Group Cohomology Lecture Notes

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Abstract

The following notes were taking during a course on Group Cohomology at the University of Washington in Spring 2014. Please send any corrections to `jps314@uw.edu`. Thanks!

Contents

April 2nd, 2014: Right Derived Functors, Examples from Groups, and Spectral Sequence Motivation

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1 Remark

We'll assume some things, like the standard homological algebra course. Can reference Weibel. We'll start with spectral sequences.

Definition 2. Recall the following definitions/results. Let $\mathcal{F} \colon \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories. (Default assumption: functors are covariant.) Also assume these categories have "enough injectives" (or projectives): this means given any object in A, we can find an injective object the given object embeds into.

If $\mathcal{F}$ is left exact, we can consider the ith right derived functor $R^i\mathcal{F}$ of $\mathcal{F}$. (Left exact means it takes a short exact sequence $0 \to A \to B \to C \to 0$ to the exact sequence $0 \to \mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(C)$.)

These functors form a "cohomological functor' in the sense of Weibel, meaning they fit into long exact sequences. How to compute/define $R^i\mathcal{F}$? Injective resolutions.

Suppose we have $A \in \mathcal{A}$. Let

$$A \to I^0 \to I^1 \to I^2 \to \cdots$$

be an injective resolution; this exists since $\mathcal{A}$ has enough injectives (construct inductively by repeatedly embedding cokernels into injectives). (To be clear, the resolution starts at I^0.) Apply $\mathcal{F}$ to the resolution to get

$$0 \to \mathcal{F}(I^0) \to \mathcal{F}(I^1) \to \mathcal{F}(I^2) \to \cdots,$$

a cochain complex. Now just take homology at each term. Then $(R^i\mathcal{F})(A) = H^i(\mathcal{F}(I^*))$.

$\mathcal{F}$ is exact if and only if each $(R^i\mathcal{F})(A) = 0$ for $i \geq 1$.

3 Example

Let G be a group. Let $\mathcal{A}$ be the category of kG modules (i.e. G-representations), where k is a field; this is a classical example of an abelian category. Consider the functor $\mathcal{F}(M) = \mathrm{Hom}_{kG}(k, M)$ from $\mathcal{A}$ to Ab, the category of abelian groups.

What can we say about this functor? $\mathrm{Hom}_G(k,-)$ is not exact, though it is left exact, so the right derived functors are interesting. (Minor note: we'll omit the kG and just write G typically.)

Definition 4.

(i) $H^n(G,M) = \mathrm{Ext}^n_G(k,M) := R^n \mathrm{Hom}_G(k,M)$.

(ii) $\mathrm{Ext}^n_G(N,M) := R^n \mathrm{Hom}_G(N,M)$ for any G-module N.
Note we can think of this as functors $\mathrm{Ext}^n_G(N,-) := R^n \mathrm{Hom}_G(N,-)$ or as $\mathrm{Ext}^n_G(-,M) := R^n \mathrm{Hom}_G(-,M)$.

5 Remark

In practice, what do we do to compute $H^n(G,M)$? By definition, we take an injective resolution

$$M \to I^0 \to I^1 \to I^2 \to \cdots,$$

apply the functor $\mathrm{Hom}_G(k,-)$ to get the cochain complex

$$\mathrm{Hom}_G(k,I^0) \to \mathrm{Hom}_G(k,I^1) \to \cdots$$

(no longer exact), and take homology to get $H^0(G,M)$, $H^1(G,M)$, etc.

6 Remark

We can also do the previous computation using a projective resolution of k rather than an injective resolution of G. Take

$$\cdots \to P_1 \to P_0 \to k,$$

a projective resolution of k (it starts at P_0), apply the $\mathrm{Hom}_G(-,M)$ functor (contravariant, this time) to get

$$\mathrm{Hom}_G(P_0,M) \to \mathrm{Hom}_G(P_1,M) \to \cdots$$

(another cochain complex). Again apply homology to get $H_0(G,M)$, $H_1(G,M)$, etc.

The bar resolution uses the projective module kG to give a very explicit version, though it gets big quickly.

7 Remark

The "minimal resolution" is something you try to construct if you really want to compute something; more on this later. Can ask software packages to compute this much of the time.

8 Remark

This gets complicated for groups with more structure. For instance, consider $GL_n(k)$; the representations form an abelian category, have enough injectives, etc. It has more structure: it's a Lie group, it's an algebraic group (meaning it's a variety with compatible multiplication). As an algebraic group, there's a different category of representations, but it doesn't have projective objects, though it does have injective objects. (As a general observation, injective objects tend to be more ubiquitous, i.e. you have them available more often).

9 Remark

It often happens that when you construct your resolution, it doesn't stop. If the group is finite, either you get no non-trivial right derived functors (occurs with semisimplicity assumptions); or infinitely many right derived functors are non-trivial.

The reason this happens for finite groups is that injective modules are the same as projective modules. No finite length resolutions over kG.

10 Proposition

$\mathrm{Hom}_G(k,-)$ *can be described as the functor of fixed points* $(-)^G$*. Hence* $H^i(G,-) = R^i(-)^G$.

PROOF Indeed,

$$\mathrm{Hom}_G(M,N)=\{f\colon M\to N \mid f(g\cdot m)=g\cdot f(m)\}$$

so $\mathrm{Hom}_G(k,M)=\{f\colon k\to M \mid f(g\cdot\lambda)=g\cdot f(\lambda)\}$. (Here we're viewing k with trivial kG-module action.) Hence $f(g\cdot\lambda)=f(\lambda)=\lambda\cdot f(1)$. Likewise $g\cdot f(\lambda)=\lambda g\cdot f(1)$. Cancelling λ and letting $f(1)=m$, we see $\mathrm{Hom}_G(k,M)$ is identified with $\{m\in M : g\cdot m=m\}$, precisely the fixed points.

11 Remark (Motivating spectral sequences)

Let $1\to K\to G\to G/K\to 1$, let M be a kG-module. Can check $M^G=(M^K)^{G/K}$. We have three functors:

$$\begin{aligned}
(-)^G &: kG\text{-Mod}\to Ab\\
(-)^K &: kK\text{-Mod}\to Ab\\
(-)^{G/K} &: k(G/K)\text{-Mod}\to Ab\,.
\end{aligned}$$

We can think of the second functor as taking kG-modules to $k(G/K)$-modules. We have

$$kG\text{-mod}\overset{(-)^K}{\to} k(G/K)\text{-mod}\overset{(-)^{G/K}}{\to} Ab$$

whose composite is $(-)^G$. More generally consider functors between abelian categories

$$\mathcal{A}\overset{\mathcal{F}}{\to}\mathcal{B}\overset{\mathcal{G}}{\to}\mathcal{C}$$

with composite $\mathcal{G}\circ\mathcal{F}=\mathcal{I}$, where $\mathcal{F},\mathcal{G},\mathcal{I}$ are left exact. What can we say about $R^i\mathcal{G}\circ R^j\mathcal{F}$? Specifically, $H^i(G/K,H^j(K,M))$ is what? Answer, roughly: there is a spectral sequence that computes this, the Lyndon-Hochschild-Serre spectral sequence. In general, $R^i\mathcal{G}\circ R^j\mathcal{F}\Rightarrow R^{i+j}(\mathcal{G}\circ\mathcal{F})$. This will be defined later; it's the Grothendieck spectral sequence.

April 4th, 2014: Filtered Complexes, Associated Graded Objects, and Spectral Sequences Defined

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12 Remark

Today's topic: defining the spectral sequence of a filtered complex. There are two standard approaches to spectral sequences. One comes from filtered complexes, perhaps favored by algebraists, and the other comes from exact pairs. We'll take the first approach. This is more or less covered in Weibel.

Definition 13. Let (C_*,d) be a chain complex (as usual, assume we're over some nice abelian category). A filtered chain complex $\{F_pC_*\}_{p\in\mathbb{Z}}$ is a collection of subcomplexes compatible with the differential d. Explicitly,

$$\cdots\subset F_{p-1}C_*\subset F_pC_*\subset F_{p+1}C_*\subset\cdots\subset C_*$$

and

$$\begin{array}{ccccccc}
\cdots & \hookrightarrow & F_pC_n & \hookrightarrow & \cdots & \hookrightarrow & C_n\\
\downarrow d & & \downarrow d & & \downarrow d & & \downarrow d\\
\cdots & \hookrightarrow & F_pC_{n-1} & \hookrightarrow & \cdots & \hookrightarrow & C_{n-1}
\end{array}$$

Definition 14. To each filtered chain complex we can define the associated graded object

$$\mathrm{gr}_* C_n := \oplus_{p\in\mathbb{Z}} \frac{F_p C_n}{F_{p-1}C_n}.$$

15 Example

PBW basis for $U(\mathfrak{g})$, the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. This basis gives a filtration for the (infinite dimensional) algebra $U(\mathfrak{g})$. The associated graded algebra is a symmetric algebra, and is very nice. Sometimes the associated graded algebra will have easy cohomology, and you want to compute the cohomology of the original object; this is where spectral sequences come in to play.

Definition 16. A (homological) spectral sequence in some abelian category $\mathcal{A}$ is the following data:

1) $E^r_{pq} \in \mathcal{A}$ for $r \geq 0$, $p, q \in \mathbb{Z}$. For fixed r, E^r_{pq} is called the rth page of the spectral sequence

2) Fixed maps $d^r_{pq} \colon E^r_{pq} \to E^r_{p-r,q+r-1}$.

3) Fixed isomorphisms

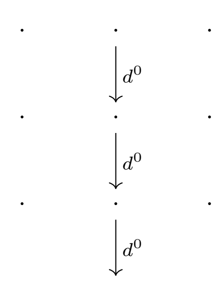
$$E^{r+1}_{pq} \cong \frac{\ker d^r_{pq}}{\operatorname{im} d^r_{p+r,q-r+1}}.$$

This is written $\{E^r_{pq}\}_{r\geq 0}$, with the rest of the maps left implicit. We say the total degree of E^r_{pq} is $p+q$.

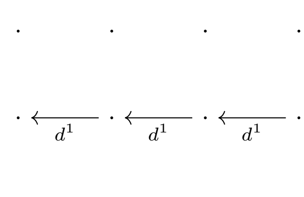
Remark: (3) implicitly requires the maps in (2) to form families of chain complexes along diagonals parallel to $(-r, r-1)$.

17 Remark

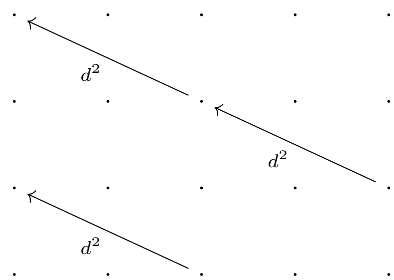
Pictures for spectral sequences. The 0th page: $d^0 \colon E^0_{pq} \to E^0_{p,q-1}$ looks like



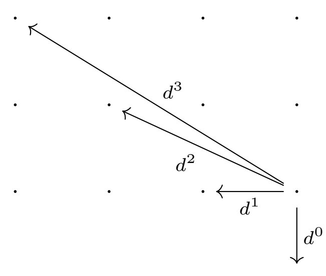
The first page: $d^1 \colon E^1_{pq} \to E^1_{p-1,q}$ gives



The second page: $d^2 \colon E^2_{pq} \to E^1_{p-2,q+1}$ gives



General pattern:

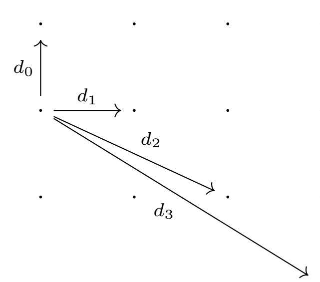


The total degree always decreases by 1 when we apply d^r.

Definition 18. A cohomological spectral sequence is like the above, but $d_r^{pq} \colon E_r^{pq} \to E_r^{p+r,q-r+1}$, and

$$E^{pq}_{r+1} \cong \frac{\ker d_r^{pq}}{\operatorname{im} d_r^{p-r,q+r-1}}$$

The general picture is



Remark: this time the total degree increases by 1 after applying d_r, and we get families of cochain complexes along diagonals parallel to $(r, 1-r)$.

Definition 19. Say a page E^r_{**} is bounded if it has finitely many non-zero terms on each diagonal ("diagonal" being page-specific). $\{E^r_{pq}\}$ is bounded if it is bounded for some r.

20 Remark

If E^r_{**} is bounded, then for all p, q, $\{E^s_{pq}\}_{s\geq r}$ has a "stable" value, i.e. for some s large enough, all higher terms are isomorphic. We denote this value by $\boxed{E^\infty_{pq}}$; this notation implies existence of a stable value (though not necessarily that a page is bounded).

Proof Sketch: if you draw the picture, eventually kernels are everything and images are nothing since they map to/from 0. Apply the isomorphism assumption (3).

21 Example

A first quadrant spectral sequence is one with $E^r_{pq} \neq 0$ only if $p, q \geq 0$. This is bounded, hence has the property above.

Definition 22. Let $\{H_n\}_{n\in\mathbb{Z}}$ be a sequence of filtered objects in $\mathcal{A}$. Assume the filtrations are finite, so

$$0 = F_t H_n \subset \cdots \subset F_s H_n \subset \cdots \subset H_n,$$

$t \leq s$. Let $\{E^r_{pq}\}$ be a homological spectral sequence. Then E^r_{pq} converges to H_{p+q}, written

$$E^r_{pq} \Rightarrow H_{p+q},$$

if E^∞_{pq} "gives the associated graded object of H_*". That is,

$$E^\infty_{pq} \cong F_p H_{p+q}/F_{p-1}H_{p+q}$$

Equivalently,

$$\mathrm{Tot}_n(E^\infty_{**}) \cong \mathrm{gr}_* H_n,$$

where

$$\mathrm{Tot}_n(E^\infty_{**}) := \oplus_{p+q=n} E^\infty_{pq}.$$

The definition for cohomological spectral sequences is similar: $E^{pq}_r \Rightarrow H^{p+q}$ means $E^{pq}_\infty \cong F_p H_{p+q}/F_{p+1}H_{p+q}$.

23 Remark

What does this mean?

(1) Look at the "$r = \infty$ page", with E^∞_{pq} at each lattice point; note that $p+q$ is constant on antidiagonals.

(2) For a particular antidiagonal with $p+q = n$, write the pieces of the associated graded object to the filtration of H_n on each lattice point. Specifically, place $\mathrm{gr}_p H_n := F_p H_n/F_{p-1}H_n$ at $(p, n-p)$.

Now $E^r_{pq} \Rightarrow H_{p+q}$ means precisely that the objects from (1) and (2) are isomorphic for all p, q. We're likely interested in computing H_n exactly, but we can in general only recover the associated graded object; frequently this is enough.

The statement $E^r_{pq} \Rightarrow H_{p+q}$ implies we've chosen some fixed isomorphisms, i.e. that E^∞_{pq} and $\mathrm{gr}_p H_n$ are not merely isomorphic, but isomorphic via some fixed (often implicit) map.

24 Example

Consider a first quadrant spectral sequence. For $E^r_{pq} \Rightarrow H_{p+q}$, the filtration we have on H_n will be

$$0 \subset F_0 H_n \subset \cdots \subset F_n H_n = H_n.$$

Here the nth diagonal connecting $(0, n)$ and $(n, 0)$ with endpoints E^∞_{0n} and E^∞_{n0} has $n+1$ objects. Have $E^\infty_{0n} \cong F_0 H_n$ and $E^\infty_{n0} = H_n/F_{n-1}H_n$. More next time.

April 7th, 2014: Edge Homomorphisms; Filtration of $H_n(C_*)$ and its Spectral Sequence for $r = 0, 1, 2$ Defined

25 Notation

"$E^2_{pq} \Rightarrow H_{p+q}$" is a minor abuse of notation meaning the spectral sequence $\{E^r_{pq}\}$ converges to H_{p+q} and E^2_{pq} is a page of particular interest. Sometimes spectral sequences don't start at 0, but that's alright.

26 Example

(Continued from last lecture.) Consider a first quadrant spectral sequence

$$E^0_{pq} \Rightarrow H_{p+q}.$$

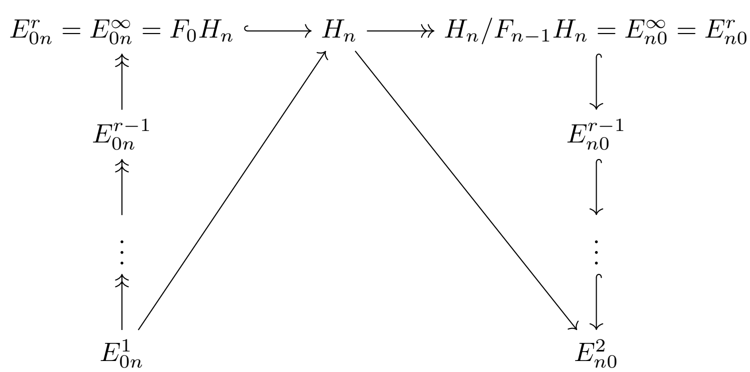
Call H_n the abutment of the spectral sequence. The abutment is just the sequence of objects $\{H_n\}_{n\in\mathbb{Z}}$; it does not know about any filtrations on those objects. Still, H_n comes with a filtration

$$0 \subset F_0H_n \subset \cdots \subset F_pH_n \subset \cdots \subset F_{n-1}H_n \subset F_nH_n = H_n.$$

By definition, $F_pH_n/F_{p-1}H_n \cong E^\infty_{pq}$; imagine writing these groups at lattice points in the first quadrant with diagonals where $p + q = n$ is constant. Connecting the diagonal $(n, 0)$ to $(0, n)$ connects objects

$$E^\infty_{n0} \cong \frac{F_nH_n}{F_{n-1}H_n}, \qquad E^\infty_{0n} \cong F_0H_n,$$

with intermediate terms $F_pH_n/F_{p-1}H_n \cong E^\infty_{pq}$. We can map F_0H_n into H_n injectively, and we can map H_n to $F_nH_n/F_{n-1}H_n$ via the quotient map; these maps are sometimes called "edge homomorphisms". More precisely, we have the top row of



for some large enough r; take $r \geq 1$. What about the rest of the diagram?

- We get E^{r+1}_{0n} by taking homology at E^r_{0n}. But for $r \geq 1$, d^r_{0n}'s target is outside of the first quadrant, so E^{r+1}_{0n} is a quotient of E^r_{0n}.
- Similarly E^{r+1}_{n0} is given by homology at E^r_{n0}, so E^{r+1}_{n0} is the kernel of d^r_{n0}, which has domain E^r_{n0}.

The diagonal maps are sometimes (also) called edge homomorphisms.

Definition 27 (Constructing the spectral sequence of a filtered complex). Let C_* be a filtered complex

$$0 = F_t C_* \subset \cdots \subset F_p C_* \subset \cdots \subset F_{s-1} C_* \subset F_s C_* \subset C_*.$$

Assume the filtration is finite. (It is enough to assume finite for any fixed n, i.e. the number of non-zero terms may or may not depend on "$*$".) Our goal is to construct a spectral sequence

$$E^r_{pq} \Rightarrow H_{p+q}(C_*).$$

(Here H_* denotes homology.) We proceed in stages.

