. Then i is a kernel for f. In particular, every arrow in $\mathbf{CH}_{\bullet,\mathbf{R}}$ has a kernel.

Proof. For $a \in A_i$, if $a \in \ker f$, then $f(d_A(a)) = d_B(f(a)) = d_B(0) = 0$ because f is a chain map. This shows that if $a \in \ker f$, $d_A(a) \in \ker f$, so restricting the differentials on $A_\bullet$ gives valid maps into the desired codomains on $K_\bullet$, and $K_\bullet$ is a valid subcomplex.

Since i maps modules in $K_\bullet$ to the kernels of f in $A_\bullet$, it's clear that $f \circ i = 0$. Let $s : C_\bullet \to A_\bullet$ be a map such that $s \circ f = 0$. This is to say that $f \circ s : C_i \to B_i$ is the zero map at each module, which is to say that $\mathrm{im} s \subset \ker f$ at each module. Then we may restrict the codomain of s to $K_\bullet$, and note that clearly $i \circ s = s$.

Also if $h : C_\bullet \to K_\bullet$ is any map such that $i \circ h = s$, then certainly h is s (with restricted codomain) since i is an inclusion at each module. We have thus proven that i is a kernel for f by definition.

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Claim 5) Let $f : A_\bullet \to B_\bullet$ be a chain map. Then the quotient map $q : B_\bullet \to C_\bullet = B_\bullet / f(A_\bullet)$ is a cokernel for f. In particular, all arrows have cokernels in $\mathbf{CH}_{\bullet,\mathbf{R}}$.

Proof. Since $C_i = B_i / f(A_i)$, and $q : B_i \to B_i / f(A_i)$ is the canonical quotient, certainly $q \circ f = 0$.

Let $D_\bullet$ be another chain complex and $q' : B_\bullet \to D_\bullet$ be a chain map such that $q' \circ f = 0$. At each i, $q'_i \circ f_i(A_i) = 0$, so $f_i(A_i) \subset \ker q'_i$. Then $q'_i : B_i \to D_i$ descends to a unique map $h_i : B_i / f_i(A_i) \to D_i$ satisfying $h_i \circ q_i = q'_i$. Thus if there exists a map $h : C_\bullet \to D_\bullet$ satisfying $h \circ q = q'$, it must be defined by h_i at each module, and thus must be unique. It solely remains to show that the h_i's assemble to form a chain map.

Let $b \in B_i$ be arbitrary and let $[b]$ be its class in $C_i = B_i / f_i(A_i)$. Then since $h_i \circ q_i = q'_i$,

$$h_{i+1}(d_C[b]) = h_{i+1}([d_B(b)]) = h_{i+1}(q_{i+1}(d_B(b))) = q'_{i+1}(d_B(b)) = d_D(q'_i(b)) = d_D(h_i(q_i(b)))$$

where the first equality holds by the definition of the differential on $C_\bullet$, and the fourth holds because q' is a chain map. The equality chain shows that h is a chain map, finishing the proof of the claim.

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Claim 6) Let $f : A_\bullet \to B_\bullet$ be a chain map. If it is injective (at each module), it is a kernel. If it is surjective (at each module), it is a cokernel.

Proof. Assume first that f is injective. Let $q : B_\bullet \to B_\bullet/f(A_\bullet)$ be the canonical quotient. At each i, q is the quotient $B_i \to B_i/f(A_i)$, so the kernel of q_i is $f(A_i)$. Since f is injective, we may identify A_i with $f(A_i) \subset B_i$ at each i. Under this identification, the differential d_A must be the restriction of the differential d_B since f is a chain map and is identified with the inclusion. Then under this identification, the map $f : A_\bullet \to B_\bullet$ is exactly the map of Claim 4 with respect to q, so f is a kernel for q.

Assume instead that f is surjective. Let $j : K_\bullet \to A_\bullet$ be the construction of Claim 4 with respect to f. Then at each i, the kernel of $f_i : A_i \to B_i$ is K_i by construction. Thus we may identify B_i with A_i/K_i because f_i is surjective. Under this identification, f_i is the canonical quotient and the differential d_B must be the descent of d_A to the quotient since f is a chain map. Thus under this identification, $f : A_\bullet \to B_\bullet$ is the construction of Claim 5 with respect to j, so it is a cokernel for j.

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Claim 7) Let $f : A_\bullet \to B_\bullet$ be a chain map. If f is monic, it's injective (at each module) and thus a kernel, and if it's epic, it's surjective (at each module) and thus a cokernel.

Proof. We prove both claims by contrapositive. Assume that f is not injective. Let $i : K_\bullet \to A_\bullet$ be the kernel construction from Claim 4. Then $f \circ i = 0$. Also $f \circ 0_K = 0$ by obvious direct computation, where 0_K is the zero map from $K_\bullet$ to $A_\bullet$. Since f is not injective, $K_\bullet$, which is the kernel of f at each module, is not the chain consisting of all zeroes. Then since i is the inclusion and 0_K is the zero map at each module, $i \neq 0_K$ because there are nonzero modules in $K_\bullet$. Thus we have $f \circ i = f \circ 0_K$ but $i \neq 0_K$, so f is not monic.

Assume instead that f is not surjective. Let $q : B_\bullet \to B_\bullet/f(A_\bullet)$ be the quotient map and let $0_B : B_\bullet \to B/f(A_\bullet)$ be the zero map. Then $q \circ f = 0_B \circ f = 0$. Since q is not surjective, $B_\bullet/f(A_\bullet)$ is nonzero at some module. Letting it be the ith module, $q_i : B_i \to B_i/f(A_i)$ and $0 : B_i \to B_i/f(A_i)$ differ, so $q \neq 0_B$ but $q \circ f = 0_N \circ f$. Thus f is not epic. This finishes proving both claims by contrapositive.

The claims about being a kernel and cokernel follow immediately from Claim 6. ■

Claim 8) $\mathbf{CH}_{\bullet,\mathbf{R}}$ is an abelian category.

Proof. Claim 1 shows that $\mathbf{CH}_{\bullet,\mathbf{R}}$ is preadditive and Claims 2,3,4,5, and 7 show the added necessary conditions for $\mathbf{CH}_{\bullet,\mathbf{R}}$ to be abelian. ■

■

2 Homework problem. Prove that the Hom-functor is left exact, and even more:

1. Show that $A \to B \to C \to 0$ is an exact sequence of R-modules if and only if for any R-module N,
$$\mathrm{Hom}(C,N) \to \mathrm{Hom}(B,N) \to \mathrm{Hom}(A,N) \to 0$$
is exact;

2. Show that $0 \to A \to B \to C$ is an exact sequence of R-modules if and only if for any R-module M
$$0 \to \mathrm{Hom}(M,A) \to \mathrm{Hom}(M,B) \to \mathrm{Hom}(M,C)$$
is exact.

For both cases give examples showing that Hom is not exact.

Proof. (JOSEPH ROGGE)

1. $(\Rightarrow)$: Let $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence of R-modules, and let N be any R-module. Then we have the sequence
$$0 \to \mathrm{Hom}(C,N) \xrightarrow{g^*} \mathrm{Hom}(B,N) \xrightarrow{f^*} \mathrm{Hom}(A,N).$$
Fix $\varphi, \psi \in \mathrm{Hom}(C,N)$, and suppose $g^*(\varphi) = g^*(\psi)$. Then for all $c \in C, \varphi(g(c)) = \psi(g(c))$. Because g is surjective, φ and ψ must agree at every point of B, hence $\varphi = \psi$, so g^* is injective.

Notice for $a \in A$, we have $f^* \circ g^*(\varphi)(a) = \varphi(g(f(a))) = \varphi(0) = 0$, so $f^* \circ g^* \equiv 0$. Namely, $\operatorname{im} g^* \subseteq \ker f^*$. Let $\sigma \in \mathrm{Hom}(B,N)$ and suppose $f^*(\sigma) = 0$. Then $\operatorname{im} f \subseteq \ker \sigma$, hence σ factors through $B/\operatorname{im} f = B/\ker g$ as $\tilde{\sigma}$. Similarly, g factors through $B/\ker g$ as $\tilde{g}$, an isomorphism with image C since g is surjective. Define a map $\tau : C \to N$ by $\tau := \tilde{\sigma} \circ \tilde{g}^{-1}$. Let π denote the projection of B onto $B/\ker g$. By construction,
$$g^*\tau = \tilde{\sigma} \circ \tilde{g}^{-1} \circ g = \tilde{\sigma} \circ \pi = \sigma.$$
Thus
$$0 \to \mathrm{Hom}(C,N) \xrightarrow{g^*} \mathrm{Hom}(B,N) \xrightarrow{f^*} \mathrm{Hom}(A,N)$$
is exact, as desired.

$(\Leftarrow)$: Now fix a (not necessarily exact) sequence of R-modules $A \xrightarrow{f} B \xrightarrow{g} C \to 0$, and suppose
$$0 \to \mathrm{Hom}(C,N) \xrightarrow{g^*} \mathrm{Hom}(B,N) \xrightarrow{f^*} \mathrm{Hom}(A,N)$$
is exact for all R-modules N. To show surjectivity of g, take $N = \operatorname{coker} g$, and let $\pi : C \to \operatorname{coker} g$ be projection. Then $\pi \circ g = 0 = 0 \circ g$, so $g^*(\pi) = g^*(0)$. But g^* is injective by exactness of
$$0 \to \mathrm{Hom}(C, \operatorname{coker} g) \xrightarrow{g^*} \mathrm{Hom}(B, \operatorname{coker} g) \xrightarrow{f^*} \mathrm{Hom}(A, \operatorname{coker} g)$$

so $\pi = 0$. Thus g is surjective.

To show exactness at B, first take $N = C$. Then $id_C \in \operatorname{Hom}(C, C)$ satisfies $0 = f^* \circ g^*(id_C) = g \circ f$, so $\operatorname{im} f \subseteq \ker g$. Now take $N = \operatorname{coker} f$, and let $\pi : B \to \operatorname{coker} f$ be the projection map. By definition $f^*\pi = 0$, so by exactness of

$$0 \to \operatorname{Hom}(C, \operatorname{coker} f) \xrightarrow{g^*} \operatorname{Hom}(B, \operatorname{coker} f) \xrightarrow{f^*} \operatorname{Hom}(A, \operatorname{coker} f),$$

there exists $h \in \operatorname{Hom}(C, \operatorname{coker} f)$ such that $g^*(h) = \pi$. This map factors through $B/\ker g$, so in particular π factors through $B/\ker g$ as well. Since the projection π factors through $B/\ker g$, we have $\ker g = \ker \pi \subseteq \operatorname{im} f$. Hence the sequence

$$A \to B \to C \to 0$$

is exact, as desired.