Definition 28. We first note the filtration on C_* induces a $\boxed{\text{filtration of } H_n(C_*)}$ for each n, as follows. The filtration on C_* is compatible with differentials, so in particular

$$\begin{array}{ccc} F_p C_{n+1} & \hookrightarrow & C_{n+1} \\ \downarrow & & \downarrow \\ F_p C_n & \hookrightarrow & C_n \\ \downarrow & & \downarrow \\ F_p C_{n-1} & \hookrightarrow & C_{n-1} \end{array}$$

From this diagram, we read off $Z_n(F_p C_*) \subset Z_n(C_*)$ and $B_n(F_p C_*) \subset B_n(C_*)$. Hence we have an induced map

$$H_n(F_p C_*) := \frac{Z_n(F_p C_*)}{B_n(F_p C_*)} \to \frac{Z_n(C_*)}{B_n(C_*)} := H_n(C_*).$$

Define $\boxed{F_p(H_n C_*)}$ as the image of $H_n(F_p C_*)$ under this map. Indeed, the above diagram actually gives $Z_n(F_p C_*) = Z_n(C_*) \cap F_p C_n$ and $B_n(F_p C_*) \subset B_n(C_*) \cap F_p C_n$. Moreover, the kernel of the above map is precisely

$$\frac{B_n(C_*) \cap F_p C_n}{B_n(C_*)},$$

and it follows that

$$F_p(H_n C_*) \cong \frac{Z_n(C_*) \cap F_p C_n}{B_n(C_*) \cap F_p C_n}.$$

For simplicity, write $Z_n := Z_n(C_*)$, $B_n := B_n(C_*)$, and $F_p H_n := F_p(H_n C_*)$. One can check this gives a triangle

$$\begin{array}{ccc} \frac{Z_n \cap F_p C_n}{B_n \cap F_p C_n} & \hookrightarrow & \frac{Z_n}{B_n} \\ \uparrow & \nearrow & \\ \frac{Z_n \cap F_{p-1} C_n}{B_n \cap F_{p-1} C_n} & & \end{array}$$

which forces the vertical arrow injective, i.e. we have $F_{p-1} H_n \hookrightarrow F_p H_n$. It turns out the associated graded object is

$$\begin{aligned} \operatorname{gr}_p H_n = \frac{F_p H_n}{F_{p-1} H_n} &\cong \frac{Z_n \cap F_p C_n}{B_n \cap F_p C_n} \Big/ \frac{Z_n \cap F_{p-1} C_n}{B_n \cap F_{p-1} C_n} \\ &\cong \frac{Z_n \cap F_p C_n}{B_n \cap F_p C_n + Z_n \cap F_{p-1} C_n} \qquad (*) \end{aligned}$$

Definition 29 ("Informal construction"). Next we construct the first few pages of a spectral sequence converging to $H_{p+q}C_*$ with the above induced filtration. Let $n = p + q$ throughout. Set

$$\boxed{E^0_{pq}} := F_pC_n/F_{p-1}C_n$$

Since d is compatible with the filtration, it induces $\boxed{d^0\colon E^0_{pq} \to E^0_{p,q-1}}$ by

$$d^0\colon \frac{F_pC_n}{F_{p-1}C_n} \to \frac{F_pC_{n-1}}{F_{p-1}C_{n-1}}.$$

(One must check this is a chain map.) Hence the 0th page looks like this:

$$\begin{array}{ccc}
\frac{F_{p-1}C_{n+1}}{F_{p-2}C_{n+1}} & & \vdots \\
\downarrow d^0 & & \downarrow d^0 \\
\frac{F_{p-1}C_n}{F_{p-2}C_n} & & \frac{F_pC_{n+1}}{F_{p-1}C_{n+1}} \\
\downarrow d^0 & & \downarrow d^0 \\
\frac{F_{p-1}C_{n-1}}{F_{p-2}C_{n-1}} & & \frac{F_pC_n}{F_{p-1}C_n} \\
\downarrow & & \downarrow d^0 \\
\vdots & & \frac{F_pC_{n-1}}{F_{p-1}C_{n-1}}
\end{array}$$

To get the next page's objects, we're forced to use

$$\boxed{E^1_{pq}} := \frac{\ker d^0_{pq}}{\operatorname{im} d^0_{p,q+1}}$$

Now we need $E^1_{p-1,q} \leftarrow E^1_{pq} : d^1$. Let $\overline{z} \in E^1_{pq}$, so $z \in F_pC_n$ and $d^0(z) = 0$. That is, $d(z) \in F_{p-1}C_{n-1}$; that is, z is a cycle modulo $F_{p-1}C_{n-1}$. Define

$$d^1_{pq}(\overline{z}) := d(z)$$

There are many checks one has to do, but they all work out. For the second page, define

$$\boxed{E^2_{pq}} := \frac{\ker d^1_{pq}}{\operatorname{im} d^1_{p+1,q}}$$

Exercise: track down d^2.

April 9th, 2014: Formal Construction of Spectral Sequence for $H_n(C_*)$; Double Complexes

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Definition 30 (Formal construction). Goal: given a filtered chain complex (C_*, d), construct a spectral sequence converging to the sequence of filtered objects $\{H_n C_*\}$ defined last time. While this won't in general recover the homology of C_*, it will recover the associated graded objects.

Throughout, $n = p + q$. Assume the filtration is finite as before. Define

$$\boxed{Z^r_{pq}} := \{z \in F_p C_n : d(z) \in F_{p-r} C_{n-1}\}.$$

Then for $r \gg 0$,

$$Z_n \cap F_p C_n = Z^\infty_{pq} = \cdots = Z^{r+1}_{pq} = Z^r_{pq} \subset Z^{r-1}_{pq} \subset \cdots \subset Z^0_{pq} = F_p C_n,$$

where $Z_n = \ker(d\colon C_n \to C_{n-1})$ as before. Recall that

$$F_p(H_n C_*) \cong \frac{Z_n \cap F_p C_n}{B_n \cap F_p C_n},$$

so "in the limit" these Z^r_{pq} give us honest cycles, which is a good sign. Similarly define

$$\boxed{B^r_{pq}} := \{z \in F_p C_n : z = d(x), x \in F_{p+r-1} C_{n+1}\}.$$

Note $B^r_{pq} = d(Z^{r-1}_{p+r-1,q-r+2})$. Indeed, for $r \gg 0$,

$$d(F_{p-1} C_{n+1}) = B^1_{pq} \subset \cdots \subset B^r_{pq} = \cdots = B^\infty_{pq} = B_n \cap F_p C_n.$$

Now set

$$\boxed{E^r_{pq}} := \frac{Z^r_{pq}}{B^r_{pq} + (Z^r_{pq} \cap F_{p-1} C_n)}.$$

Note

- $d(Z^r_{pq}) \subset Z^r_{p-r,q+r-1} \subset F_{p-r} C_{n-1}$.
- $d(B^r_{pq}) = 0$
- $Z^r_{pq} \cap F_{p-1} C_n = \{z \in F_{p-1} C_n : d(z) \in F_{p-r} C_{n-1}\} = Z^{r-1}_{p-1,q+1}$. Hence $d(Z^r_{pq} \cap F_{p-1} C_n) = d(Z^{r-1}_{p-1,q+1}) = B^r_{p-r,q+r-1}$.

From these observations, it follows that we can define differentials $\boxed{d^r_{pq}\colon E^r_{pq} \to E^r_{p-r,q+r-1}}$ by

$$d^r_{pq}\colon \frac{Z^r_{pq}}{B^r_{pq} + (Z^r_{pq} \cap F_{p-1} C_n)} \to \frac{Z^r_{p-r,q+r-1}}{B_{p-r,q+r-1} + (Z^r_{p-r,q+r-1} \cap F_{p-r-1} C_{n-1})}.$$

Exercise ("for strong-willed people"): show

$$E^{r+1}_{pq} \cong \frac{\ker d^r_{pq}}{\operatorname{im} d^r_{p+r,q-r+1}}.$$

(Why does any of this work, and what is going on here? Very roughly, we're trying to make successive approximations of $F_p(H_n C_*)$ where the next approximation is given by taking homology of the previous approximation. This must have been motivated by some particular objects of interest originally, but this particular construction is very general, hence abstract. "In the limit" the approximations converge to the quantities we were after, which is shown formally in the next remark.)

31 Remark

What happens at ∞? For r large compared to p, Z^r_{pq} is just cycles since $F_{p-r}C_{n-1} = 0$. We have

$$E^\infty_{pq} = \frac{Z^\infty_{pq}}{B^\infty_{pq} + (Z^\infty_{pq} \cap F_{p-1}C_n)} = \frac{Z_n \cap F_pC_n}{B_n \cap F_pC_n + Z_n \cap F_{p-1}C_n} \cong \mathrm{gr}_p\, H_n(C_*),$$

where the last equality comes from formula $(*)$ from last time. Typically this construction is not useful for actually computing things, but its existence is very useful for proving general properties.

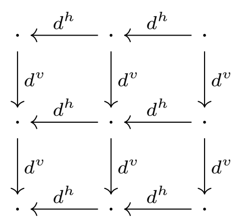
Definition 32. A **double complex** C_{**} is a collection of objects $\{C_{pq}\}_{p,q\in\mathbb{Z}}$ in some abelian category together with differentials

$$d^v_{pq} \colon C_{pq} \to C_{p,q-1}$$

and

$$d^h_{pq} \colon C_{pq} \to C_{p-1,q}$$

where $d^hd^h = 0, d^vd^v = 0$, and $d^hd^v + d^vd^h = 0$. (Note: we can get an equivalent definition by toggling the sign on every other row, which gives commutativity of each square rather than anticommutativity.) We think of the complex as



Call $p + q$ the **total degree** of C_{pq}. Total degree is constant along antidiagonals.

Definition 33. The **total complex** associated to a double complex C_{**} is

$$\boxed{\mathrm{Tot}_n(C_{**})} := \bigoplus_{p+q=n} C_{pq}$$

with

$$d\colon \mathrm{Tot}_n(C_{**}) \to \mathrm{Tot}_{n-1}(C_{**})$$

given by $d^h + d^v$ in each summand. Pictorially, we imagine an antidiagonal with total degree $p + q = n$ as the nth graded piece, and the differential maps from this antidiagonal to one immediately below it "both horizontally and vertically". Note that indeed $d^2 = (d^h + d^v)^2 = 0$, which partially justifies the anticommutativity in the definition of a double complex.

34 Remark

Preview of next time: there exist two filtrations on $\mathrm{Tot}_*(C_{**})$ that leads to two spectral sequences converging to the same abutment $H_n(\mathrm{Tot}_*(C_{**}))$. In practice, one way tends to be easy and the other way is what you're actually interested in.

April 11th, 2014: Row and Column Filtrations of Double Complexes; the Snake Lemma

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35 Remark

Math Olympiad on first Sunday of June. If you're able to help out, let Julia know. Time committment: 9-1pm on Sunday, June 1st; 3-5pm Friday, May 30th; get problems Monday, May 26th.

Today, we'll get two filtrations on $\operatorname{Tot}(C_{**})$. They'll come from two filtrations on C_{**} itself. As application, we'll use them to prove the Snake lemma.

Definition 36 (Column Filtration). I. Given a double complex C_{**}, consider the truncated complex which is visually given by putting 0's to the right of column p. Formally, let

$$\boxed{\tau_{\leq p} C_{**}} := \begin{cases} C_{p',q} & p' \leq p \\ 0 & p' > p \end{cases}$$

Define a filtration on the chain complex $\operatorname{Tot}_n(\tau_{\leq p} C_{**})$ by

$$\boxed{F_p \operatorname{Tot}_n(C_{**})} := \operatorname{Tot}_n(\tau_{\leq p} C_{**}) = \bigoplus_{p'+q=n, p' \leq p} C_{p',q}.$$

(Pictorially, we just add up the $p'+q=n$ antidiagonal with 0's to the right of column p.) This is the $\boxed{\text{column filtration of } C_{**}}$.

37 Proposition

Let $\boxed{{}^{I}E^r_{pq}}$ *be the spectral sequence associated to the column filtration of* C_{**}. *We'll compute the first three pages explicitly,* ${}^{I}E^r_{pq}$ *for* $r = 0, 1, 2$.

- *We see*

$${}^{I}E^0_{pq} = \frac{F_p \operatorname{Tot}_n(C_{**})}{F_{p-1} \operatorname{Tot}_n(C_{**})} = C_{pq},$$

and indeed $d^0 = d^v$ *from the double complex. So,* ${}^{I}E^0_{pq}$ *reconstructs (the vertical portion of) the double complex.*

- *To get the next page, take "vertical homology",* ${}^{I}E^1_{pq} = H_q(C_{p*,d^v})$. *For convenience, abuse notation and say* ${}^{I}E^1_{pq} = H^v(C_{**})$. *It's not hard to check that* d^1_{pq} *is induced by* d^h *in* C_{**}, *giving* $d^1_{pq} \colon {}^{I}E^1_{pq} \to {}^{I}E^1_{p-1,q}$.

- *To get the next page, take "horizontal homology",* ${}^{I}E^2_{pq} = H^h({}^{I}E^1_{pq}) = H^h(H^v(C_{**}))$.

We know that ${}^{I}E^2_{pq} \Rightarrow H_{p+q}(\operatorname{Tot}(C_{**}))$. *In summary, we start with the double complex with the vertical arrows, take vertical homology to get the horizontal arrows, and take horizontal homology of that.*

Definition 38 (Row Filtration). II. Let $\boxed{C^T_{pq}} := C_{qp}$, i.e. flip the double complex through the main diagonal. Note that the total complex is unchanged and that horizontal differentials become vertical differentials and vice-versa. We can do the same to a spectral sequence, where we also flip the differentials through the main diagonal.

The $\boxed{\text{row filtration of } C_{**}}$ is the column filtration of C^T_{**}. Intuitively, we imagine putting 0's in all rows above row q to obtain this filtration.

Let $\boxed{{}^{II}E^r_{pq}}$ be the spectral sequence associated with the column filtration on C^T_{**}.

39 Proposition

We have

$${}^{II}E^2_{pq} = (H^v H^h(C_{**}))^T \Rightarrow H_{p+q}(\operatorname{Tot}(C_{**})).$$

40 Remark

A (homological) spectral sequence's differentials go downward and then left, while we're taking homology in the other order: horizontal, then vertical. This explains the ^{T}: we must transpose our spectral sequence to interpret the second page in this way. Since the notation is already so verbose, the ^{T} will typically be dropped, though be careful. For instance, ${}^{II}E^0_{pq} = C_{qp}$, while $({}^{II}E^0_{pq})^T = C_{pq}$.

41 Remark

The bottom line is that we get two different spectral sequences ${}^{I}E^r_{pq}$ and ${}^{II}E^r_{pq}$ converging to the same abutment $H_{p+q}(\mathrm{Tot}(C_{**}))$, where the first two pages are computed by taking horizontal or vertical homology starting with the original double complex.

Warning: While the abutments are the same, the induced filtrations on that abutment are in general different, so for instance it is not generally the case that ${}^{I}E^\infty_{pq} = {}^{II}E^\infty_{pq}$.

42 Lemma (Snake Lemma)

We start with

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle\alpha} & & \big\downarrow{\scriptstyle\beta} & & \big\downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \end{array}$$

We consider this as part of a double complex (pq written as a superscript)

$$\begin{array}{ccccc} C^{(01)} & \longleftarrow & B^{(11)} & \longleftarrow & A^{(21)} \\ \big\downarrow{\scriptstyle\gamma} & & \big\downarrow{\scriptstyle\beta} & & \big\downarrow{\scriptstyle\alpha} \\ C'^{(00)} & \longleftarrow & B'^{(10)} & \longleftarrow & A'^{(20)} \end{array}$$

(Technically, we've replaced β with $-\beta$ to get commutativity. Kernels and cokernels are unaffected.) We have two spectral sequences associated to this complex, ${}^{I}E^2_{pq} = H^h(H^v(C_{**}))$ and ${}^{II}E^2_{pq} = H^v(H^h(C_{**}))$. The horizonal-first sequence is very nice, since our rows are exact, giving $H^h = 0$:

$${}^{II}E^2_{pq} = H^v(H^h(C_{**})) = 0.$$

Hence $E^\infty_{pq} = 0$, so the abutment is zero, whence $H_*(\mathrm{Tot}(C_{**})) = 0$. Therefore

$${}^{I}E^2_{pq} = H^h(H^v(C_{**})) \Rightarrow 0.$$

What does this tell us about the ${}^{I}E^2_{pq}$ page?

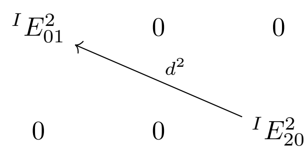
- ${}^{I}E^1_{pq} = H^v(C_{**})$*; this is a very small complex, so we can do it explicitly. We get*

$$\ker\gamma \xleftarrow{d^1} \ker\beta \xleftarrow{d^1} \ker\alpha$$

$$\operatorname{coker}\gamma \xleftarrow{d^1} \operatorname{coker}\beta \xleftarrow{d^1} \operatorname{coker}\alpha$$

where the horizontal differentials are induced by the original horizontal differentials.

- *Next, ${}^I E^2_{pq}$ is obtained by taking homology of the previous page. Consider the upper right corner, $\ker\alpha$ at (21). We know from the other sequence that $E^\infty_{21} = 0$, but it's easy to see that E_{21} stabilizes at the page $r = 2$. Likewise ${}^I E^2_{pq} = 0$ for $pq = 00, 10, 11$, leaving*

  Hence, in the ${}^I E^1_{pq}$ page, the top row is left exact (i.e. exact at $\ker\alpha$ and $\ker\beta$), and the bottom row is right exact (i.e. exact at $\operatorname{coker}\gamma$, $\operatorname{coker}\beta$), so we have

  $$0 \to \ker\alpha \to \ker\beta \to \ker\gamma \qquad \operatorname{coker}\alpha \to \operatorname{coker}\beta \to \operatorname{coker}\gamma \to 0$$

  exact. Moreover, there are no nontrivial differentials in the $r = 3$ or higher page, so taking homology of the $r = 2$ page gives zero in the two remaining places. Hence the kernel of d^2 is trivial and the image of d^2 is everything, so $d^2 : {}^I E^2_{20} \to {}^I E^2_{01}$ is an isomorphism. More explicitly,

  $$d^2 \colon \ker(\operatorname{coker}\alpha \to \operatorname{coker}\beta) \xrightarrow{\cong} \operatorname{coker}(\ker\beta \to \ker\gamma).$$

  Taking the inverse gives a surjective map

  $$\ker\gamma \to \ker(\operatorname{coker}\alpha \to \operatorname{coker}\beta)$$

  with kernel $\operatorname{im}(\ker\beta \to \ker\gamma)$, which gives us the last (middle) map in the Snake lemma's long exact sequence. (Lingering question: what about naturality?)