To see that contravariant Hom is not exact, consider the sequence

$$0 \to \mathbb{Z}/3\mathbb{Z} \xrightarrow{\cdot 3} \mathbb{Z}/9\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/3\mathbb{Z} \to 0$$

where π is projection onto the cokernel of $\cdot 3$. Applying $\operatorname{Hom}(-, \mathbb{Z}/3\mathbb{Z})$, we obtain

$$0 \to \operatorname{Hom}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \xrightarrow{(\cdot 3)_*} \operatorname{Hom}(\mathbb{Z}/9\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \xrightarrow{\pi_*} \operatorname{Hom}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \to 0.$$

Because $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/9\mathbb{Z}$ are cyclic, any homomorphism out of each space is determined by the image of 1. Therefore, $\operatorname{Hom}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$ as 1 can map to $0, 1$, or 2. Similarly, $\operatorname{Hom}(\mathbb{Z}/9\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$ as 1 can map to $0, 1$, or 2 (all have order 3, which divides 9). We know $(\cdot 3)^*$ is surjective since $\operatorname{Hom}(-, \mathbb{Z}/3\mathbb{Z})$ is left exact, so in fact it is an isomorphism. Hence $\pi^* = 0$, but $\operatorname{Hom}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \neq 0$, so the sequence

$$0 \to \operatorname{Hom}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \xrightarrow{(\cdot 3)_*} \operatorname{Hom}(\mathbb{Z}/9\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \xrightarrow{\pi_*} \operatorname{Hom}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \to 0.$$

is not exact.

2. $(\Rightarrow)$: Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ be a left exact sequence of R-modules and let N be any R-module. We want to show

$$0 \to \operatorname{Hom}(N, A) \xrightarrow{f_*} \operatorname{Hom}(N, B) \xrightarrow{g_*} \operatorname{Hom}(N, C)$$

is exact. Fix $\varphi, \psi \in \operatorname{Hom}(N, A)$, and suppose $f_*(\varphi) = f_*(\psi)$. Then for all $n \in N$,

$$f(\varphi(n)) = f_*(\varphi)(n) = f_*(\psi)(n) = f(\psi(n)).$$

By injectivity of f, we have $\varphi(n) = \psi(n)$, so $\varphi = \psi$. Thus f_* is injective.

Because $g \circ f \cong 0$, for all $\varphi \in \operatorname{Hom}(N, A), n \in N$, we have $g_* \circ f_*(\varphi)(n) = g(f(\varphi(n))) = 0$, so $\operatorname{im} f_* \subseteq \ker g_*$. To show the reverse containment, suppose $\sigma \in \operatorname{Hom}(N, B)$ and $g_*(\sigma) = 0$. Then $\operatorname{im} \sigma \subseteq \ker g = \operatorname{im} f$, hence for all $n \in N$, there exists $a \in A$ such that $f(a) = \sigma(n)$. By injectivity of f, this

a is unique. Define a map $\varphi : N \to A$ by $\tau(n) := f^{-1}(\sigma(n))$. Observe $f^{-1}(\sigma(0)) = f^{-1}(0) = 0$. Moreover,

$$f(f^{-1}(\sigma(m))f^{-1}(\sigma(n))) = \sigma(m)\sigma(n) = \sigma(mn) = f(f^{-1}(\sigma(mn)),$$

so by injectivity of f, we have $\tau(mn) = \tau(m)\tau(n)$. Thus τ is indeed a homomorphism, and by definition $f_*(\tau) = \sigma$. Thus the sequence

$$0 \to \mathrm{Hom}(N, A) \xrightarrow{f_*} \mathrm{Hom}(N, B) \xrightarrow{g_*} \mathrm{Hom}(N, C)$$

is exact.

$(\Leftarrow)$: Now suppose $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ is a (not necessarily exact) sequence of R-modules, and for all R-modules N, the sequence

$$0 \to \mathrm{Hom}(N, A) \xrightarrow{f_*} \mathrm{Hom}(N, B) \xrightarrow{g_*} \mathrm{Hom}(N, C)$$

is exact. Recall $\mathrm{Hom}(R, M)$ is naturally isomorphic to M, and under this isomorphism, pushforwards φ_* becomes the original map φ. Taking $N = R$, we have an exact sequence naturally isomorphic to $0 \to A \xrightarrow{f} B \xrightarrow{g} C$, so in particular the original sequence is exact.