April 14th, 2014: The Five Lemma; Maps of Spectral Sequences

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43 Lemma

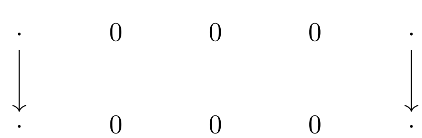
The Five Lemma states the following. Suppose we a commutative diagram where the two rows are exact sequences:

$$\begin{array}{ccccccccc}
E & \longleftarrow & D & \longleftarrow & C & \longleftarrow & B & \longleftarrow & A \\
\downarrow & & \downarrow & & \downarrow\gamma & & \downarrow & & \downarrow \\
E' & \longleftarrow & D' & \longleftarrow & C' & \longleftarrow & B' & \longleftarrow & A'
\end{array}$$

Assume all vertical maps except the middle are isomorphisms. Then the middle is also an isomorphism.

Proof Consider the diagram as a double complex. (To get anticommutativity, toggle the signs of the second and fourth vertical arrows without loss of generality.) As before, there are two spectral sequences associated with it, depending on the order in which we take horizontal and vertical homology.

For ${}^{II}E^2_{pq} = H^v(H^h(C_{**}))$, the $r = 1$ page is



where $\cdot$ represents the kernel or cokernel of the appropriate map. Here the down arrows are induced by the original down arrows. The remaining pages do not change the $n = 2$ and $n = 3$ antidiagonals, hence they preserve these pieces of the total complex, which are trivial. (Minor note: we've transposed our diagram, which is why the arrows are down at this step rather than left.)

Now compute ${}^I E^1_{pq}$: we get kernels in the top row and cokernels in the bottom row. Since all maps except γ are isomorphisms, these are almost all zero:

$$\begin{array}{ccccc} 0 & 0 \longleftarrow & \ker\gamma & \longleftarrow 0 & 0 \\ & & & & \\ 0 & 0 \longleftarrow & \operatorname{coker}\gamma & \longleftarrow 0 & 0 \end{array}$$

There are no non-trivial differentials left, so we've stabilized in every place. Hence the $n = 2$ and $n = 3$ antidiagonals give the abutment, which we saw from the other sequence are both zero. Hence $\ker\gamma = \operatorname{coker}\gamma = 0$, and γ is an isomorphism. Check!

44 Remark

What are the weaker versions of the five lemma? Tweak the above proof to get them.

Definition 45. Let E^r_{pq} and ${E'}^r_{pq}$ be two spectral sequences. A map of spectral sequences is a map of each object of each page $f^r_{pq} \colon E^r_{pq} \to {E'}^r_{pq}$ such that

1. We can apply f^r followed by a differential, or a differential followed by f^r; i.e. the obvious parallelograms commute for each page.
2. f^{r+1} is given by

$$\begin{array}{ccc} E^{r+1}_{pq} & \overset{\cong}{\longleftrightarrow} & \frac{\ker d^r}{\operatorname{im} d^r} \\ {\scriptstyle f^{r+1}} \downarrow & & \downarrow \\ {E'}^{r+1}_{pq} & \overset{\cong}{\longleftrightarrow} & \frac{\ker {d'}^r}{\operatorname{im} {d'}^r} \end{array}$$

where the vertical arrow on the right is induced by f^r. (This says a map of spectral sequences is determined on the first page.)

46 Proposition

Let $f \colon C_* \to C_*$ be a map of filtered complexes (that is, $f|_{F_p C_*} \to F_p C'_*$). Then f induces a map of the associated spectral sequences

$$f^r \colon E^r_{pq}(C_*) \to E^r_{pq}(C'_*)$$

47 Theorem (Mapping Lemma)

Let $E^*_{pq} \Rightarrow H_{p+q}$, ${E'}^*_{pq} \Rightarrow H'_{p+q}$, and suppose we have maps

$$h \colon H \to H', \qquad f^* \colon E^*_{pq} \to {E'}^*_{pq}.$$

Suppose they are "compatible" in the sense that the map between E^∞_{pq} and ${E'}^\infty_{pq}$ induces the same map between graded pieces as h does.

If f^r is an isomorphism for some $r > 0$, then $h \colon H \overset{\cong}{\to} H'$ is an isomorphism.

Proof Exercise; use induction; assume the filtrations are finite as before. We implicitly used an easy version of this in the proof of the snake lemma.

48 Corollary (Eilenberg-Moore Comparison Theorem)

Let $f\colon B \to C$ be a map of filtered complexes (assume filtrations are finite; "locally finite" may be a good term, meaning for each n it's finite, but the lengths could vary with n). Suppose for some $r > 0$, $f^r\colon E^r_{pq}(B) \to E^r_{pq}(C)$ is an isomorphism. Then $f\colon B \to C$ is an isomorphism.

Note: This is incorrect; see next lecture's opening remark.

April 16th, 2014: The Universal Coefficient Theorem

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49 Remark

The comparison theorem from the end of class last time was "way too strong". The actual conclusion is just that $H_*(f)\colon H_*(B) \overset{\cong}{\to} H_*(C)$.

For a counterexample of the original statement that $f\colon B \overset{\cong}{\to} C$, take C and add an exact complex; it will die right away, so essentially the spectral sequence doesn't see it.

50 Example

Motivation for the universal coefficient theorem: consider singular homology of a topological space X. Recall this is defined by looking at the free $\mathbb{Z}$-modules of maps from simplices to the space, turned into a chain complex using certain boundary maps ("boundary" map is literal in this case). Singular homology with $\mathbb{Z}$-coefficients is the homology of this chain complex.

More generally, suppose M is an abelian group. We can do the same construction except using coefficients in M rather than $\mathbb{Z}$, denoted $H_*(X, M)$. (Explicitly, apply the $- \otimes_{\mathbb{Z}} M$ functor to the previous chain complex and take homology.) If we've computed $H_*(X) := H_*(X, \mathbb{Z})$, how can we use that information to compute $H_*(X, M)$?

51 Theorem (Universal Coefficient Theorem for *R*-modules over a PID)

Let R be a PID, let C_* be a chain complex of free R-modules, and let M an R-module. We have a short exact sequence

$$0 \to H_n(C_*) \otimes M \to H_n(C_* \otimes M) \to \mathrm{Tor}_1^R(H_{n-1}(C_*), M) \to 0.$$

There is a version for cohomology,

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(C), M) \to H^n(\mathrm{Hom}_R(C_*, M)) \to \mathrm{Hom}_R(H_n(C), M) \to 0.$$

The general (homology) conclusion for an arbitrary ring R is that we have a spectral sequence

$$\mathrm{Tor}_q^R(H_p(C_*), M) \Rightarrow H_{p+q}(C_* \otimes M).$$

52 Remark

This is a special case of the Künneth formula. The universal coefficient theorem (for homology or cohomology) typically says there is a non-canonical (right-)splitting. Julia thinks this can be shown with spectral sequences as well, but she didn't want to open that can of worms.

The reason we get a short exact sequence is that, for a PID, a projective (really, free) resolution has two terms, the first corresponding to a generating set and the second corresponding to relations between those generators, which forms a submodule of a free module, which is thus free. This is why $\mathrm{Tor}^i_R = 0$ for $i > 1$.

We'll also assume we're in the category of finitely generated R-modules for today, though there are more general statements. (Probably completely unnecessary: see opening remark of next lecture.)

PROOF Let $P_* \to M$ be a projective resolution of M. Let $C_* \otimes P_*$ be the double complex obtained by tensoring these two complexes, as follows. (For this construction we consider P_* as $\cdots \to P_1 \to P_0 \to 0$, so that taking homology at P_0 gives us M.)

Definition 53. The tensor product of chain complexes C_*, P_* is the double complex $C_* \otimes P_*$ with objects

$$(C_* \otimes P_*)_{pq} = C_p \otimes P_q,$$

and differentials

$$\begin{array}{ccc} C_{p-1}\otimes P_q & \xleftarrow{d_p^C\otimes 1} & C_p\otimes P_q \\ \Big\downarrow{\scriptstyle 1\otimes d_q^P} & & \Big\downarrow{\scriptstyle 1\otimes d_q^P} \\ C_{p-1}\otimes P_{q-1} & \xleftarrow[d_p^C\otimes 1]{} & C_p\otimes P_{q-1}\end{array}$$

As usual, this actually commutes, so toggle the sign of the vertical differentials on every other column. (Note: most sources seem to define the tensor product of chain complexes as the total complex of this double complex.)

Taking vertical homology of $C_* \otimes P_*$ gives $H_q(C_p \otimes P_*) \cong C_p \otimes H_q(P_*)$ since the $C_p \otimes -$ functor is exact (C_p being free, hence projective, hence flat) and exact functors commute with homology. Likewise, taking horizontal homology gives $H_p(C_*) \otimes P_q$. Thus ${}^{I}E^1_{pq} = H^v(C_* \otimes P_*) = C_p \otimes H_q(P_*)$ and ${}^{II}E^1_{pq} = H^h(C_* \otimes P_*) = H_p(C_*) \otimes P_q$.

P_* is exact except at $q = 0$, so $H_q(P_*) = 0$ except for $H_0(P_*) \cong M$. That is, ${}^{I}E^1_{pq}$ is just $C_* \otimes M$ on the x-axis with zeros elsewhere. After taking homology, ${}^{I}E^2_{pq}$ collapses, meaning all differentials are zero at and after this page, so ${}^{I}E^\infty_{pq}$ is just $H_p(C_* \otimes M)$ for $q = 0$ and 0 otherwise. Hence we can just say ${}^{I}E^r_{pq} \Rightarrow H_{p+q}(C_* \otimes M)$. In this case we can determine the filtration on the right-hand side exactly: there is precisely one non-zero element on each antidiagonal $p + q = n$, namely for $q = 0$, forcing the filtration to have just one non-zero term, namely $F_p H_p(C_* \otimes M) = H_p(C_* \otimes M)$, as you'd expect.

On the other hand, the columns of ${}^{II}E^1_{pq}$ look like

$$\begin{array}{c} H_p(C_*) \otimes P_q \\ \Big\downarrow{\scriptstyle 1\otimes d_q^P} \\ H_p(C_*) \otimes P_{q-1} \\ \Big\downarrow{\scriptstyle 1\otimes d_{q-1}^P} \\ \vdots \\ \Big\downarrow \\ H_p(C_*) \otimes P_0 \end{array}$$

Recall how $\mathrm{Tor}^q_R(A, M)$ was defined: take a projective resolution $P_* \to M$ and set $\mathrm{Tor}^q_R(A, M) := H_q(A \otimes_R P_*)$. This is precisely what we're doing here, i.e.

$${}^{II}E^2_{pq} = \mathrm{Tor}^q_R(H_p(C_*), M).$$

Since this must converge to $H_{p+q}(C_* \otimes M)$, the statement for general R follows.

Now suppose R is a PID, so $\mathrm{Tor}_R^q(H_p(C_*), M)$ has only the $q = 0$ term $H_p(C_*) \otimes M$ and the $q = 1$ term $\mathrm{Tor}_R^1(H_p(C_*), M)$. Hence ${}^{II}E_{pq}^2$ is (after transposing)

$$\begin{array}{cccc}
0 & H_{p+1}(C_*) \otimes M & \mathrm{Tor}_R^1(H_{p+1}(C_*), M) & 0 \\
0 & H_p(C_*) \otimes M & \mathrm{Tor}_R^1(H_p(C_*), M) & 0 \\
0 & H_{p-1}(C_*) \otimes M & \mathrm{Tor}_R^1(H_{p-1}(C_*), M) & 0
\end{array}$$

The sequence thus collapses at this page. There are precisely two nonzero elements on each antidiagonal $p + q = n$, and this spectral sequence converges to $H_{p+q}(C_* \otimes M)$ as before. Since there are only two columns, we can again determine the filtration exactly:

$$\begin{array}{cccc}
0 & F_0 H_{p+1}(C_* \otimes M) & \frac{F_1 H_{p+2}(C_* \otimes M)}{F_0 H_{p+2}(C_* \otimes M)} & 0 \\
0 & F_0 H_p(C_* \otimes M) & \frac{F_1 H_{p+1}(C_* \otimes M)}{F_0 H_{p+1}(C_* \otimes M)} & 0 \\
0 & F_0 H_{p-1}(C_* \otimes M) & \frac{F_1 H_p(C_* \otimes M)}{F_0 H_p(C_* \otimes M)} & 0
\end{array}$$

(Dotted lines indicate antidiagonals.) Hence $F_1 H_p(C_* \otimes M) = H_p(C_* \otimes M)$ and $F_0 H_p(C_* \otimes M) = H_p(C_*) \otimes M$, where crucially

$$\mathrm{Tor}_R^1(H_{p-1}(C_*), M) = \frac{H_p(C_* \otimes M)}{H_p(C_*) \otimes M}.$$

It follows that we have a short exact sequence

$$0 \to H_p(C_*) \otimes M \to H_p(C_* \otimes M) \to \mathrm{Tor}_R^1(H_{p-1}(C_*), M) \to 0.$$

Questions: (1) Is it enough to assume what we assumed to pull the modules out of the tensor products/homologies? (2) Is there a non-canonical splitting in the general version of UCT? What do you need to assume to get that?

April 18th, 2014: Cartan-Eilenberg Resolutions; Projects

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54 Remark
Some comments about last lecture:

1) We used that a submodule of a free module is free over a PID ; this is always true (even in the infinitely generated case and is proved in eg. Rotman's *Advanced Modern Algebra*. It uses the axiom of choice.

2) UCT for homology holds for any coefficients.

3) One can formulate UCT more generally assuming that C_* is flat and $d(C_*) = B_*$ is flat. (The cycles will then also be flat since we'll have short exact sequences with the correct two of the three terms flat, forcing the third flat using the induced long exact sequence. This generalizes to cohomological functors.)

4) The second question at the end of last lecture doesn't really make sense unless you look at the $r = \infty$ term, but then it's probably hopeless to answer.

55 Motivation

Informally, we want a simultaneous projective resolution of a chain complex C_*

$$\begin{array}{ccccc} P_{p-1,1} & \longleftarrow & P_{p,1} & \longleftarrow & P_{p+1,1} \\ \downarrow & & \downarrow & & \downarrow \\ P_{p-1,0} & \longleftarrow & P_{p,0} & \longleftarrow & P_{p+1,0} \\ \downarrow \epsilon & & \downarrow \epsilon & & \downarrow \epsilon \\ C_{p-1} & \longleftarrow & C_p & \longleftarrow & C_{p+1} \end{array}$$

where P_{**} is a double chain complex. This isn't quite strong enough, but it's close. The statement $\boxed{P_{**} \to C_*}$ should be interpreted as meaning the above diagram holds.

Definition 56 (Cartan-Eilenberg Resolutions). Let $\mathcal{A}$ be an abelian category with enough projectives. Let C_* be a complex in $\mathrm{Ch}(\mathcal{A})$, the category of chain complexes over $\mathcal{A}$ with obvious morphisms. A double complex (not a priori projective resolutions in any sense)

$$P_{**} \xrightarrow{\epsilon} C_*$$

is a $\boxed{\text{Cartan-Eilenberg Resolution}}$ if

(1) $P_{p,*} = 0$ if $C_p = 0$, and

(2) $H_p(P_{**}, d^h) \overset{H_p(\epsilon)}{\to} H_p(C_*)$, $B_p(P_{**}, d^h) \overset{B_p(\epsilon)}{\to} B_p(C_*)$ are projective resolutions.

($H_p(P_{**}, d^h)$ refers to horizontal homology, $B_p(P_{**}, d^h)$ refers to taking horizontal boundaries, and $Z_p(P_{**}, d^h)$ below refers to taking horizontal cycles.)

57 Proposition

Let $P_{**} \to C_*$ be a Cartan-Eilenberg resolution. Then $Z_p(P_{**}, d^h) \to Z_p(C_*)$ and $P_{p*} \to C_p$ are projective resolutions for all p. (Part of the conclusion is that the first resolution is well-defined.)

Proof Consider

$$0 \to B_p(P_{*q}) \to Z_p(P_{*q}) \to H_p(P_{*q}) \to 0.$$

This splits since $H_p(P_{*q}) = H_p(P_{*q}, d^h)$ is projective. Hence $Z_{pq} = H_{pq} \oplus B_{pq}$ is projective. Now consider

$$0 \to Z_{pq} \to P_{pq} \xrightarrow{d^h} B_{p-1,q} \to 0,$$

which splits again, so P_{p*} is projective. As for exactness, consider

$$\begin{array}{ccccc}
B_p(P_{*q}) & \longrightarrow & Z_p(P_{*q}) & \longrightarrow & H_p(P_{*q}) \\
\downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow \\
B_p(C_*) & \longrightarrow & Z_p(C_*) & \longrightarrow & H_p(C_*)
\end{array}$$

Use the following fact: if $A_* \to B_* \to C_*$ is a short exact sequence of complexes, then it induces a long exact sequence on homology. (This will be a homework problem.) The "correct" two (here, outside) sequences are exact here, so this argument forces the middle sequence exact, so $Z_p(P_{**}) \to Z_p(C_*)$ is exact. Use a similar argument for $P_{p*} \to C_*$

58 Lemma (Horseshoe Lemma)

(This is Weibel, 2.2.8.) Suppose we have a short exact sequence with projective resolutions on the left and right terms. Then there is a projective resolution of the middle term commuting with everything in sight:

$$\begin{array}{ccccccccc}
 & & P'_* & \overset{\exists}{\dashrightarrow} & \exists P_* & \overset{\exists}{\dashrightarrow} & P''_* & & \\
 & & \downarrow & & \downarrow \exists & & \downarrow & & \\
0 & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & C'' & \longrightarrow & 0
\end{array}$$

59 Theorem

Let C_* be a complex in $\mathrm{Ch}(\mathcal{A})$, where $\mathcal{A}$ is an abelian category with enough projectives. Then there exists a Cartan-Eilenberg resolution $P_{**} \to C_*$.

Proof We have enough projectives to create projective resolutions of the left and right terms of the diagram below, so from the Horseshoe Lemma, we have in all

$$\begin{array}{ccccccccc}
 & & P^B_{p*} & \longrightarrow & P^Z_{p*} & \longrightarrow & P^H_{p*} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B_p(C_*) & \longrightarrow & Z_p(C_*) & \longrightarrow & H_p(C_*) & \longrightarrow & 0
\end{array}$$

Doing the same thing again,

$$\begin{array}{ccccccccc}
 & & P^Z_{p*} & \longrightarrow & P_{p*} & \longrightarrow & P^B_{p-1,*} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_p(C_*) & \longrightarrow & C_p & \longrightarrow & B_{p-1}(C_*) & \longrightarrow & 0
\end{array}$$

This gives us a double complex $P_{**} \to C_*$; we just need it to satisfy the Cartan-Elienberg properties. (Minor note: toggle signs on every other column to get anticommutativity.) Note that

$$d^h_{pq} \colon P_{pq} \twoheadrightarrow P^B_{p-1,q} \hookrightarrow P^Z_{p-1,q} \hookrightarrow P_{p-1,q}.$$

Exercise: convince yourself everything works from here.