To see that covariant Hom is not exact, consider the sequence

$$0 \to \mathbb{Z}/3\mathbb{Z} \xrightarrow{\cdot 3} \mathbb{Z}/9\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/3\mathbb{Z} \to 0$$

where π is projection onto the cokernel of $\cdot 3$. Applying $\mathrm{Hom}(\mathbb{Z}/3\mathbb{Z}, -)$, we obtain

$$0 \to \mathrm{Hom}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \xrightarrow{(\cdot 3)_*} \mathrm{Hom}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/9\mathbb{Z}) \xrightarrow{\pi_*} \mathrm{Hom}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \to 0.$$

Because $\mathbb{Z}/3\mathbb{Z}$ is cyclic, any homomorphism is determined by the image of 1, so $\mathrm{Hom}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$ as 1 can map to $0, 1$, or 2. Similarly, $\mathrm{Hom}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/9\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$ as 1 can map to $0, 3$, or 6. We know $(\cdot 3)_*$ is injective since $\mathrm{Hom}(\mathbb{Z}/3\mathbb{Z}, -)$ is left exact, so in fact it is an isomorphism. Hence $\pi_* = 0$, but $\mathrm{Hom}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \neq 0$, so the sequence

$$0 \to \mathrm{Hom}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \xrightarrow{(\cdot 3)_*} \mathrm{Hom}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/9\mathbb{Z}) \xrightarrow{\pi_*} \mathrm{Hom}(\mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/3\mathbb{Z}) \to 0.$$

is not exact.

■

3 Homework problem. Show that an R-module P is projective (satisfies the lifting property) if and only if the functor $\mathrm{Hom}_R(P, -)$ is exact.

Proof. Take a projective R-module P. This module satisfies the following lifting property: for every surjection of R-modules $B \twoheadrightarrow C$ and R-module map $P \to C$, there is a map $P \to B$ such that the appropriate diagram commutes:

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d] \\ B \arrow[r, two heads] & C. \end{tikzcd}$$

This precisely encodes the condition that $\mathrm{Hom}_R(P, -)$ is exact. The functor $\mathrm{Hom}_R(P, -)$ is always left exact, and the lifting property tells us that for every map $P \to C$, we can pull back to a map $P \to B$; i.e., this tells us that the induced map

$$\mathrm{Hom}_R(P, B) \to \mathrm{Hom}_R(P, C)$$

is surjective, hence $\mathrm{Hom}_R(P, -)$ is exact. This argument holdsin the opposite direction. (JACKSON MORRIS) ■

4 Homework problem. Show a null homotopic chain complex is split exact.

Proof. (Ansel Goh) Assume that a chain complex $C_\bullet$ is null homotopic. Then, there exists $\{s_n : C_n \to C_{n+1}\}$ such that $sd + ds = \mathrm{id}_C$. We want to show that for each n, the short exact sequence

$$0 \to B_n(C) \hookrightarrow C_n \twoheadrightarrow B_{n-1} \to 0$$

splits. We will do this by showing that ds gives a split injection, meaning that if f is the inclusion map from B_n to C_n, then $dsf = \mathrm{id}_{B_n}$. Note that s maps from C_n into C_{n+1} and by definition, $B_n = \mathrm{im}(d_{n+1})$ so ds does in fact map from C_n to B_n. Then, we know that $sd + ds = \mathrm{id}_C$ so $dsf = \mathrm{id}_C f - sdf = \mathrm{id}_{B_n} - sdf$. Therefore, we simply need to show that $sdf = 0$. Note that for all $x \in B_n(C)$, $x \in \mathrm{im}(d_{n+1}) \subset \ker(d_n)$. So, $sdf(x) = sd(x) = s(0) = 0$. Thus, $dsf = \mathrm{id}_{B_n}$ and ds gives us a split injection. As a result, the short exact sequence

$$0 \to B_n(C) \hookrightarrow C_n \twoheadrightarrow B_{n-1} \to 0$$

splits for each n and $C_\bullet$ is split exact. ■

5 Homework problem. Complete the proof of Proposition 4.1.

6 Homework problem. Prove Lemma 4.6

Proof. ($\implies$) Assume $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is exact. Consider

$$0 \to \mathrm{Hom}(M, A) \xrightarrow{\alpha_*} \mathrm{Hom}(M, B) \xrightarrow{\beta_*} \mathrm{Hom}(M, C).$$

Note $\alpha_*(f) = 0$ iff $\alpha \circ f = 0$ iff f factors through $\ker \alpha$ which is 0 by assumption. It follows $f = 0$ and $\ker \alpha_* = 0$. This proves exactness of hom sequence at $\mathrm{Hom}(M, A)$. Also note $g \in \ker \beta_*$ iff $\beta \circ g = 0$ iff g factors through $\ker \beta = \mathrm{im}\, \alpha$. But α is a monomorphism in some abelian category and hence $\mathrm{im}\, \alpha = \alpha$. So $g \in \ker \beta_*$ iff g factors through α iff $g \in \mathrm{im}\, \alpha_*$. That is $\mathrm{im}\, \alpha_* = \ker \beta_*$. (Recall for a monomorphism α in some abelian category, by axiom $\alpha = \ker \mathrm{coker}\, \alpha$ and by definition $\mathrm{im}\, \alpha = \ker \mathrm{coker}\, \alpha$ Weibel p.6.) This proves exactness of hom sequence at $\mathrm{Hom}(M, B)$.

($\impliedby$) Assume $0 \to \mathrm{Hom}(M, A) \xrightarrow{\alpha_*} \mathrm{Hom}(M, B) \xrightarrow{\beta_*} \mathrm{Hom}(M, C)$ for all M. Consider

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C.$$

Note $\alpha \circ f = 0$ iff $\alpha_*(f) = 0$ iff $f = 0$ since α_* is injection. So $\ker\alpha = 0$ (so α is monomorphism) and the sequence is exact at A. Also $\operatorname{im}\alpha_* = \ker\beta_*$ implies for all $M \xrightarrow{g} B$ such that $\beta \circ g = 0$, there is a $M \xrightarrow{f} A$ such that $g = \alpha \circ f$. In commutative diagram, we get

$$\begin{tikzcd}
 & & M \arrow[dl, dotted, bend right, "f"'] \arrow[d, "g"] \arrow[dr, bend left, "0"] & \\
0 \arrow[r] & A \arrow[r, "\alpha"] & B \arrow[r, "\beta"] & C
\end{tikzcd}.$$

This is exactly that $\alpha = \ker\beta$. Since α is monomorphism, $\alpha = \operatorname{im}\alpha$. It follows $\operatorname{im}\alpha = \ker\beta$ and the sequence is exact at B. (ERIC ZHANG). ■

7 Homework problem. Prove the Yoneda lemma (Proposition 4.2).

Proof. First, note that there are some subtleties regarding the exact statement of the lemma. We can either proceed by assuming that $\mathscr{A}$ is an arbitrary category (in which case its hom sets need not have the structure of an abelian group) and $\mathscr{F}$ is a functor $\mathscr{A} \to \mathrm{Set}$, and show that the collection of natural transformations from $h_A : \mathscr{A} \to \mathrm{Set}$ to $\mathscr{F} : \mathscr{A} \to \mathrm{Set}$ is isomorphic to $\mathscr{F}(A)$ as sets, or we can assume that $\mathscr{A}$ is at least preadditive (so that the hom functors $\mathscr{A} \to \mathrm{Set}$ are actually functors $\mathscr{A} \to \mathrm{Ab}$) and that the functor $\mathscr{F} : \mathscr{A} \to \mathrm{Ab}$ is additive (so that all of the maps $\mathrm{Hom}_{\mathscr{A}}(X, Y) \to \mathrm{Hom}_{\mathrm{Ab}}(\mathscr{F}(X), \mathscr{F}(Y))$ are group homomorphisms), and show that the collection of natural transformations from $h_A : \mathscr{A} \to \mathrm{Ab}$ to $\mathscr{F} : \mathscr{A} \to \mathrm{Ab}$ is isomorphic to $\mathscr{F}(A)$ as abelian groups. We will refer to the latter case as "case 2" below, so that we can make the necessary modifications to the proof as appropriate.

Then, fixing any object $A \in \mathscr{A}$ and any contravariant functor $\mathscr{F} : \mathscr{A}^{\mathrm{op}} \to \mathrm{Set}$ (or $\mathscr{A}^{\mathrm{op}} \to \mathrm{Ab}$ in case 2), we recall that by definition a natural transformation $\Phi \in \mathrm{Nat}(h_A, \mathscr{F})$ is a collection of functions (homomorphisms of abelian groups in case 2) $\Phi(X) : h_A(X) = \mathrm{Hom}_{\mathscr{A}}(X, A) \to \mathscr{F}(X)$ for each object $X \in \mathscr{A}$ such that the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{A}}(Y, A) \arrow[r, "g^*"] \arrow[d, "\Phi(Y)"'] & \mathrm{Hom}_{\mathscr{A}}(X, A) \arrow[d, "\Phi(X)"] \\
\mathscr{F}(Y) \arrow[r, "\mathscr{F}(g)"'] & \mathscr{F}(X)
\end{tikzcd}$$

for any objects $X, Y \in \mathscr{A}$ and any morphism $g \in \mathrm{Hom}_{\mathscr{A}}(X, Y)$. In particular, using the map $\Phi(A) : \mathrm{Hom}_{\mathscr{A}}(A, A) \to \mathscr{F}(A)$, we can define the element $a_\Phi := \big(\Phi(A)\big)(\mathrm{id}_A) \in \mathscr{F}(A)$, in which case for any object $X \in \mathscr{A}$ any element $f \in \mathrm{Hom}_{\mathscr{A}}(X, A)$ the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{A}}(A, A) \arrow[r, "f^*"] \arrow[d, "\Phi(Y)"'] & \mathrm{Hom}_{\mathscr{A}}(X, A) \arrow[d, "\Phi(X)"] \\
\mathscr{F}(A) \arrow[r, "\mathscr{F}(f)"'] & \mathscr{F}(X)
\end{tikzcd}$$

gives us that

$$\begin{aligned}\big(\Phi(X)\big)(f) &= \big(\Phi(X)\big)(\mathrm{id}_A \circ f) = \big(\Phi(X)\big)\big(f^*(\mathrm{id}_A)\big)\\ &= \big(\mathscr{F}(f)\big)\big((\Phi(Y))(\mathrm{id}_A)\big) = \big(\mathscr{F}(f)\big)(a_\Phi).\end{aligned}$$