60 Remark

Good thing to think about: what are the projective objects in the category of chain complexes? Is a Cartan-Eilenberg resolution just a projective object in the category? Is it more, less, or none of the above? What about $\mathrm{Ch}_{\geq 0}(\mathcal{A})$ (chain complexes bounded below, i.e. $C_p = 0$ for $p \ll 0$)?

61 Fact

Assume C_* is bounded below. Then $P_{p*} = 0$ for $p \ll 0$ in the above construction. Hence we can form $\mathrm{Tot}(P_{**})$, with a quasi-isomorphism $\mathrm{Tot}(P_{**}) \xrightarrow{\sim} C_*$, meaning it induces an isomorphism on homology.

[If you don't put a boundedness condition, there are a couple of choices for what you mean by $\mathrm{Tot}(P_{**})$. You can take the infinite direct sum, but it may not exist, eg. over the category of finitely generated R-modules. Weibel discusses this at length.]

This says the category of chain complexes is somewhat different from the category of, say, R-modules, since if we pass to the derived category, where we replace the isomorphisms with quasi-isomorphisms, we can still use projective chain complexes, so we "don't leave" the category.

62 Remark

Projects for the end of the quarter:

I Applications of spectral sequences to Algebraic Topology; Leray-Serre spectral sequence.

(1) (Serre) Fibrations
(2) Construction of Leray-Serre spectral sequence H^*
(3) Multiplicative structure
(4) Applications: $H^*(\mathbb{C}P^n, R)$
(5) Cohomology with local coefficients
(6) $H^*(SU(n), R)$ and $H^*(SO(n), F_2)$.

The first four are in the simply-connected case. References: Weibul; McCleary.

II Finite generation of $H^*(G, k)$; Venkov, 1959; Benson, II.3.30.

III Lie algebra cohomology, Koszul complex, Chevalley-Eilenberg ex. Reference: Weibel §7.7.

April 21st, 2014: Left Hyper-derived Functors and their Spectral Sequences; Injective Cartan-Eilenberg Resolutions

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63 Remark

Given two complexes C_*, D_* (in some abelian category), we can take the tensor product $C_* \otimes D_*$ as before. A potential abuse of notation is $H_n(C_* \otimes D_*) := H_n(\mathrm{Tot}(C_* \otimes D_*))$.

Definition 64 (Left Hyper-derived Functors). Let $F\colon \mathcal{A} \to \mathcal{B}$ be a right exact functor, $\mathcal{A}, \mathcal{B}$ abelian, $C_* \in \mathrm{Ch}(\mathcal{A})$. The ith $\boxed{\text{left hyper-derived functor}}$ of F is

$$\boxed{\mathbb{L}_i F(C_*)} := H_i \,\mathrm{Tot}(F(P_{**}))$$

where $P_{**} \to C_*$ is a Cartan-Eilenberg resolution of C_*. $\mathbb{L}_i F\colon \mathrm{Ch}(\mathcal{A}) \to \mathcal{B}$ is a functor when it exists.

Why might it not exist? The total complex may not exist since it might require infinite direct sums. By contrast, $\mathbb{L}_i F\colon \mathrm{Ch}_{\geq 0}(\mathcal{A}) \to \mathcal{B}$ does exist; left hyper-derived functors exist more generally when $\mathcal{B}$ is cocomplete. We don't really need to assume $\mathcal{B}$ is abelian; weaker assumptions suffice.

Definition 65 (Injective Cartan-Eilenberg Resolutions). If $\mathcal{A}$ is an abelian category with enough injectives, then we can construct injective Cartan-Eilenberg resolutions, meaning the following. Let $C^* \in \mathrm{Ch}^{\cdot}(\mathcal{A})$ be a cochain complex. $C^* \to I^{**}$ is an $\boxed{\text{injective Cartan-Eilenberg resolution}}$ if

(1) I^{pq} is injective for all p, q;

(2) $C^* \to I^{**}$ is a double complex, $I^{pq} = 0$ for $q < 0$

(3) $I^{pq} = 0$ if $C^p = 0$

(4) $B^p(C^*) \to B^p(I^{*q}, d^h)$ and $H^p(C^*) \to H^p(I^{*q}, d^h)$ are injective resolutions.

66 Remark

This is slightly different from the projective version, in particular we didn't require (1), though (1) was a consequence of the analog of (3) and (4); probably still the case.

Injective Cartan-Eilenberg resolutions exist. If $C^* \to I^{**}$ is an injective Cartan-Eilenberg resolution, then C^* is quasiisomorphic to $\mathrm{Tot}(I^{**})$ (for instance this works nicely in $\mathrm{Ch}^{\geq 0}(\mathcal{A})$).

67 Remark

Left (and right) hyper-derived functors are well-defined. The "hyper" fundamental theorem of homological algebra says that any two Cartan-Eilenberg resolutions of the same complex are chain homotopic. See Weibel for a careful statement.

Indeed, if $P_{**} \to C_*$ and $Q_{**} \to C_*$ are two Cartan-Eilenberg resolutions, then

$$\mathrm{Tot}(F(P_{**})), \mathrm{Tot}(F(Q_{**}))$$

are chain homotopic, so $\mathbb{L}_i F$ is independent of the choice of resolution.

Also, we didn't actually use right (or left) exactness in the definition of hyper-derived functors. (We may need additivity, which is weaker.)

68 Remark

Let C_* be concentrated in degree 0,

$$\cdots \to 0 \to C_0 \to 0 \to \cdots,$$

and let F be right exact. Then $\mathbb{L}_i F(C_*) \cong L_i F(C_0)$. What if it was concentrated in degree n?

Definition 69. Define $\boxed{\text{dimension shifting}}$ of a chain complex by

$$\boxed{C_*[n]}_i := C_{i+n}.$$

Hence

$$(\mathbb{L}_i F)(C_*[n]) = \mathbb{L}_{i+n} F(C_*).$$

70 Theorem (Spectral sequences for left hyper-derived functors)

Let C_* be a complex in $\mathcal{A}$, $F \colon \mathcal{A} \to \mathcal{B}$ a right exact functor. There exist two spectral sequences converging to the same thing,

$${}^{I}E^2_{pq} = H_p(L_q F(C_*)) \Rightarrow \mathbb{L}_{p+q} F(C_*)$$

and

$${}^{II}E^2_{pq} = (L_q F)(H_p(C_*)) \Rightarrow \mathbb{L}_{p+q} F(C_*).$$

(Actually, the first one converges under the assumption C_* is bounded.)

Proof Recall $\mathbb{L}_{p+q}F(C_*)$ is the homology of the total complex of the double complex $F(P_{**})$, so we can employ our previous machinery exactly. Let $P_{**} \to C_*$ be a Cartan-Eilenberg resolution. We have

$$\begin{array}{c} FP_{p2} \\ \downarrow \\ FP_{p1} \\ \downarrow \\ FP_{p0} \\ \downarrow \\ FC_p \end{array}$$

so ${}^IE^1_{pq} = H^v_q(F(P_{p*})) = L_qF(C_p)$. Hence

$${}^IE^2_{pq} = H_p(L_qF(C_*)).$$

In the other direction, ${}^{II}E^1_{pq} = H^h_p(F(P_{*q}))$, which we claim (to be proved next lecture) is $F(H^h_p(P_{*q}))$; let's call $H_{pq} := H^h_p(P_{*q})$. Given the claim,

$${}^{II}E^2_{pq} = H^v_q(F(H^h_p(P_{*q}))) = (L_qF)(H_p(C_*))$$

since $H^h_p(P_{*q}) \to H_p(C_*)$ is a projective resolution.

Notes on convergence: a priori (I) is an upper half plane spectral sequence so convergence is not guaranteed, but if C_* is bounded below, the double complex is in a (correct) quarter plane, so (I) converges. Exercise: why does ${}^{II}E^2_{pq}$ converge even without a boundedness assumption? (This is also discussed in the next lecture.)

April 23rd, 2014: Right Hyper-Derived Functors; Hypertor

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71 Remark

Unproved claim from last time: $H^h_p(F(P_{*q})) = F(H^h_p(P_{*q}))$ where F is a right exact functor and $P_{**} \to C_*$ is a Cartan-Eilenberg resolution.

For context, recall we were showing

$${}^{II}E^2_{pq} = (L_pF)(H_q(C_*)) \Rightarrow (\mathbb{L}_{p+q}F)(C_*),$$

and had completed the proof assuming the claim.

Proof Consider

$$\begin{array}{ccccc} Z_{pq} & \hookrightarrow & P_{pq} & \xrightarrow{d^h} & P_{p-1,q} \\ & & & \searrow\!\!\!\!\!\rightarrow & \uparrow \\ & & & & d^h(P_{pq}) = B_{p-1,q} \end{array}$$

This sequence splits as $P_{pq} \cong Z_{pq} \oplus P'_{pq}$ with $P'_{pq} \cong d^h(P_{pq})$. Similarly

$$d^h(P_{p+1,q}) = B_{pq} \hookrightarrow Z_{pq} \twoheadrightarrow H_{pq} = H_p(P_{*q}, d^h)$$

splits with $Z_{pq} \cong B_{pq} \oplus H_{pq}$. Back to our original sequence, we have

$$\begin{array}{ccccccccc} \cdots & \xleftarrow{d^h} & P_{p-1,q} & \xleftarrow{d^h} & P_{p,q} & \xleftarrow{d^h} & P_{p+1,q} & \longleftarrow & \cdots \\ & & & & \updownarrow = & & & & \\ & & & & B_{pq} \oplus H_{pq} \oplus B_{p-1,q} & & & & \end{array}$$

Apply F to get

$$\begin{array}{ccccccc} \cdots & \longleftarrow & F(P_{p-1,q}) & \xleftarrow{F(d^h)} & F(P_{p,q}) & \longleftarrow & \cdots \\ & & \updownarrow = & & \updownarrow = & & \\ & & F(B_{p-1,q}) \oplus F(H_{p-1,q}) \oplus F(B_{p,q}) & & F(B_{pq}) \oplus F(H_{pq}) \oplus F(B_{p+1,q}) & & \end{array}$$

Here we've used the additivity of F to distribute over sums. The induced map is

$$F(B_{pq}) \oplus F(H_{pq}) \oplus F(B_{p+1,q}) \to F(B_{p-1,q}) \oplus F(H_{p-1}, q) \oplus F(B_{pq})$$

with $F(B_{pq}) \to F(B_{pq})$ the identity, and it follows that

$$H_p(F(P_{*q}), F(d^h)) \cong F(H_{pq}) = F(H_p(P_{*q}), d^h),$$

as desired.

72 Remark

On convergence of the spectral sequences from last time: they were

$${}^I E^2_{pq} = H_p(L_q F(C_*)) \Rightarrow \mathbb{L}F(C_*)$$

and

$${}^{II} E^2_{pq} = L_p F(H_q(C_*)) \Rightarrow \mathbb{L}F(C_*).$$

Weibel says the following. In ${}^{II}E^2_{pq}$, the row filtration used to construct it is (1) bounded below and (2) "exhaustive". Hence he can apply the "classical convergence theorem" 5.5.1 to conclude that ${}^{II}E^2_{pq}$ converges. Read about it in Weibel if you're interested in a rigorous discussion; what follows is a summary.

How are ${}^I E$ and ${}^{II} E$ for the hyperderived functor spectral sequences different? We start with a complex and then we resolve it. A priori we get a double complex in the upper half plane, which remains in the upper half plane after applying the functor. The column filtration used in (I) is hence not bounded below in general.

The row filtration for (II) is much nicer: we start at 0, take the first row, take the second row, etc. It's bounded below since we started in the upper half plane. Now after transposing the spectral sequence lives in the right half-plane. It follows that at each point the cycles stabilize (since the differentials starting there will eventually land in the left half-plane). Hence eventually the objects give a sequence of quotients. According to Weibel, this is enough.

Definition 73. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a left-exact functor between abelian categories with enough injectives. The ith right hyper-derived functor of F is

$$\boxed{\mathbb{R}^i F(C^*)} := H^i(\mathrm{Tot}(F(I^{**})))$$

where $C^* \to I^{**}$ is an injective Cartan-Eilenberg resolution. As before this is a functor $\mathbb{R}^i F\colon \mathrm{Ch}(\mathcal{A}) \to \mathcal{B}$ when it exists.

Again, this is perfectly well-defined in general if C^* is bounded below. Otherwise consult Weibel. It is independent of the choice of resolution.

74 Proposition

Let $C \in \mathrm{Ch}^{\geq 0}(\mathcal{A})$, $F\colon \mathcal{A} \to \mathcal{B}$ a left-exact functor between abelian categories with enough injectives. Then we have two convergent (cohomological) spectral sequences,

$$^{I}E_2^{pq} = H^p(R^q F(C^*)) \Rightarrow \mathbb{R}^{p+q} F(C^*)$$

and

$$^{II}E_2^{pq} = R^p F(H^q(C^*)) \Rightarrow \mathbb{R}^{p+q} F(C^*).$$

75 Example

Let C_*, D_* be complexes of R-modules. (If R is non-commutative, we have to consider C_* as consisting of right R-modules, D_* as consisting of left R-modules. Assume R is commutative so we don't have to worry.) Form a functor $\mathrm{Tot}(- \otimes_R D_*)\colon \mathrm{Ch}_{\geq 0}(\mathcal{A}) \to \mathrm{Ch}_{\geq 0}(\mathcal{A})$. We define hypertor as

$$\boxed{\mathbb{T}\mathrm{or}_i^R(C_*, D_*)} := L_i(\mathrm{Tor}(C_* \otimes_R D_*)) = H_i(\mathrm{Tot}(P_{\cdot *} \otimes_R D_*)).$$

This makes sense: it's just the left hyper-derived functor of the above tensor product functor.

April 25th, 2014: Derived Functors; Grothendieck Spectral Sequence

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76 Remark

Given an abelian category $\mathcal{A}$, there is a derived category $D(\mathcal{A})$. We won't take the time to define this properly, but it is roughly the category $\mathrm{Ch}(\mathcal{A})$ of chain complexes over $\mathcal{A}$ where quasiisomorphisms (meaning maps between chain complexes which induce isomorphisms in homology) are turned into isomorphisms. Recall that we had asserted the total complex of an injective Cartan-Eilenberg resolution of a chain complex is quasiisomorphic to that chain complex, so in the derived category, these things are isomorphic.

There is a similar notion, the derived category bounded below, $D^+(\mathcal{A})$, using chain complexes which are bounded below.

Sadly (or happily?), the derived category is not in general abelian.

Definition 77. Let $F\colon \mathcal{A} \to \mathcal{A}$ be a functor, and take $\mathcal{A}$ abelian. We define a functor $\boxed{\mathbb{R}^* F}\colon \mathrm{Ch}^+(\mathcal{A}) \to \mathrm{Ch}^+(\mathcal{A})$ or $\mathbb{R}^* F\colon D^+(\mathcal{A}) \to D^+(\mathcal{A})$ as follows.

Let $\mathrm{Ch}^+(F)\colon \mathrm{Ch}^+(\mathcal{A}) \to \mathrm{Ch}^+(\mathcal{A})$ be the induced functor on chain complexes. Let $\mathbb{R}^* F\colon \mathrm{Ch}^+(\mathcal{A}) \to \mathrm{Ch}^+(\mathcal{A})$ be the (0th) right derived functor of $\mathrm{Ch}^+(F)$. Suppose $\mathrm{Ch}^+(F)$ descends to the derived category, so also $\mathbb{R}^* F\colon D^+(\mathcal{A}) \to D^+(\mathcal{A})$.

Let $C^* \to I^{**}$ be an injective Cartan-Eilenberg resolution of C^*. Letting $I^* = \mathrm{Tot}(I^{**})$, as noted above, $C^* \xrightarrow{\sim} I^*$ in $D^+(\mathcal{A})$. Hence we have a (one term) injective resolution of $C^* \in D^+(\mathcal{A})$, so that

$$\mathbb{R}^* F(C^*) = F(I^*).$$

Moreover, we recover our earlier hyper-derived functors as

$$H^i(\mathbb{R}^* F(C^*)) = \mathbb{R}^i F(C^*).$$

Definition 78. Let $C_*, D_* \in \mathrm{Ch}^+(\mathcal{A})$; say $A = R$-mod, $\mathcal{O}_X$-mod, $\mathrm{QCoh}(X)$, ... We define the derived tensor product

$$\boxed{C_* \otimes^{\mathbb{L}}_R D_*} := \mathbb{L}^*(C_* \otimes_R D_*) := \mathrm{Tot}(P_* \otimes D_*) \cong \mathrm{Tot}(C_* \otimes_R Q_*) \cong \mathrm{Tot}(P_* \otimes_R Q_*)$$

where $P_* \xrightarrow{\sim} C_*$ is a quasiisomorphism, say $P_* = \mathrm{Tot}(P_{**})$ for a Cartan-Eilenberg resolution $P_{**} \to C_*$, and similarly with $D_* \xrightarrow{\sim} Q_*$. One defines $\mathrm{RHom}\colon D^+(\mathcal{A}) \to D^+(\mathcal{A})$ similarly.

Definition 79. Let $\mathcal{A}, \mathcal{B}$ be abelian categories where $\mathcal{A}$ has enough injectives. Suppose $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ is a left exact functor. We'll say that $X \in \mathcal{A}$ is $\mathcal{F}$-acyclic if $R^i\mathcal{F}(X) = 0$ for $i > 0$.

80 Example

Working entirely in the category of R-modules, let $\mathcal{F} = - \otimes_R Y$. Certainly flat (or free, or projective) X are acyclic. By definition, X is $\mathcal{F}$-acyclic iff $\mathrm{Tor}_i^R(X, Y) = 0$ for $i > 0$. Note Y is fixed here, so flatness is much stronger than necessary, eg. take $Y = 0$.

Note: an injective object is always $\mathcal{F}$-acyclic, since it has an injective resolution of length 1.

81 Theorem (Grothendieck spectral sequence)

Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories. Suppose

$$\mathcal{A} \xrightarrow{\mathcal{G}} \mathcal{B} \xrightarrow{\mathcal{F}} \mathcal{C}$$

are left exact functors. Suppose (1) $\mathcal{A}$ has enough injectives, and (2) $\mathcal{G}$ takes injective objects in $\mathcal{A}$ to $\mathcal{F}$-acyclic objects in $\mathcal{B}$. (If $\mathcal{G}$ takes injectives to injectives, (2) is trivially satisfied.) Then there is a first quadrant spectral sequence

$$R^p\mathcal{F} \circ R^q\mathcal{G}(X) \Rightarrow R^{p+q}(\mathcal{F} \circ \mathcal{G})(X)$$

for any $X \in \mathcal{A}$. There is a similar sequence for left derived functors.

Proof Let $X \to I^*$ be an injective resolution in $\mathcal{A}$. Consider

$$\mathcal{G}(X) \to \mathcal{G}(I^*) \in \mathrm{Ch}^+(\mathcal{B})$$

Recall the spectral sequences for right-derived functors of $\mathcal{F}$. In particular, for $C^* \in \mathrm{Ch}^+(\mathcal{B})$, we have

$$^{I}E_2^{pq} = H^p(R^q\mathcal{F}(C^*)), {}^{II}E_2^{pq} = R^p\mathcal{F}(H^q(C^*)) \qquad \Rightarrow \mathbb{R}^{p+q}\mathcal{F}(C^*).$$

(Julia described this as using two "hyper-cohomology spectral sequences" associated to the complex $\mathcal{G}(I^*)$ and the functor $\mathcal{F}$.) Since $\mathcal{G}$ takes injectives to $\mathcal{F}$-acyclics, the first one collapses:

$$^{I}E_2^{pq} = H^p(R^q\mathcal{F}(\mathcal{G}(I^*))) = \begin{cases} H^p(\mathcal{F}(G(I^*))) & q = 0 \\ 0 & q > 0 \end{cases}$$

Note that $H^p(\mathcal{F}(\mathcal{G}(I^*))) = H^p((\mathcal{F} \circ \mathcal{G})(I^*)) = R^p(\mathcal{F} \circ \mathcal{G})(X)$. Since the $^{I}E_2^{pq}$ spectral sequence collapses to one non-zero row, we recover the abutment exactly, i.e. the filtration must be trivial, so $R^p(\mathcal{F} \circ \mathcal{G})(X) \cong \mathbb{R}^p\mathcal{F}(\mathcal{G}(I^*))$.

For $^{II}E_2^{pq}$, we have

$$^{II}E_2^{pq} = R^p\mathcal{F}(H^q(\mathcal{G}(I^*))) = R^p\mathcal{F}(R^qG(X)) = R^p\mathcal{F} \circ R^q\mathcal{G}(X).$$

And (amazingly) that's it.

82 Remark

If

$$\mathcal{A} \stackrel{\mathcal{G}}{\to} \mathcal{B} \stackrel{\mathcal{F}}{\to} \mathcal{C}$$

induce

$$D^+(\mathcal{A}) \stackrel{\mathbb{R}^*\mathcal{G}}{\to} D^+(\mathcal{B}) \stackrel{\mathbb{R}^*\mathcal{F}}{\to} D^+(\mathcal{C}),$$

then the Grothendieck spectral sequence tells us precisely that there is an isomorphism of functors

$$\mathbb{R}^*\mathcal{F} \circ \mathbb{R}^*\mathcal{G} \stackrel{\sim}{\to} \mathbb{R}^*(\mathcal{F} \circ \mathcal{G}).$$

Exercise: explain this.

April 28th, 2014: Lyndon-Hochschild-Serre Spectral Sequence

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83 Remark

Today, we'll construct the Lyndon-Hochschild-Serre spectral sequence. Other names: Lyndon spectral sequence; Hochschild-Serre spectral sequence.

84 Theorem

Suppose G is a group with normal subgroup H. Let M be a representation of G (i.e. a kG-module). Then there is a first quadrant spectral sequence

$$H^p(G/H, H^q(H, M)) \Rightarrow H^{p+q}(G, M).$$

($k = \mathbb{Z}$ also works.)

85 Remark

(1) There is a "dual" spectral sequence for group homology H_*.

(2) E_2^{pq} is a "multiplicative" spectral sequence. More on this later.

(3) Exists in other contexts such as

I. Hopf algebras (much weaker structure works: augmented algebras, i.e. have $k \to A \to k$)

II. Lie algebras

III. Algebraic/Lie groups. (We don't think of them as discrete groups, so we work in a different category when taking cohomology. Hence this will be similar to but not the same as the version in the theorem.)

Proof This is a Grothendieck spectral sequence. Recall $H^n(G, M) = \mathrm{Ext}^n_G(k, M) = R^n \mathrm{Hom}_G(k, M) = R^n(M)^G$, or in other words, group cohomology is just computing right-derived functors of the G-invariants functor $(-)^G$. Letting $\mathcal{G} = (-)^H$ and $\mathcal{F} = (-)^{G/H}$, we have

$$G\text{–mod} \stackrel{\mathcal{G}}{\to} G/H\text{–mod} \stackrel{\mathcal{F}}{\to} k\text{–mod}$$

For the Grothendieck spectral sequence, we need to check

(1) $\mathcal{F} \circ \mathcal{G} = (-)^G$

(2) G–mod and H–mod have enough injectives. (Cohomological version needs injectives.)

(3) $\mathcal{G} = (-)^H$ takes injective G-modules to ($\mathcal{F} = (-)^{G/H}$)-acyclic G/H-modules.

(1) is clear; see first lecture. (2): recall how to construct enough injective modules. Over $\mathbb{Z}$, we can create injective $\mathbb{Z}$-modules by considering $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)$ where A is a divisible group. Over a field k, we can create an injective G-module by considering $\mathrm{Hom}_k(kG, V)$ for any vector space V over k viewed as a trivial G-module.

How does one show an object is injective? Can use the definition showing certain things lift; can show Ext^1 vanishes; can show Hom into that module is an exact functor. We'll take the last approach. In this case, we want to show $\mathrm{Hom}_G(-, \mathrm{Hom}_k(kG, V))$ is an exact functor of G-modules. By ajointness in the first isomorphism and "cancelling" the kG in the second, we have

$$\mathrm{Hom}_G(M, \mathrm{Hom}_k(kG, V)) \xrightarrow{\sim} \mathrm{Hom}_G(M \otimes_k kG, V) \cong \mathrm{Hom}_k(M, V),$$

which is exact since Hom functors over k are exact. The same proof goes through for $\mathbb{Z}$, except V needs to be an injective $\mathbb{Z}$-module, i.e. needs to be divisible.

Claim: this construction gives us enough injectives. For any G-module V, we have an embedding $V \hookrightarrow \mathrm{Hom}_k(kG, V)$ where $v \mapsto \phi_v$ with $\phi_v(g) = gv$. This is an injection. Hence any G-module embeds into an injective G-module. Technical note: we need to check $V \hookrightarrow \mathrm{Hom}_k(kG, V)$ is a G-map. (This is an analogue of the "tensor identity": given a G-module M, we can construct the G-module $kG \otimes_k M$ using the diagonal action; there is an isomorphism $kG \otimes_k M \xrightarrow{\sim} kG \otimes_k M_{\mathrm{triv}}$ where in the second case M has trivial G-action. This will go into the next homework set.)

For (3), we claim it suffices to show that for any G-module V, $\mathrm{Hom}_k(kG, V)^H$ is injective as a G/H-module. One option is to weaken the assumptions on the Grothendieck spectral sequence slightly and not require *everything* to be $\mathcal{F}$-acyclic. In this case, however, the original assumptions actually follow: given an arbitrary injective V, embed it into $\mathrm{Hom}_k(kG, V)$ and exhibit this as a direct sum. To finish it off, distribute over sums.

We have $\mathrm{Hom}_k(kG, V)^H \cong \mathrm{Hom}_k(k(G/H), V)$ since we take the trivial action on V; since this is an injective G/H-module, $(-)^H$ takes injective G-modules to injective G/H-modules, as required.

The Grothendieck spectral sequence in this situation is thus

$$R^p(-)^{G/H} \circ R^q(M)^H \Rightarrow R^{p+q}(M)^G$$
$$R^p \mathrm{Hom}_{G/H}(k, R^q \mathrm{Hom}_H(k, M)) \Rightarrow R^{p+q}(\mathrm{Hom}_G(k, M))$$
$$H^p(G/H, H^q(H, M)) \Rightarrow H^{p+q}(G, M)$$

Exercise: a projective kG-module is projective as a kH-module. Equivalently, show kG is free as a kH-module. (Hint: break it up using cosets G/H.)

April 30th, 2014: Cross Products, Cup Products, and Group Cohomology Ring Structure

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86 Remark

Today we'll review product structures in group cohomology. We'll work over a field k, though things also work over $\mathbb{Z}$.

Definition 87. Let G, H be groups. The $\boxed{\text{cross product}}$ (or $\boxed{\text{external product}}$) of H and G is a map

$$\boxed{\times}\colon H^p(G, k) \times H^q(H, k) \to H^{p+q}(G \times H, k).$$

Recall $H^p(G,k) := \mathrm{Ext}^p_{kG}(k,k)$; here G acts trivially on k. Let $P_* \to k$ be a projective resolution over G and $Q_* \to k$ a projective resolution over H, so $H^p(G,k) := H_p(\mathrm{Hom}_G(P_*,k))$, $H^q(H,k) := H_q(\mathrm{Hom}_G(Q_*,k))$. Form $P_* \otimes Q_*$: since the modules are in different categories, we're tensoring over k and taking the total complex. We then give a $G \times H$-module structure using a diagonal action. This gives a projective resolution $P_* \otimes Q_* \to k$ in the category of $G \times H$-modules. Hence $H^{p+q}(G \times H, k) := H_{p+q}(\mathrm{Hom}_G(P_* \otimes Q_*, k))$. We define a family of maps

$$\mathrm{Hom}_G(P_p,k) \times \mathrm{Hom}_H(Q_q,k) \to \mathrm{Hom}_{G\times H}((P_* \otimes Q_*)_{p+q}, k)$$
$$(\mu,\nu) \mapsto \widetilde{\mu \times \nu} \colon (P_* \otimes Q_*)_{p+q} \to k$$

where

$$\widetilde{\mu \times \nu}(x \otimes y) = \mu(x)\nu(y)$$

if $x \in P_p, y \in Q_q$, and which is zero otherwise. We would hope this map descends to homology, and indeed it does, giving $\times$. (This is independent of the choice of resolutions.) Moreover, this induces a map $H^p(G,k) \otimes H^q(H,k) \to H^{p+q}(G \times H, k)$, which gives the map from the Künneth formula

$$\bigoplus_{p+q=n} H^p(G,k) \otimes H^q(H,k) \xrightarrow{\sim} H^n(G \times H, k).$$

(The usual Tor term disappears here giving an isomorphism, since our coefficients are in k.)

88 Remark

In the above,

$$H^p(G,k) := H^p(\mathrm{Hom}_G(P_*,k)) \cong H^p(\mathrm{Hom}_k(P_*,k)^G).$$

The isomorphism comes from the fact that

$$\mathrm{Hom}_G(M,N) \cong \mathrm{Hom}_k(M,N)^G,$$

where G acts on $\mathrm{Hom}_k(M,N)$ as follows: if $f \colon M \to N$ is k-linear, let

$$g \cdot f = (m \mapsto gf(g^{-1}m)).$$

Definition 89. Given groups G, H, a G-module M, and an H-module N (both over a field k), define the external tensor product $M \boxtimes N$ as the vector space $M \otimes_k N$ with a diagonal $G \times H$-module action

$$(g,h) \cdot (m \otimes n) = gm \otimes hn := gm \boxtimes hn.$$

For instance, $k \boxtimes k \cong k$, using trivial G, H, and $G \times H$ actions.

Definition 90. The cross product above can given with general coefficients as follows. Let M be a G-module and N an H-module. Define

$$\times \colon H^p(G,M) \otimes H^q(H,N) \to H^{p+q}(G \times H, M \boxtimes N)$$

by using

$$\widetilde{\mu \times \nu}(x \otimes y) = \mu(x) \boxtimes \nu(y)$$

in the above construction. (We can freely pass from $\times$ to $\otimes$ in the domain.)

Definition 91. We next define the cup product or internal product. Let $\Delta \colon G \to G \times G$ be the diagonal map $g \mapsto (g,g)$, which induces a map on cohomology in the other direction. The cup product is defined to be the composite of the cross product and this induced map:

$$\boxed{\smile} \colon H^p(G,k) \otimes H^q(G,k) \xrightarrow{\times} H^{p+q}(G \times G, k) \xrightarrow{\Delta^*} H^{p+q}(G,k).$$

92 Remark

$\Delta\colon G \to G \times G$ corresponds to $\Delta\colon kG \to kG \otimes kG$, the standard coproduct on the group algebra, which is part of the standard Hopf algebra struture on kG. More generally, we can construct the cup product in cohomology for any Hopf algebra with coproduct $\Delta\colon A \to A \otimes A$, since it induces

$$H^*(A,k) \otimes H^*(A,k) \to H^*(A \otimes A, k) \xrightarrow{\Delta^*} H^*(A,k).$$

Here the $\boxed{\text{cohomology of a Hopf algebra}}$ is defined by $\boxed{H^n(A,k)} := \operatorname{Ext}^n_A(k,k) = R^n \operatorname{Hom}_A(k,k)$. Indeed, we can do it for "augmented algebras".

Definition 93. Suppose $G \xrightarrow{f} G'$ is a map of groups and M is a G'-module. We claim $\boxed{H^*(G',M) \xrightarrow{f^*} H^*(G,M)}$ is a contravariant functor. (I was unable to make complete sense of this as a functor: we seem to need G' before we can fix M ....) Here $H^n(G,M)$ is computed by considering M as a G-module via pullback through f.

Let $P_{G',*} \to k$ be a projective resolution over G', and let $P_{G,*} \to k$ be a projective resolution over G. Consider the $P_{G',*}$ as G-modules via pullback, so $P_{G',*} \to k$ is an exact sequence of G-modules (though they're almost certainly not projective). By standard homological algebra, since the $P_{G,*}$'s are indeed projective, we can lift to get a map F of chain complexes (of G-modules):

$$\begin{array}{ccc} P_{G,*} & \longrightarrow & k \\ \big\downarrow F & & \big\updownarrow = \\ P_{G',*} & \longrightarrow & k \end{array}$$

This gives the F^* arrow in the following diagram:

$$\begin{array}{ccc} \operatorname{Hom}_{G'}(P_{G',*}, M) & \overset{\exists}{\dashrightarrow} & \operatorname{Hom}_G(P_{G,*}, M) \\ & \searrow \quad \nearrow_{F^*} & \\ & \operatorname{Hom}_G(P_{G',*}, M) & \end{array}$$

Given $G \to G'$, $N^{G'} \hookrightarrow N^G$, which gives us the left arrow: the $\operatorname{Hom}_{G'}$ are the G' invariants, whereas the Hom_G are the G-invariants; apply the remark following the definition of the cross product. The dashed arrow is simply the composite. Take homology to get

$$H^n(G',M) \xrightarrow{f^*} H^n(G,M).$$

This also shows us that in some sense cohomology is not the same as homology. For homology, you can think of it as a bifunctor, complete with induced maps. Cohomology is strange in that it's covariant in one variable and contravariant in the other variable.

Definition 94. We define a graded k-algebra structure on $\boxed{H^*(G,k)}$. The grading is given by $H^*(G,k) = \oplus_{n \geq 0} H^n(G,k)$. The cup product induces

$$\boxed{\smile}\colon H^*(G,k) \otimes H^*(G,k) \to H^*(G,k)$$

since this is

$$\smile\colon \bigoplus_{n \geq 0} \left(\bigoplus_{p+q=n} H^p(G,k) \otimes H^q(G,k) \right) \to \bigoplus_{n \geq 0} H^n(G,k).$$

95 Theorem

The cup product $\smile$ makes $H^*(G,k)$ a graded-commutative associative unital k-algebra. It is natural with respect to the maps of groups defined above.

96 Remark

The same is true with G replaced by any Hopf algebra A.

Here "graded-commutative" mean "commutative in the graded sense," so commutative up to a particular sign. Indeed, the 0th graded piece will be k, which gives the k-algebra structure. See next lecture.

May 2nd, 2014: The Yoneda Product; Multiplicative Spectral Sequences

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Definition 97. Let $A = \oplus_{n\geq 0} A_n$ be a graded algebra, so we have multiplication maps $A_m \otimes A_n \stackrel{\times}{\to} A_{n+m}$. A is graded-commutative if

$$xy = (-1)^{\deg x \deg y} yx.$$

In particular, for $H^*(G,k)$ defined last time has this property. Indeed, $H^{\text{even}}(G,k)$ (with the obvious definition) is commutative. Of course, if char $k = 2$, graded-commutative and commutative are the same.

98 Remark

What is involved in showing the associativity of the cup product? Recall that $G \stackrel{\Delta}{\to} G \times G$ induces a map $P_* \stackrel{\tilde{\Delta}}{\to} P_* \otimes P_*$, called a diagonal approximation; here $P_* \to G$ and $P_* \otimes P_* \to G \times G$ are projective resolutions, as above. Associativity boils down to the commutativity of the following diagram

$$\begin{array}{ccc} P_* & \xrightarrow{\tilde{\Delta}} & P_* \otimes P_* \\ {\scriptstyle \tilde{\Delta}}\downarrow & & \downarrow{\scriptstyle \tilde{\Delta}\otimes 1} \\ P_* \otimes P_* & \xrightarrow[1\otimes\tilde{\Delta}]{} & P_* \otimes P_* \otimes P_* \end{array}$$

Indeed, we only need it to commute "up to homotopy", that is, the two maps $P_* \to P_* \otimes P_* \otimes P_*$ are homotopic. It's possible to write down a particular diagonal approximation called the Alexander-Whitney diagonal approximation which makes this diagram commute on the nose; this comes from the bar resolution.

(Question: can we just say the two ways of going around that diagram are lifts of the map $G \to G \times G \times G$, which then must be homotopic by some standard uniqueness result?)

99 Remark

What is involved in showing the graded-commutativity of the cup product? It does not come from the level of cochains. Here's a sketch of one proof. $H^*(G,k)$ has another way to multiply things:

Definition 100 (Ext via Extensions). Recall we defined $H^n(G,k) := \text{Ext}^n_G(k,k)$. We can define $\text{Ext}^n_G(M,N)$ in a different way than the usual projective resolution method by looking at equivalence classes of length n extensions from N to M,

$$0 \longrightarrow N \longrightarrow X_1 \longrightarrow \cdots \longrightarrow X_n \longrightarrow M \longrightarrow 0$$

modulo some equivalence relation. The **Baer sum** gives an abelian group structure.

There is a book by D. Benson, "Representations and Cohomology". Great reference book, not a textbook. There's also a book by L. Evens, "Cohomology of Groups". Kenneth Brown's "Cohomology of Groups" is probably the standard reference.

Definition 101 (Yoneda Product). The Yoneda product is a map

$$\mathrm{Ext}^n_G(M,N) \times \mathrm{Ext}^m_G(L,M) \to \mathrm{Ext}^{n+m}_G(L,N)$$

defined using the extension interpretation above. We literally "splice" two long exact sequences:

$$\begin{aligned} 0 \to N \to \cdots_1 \to M \to 0 \qquad &\times \qquad 0 \to M \to \cdots_2 \to L \to 0 \\ = \quad 0 \to N \to \cdots_1 &\to \cdots_2 \to M \to 0 \end{aligned}$$

There are a number of checks to make, particularly that this is well-defined under the equivalence relation, and that it respects the group structure. Associativity is clear. Graded-commutativity in the case of $M = N = k$ is not at all clear.

102 Proposition (Eckmann-Hilton argument)

Let X be a set with two operations $*\colon X \times X \to X$ and $\cdot\colon X \times X \to X$ such that (1) they share a two-sided unit, and (2) they are compatible in the sense that

$$(a * b) \cdot (c * d) = (a \cdot c) * (b \cdot d).$$

Then the operations are the same and are commutative.

Proof Have fun.

Apply a slightly modified version of the Eckmann-Hilton argument to the cup product and the Yoneda product on $H^*(G,k)$ to get that they are the same, and are both graded-commutative.