That is, this tells us that the map $\mathrm{Nat}(h_A, \mathscr{F}) \to \mathscr{F}(A)$ sending $\Phi \mapsto a_\Phi$ is injective, since if we have any natural transformations $\Phi, \Psi \in \mathrm{Nat}(h_A, \mathscr{F})$ such that $a_\Phi = a_\Psi$, then for every object $X \in \mathscr{A}$ we have that the maps $\Phi(X), \Psi(X) : \mathrm{Hom}_{\mathscr{A}}(X, A) \to \mathscr{F}(X)$ are equal because

$$\forall f \in \mathrm{Hom}_{\mathscr{A}}(X, A) : \quad \big(\Phi(X)\big)(f) = \big(\mathscr{F}(f)\big)(a_\Phi) = \big(\mathscr{F}(f)\big)(a_\Psi) = \big(\Psi(X)\big)(f),$$

in which case $\Phi = \Psi$. To see that the given map $\mathrm{Nat}(h_A, \mathscr{F}) \to \mathscr{F}(A)$ is also surjective, take any arbitrary element $a \in \mathscr{F}(A)$, and define a collection of maps $\Phi(X) : \mathrm{Hom}_{\mathscr{A}}(X, A) \to \mathscr{F}(X)$ for each object $X \in \mathscr{A}$ by

$$\big(\Phi(X)\big)(f) = \big(\mathscr{F}(f)\big)(a) \qquad \text{for} \quad f \in \mathrm{Hom}_{\mathscr{A}}(X, A).$$

Note that in case 2, this is in fact a group homomorphism $\mathrm{Hom}_{\mathscr{A}}(X, A) \to \mathscr{F}(X)$, because for any elements $f_1, f_2 \in \mathrm{Hom}_{\mathscr{A}}(X, A)$ we have that

$$\begin{aligned}\big(\Phi(X)\big)(f_1 + f_2) &= \big(\mathscr{F}(f_1 + f_2)\big)(a) = \big(\mathscr{F}(f_1) + \mathscr{F}(f_2)\big)(a)\\ &= \big(\mathscr{F}(f_1)\big)(a) + \big(\mathscr{F}(f_2)\big)(a) = \big(\Phi(X)\big)(f_1) + \big(\Phi(X)\big)(f_2).\end{aligned}$$

Then this prescription in fact defines a natural transformation $\Phi \in \mathrm{Nat}(h_A, \mathscr{F})$, since for any objects $X, Y \in \mathscr{A}$ and any morphism $g \in \mathrm{Hom}_{\mathscr{A}}(X, Y)$, we can verify that the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{A}}(Y, A) \arrow[r, "g^*"] \arrow[d, "\Phi(Y)"'] & \mathrm{Hom}_{\mathscr{A}}(X, A) \arrow[d, "\Phi(X)"] \\
\mathscr{F}(Y) \arrow[r, "\mathscr{F}(g)"'] & \mathscr{F}(X)
\end{tikzcd}$$

commutes. Indeed, given any arbitrary element $f \in \mathrm{Hom}_{\mathscr{A}}(Y, A)$, we know on one hand that

$$\big(\Phi(X)\big)\big(g^*(f)\big) = \big(\Phi(X)\big)(f \circ g) = \big(\mathscr{F}(f \circ g)\big)(a),$$

and on the other hand we have that

$$\big(\mathscr{F}(g)\big)\big((\Phi(Y))(f)\big) = \big(\mathscr{F}(g)\big)\big((\mathscr{F}(f))(a)\big) = \big(\mathscr{F}(g) \circ \mathscr{F}(f)\big)(a) = \big(\mathscr{F}(f \circ g)\big)(a)$$

as desired. Since the given natural transformation Φ satisfies

$$a_\Phi = \big(\Phi(A)\big)(\mathrm{id}_A) = \big(\mathscr{F}(\mathrm{id}_A)\big)(a) = \mathrm{id}_{\mathscr{F}(A)}(a) = a,$$

we have produced the desired element $\Phi \in \mathrm{Nat}(h_A, \mathscr{F})$ such that $a_\Phi = a$ for the arbitrary element $a \in \mathscr{F}(A)$. Therefore the map $\mathrm{Nat}(h_A, \mathscr{F}) \to \mathscr{F}(A), \Phi \mapsto a_\Phi$ is bijective and gives an isomorphism

$$\mathrm{Nat}(h_A, \mathscr{F}) \cong \mathscr{F}(A)$$

as sets. In case 2, we note that $\mathrm{Nat}(h_A, \mathscr{F})$ actually has the structure of an abelian group with respect to object-wise addition. That is, given $\Phi, \Psi \in \mathrm{Nat}(h_A, \mathscr{F})$ we define $\Phi + \Psi$ by taking the map $(\Phi + \Psi)(X) : \mathrm{Hom}_{\mathscr{A}}(X, A) \to \mathscr{F}(X)$ to be the pointwise sum of $\Phi(X), \Psi(X) : \mathrm{Hom}_{\mathscr{A}}(X, A) \to \mathscr{F}(X)$ for any object $X \in \mathscr{A}$, which is indeed a natural transformation because we have that

$$\begin{aligned}\mathscr{F}(g) \circ (\Phi + \Psi)(Y) &= \mathscr{F}(g) \circ \big(\Phi(Y) + \Psi(Y)\big) = \mathscr{F}(g) \circ \Phi(Y) + \mathscr{F}(g) \circ \Psi(Y) \\ &= \Phi(X) \circ g^* + \Psi(X) \circ g^* = \big(\Phi(X) + \Psi(X)\big) \circ g^* \\ &= (\Phi + \Psi)(X) \circ g^*\end{aligned}$$