Definition 103 (Multiplicative Structure of Spectral Sequences). Suppose E_r^{pq} is a cohomological spectral sequence. It is **multiplicative** if we have a map

$$\cdot\colon E_r^{pq} \times E_r^{p',q'} \to E_r^{p+p',q+q'}$$

such that $d_r(x \cdot y) = d_r(x) \cdot y + (-1)^p x \cdot d_r(y)$. (Here we consider $\deg x = p$ for $x \in E_r^{p,q}$.)

104 Remark

If we have such a multiplication on a single page, it induces multiplications on subsequence pages. Frequently the multiplicativity starts at the $r = 2$ page, for whatever reason.

105 Proposition

Suppse H is a normal subgroup of G. The Lyndon-Hochschild-Serre spectral sequence is multiplicative.

Proof Recall this sequence is

$$E_2^{pq} = H^p(G/H, H^q(H,k)) \Rightarrow H^{p+q}(G,k).$$

The multiplicative structure on the $r = 2$ page is given by

$$\begin{array}{c} H^p(G/H, H^q(H,k)) \otimes H^{p'}(G/H, H^{q'}(H,k)) \\ \big\downarrow \smile_{G/H} \\ H^{p+p'}(G/H, H^q(H,k) \boxtimes H^{q'}(H,k)) \\ \big\downarrow \text{“}\smile\text{”}_H \\ H^{p+p'}(G/H, H^{q+q'}(H,K)) \end{array}$$

With tedious work, one can check the differential condition.

(Whenever we have a "multiplication on k", say replacing k by a k-algebra, we have a similar statement.)

May 5th, 2014: Group Functors—Restriction, Induction, and Coinduction

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106 Remark

Friday will be a short lecture, from 12:30 to 1pm. Today: induction, coinduction, restriction, and other functors on G-modules. G will be a discrete group.

Definition 107. Let H be a subgroup of G (or more generally $i\colon H \hookrightarrow G$ an embedding). There is an obvious restriction functor

$$\mathrm{Res}^G_H\colon G\text{–mod} \to H\text{–mod},$$

which is really the pullback functor associated to $i\colon H \hookrightarrow G$. If M is a G-module, this is written $\boxed{M{\downarrow_H}} := \mathrm{Res}^G_H M$. This functor is covariant and exact. Wonderfully, it has a left *and* right adjoint. (For instance, it is thus exact.)

Definition 108. Let M be an H-module with H a subgroup of G. We can extend an H-module M to a G-module via extension of scalars,

$$\boxed{\mathrm{Ind}^G_H M} := kG \otimes_{kH} M.$$

This process is called induction. In detail, kG is a kG-kH bimodule, so we indeed get a kG-module structure on the tensor product. Sometimes this is called tensor induction; this is left adjoint to the restriction functor, as detailed below.

Similarly, we can define coinduction by

$$\boxed{\mathrm{Coind}^G_H M} := \mathrm{Hom}_{kH}(kG, M).$$

We give the right-hand side a G-module structure as follows: if $f \in \mathrm{Hom}_{kH}(kG, M)$, then $(g \cdot f)(-) := f(-g)$. Note that

$$g_1 \cdot (g_2 \cdot f)(-) = (g_2 \cdot f)(-g_1) = f(-g_1 g_2) = (g_1 g_2) \cdot f(-),$$

which explains the use of right multiplication. Moreover, this gives an H-module morphism:

$$(g \cdot f)(h-) = f(h-g) = h(f(-g)) = h((g \cdot f)(-)).$$

This operation is sometimes called hom induction; it is right adjoint to the restriction functor, as detailed below.

Warning: "induction" and "coinduction" mean different things to different people in different contexts (they frequently flip), so be careful! The "tensor"/"hom" alternatives are unambiguous.

109 Proposition (Adjunction in general)

Let R, S be rings, X an R, S-bimodule, M an R-module, N an S-module. Consider the functor $\mathrm{Hom}_R(X, -)\colon R\text{–mod} \to S\text{–mod}$. *Here the S-module structure is defined as for coinduction, which* *uses the action of S on X. Also consider the functor $X \otimes_S -: S$–mod $\to R$–mod. These functors form an adjunction (i.e. an adjoint pair):*

$$\mathrm{Hom}_R(X \otimes_S N, M) \cong \mathrm{Hom}_S(N, \mathrm{Hom}_R(X, M)),$$

where the bijection is functorial.

110 Proposition

There exist natural isomorphisms of k-vector spaces such that for any G-module N,

$$\mathrm{Hom}_G(kG \otimes_{kH} M, N) \cong \mathrm{Hom}_H(M, N{\downarrow_H}).$$

and similarly

$$\mathrm{Hom}_H(N{\downarrow_H}, M) \cong \mathrm{Hom}_G(N, \mathrm{Hom}_H(kG, M)).$$

Proof For the first, use $S = kH, R = kG$, and $X = kG$ in the notation of the previous proposition. Then we have

$$\mathrm{Hom}_{kG}(kG \otimes_{kH} N, M) \cong \mathrm{Hom}_{kH}(N, \mathrm{Hom}_{kG}(kG, M)).$$

However, $\mathrm{Hom}_{kG}(kG, M) \cong M{\downarrow_H}$ through the H-module map $f \mapsto f(1)$. (Note $h \cdot f \mapsto (h \cdot f)(1) = f(1) = hf(1)$.) For the second, let $S = kG$, $R = kH$, and $X = kG$. We have

$$\mathrm{Hom}_{kH}(kG \otimes_{kG} N, M) \cong \mathrm{Hom}_{kG}(N, \mathrm{Hom}_{kH}(kG, M)),$$

but $kG \otimes_{kG} N \cong N{\downarrow_H}$ as an H-module.

Let's write down the maps explicitly. We'll do the bijection

$$\mathrm{Hom}_{kG}(kG \otimes_{kH} M, N) \xrightarrow{\sim} \mathrm{Hom}_{kH}(M, N{\downarrow_H}).$$

Given a G-module morphism $f: kG \otimes_{kH} M \to N$, construct an H-module morphism $f \circ i: M \to N{\downarrow_H}$ where $i: M \to kG \otimes_{kH} M$ is the natural map $m \mapsto 1 \otimes m$. To go the other way, suppose $F: M \to N{\downarrow_H}$ is an H-module morphism. Construct $\tilde{F}: kG \otimes_{kH} M \to N{\downarrow_H}$ via $\tilde{F}(g \otimes m) = gf(m)$. (Of course, one must check these maps are well-defined, are mutual inverses, are functorial, etc., but it all works out and is really embedded in the proof of the previous proposition.)

May 7th, 2014: Frobenius Reciprocity; Coinduction and Induction Identities

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111 Theorem (Frobenius Reciprocity / Shapiro's Lemma / Eckmann-Shapiro Lemma)

Let H be a subgroup of G, M an H-module, N a G-module. Then

$$\mathrm{Ext}^*_G(N, \mathrm{Coind}_H^G M) \xrightarrow{\sim} \mathrm{Ext}^*_H(N{\downarrow_H}, M)$$
$$\mathrm{Ext}^*_G(\mathrm{Ind}_H^G M, N) \xrightarrow{\sim} \mathrm{Ext}^*_H(M, N{\downarrow_H})$$

(These are isomorphisms of k-vector spaces. Formally we view $\mathrm{Hom}_G(-,-)$ as mapping into k-vector spaces.)

Proof $\mathrm{Ind}_H^G(-) = kG \otimes_{kH} -$, and kG is a free kH-module, hence is flat, so $\mathrm{Ind}_H^G(-)$ is exact. (Indeed, $kG = \oplus t\, kH$ where the sum is over a set of coset representatives $t \in G/H$.) Similarly $\mathrm{Coind}_H^G(-) = \mathrm{Hom}_{kH}(kG, -)$ is exact since kG is free, hence projective.

Indeed, Coind_H^G takes injective kH-modules to injective kG-modules, which is a general property of right adjoint functors whose left adjoint is exact. (It's a good "two-line exercise".) Likewise, Ind_H^G takes projectives to projectives, which is a general property of left adjoint functors whose right adjoint is exact. We can apply the Grothendieck spectral sequence to the composition of functors $\mathrm{Hom}_G(N, \mathrm{Coind}_H^G(M)) \cong \mathrm{Hom}_H(N{\downarrow_H}, M)$, which gives

$$\mathrm{Ext}_G^i(N, R^j \,\mathrm{Coind}_H^G(M)) \Rightarrow \mathrm{Ext}_H^{i+j}(N{\downarrow_H}, M).$$

Since $\mathrm{Coind}_H^G(-)$ is exact, its right-derived functors are trivial for $j > 0$, so we have only one line and the spectral sequence collapses to give

$$\mathrm{Ext}_G^i(N, \mathrm{Coind}_H^G(M)) \cong \mathrm{Ext}_H^i(N{\downarrow_H}, M).$$

The proof for $\mathrm{Ind}_H^G(-)$ is almost the same:

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G M, N) \xrightarrow{\sim} \mathrm{Hom}_H(M, N{\downarrow_H}).$$

However, $\mathrm{Ind}_H^G(-)$ would naturally use left derived functors, whereas $\mathrm{Hom}_G(-, N)$ would naturally use right derived functors, so you have to be careful. Since $\mathrm{Ind}_H^G(-)$ is exact, you can also just use right derived functors.

112 Corollary

$H^*(G, \mathrm{Coind}_H^G M) \xrightarrow{\sim} H^*(H, M)$.

113 Theorem

If $[G : H] < \infty$, *then* $\mathrm{Coind}_H^G M \xrightarrow{\sim} \mathrm{Ind}_H^G M$, *and these isomorphisms are functorial.*

114 Remark

Let T be a fixed set of coset representatives for G/H. We find

$$\mathrm{Ind}_H^G M = kG \otimes_{kH} M = \oplus_T tkH \otimes_{kH} M = \oplus_T t \otimes_k M$$
$$\mathrm{Coind}_H^G M = \mathrm{Hom}_{kH}(\oplus t\; kH, M) = \oplus_T \mathrm{Hom}_k(kt, M) \cong \oplus_T tM.$$

(These isomorphisms are non-canonical, and we used the fact that finite products are finite sums. This isomorphism explain why we probably haven't heard of coinduction before—it gives no new content in this case.)

Proof Define $\phi\colon kG \otimes_{kH} M \to \mathrm{Hom}_{kH}(kG, M)$ as follows. For $\tilde{m} \in kG \otimes_{kH} M$, write $\tilde{m} = \sum_T t \otimes m_t$. Define $f_{\tilde{m}}\colon G \to M$ via $f_m(g) = gt^{-1}m_t$. Set $\phi(\tilde{m}) = f_{\tilde{m}}$. Define a map ψ in the opposite direction by sending $f\colon kG \to M$ to $\sum_T t \otimes f(t^{-1})$. We also need to check (1) these maps are independent of coset representatives; (2) $f_{\tilde{m}}$ is H-equivariant; (3) ψ, ϕ are G-equivariant; (4) ψ, ϕ are mutual inverses. (Note: Julia didn't check these maps carefully; it's a straightforward if tedious exercise to do so and correct any mistakes.)

115 Remark

$kG^{\#} := \mathrm{Hom}_k(kG, k)$ is a G-module. A special case of the previous isomorphism is that $kG^{\#} \xrightarrow{\sim} kG$ as G-modules when G is a finite group. This implies kG is a self-injective algebra (i.e. kG is an injective kG-module; contrast with e.g. $\mathbb{Z}$). Hence projective and injective modules are the same in G–mod (even infinite dimensional ones!). A category for which this holds is sometimes called a quasi-Frobenius category, which is related to Frobenius algebras.

116 Remark

Cohomology is only interesting in the modular case for finite G, since otherwise kG is semisimple, so k is projective, and cohomology is quite trivial. It does get interesting for $k = \mathbb{Z}/(p)$ or $k = \overline{\mathbb{F}_p}$, for instance.

117 Theorem (⊗-identity)

Let H be a subgroup of G. Then

$$\operatorname{Coind}_H^G(M \otimes_k \operatorname{Res}_H^G N) \cong \operatorname{Coind}_H^G(M) \otimes_k N.$$

Proof This will be on the next homework assignment.

118 Corollary

Let N be a G-module. Then $\operatorname{Coind}_H^G(N) \cong k(G/H) \otimes_k N$, where $k(G/H) = \operatorname{Coind}_H^G(k)$. (We can give a G-module structure to $k(G/H)$ even when H is not normal in an obvious way.)

119 Lemma

We have canonical maps

$$\begin{array}{cc} \operatorname{Coind}_H^G M \xrightarrow{e} M & M \xrightarrow{f} \operatorname{Coind}_H^G M \\ (h\colon G \to M) \mapsto h(1) & m \mapsto (f_m\colon g \mapsto gm) \end{array}$$

Similarly we have $\operatorname{Ind}_H^G M \to M$ via $g \otimes m \to gm$. The following diagram commutes:

$$\begin{array}{ccc} H^*(G,M) & \xrightarrow{\text{Res}^*} & H^*(H,M) \\ & {\scriptstyle f^*} \searrow & \downarrow {\scriptstyle \textit{Frob.}} \, \wr \\ & & H^*(G, \operatorname{Coind}_H^M M) \end{array}$$

Proof This will be on the next homework assignment.

May 9th, 2014: Corestriction, the Transfer Map, and $\operatorname{Cor}_H^G \operatorname{Res}_H^G = [G : H]$

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120 Remark

This lecture uses some notation Weibel does not use and which may not be terribly standard for corestriction.

Definition 121. Let H be a subgroup of G, M a G-module. Corestriction is a map

$$\operatorname{Cor}_i\colon H^i(H,M) \to H^i(G,M);$$

contrast this with the map induced by restriction, which goes the other way.

More precisely, suppose $[G : H] < \infty$. Consider

$$\begin{array}{ccc} H^i(H,M) & \xrightarrow[\text{Frob.}]{\sim} & H^i(G, \operatorname{Coind} M) \\ {\scriptstyle \operatorname{Cor}_i} \downarrow & & \downarrow {\scriptstyle [G:H]<\infty} \, S \\ H^i(G,M) & \xleftarrow[\operatorname{Ind} M \to M]{} & H^i(G, \operatorname{Ind} M) \end{array}$$

(where recall $\operatorname{Ind} M \to M$ is given by $g \otimes m \mapsto gm$).

Definition 122. Let H be a subgroup of G, M a G-module. Suppose $[G:H] < \infty$. The transfer map

$$\operatorname{Tr}_*\colon H^*(H,M) \to H^*(G,M)$$

is the map induced by $\operatorname{Tr}\colon M^H \to M^G$ where $m \mapsto \sum tm$ with the sum over a set of coset representatives of G/H.

123 Lemma

Cor_* and Tr_* from $H^*(H,M) \to H^*(G,M)$ are the same maps.

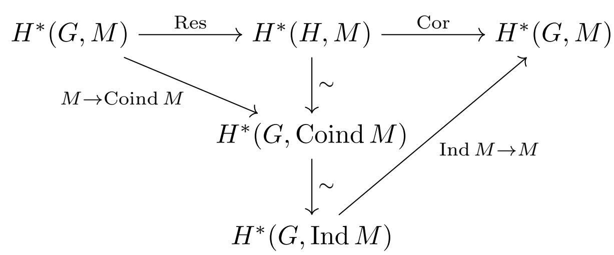
124 Proposition

The composite

$$\operatorname{Cor}_H^G \operatorname{Res}_H^G\colon H^*(G,M) \to H^*(H,M) \to H^*(G,M)$$

is the multiply-by-$[G:H]$-map.

Proof Write the composite as



We wish to compute $M \to \operatorname{Coind} M \overset{\sim}{\to} \operatorname{Ind} M \to M$ (and then apply the $H^*(G,-)$ functor). Say T is a fixed set of coset representatives of G/H. This composite is then

$$m \mapsto (f_m\colon g \mapsto gm) \mapsto \sum_T t \otimes f_m(t^{-1}) \mapsto \sum_T t f_m(t^{-1}) = \sum_T tt^{-1}m = [G:H]m.$$

(If the map before taking cohomology is multiplication by an integer, the same is true after taking cohomology.)

125 Corollary

Let G be a finite group of order n. Then $n \cdot H^i(G,M) = 0$ for any G-module M, i.e. cohomology is always $|G|$-torsion, for $i > 0$.

Proof Consider $\{e\} \le G$ and consider the composite

$$H^i(G,M) \overset{\operatorname{Res}}{\to} H^i(e,M) \overset{\operatorname{Cor}}{\to} H^i(G,M)$$

which from the above proposition is multiplication by n. However, $H^i(e,M) = 0$, so multiplication by n factors through 0, hence is 0.

126 Corollary

Let G be finite of order n, $\operatorname{Char} k = p \ge 0$. If $(n,p) = 1$ or $p = 0$, then $H^i(G,M) = 0$ for $i > 0$.

Proof By the previous corollary, multiplication by n kills $H^i(G,k)$. If they're relatively prime or $p = 0$, n is invertible in the underlying field, so $H^i(G,k) = 0$.

127 Corollary

Reduction to Sylow subgroups—class cut short today, see next lecture.

May 12th, 2014: Sylow Subgroups; the Double Coset Formula; Commuting $\mathrm{Res}, \mathrm{Ind}, \mathrm{Cor}$

Summary Let H be a subgroup of G of finite index. Recall we showed the composite

$$H^n(G,M) \overset{\mathrm{Res}}{\to} H^n(H,M) \overset{\mathrm{Cor}=\mathrm{Tr}}{\to} H^n(G,M)$$

is multiplication by $[G:H]$.

128 Corollary

Suppose $p \mid |G|$. Let G_p be a Sylow p-subgroup of G. Then

$$\mathrm{Res}_{(p)} \colon H^n(G,M)_{(p)} \hookrightarrow H^n(G_p, M),$$

where $H^n(G,M)_{(p)}$ means we localize the G-module $H^n(G,M)$ at the element $p \in kG$ (which may be zero).

Proof Apply the proposition to $H = G_p$ to get

$$H^n(G,M) \overset{\mathrm{Res}}{\to} H^n(G_p,M) \overset{\mathrm{Cor}}{\to} H^n(G,M),$$

where the composite is multiplication by $[G:G_p]$. Apply the localize-at-p functor, and the resulting composite is an isomorphism since $([G:G_p],p)=1$, so $[G:G_p]$ is invertible over $G_{(p)}$. Note that $H^n(G_p,M)_{(p)} = H^n(G_p,M)$. The claim follows.

129 Corollary

If $\mathrm{Char}\, k = p$ and $p \mid |G|$, then

$$\mathrm{Res} \colon H^n(G,M) \hookrightarrow H^n(G_p,M).$$

Proof $[G\colon G_p]$ is invertible in kG, so the original composite from the previous corollary is an isomorphism.

130 Remark

What is $\mathrm{im}(\mathrm{Res}\colon H^n(G,M) \to H^n(G_p,M))$? We'll first study the composite

$$\mathrm{Res} \circ \mathrm{Cor} \colon H^n(H,M) \overset{\mathrm{Tr}}{\to} H^n(G,M) \overset{\mathrm{Res}}{\to} H^n(H,M).$$

Definition 131 (**Double Cosets**). Let K, H be subgroups of G. The set $K\backslash G/H$ is $\{KxH : x \in G\}$.

Note that G/H consists of the sets xH for $x \in G$. Act by K on the left to split into further (double) cosets.

132 Example

$D_4 = C_4 \rtimes C_2 = \langle r, s : r^4 = s^2 = 1, sr = r^{-1}s\rangle$.

Let $K = H = C_2 = \langle s \rangle$. What is $C_2\backslash D_4/C_2$? $D_4 = C_2 \coprod rC_2 \coprod r^2C_2 \coprod r^3C_2$. Now act by s on the left to get $sC_2 = C_2$, $s(rC_2) = r^3sC_2 = r^3C_2$, and $s(r^2C_2) = r^2C_2$. Hence $D_4 = C_2 \coprod (rC_2 \coprod r^3C_2) \coprod r^2C_2$, or equivalently $C_2 \coprod C_2rC_2 \coprod C_2r^2C_2$. Note that the sizes of double cosets are not generally all the same.

133 Proposition (Double Coset Formula)

Let H, K be subgroups of G, and assume $H, K, H\cap K$ are finite index in G. Then

$$\mathrm{Res}^G_K \mathrm{Ind}^G_H M \cong \oplus_{x\in X} \mathrm{Ind}^K_{K\cap xHx^{-1}} \mathrm{Res}^{xHx^{-1}}_{K\cap xHx^{-1}} xM,$$

where the sum is over a fixed set X of double coset representatives of $K\backslash G/H$, and the isomorphism is as K-modules. For convenience, write ${}^xH := xHx^{-1}$. As a vector space, $xM \cong M$, though we view it as an ${}^xH$-module, since $xhx^{-1}\cdot(xm) = x(hm)$.

134 Example

Let $K = H$ be a normal subgroup of G. The double cosets HxH are just the left (or right) cosets since H is normal. The formula then reads

$$\operatorname{Res}_H^G \operatorname{Ind}_H^G \cong \oplus_{x \in G/H} xM.$$

135 Proposition

Let M be a G-module, K, H subgroups of G of finite index, and $z \in H^n(H, M{\downarrow_H})$. Consider

$$H^n(H,M) \xrightarrow[\text{Tr}]{\text{Cor}} H^n(G,M) \xrightarrow{\text{Res}} H^n(K,M)$$

This composite is given by

$$\operatorname{Res}_K^G \circ \operatorname{Cor}_H^G(z) = \sum_{x \in X} \operatorname{Cor}_{K \cap ({}^xH)}^K \operatorname{Res}_{K \cap ({}^xH)}^{({}^xH)} xz,$$

where the sum is over a fixed set of double coset representatives X for $K \backslash G/H$.

136 Remark

Here

$$z \in H^n(H,M) = \operatorname{Ext}_H^n(k,M) = H^n(\operatorname{Hom}_k(P_n,M)^H),$$

but what do we mean by $xz \in H^n({}^xH, xM)$? First pick $\tilde{z} \in \operatorname{Hom}_k(P_n,M)^H$ representing z; this is of course a map $P_n \to M$. Construct the map $x\tilde{z} \in \operatorname{Hom}_k(P_*, xM)^{{}^xH}$ by pointwise multiplication by x on the left, and define xz as the homology class of $x\tilde{z}$.

Minor note: we need the P_n's to be G-modules; we can take a projective resolution of G-modules, and that is a projective resolution of H-modules, and even ${}^xH$-modules, so this is fine.

Proof See next lecture.

May 14th, 2014: Dimension Shifting and Syzygies; $\operatorname{Res}_{G_p}^G$ surjects onto G-invariants of $H^n(G_p, M)$

Proof (of proposition from the very end of last lecture). We'll give two proofs. The first one uses the explicit transfer map. Let $P_* \to k$ be a projective resolution over G, which remains a projective resolution over H and K. We have

$$\operatorname{Hom}_k(P_n,M)^H \xrightarrow{\text{Tr}} \operatorname{Hom}_k(P_n,M)^G \hookrightarrow \operatorname{Hom}_k(P_n,M)^K$$

where Tr acts by sending f to $\sum_{t \in \tau} tf$ with τ a fixed set of coset representatives of G/H. Roughly, we wish to break up the sum over τ into a sum over double coset representatives KxH, but this is not a bijection, so in addition to double cosets, there is an inner sum over cosets of $K/K \cap ({}^xH)$. More formally, we have the following:

137 Proposition

Let K, H be subgroups of G. There exist sets τ, χ such that:

(i) τ is a complete set of distinct coset representatives for G/H;

(ii) *χ is a complete set of distinct double coset representatives for $K\backslash G/H$;*

(iii) *for each $x \in \chi$, there is a set ϵ_x which is a complete set of distinct coset representatives for $K/(K \cap {}^xH)$;*

(iv) *there is a bijection between $t \in \tau$ and pairs $(x \in \chi, e \in \epsilon_x)$, where $t = ex$ and $tH \subset KxH$.*

Hence

$$\sum_{t\in\tau} tf = \sum_{x\in\chi}\left(\sum_{e\in\epsilon_x} exf\right)$$

We can think of $xf \in \mathrm{Hom}(P_n, M)^{{}^xH}$. It follows that

$$\sum_{t\in T} tf = \sum_{x\in X} \mathrm{Cor}^{K}_{K\cap({}^xH)} \mathrm{Res}^{{}^xH}_{K\cap({}^xH)} xf.$$

138 Remark

M is a G-module, so fix $g \in G$ and consider the map $M \xrightarrow{g\cdot} M$ given by $m \mapsto gm$. The map $g\cdot$ is not in general H-equivariant, but it's close. Note that $hm \mapsto ghm = ghg^{-1}gm$. Hence if we give the second M an H-action of $h \cdot m = ghg^{-1}m$, $g\cdot$ is in fact H-equivariant.

The second proof uses "dimension shifting".

Definition 139. Consider $\mathrm{Ext}^*_G(N, M)$. Let $P_* \to N$ be a minimal resolution (meaning every other resolution factors through this one). Let $\boxed{\Omega N} := \ker(P_0 \twoheadrightarrow N)$ be a $\boxed{\text{syzygy}}$ (sometimes called $\boxed{\text{Heller shift}}$). More generally,

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & & & P_1 & \longrightarrow & & & P_0 & \longrightarrow & N \\ & \searrow & & \nearrow & & \searrow & & \nearrow & & & \\ & & \Omega^2 N & & & & \Omega^1 N & & & & \end{array}$$

where $\boxed{\Omega^n N} := \ker(P_{n-1} \to P_{n-2})$, $\Omega^{-1}N := \mathrm{coker}(N \hookrightarrow I(N))$.

140 Lemma

Aside: if we construct syzygies out of arbitrary projective resolution, they will differ only by a projective summand.

141 Lemma

Consider $\mathrm{Ext}^n_G(N, M)$ *for* $n \geq 1$. *Then*

$$\mathrm{Ext}^n_G(N, M) \cong \mathrm{Hom}_G(\Omega^n N, M).$$

Proof Homework. There is a dual version where we use negative syzygies and replace M rather than N.

For the proof of the double coset formula for cohomology, it suffices to consider

$$\begin{array}{ccccc} \mathrm{Hom}_H(N, M) & \xrightarrow{\mathrm{Cor}} & \mathrm{Hom}_G(N, M) & \xhookrightarrow{\mathrm{Res}} & \mathrm{Hom}_k(N, M) \\ \downarrow \wr & & & \nearrow_{h\otimes m \mapsto hm} & \\ \mathrm{Hom}_G(N, \mathrm{Ind}\, M) & \hookrightarrow & \mathrm{Hom}_k(N, \mathrm{Ind}\, M) & & \end{array}$$

where in the diagonal arrow we use the double coset formula for modules. This is essentially the proof.

Definition 142. Let H be a subgroup of G, M a G-module. Fix $g \in G$ and define the top arrow in the following (not necessarily commutative!) diagram as above:

$$\begin{array}{ccc} H^n(H, M) & \xrightarrow{\quad g\cdot \quad} & H^n({}^gH, gM) \\ & {\scriptstyle\text{Res}}\searrow \quad \swarrow{\scriptstyle\text{Res}} & \\ & H^n(H \cap ({}^gH), M) & \end{array}$$

We say that $z \in H^n(H, M)$ is $\boxed{G\text{-invariant}}$ if this diagram commutes when we run z through it, for all $g \in G$.

If H is a normal subgroup of G, then this is the "usual" invariance under the action of G on $H^n(H, M)$.

143 Proposition

Suppose G is a fnite group, G_p is a Sylow p-subgroup, and $\operatorname{Char} k = p$. *Then*

$$\operatorname{Res}^G_{G_p} : H^n(G, M) \hookrightarrow H^n(G_p, M)$$

surjects onto G-invariants of $H^n(G_p, M)$.

Proof Pick $z \in H^n(G_p, M)^G$. We want to show z is in the image of $\operatorname{Res}^G_{G_p}$. Let $\zeta = \operatorname{Cor}^G_{G_p} z \in H^n(G, M)$. We compute

$$\begin{aligned} \operatorname{Res}^G_{G_p} \zeta = \operatorname{Res}^G_{G_p} \operatorname{Cor}^G_{G_p} z &= \sum_{x \in G_p \backslash G / G_p} \operatorname{Cor}^{G_p}_{G_p \cap {}^xG_p} \left[\operatorname{Res}^{{}^xG_p}_{G_p \cap {}^xG_p} xz\right] \\ &= \sum_{x \in G_p \backslash G / G_p} \operatorname{Cor}^{G_p}_{G_p \cap {}^xG_p} \left[\operatorname{Res}^{G_p}_{G_p \cap {}^xG_p} z\right] = \sum_{x \in G_p \backslash G / G_p} [G_p : (G_p \cap {}^xG_p)] z \\ &= [G : G_p] z \in \operatorname{im} \operatorname{Res}^G_{G_p}. \end{aligned}$$

But $[G : G_p]$ is invertible since $(p, [G : G_p]) = 1$, so $z \in \operatorname{im} \operatorname{Res}^G_{G_p}$. Next time we'll show G acts trivially on its own cohomology, which will say that the image is no more than the G-invariants, completing the proposition.

May 16th, 2014: Conjugation of G on $H^*(G, M)$ is Trivial; Center Kills Argument; Algebraic Groups

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144 Remark

Last time we asserted $\operatorname{Res} : H^*(G, M) \hookrightarrow H^*(G_p, M)$ surjects onto the G-invariants $H^*(G_p, M)^G$. We showed the image contains the G-invariants, and will give the reverse inclusion today. Indeed, it is a corollary of the next proposition:

145 Proposition

The action (by conjugation) of G on $H^*(G, M)$ is trivial.

146 Remark

This means the following. Let H be a subgroup of G and fix $g \in G$. Consider the map $H \to {}^gH$ given by $\alpha_g : h \mapsto ghg^{-1}$ for some $g \in G$. Let ${}^gM$ be just M as a set, but give it an H-action through α_g, namely $h \cdot m = ghg^{-1}m$. Then the map $f_g \colon M \to {}^gM$ given by $m \mapsto gm$ is H-equivariant:

$$hm \mapsto ghm = (ghg^{-1})(gm) = h \cdot (gm).$$

We get

$$(\alpha_g^*, f_{g*}) \colon H^*({}^gH, M) \xrightarrow{f_{g*}} H^*({}^gH, {}^gM) \xrightarrow{\alpha_g^*} H^*(H, M).$$

For $H = G$, we get a map $H^*(G, M) \to H^*(G, M)$. After all this trouble, we'll show this map is trivial.

Proof How do we actually compute the induced map on cohomology $H^*({}^gH, M) \to H^*(H, M)$? We can interpret this as $\mathrm{Ext}^*_{{}^gH}(k, M) \to \mathrm{Ext}^*_H(k, M)$ and compute these with a fixed projective resolution. Let $P_* \to k$ be a projective resolution of G-modules. Before taking homology, we want $\mathrm{Hom}_{{}^gH}(P_*, M) \to \mathrm{Hom}_H(P_*, M)$. Suppose $f \colon P_* \to M$ as ${}^gH$-modules. Construct a map of H-modules as the composite

$$P_* \xrightarrow{g\cdot} P_* \xrightarrow{f} M \xrightarrow{g^{-1}\cdot} M.$$

For $H = G$, this map is $- \mapsto g- \mapsto f(g-) = gf(-) \mapsto g^{-1}gf(-) = f(-)$. Hence the map is the identity map on the level of complexes before we take homology, so the induced map is certainly the identity.

147 Corollary
$\mathrm{im}(\mathrm{Res}^G_{G_p} \colon H^*(G, M) \to H^*(G_p, M)) = H^*(G_p, M)^G$.

148 Remark
If G_p is normal, then this is $H^*(G_p, M)^{G/G_p}$, using "honest invariants".

149 Proposition (Center kills argument "Center kills" argument)
$H^*(\mathrm{GL}_n(k), k^n) = 0$, so long as k is any field different from $\mathbb{F}_2$. (Here k^n is a $\mathrm{GL}_n(k)$-module using the usual left multiplication by a matrix of a column vector.)

Proof The general idea is the following. Given a group G acting on M, the map $g \colon M \to M$ given by $m \mapsto gm$ is not a G-map in general, though it *is* if $g \in Z(G)$ is in the center of G. As above, g induces an action $(\alpha_g^*, f_{g*}) \colon H^*(G, M) \to H^*(G, M)$ which is the trivial (identity) map. Since g is central, $\alpha_g \colon G \to G$ given by $x \mapsto gxg^{-1}$ is the identity. So, the "conjugation" action by g on $H^*(G, M)$ is induced just by $f_g \colon M \xrightarrow{g\cdot} M$:

$$(\alpha_g^*, f_{g*}) = (\mathrm{id}, f_{g*}) = f_{g*} \colon H^*(G, M) \xrightarrow{\mathrm{id}} H^*(G, M).$$

Hence let $\lambda I_n \in Z(\mathrm{GL}_n(k))$. The induced map $k^n \to k^n$ is just multiplication by λ. Exercise: the induced map on cohomology $H^*(\mathrm{GL}_n(k), k^n)$ is multiplication by the same scalar,

$$H^i(\mathrm{GL}_n(k), k^n) \xrightarrow{\lambda\cdot} H^i(\mathrm{GL}_n(k), k^n).$$

Take $\lambda \neq 1, 0$ to see that $H^i(\mathrm{GL}_n(k), k^n)$ is annihilated by $\lambda - 1$, so $H^i(\mathrm{GL}_n(k), k^n) = 0$.

150 Remark
No lecture next week or the following Monday. Student lectures for the following few sessions; we'll see if we want to meet the last week of classes.

Definition 151. An algebraic group is roughly a variety (of finite type) with a group structure given by regular functions. An alternate (more general) definition is the following. Fix a field k. An algebraic group G is a representable functor from k-algebras to groups. That is, there is a k-algebra we'll call $k[G]$ such that $G(R) = \mathrm{Hom}_{k\text{-alg}}(k[G], R)$.

For an algebraic group, $k[G]$ is a Hopf algebra. The group multiplication from $G(R)$ gives the comultiplication on $k[G]$ and the inverse gives the antipode. A $k[G]$-comodule M is defined using an action map $M \to M \otimes_k k[G]$. The category of comodules has enough injectives (but not in general projectives), so you can apply much of homological algebra. The tensor identity still holds (you can tensor over k) using the comultiplication of $k[G]$.

(Minor note: the tensor identity is in Jantzen's "Representations of Algebraic Groups," however he uses a scheme-theoretic generalization and representable functors, so there is a fair amount of translation required.)

May 28th, 2014: Fibrations and the Leray-Serre Spectral Sequence

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Summary James and Nick are going to discuss the Leray-Serre spectral sequence for computing cohomology rings.

152 Remark
Nick is up first. He'll briefly introduce fibrations.

153 Remark
Let $f\colon X \to Y$ be a continuous map. Roughly, we try to relate the cohomology of X to the cohomology of Y. For singular cohomology, we have the Leray-Serre spectral sequence, which in particular applies to "Serre fibrations." For sheaf cohomology, we have the Leray spectral sequence, which applies when f is a proper map.

Definition 154. A Serre fibration is a continuous map $E \to B$ which has the homotopy lifting property for all closed unit balls $e_n \in \mathbb{R}^n$ (equivalently, all CW-complexes). See beginning of next lecture for a rigorous definition.

155 Example
First main example: path fibrations. Let (B, b) be a pointed space. Give the set E of continuous maps $[0, 1] \to Y$ sending 0 to b the compact-open topology, and let $\pi\colon E \to B$ be evaluation at the end point. π is a Serre fibration.

156 Example
Second main example: fiber bundles. Let $\pi\colon E \to B$ be a surjective continuous map which is "locally trivial", meaning each point in B has a neighborhood N such that projection from $\pi^{-1}(N) \cong N \times F$ to N is "equivalent" to E.

157 Example
Let G be a Lie group, H a compact subgroup. Then $G \to G/H$ can be seen as a morphism of smooth manifolds, and in particular a fiber bundle with fiber H.

158 Remark
James is up next. His main reference is Spanier, and he'll discuss some high points for a proof of the Serre spectral sequence.

Definition 159. A fibration $\rho\colon E \to B$ is orientable if for loops $\gamma\colon [0, 1] \to B$, the induced map in homology $h[\gamma]_*\colon H_*(\rho^{-1}(\gamma(0))) \to H_*(\rho^{-1}(\gamma(1)))$ is the identity.

160 Example
If the base B of the fibration is simply-connected, then the fibration is orientable. In the absence of this or a similar assumption, the following spectral sequence becomes significantly more complicated.

161 Theorem (Leray-Serre Spectral Sequence)

If $\rho\colon E \to B$ is an orientable fibration, there is a first quadrant cohomological spectral sequence converging to $H^*(E)$ with

$$E_2^{p,q} = H^p(B, H^q(F)) \Rightarrow H^*(E).$$

(The underlying coefficient ring R is understood throughout. For instance, we can use $R = \mathbb{Z}$.)

162 Proposition

Some nice properties:

1) Each page has a ring structure,

$$h\colon E_r^{p,q} \otimes E_r^{p',q'} \to E_r^{p+p',q+q'}$$

2) The ring structure on the E_2 page is induced by the cup product, and the same holds on the E_∞ page. (Moreover, the ring structure on the $(r+1)$st page is induced by the ring structure on the rth page.)

3) Each d_r is a (graded-commutative) derivation.

4) For each r, $d_r\colon E_r^{0,r-1} \to E_r^{r,0}$ (going from the first column to the first row) is called the *transgression*. *It can be constructed from the long exact sequences of the pairs (E, F) and (B, b_0) where b_0 is some fixed base point.*

163 Remark

Outline of proof of theorem: let $\rho\colon E \to B$ be a fibration and assume B is a simply-connected CW complex. The simply-connected assumption is so we can assume the fibration is orientable; the CW complex assumption is theoretically harmless by CW approximation results.