for all $X, Y \in \mathscr{A}$ and $g \in \mathrm{Hom}_{\mathscr{A}}(X, Y)$. We can similarly verify that the identity element in $\mathrm{Nat}(h_A, \mathscr{F})$ is the zero natural transformation given by taking all of the maps $\mathrm{Hom}_{\mathscr{A}}(X, A) \to \mathscr{F}(X)$ to be identically zero, and additive inverses are defined by setting $(-\Phi)(X) = -\big(\Phi(X)\big)$. In this case, we see that the bijection $\mathrm{Nat}(h_A, \mathscr{F}) \to \mathscr{F}(A), \Phi \mapsto a_\Phi$ is in fact a group homomorphism, because for all $\Phi, \Psi \in \mathrm{Nat}(h_A, \mathscr{F})$

$$\begin{aligned}a_{(\Phi+\Psi)} &= \big((\Phi + \Psi)(X)\big)(\mathrm{id}_A) = \big(\Phi(X) + \Psi(X)\big)(\mathrm{id}_A) \\ &= \big(\Phi(X)\big)(\mathrm{id}_A) + \big(\Psi(X)\big)(\mathrm{id}_A) = a_\Phi + a_\Psi.\end{aligned}$$

(BASHIR ABDEL-FATTAH) ■

8 Homework problem. Prove Proposition 6.6. Let $(\mathcal{F}, \mathcal{G})$ be an adjoint pair of functors. Then $\mathcal{F}$ is right exact, and $\mathcal{G}$ is left exact.

Proof. Let $(\mathcal{F}, \mathcal{G})$ be an adjoint pair of functors with $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{B} \to \mathcal{A}$. We first show $\mathcal{G}$ is left exact. Indeed, let $0 \to B_1 \to B_2 \to B_3 \to 0$ be a short exact sequence in $\mathcal{B}$. For every $A \in \mathcal{A}$, the functor $\mathrm{Hom}_{\mathcal{B}}(\mathcal{F}(A), -)$ is left exact (Example 4.5), so

$$0 \to \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}(A), B_1) \to \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}(A), B_2) \to \mathrm{Hom}_{\mathcal{B}}(\mathcal{F}(A), B_3)$$

is an exact sequence. Since $(\mathcal{F}, \mathcal{G})$ is an adjoint pair, we have natural isomorphism $\mathrm{Hom}_{\mathcal{B}}(\mathcal{F}(A), B_i) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{G}(B_i)$ for $i = 1, 2, 3$, so

$$0 \to \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{G}(B_1) \to \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{G}(B_2) \to \mathrm{Hom}_{\mathcal{A}}(A, \mathcal{G}(B_3))$$

is also an exact sequence. By Lemma 4.6, since $A \in \mathcal{A}$ was arbitrary, we obtain that $0 \to \mathcal{G}(B_1) \to \mathcal{G}(B_2) \to \mathcal{G}(B_3)$ is exact as desired. Therefore, $\mathcal{G}$ is a left exact functor.

To see that $\mathcal{F}$ is right exact, we simply observe that $(\mathcal{G}^{op}, \mathcal{F}^{op})$ is an adjoint pair. Thus, $\mathcal{F}^{op}$ is a left exact functor by above. Therefore, $\mathcal{F}$ is a right exact functor. (NATHAN LOUIE)

■

9 Homework problem. Compute $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, N)$

Notes/Hints: Tor commutes with direct limits. $\mathbb{Q}/\mathbb{Z} \cong \varinjlim \mathbb{Z}/n\mathbb{Z}$

Proof. As the hint says, Tor commutes with direct limits, and $\mathbb{Q}/\mathbb{Z} \cong \varinjlim \mathbb{Z}/n\mathbb{Z}$, we see that

$$\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, N) = \varinjlim \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, N) = \varinjlim {}_nN = \mathrm{Tor}(N)$$

where $\mathrm{Tor}(N)$ denotes the torsion subgroup of N. ■

10 Homework problem. By Example 10.1, $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/p, \mathbb{Z}/p) \simeq \mathbb{Z}/p$, so there should be p isomorphism classes of length-1 extensions (i.e. short exact sequences) of the form

$$0 \to Z/p \hookrightarrow E \twoheadrightarrow \mathbb{Z}/p \to 0$$

for some abelian group E. Find them all. (Hint: there is one split exact sequence that corresponds to $0 \in \mathbb{Z}/p$, while another choice of E will give you the $p-1$ short exact sequences that do not split.)

Proof. Note that we must have that $|E| = p^2$, and we know that any group of order p^2 for p a prime is isomorphic to either $\mathbb{Z}/p \oplus \mathbb{Z}/p$ or to $\mathbb{Z}/p^2$.

We claim that the desired p isomorphism classes of length-1 extensions are represented by the short exact sequences

$$0 \to \mathbb{Z}/p \overset{\iota}{\hookrightarrow} \mathbb{Z}/p \oplus \mathbb{Z}/p \overset{\pi}{\twoheadrightarrow} \mathbb{Z}/p \to 0$$

and

$$0 \to \mathbb{Z}/p \overset{p}{\hookrightarrow} \mathbb{Z}/p^2 \overset{i}{\twoheadrightarrow} \mathbb{Z}/p \to 0$$

for $1 \le i \le p-1$.

The split extension above is an extension of $\mathbb{Z}/p$ by $\mathbb{Z}/p$, and it is the only possible extension with $E = \mathbb{Z}/p \oplus \mathbb{Z}/p$, so we fix $1 \le i \le p-1$ and show that

$$0 \to \mathbb{Z}/p \overset{p}{\hookrightarrow} \mathbb{Z}/p^2 \overset{i}{\twoheadrightarrow} \mathbb{Z}/p \to 0$$

is really an extension of $\mathbb{Z}/p$ by $\mathbb{Z}/p$.

Note that this is in fact the case since the map $\mathbb{Z}/p \overset{p}{\hookrightarrow} \mathbb{Z}/p^2$ is injective, and the map $\mathbb{Z}/p^2 \overset{i}{\twoheadrightarrow} \mathbb{Z}/p$ is surjective since i is non-zero and thus has a non-zero element of $\mathbb{Z}/p$ in its image, which generates all of $\mathbb{Z}/p$.

For $E = \mathbb{Z}/p^2$, the left homomorphism must be injective, so it must send $1 \in \mathbb{Z}/p$ to an element of order p (otherwise, we get either the trivial homomorphism or all of $\mathbb{Z}/p^2$ and hence not an injective map). We choose this map to be p without loss of generality, since the image of p is the same, the unique subgroup in $\mathbb{Z}/p^2$ of order p, as that of $2p, 3p, \ldots, (p-1)p$.

The right map we take to be one of $i = 1, \ldots, p-1$ without loss of generality, since these yield a surjection onto $\mathbb{Z}/p$ with the subgroup of $\mathbb{Z}/p^2$ of order p as kernel, and 1 yields the same map as $p+1$, 2 the same as $p+2$, etc.

For $1 \le i \ne j \le p-1$, note that

$$0 \to \mathbb{Z}/p \overset{p}{\hookrightarrow} \mathbb{Z}/p^2 \overset{i}{\twoheadrightarrow} \mathbb{Z}/p \to 0$$

and

$$0 \to \mathbb{Z}/p \overset{p}{\hookrightarrow} \mathbb{Z}/p^2 \overset{j}{\twoheadrightarrow} \mathbb{Z}/p \to 0$$

give inequivalent extensions of $\mathbb{Z}/p$ by $\mathbb{Z}/p$, since the right square in

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}/p \arrow[r, hook, "p"] \arrow[d, dashed, "="] & \mathbb{Z}/p^2 \arrow[r, two heads, "i"] \arrow[d, dashed, "f"] & \mathbb{Z}/p \arrow[r] \arrow[d, dashed, "="] & 0 \\
0 \arrow[r] & \mathbb{Z}/p \arrow[r, hook, "p"] & \mathbb{Z}/p^2 \arrow[r, two heads, "j"] & \mathbb{Z}/p \arrow[r] & 0
\end{tikzcd}$$

cannot commute, since $f(1) :\equiv n \mod p^2 \equiv ij^{-1} \mod p$, and the left square commutes, and we have that $pm \equiv npm \mod p$ for all $m \in \mathbb{Z}/p$. So $n \equiv 1 \mod p$, and thus $n \equiv ij^{-1} \mod p \equiv 1 \mod p$, and so $i \equiv j \mod p$, a contradiction to our choice of $1 \le i \ne j \le p-1$.

(WILLIAM DUDAROV) ■

11 Homework problem. Given an abelian group A, compute $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n, A)$. (This should be similar to computing Tor, except the arrows are reversed.)

Proof. (Gavin Pettigrew). Recall that $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n, A)$ may be computed using either an injective resolution of A or a projective resolution of $\mathbb{Z}/n$. We will use the latter method. Letting $\varphi : \mathbb{Z} \to \mathbb{Z}$ be the map $\varphi(x) = nx$, observe that

$$0 \to \mathbb{Z} \xrightarrow{\varphi} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/n \to 0$$

is a projective resolution of $\mathbb{Z}/n$. Applying the contravariant functor $\mathrm{Hom}_{\mathbb{Z}}(-, A)$, we obtain a cochain complex

$$0 \leftarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, A) \xleftarrow{\varphi^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, A),$$

where φ^* is the map $\varphi^*(f) = f \circ \varphi = n \cdot f$. But since $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, A)$ is naturally isomorphic to A, this complex is isomorphic to

$$0 \leftarrow A \leftarrow A,$$

where the map $A \to A$ is still multiplication by n. Therefore, the first cohomology group is

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/n, A) = A/nA.$$

■

12 Homework problem. Check that the restricted enveloping algebra $\mathcal{U}^{[p]}(\mathfrak{gl}_n)$ for $\mathfrak{gl}_n$ is Frobenius, by constructing the form using Poincaré-Birkhoff-Witt.

13 Homework problem. Write the normalized Bar resolution $\overline{B}_\bullet$ for $G = \mathbb{Z}/p$, even $\mathbb{Z}/2$, and compare to the periodic resolution.