- Step 1: filter E by setting $E_s = \rho^{-1}(B^s)$, where B^s is the s-skeleton of the CW complex B. This induces a filtration on singular cochains $S^*(E)$: set $F^s(S^*(E)) = \ker(S^*(E) \to S^*(E_{s-1}))$. The filtration on cohomology is given by $F^s(H^*(E)) = \ker(H^*(E) \to H^*(E_{s-1}))$. Note $F^s(S^*(E)) \supset F^{s+1}(S^*(E))$ since

$$\begin{array}{ccc} H^*(E) & & \\ \downarrow & \searrow & \\ H^*(E_{s+1}) & \longrightarrow & H^*(E_s) \end{array}$$

- Step 2:

  164 Proposition

  There is an E_1 spectral sequence converging to the cohomology $H^*(E)$ with $E_1^{s,t} = H^{s+t}(E_s, E_{s-1})$ (using the relative singular cohomology here), with d_1 corresponding to the boundary map associated to the triple (E_s, E_{s-1}, E_{s-2}).

  Proof We know there is an E_1 spectral sequence with $E_1^{s,t}$ given by

  $$H^{s+t}(F^s(S^*E)/F^{s+1}(S^*E)).$$

  The result follows since

  $$F^s(S^*E)/F^{s+1}(S^*E) = \mathrm{Hom}(S(E_s)/S(E_{s-1}); R).$$

- Step 3: relate the E_1 page with the singular cochain complex of B. We hope we have isomorphisms given by the dashed lines below:

$$\begin{array}{ccc} H^{s+t}(E_s, E_{s-1}) & \overset{\psi}{\dashrightarrow} & H^s(B^s, B^{s-1}; H^tF) \\ \downarrow d_1 & & \downarrow \delta \\ H^{s+t+1}(E_{s+1}, E_s) & \overset{\psi}{\dashrightarrow} & H^{s+1}(B^{s+1}, B^s; H^tF) \end{array}$$

where δ corresponds to the boundary map of the triple (B^{s+1}, B^s, B^{s-1}). Taking cohomology of the right-hand column is precisely how we compute the cellular cohomology of B, $H^s_{\text{cellular}}(B; H^t(F))$, but cellular and singular cohomology agree here.

165 Lemma

Let Δ^s be an s-simplex, $\partial\Delta^s$ its boundary. Let $E \to B$ be an orientable fibration. Given a map $\sigma\colon (\Delta^s, \partial\Delta^s) \to (B^s, B^{s-1})$, we get a lifting $\widetilde{\sigma}\colon (\Delta^s, \partial\Delta^s) \times F \to (E^s, E^{s-1})$ such that the induced map on cohomology

$$\widetilde{\sigma}^*\colon H^*(E^s, E^{s-1}) \to H^*((\Delta^s, \partial\Delta^s) \times F)$$

depends only on σ.

166 Lemma

If ζ^* is a fixed generator of $H^s(\Delta^s, \partial\Delta^s)$, there is an isomorphism

$$H^s(F) \xrightarrow{\sim} H^{q+s}(\Delta^s, \partial\Delta^s) \times F$$

given by $v \mapsto \zeta^* \times v$.

The ψ above is characterized by the equation

$$\zeta^* \times \langle \psi(u), \sigma \rangle = \widetilde{\sigma}^*(u),$$

using the Kronecker pairing.

May 30th, 2014: Using the Serre Spectral Sequence: $H^*(\mathbb{C}P^\infty; R) \cong R[x]$ and the Gysin Sequence

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167 Remark

Josh is presenting today; his lecture notes are below; he'll finish on Monday.

Summary The Serre spectral sequence effectively computes cohomology rings for numerous classical spaces. As a sample application, we compute $H^*(\mathbb{C}P^\infty; R) \cong R[x]$ where $\deg x = 2$ and we prove the Gysin sequence.

The main reference is McCleary's "A User's Guide to Spectral Sequences."

168 Notation

R will refer to a commutative unital ring.

If X is a topological space, $H^*(X; R)$ denotes the singular cohomology of X with coefficients in R, which is a graded, graded-commutative R-algebra using the cup product for multiplication.

Definition 169. A map $\pi\colon E \to B$ of topological spaces has the homotopy lifting property with respect to a space Y if, given any homotopy $G\colon Y \times I \to B$ and an "initial lift" $\ell\colon Y \to E$ (meaning $\pi\ell\colon Y \to B$ is $G(-, 0)\colon Y \to B$), there is a "full lift" $\tilde{G}\colon Y \times I \to E$ (meaning $\pi\tilde{G} = G$) starting at ℓ (meaning $\tilde{G}(-, 0) = \ell$).

Definition 170. A map $\pi\colon E \to B$ with the homotopy lifting property with respect to all spaces is a $\boxed{\text{Hurewicz fibration}}$ or just a $\boxed{\text{fibration}}$. If it only has the property with respect to closed unit spheres in $\mathbb{R}^n$ (equivalently, finite CW complexes) it is a $\boxed{\text{Serre fibration}}$.

E is called the $\boxed{\text{total space}}$ and B is called the $\boxed{\text{base space}}$.

171 Remark

Suppose $\pi\colon E \to B$ is a fibration. Let $F_b := \pi^{-1}(b)$ for $b \in B$. If B is path-connected, each F_b has the same homotopy type (eg. $H^*(F_b; R)$ is constant up to isomorphism). In this case, we write $\boxed{F \hookrightarrow E \xrightarrow{p} B}$ and call F the $\boxed{\text{fiber}}$, without having any particular F_b in mind.

172 Theorem (Cohomological Serre Spectral Sequence)

Suppose we have a fibration $F \hookrightarrow E \xrightarrow{\pi} B$ where B is path-connected and F is connected. Further suppose B is simply-connected. Then there exists a first quadrant spectral sequence of algebras with

$$E_2^{p,q} \cong H^p(B; H^q(F; R)) \Rightarrow H^*(E; R).$$

Indeed, we have a multiplicative structure on $E_2^{*,*}$:

$$\begin{aligned} E_2^{p,q} \otimes E_2^{p',q'} &\to E_2^{p+p',q+q'} \\ u \otimes v &\mapsto u \cdot_2 v = (-1)^{pq'} u \smile v \end{aligned}$$

where $\smile$ is given as before by

$$\begin{array}{c} H^p(B; H^q(F; R)) \otimes H^{p'}(B; H^{q'}(F; R)) \\ \downarrow \smile_B \\ H^{p+p'}(B; H^q(F; R) \otimes H^{q'}(F; R)) \\ \downarrow \smile_F \\ H^{p+p'}(B; H^{q+q'}(F; R)). \end{array}$$

173 Remark

The convergence is as algebras, which roughly means that each page of the spectral sequence has a differential bigraded algebra structure which induces the next pages' structure, and the E_∞ page is isomorphic to the induced algebra of the associated graded object of $H^*(E; R)$. See McCleary for details.

174 Proposition

$E_2^{*,0} \cong H^*(B; R)$ and $E_2^{0,*} \cong H^*(F; R)$ as algebras, using the product structure on $E_2^{*,*}$ on the left and the cup product structures on the right.

175 Example

James will briefly compute $H^*(\mathbb{C}P^\infty; R) = R[x]$ where $\deg x = 2$. Here $\mathbb{C}P^\infty$ is the colimit (union) of the natural inclusions $\mathbb{C}P^n \hookrightarrow \mathbb{C}P^{n+1}$. He begins with a fibration

$$\mathbb{C}^\times \to \mathbb{C}^\infty - \{0\} \to \mathbb{C}P^\infty.$$

(It happens that $\mathbb{C}P^\infty$ is $K(\mathbb{Z}, 2)$, an Eilenberg-MacLane space. Hence we can also use the path-loop fibration

$$\Omega K(\mathbb{Z}, 2) \cong S^1 \to PK(\mathbb{Z}, 2) \to K(\mathbb{Z}, 2),$$

which is really the same as the previous fibration, and allows us to easily see $\mathbb{C}^\infty - \{0\}$ is contractible.)

The E_2 page of the Serre spectral sequence is quite nice since $H^*(S^1;R)$ is so simple. We have only two non-zero rows, both consisting of $H^p(\mathbb{C}P^\infty;R)$, for $q=0,1$. The sequence collapses after the E_2 page. Also, the E_∞ page is empty since our total space is contractible, except for the $(0,0)$ term, from which we can deduce all the differentials from $q=1$ to $q=0$ on the E_2 page are isomorphisms, and $E_2^{1,0}=0$. The suggested graded structure follows. The ring structure also agrees, using the same type of argument as in the Gysin sequence below. Back to Josh.

176 Theorem (The Gysin Sequence)

Suppose $F \hookrightarrow E \xrightarrow{\pi} B$ is a fibration, where B is path-connected and simply connected. If F is a homology n-sphere *for $n \geq 1$, then there is an exact sequence*

$$\cdots \to H^k(B;R) \xrightarrow{\gamma} H^{n+1+k}(B;R) \xrightarrow{\pi^*} H^{n+1+k}(E;R) \to H^{k+1}(B;R) \to \cdots,$$

where indeed $\gamma(-) = z \smile -$ for some $z \in H^{n+1}(B;R)$. Moreover, if n is even, then in fact $2z=0$.

177 Remark

z in the theorem is called an Euler class. If we have a sphere bundle $S^n \hookrightarrow E \xrightarrow{\pi} B$ and happen to know $H^{n+1}(B;R)$ has trivial 2-torsion (for $n \geq 1$ even), then $z=0$ so $\gamma=0$. This severely restricts the possible sphere bundles of spheres over spheres; indeed, the four Hopf fibrations corresponding to the division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ give the only possible dimensions of the various spheres involved.

Proof By definition, a homology n-sphere is a space X where $H_k(X;\mathbb{Z})$ is 0, except we get one copy of $\mathbb{Z}$ for $k=n$ and one more for $k=0$. (Hence $H_*(X;\mathbb{Z}) = H_*(S^n;\mathbb{Z})$). It follows that $H^k(X;R)$ is 0, except we get one copy of R for $k=n$ and for $k=0$. (Actually, this step is unclear: does it really follow, or do we need to assume F is a "cohomology n-sphere with R coefficients"? The theorem statement is McCleary's.) The Serre spectral sequence associated to this fibration is thus mostly zero, since $H^k(F;R)=0$ unless $k=0,n$. Indeed, the E_2 page is in part

$$\begin{array}{cccc}
q=n: & H^0(B;R) & H^1(B;R) & H^2(B;R) \\
 & \vdots & \vdots & \vdots \\
q=0: & H^0(B;R) & H^1(B;R) & H^2(B;R)
\end{array}$$

It follows that $E_2 \cong \cdots \cong E_{n+1}$ and $H(E_{n+1}, d_{n+1}) \cong E_{n+2} \cong \cdots \cong E_\infty$. That is, the E_∞ page is

$$\begin{array}{cccc}
q=n: & \begin{array}{c}\ker d_{n+1} \\ \cong H^n/F^1H^n\end{array} & \begin{array}{c}\ker d_{n+1} \\ \cong F^1H^{n+1}/F^2H^{n+1}\end{array} & \begin{array}{c}\ker d_{n+1} \\ \cong F^2H^{n+2}/F^3H^{n+2}\end{array} \\
 & \vdots & \vdots & \vdots \\
q=0: & \begin{array}{c}H^0(B;R)/\operatorname{im} d_{n+1} \\ \cong H^0\end{array} & \begin{array}{c}H^1(B;R)/\operatorname{im} d_{n+1} \\ \cong F^1H^1\end{array} & \begin{array}{c}H^2(B;R)/\operatorname{im} d_{n+1} \\ \cong F^2H^2\end{array}
\end{array}$$

Note that for $k \geq 0$,

$$0 \to E_\infty^{k,n} \to E_2^{k,n} \overset{d_{n+1}^{k,n}}{\to} E_2^{n+1+k,0} \to E_\infty^{n+1+k,0} \to 0$$

is an exact sequence, since

$$0 \to \ker d_{n+1}^{k,n} \hookrightarrow E_2^{k,n} \to E_2^{n+1+k,0} \twoheadrightarrow E_2^{n+1+k,0} / \operatorname{im} d_{n+1} \to 0.$$

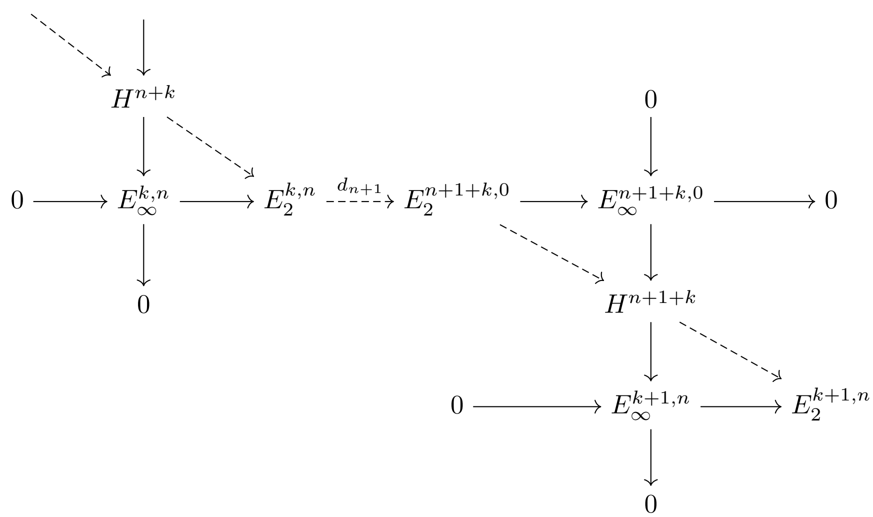
Moreover, looking at a fixed antidiagonal of degree $n+k$ for $k \geq 0$, there are two places where the containments of the filtration F^*H^{n+k} might be proper, namely at $F^kH^{k+n} \supset F^{k+1}H^{k+n}$ and at $F^{k+n}H^{k+n} \supset 0$. Thus

$$0 \to E_\infty^{n+k,0} \to H^{n+k} \to E_\infty^{k,n} \to 0$$

is an exact sequence, since

$$0 \to F^{k+n}H^{k+n} \hookrightarrow F^kH^{k+n} \twoheadrightarrow F^kH^{k+n}/F^{k+1}H^{k+n} \to 0.$$

Splice these two sequences together via



to get the long exact sequence

$$\cdots \to H^{n+k} \to E_2^{k,n} \overset{d_{n+1}}{\to} E_2^{n+1+k,0} \to H^{n+1+k} \to \cdots$$

which in our case is

$$\cdots \to H^{n+k}(E;R) \to H^k(B;R) \overset{d_{n+1}}{\to} H^{n+1+k}(B;R) \to H^{n+1+k}(E;R) \to \cdots .$$

To describe d_{n+1}, let $h \in H^n(F;R) \cong R$ be a generator. Identify $E_2^{*,n} = H^*(B;R) \otimes h$ and $E_2^{*,0} = H^*(B;R) \otimes 1$ (formally, use the Universal Coefficient Theorem to get $H^n(B;H^n(F;R)) \cong H^n(B;R) \otimes H^n(F;R)$, which works here since $H^n(F;R) \cong R$). Hence (for instance)

$$\smile : E_2^{0,n} \otimes E_2^{k,0} \to E_2^{k,n}$$
$$(1 \otimes h) \smile (x \otimes 1) = (-1)^{n \deg x}(x \otimes h).$$

d_{n+1} is a differential in the multiplicative sense, i.e. it satisfies a Leibniz rule (up to a sign) with the multiplication $\cdot$ on $E_2^{*,*}$. Since $\smile$ is just $\cdot$ up to a sign, d_{n+1} satisfies a Leibniz rule with respect to $\smile$. Letting $d_{n+1}(1 \otimes h) = z \otimes 1$ for some $z \in H^{n+1}(B; R)$ and putting it all together, we now compute

$$\begin{aligned}(-1)^{n \deg x} d_{n+1}(x \otimes h) &= d_{n+1}((1 \otimes h) \smile (x \otimes 1)) \\ &= [d_{n+1}(1 \otimes h)] \smile (x \otimes 1) + (-1)^n (1 \otimes h) \smile [d_{n+1}(x \otimes 1)] \\ &= (z \otimes 1) \smile (x \otimes 1) + 0 \\ &= (z \smile x) \otimes 1.\end{aligned}$$

(Here $d_{n+1}(x \otimes 1) = 0$ since it lands below the x-axis.) Let $\gamma(x) = (-1)^{n \deg x} d_{n+1}(x \otimes h)$ to get the map in the theorem statement.

Finally, if n is even, since $h \smile h \in H^{2n}(F; R) = 0$, we have

$$\begin{aligned}0 &= d_{n+1}(1 \otimes (h \smile h)) \\ &= d_{n+1}((1 \otimes h) \smile (1 \otimes h)) \\ &= (z \otimes 1) \smile (1 \otimes h) + (-1)^n (1 \otimes h) \smile (z \otimes 1) \\ &= (2z) \otimes h.\end{aligned}$$

Hence $2z = 0$. (If $2 = 0$ in R, this is trivial.)

June 2nd, 2014: $H^*(\mathbb{C}P^n; R) \cong R[x]/(x^{n+1})$; Cohomology of Homogeneous Spaces: The Flag Manifold and Grassmannians

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Summary Josh is finishing today. He'll use the Gysin sequence from last time to compute $H^*(\mathbb{C}P^n; R) \cong R[x]/(x^{n+1})$ ($\deg x = 2$) explicitly. He'll also summarize numerous related computations taken from McCleary. He'll finish with a discussion of cohomology of homogeneous spaces, with the flag manifold as a running example, and will end with some remarks on the cohomology of Grassmannians.

178 Example ($H^*(\mathbb{C}P^n; R)$)
Claim: $H^*(\mathbb{C}P^n; R) \cong R[x]/(x^{n+1})$ with $\deg x = 2$, for $n \geq 0$.

Proof The $n = 0$ case is trivial. Since $\mathbb{C}P^1 \cong S^2$ via stereographic projection, the graded structure of $H^*(S^2; R)$ forces the ring structure to be trivial. So, take $n > 1$.

The quotient $\mathbb{C}^{n+1} - \{0\} \twoheadrightarrow (\mathbb{C}^{n+1} - \{0\})/\!\sim\, := \mathbb{C}P^n$ can be interpreted as

$$S^1 \hookrightarrow S^{2n+1} \to \mathbb{C}P^n,$$

a fibration (indeed, a fiber bundle). $\mathbb{C}P^n$ is in general simply-connected (and path-connected), so the Gysin sequence applies. It starts with

$$\begin{aligned}0 \to H^0(\mathbb{C}P^n; R) &\xrightarrow{\gamma} H^2(\mathbb{C}P^n; R) \to H^2(S^{2n+1}; R) \\ \to H^1(\mathbb{C}P^n; R) &\xrightarrow{\gamma} H^3(\mathbb{C}P^n; R) \to H^3(S^{2n+1}; R) \to \cdots\end{aligned}$$

Since $2n + 1 \geq 5$, $H^2(S^{2n+1}; R) = H^3(S^{2n+1}; R) = 0$. Hence $x := \gamma(1) \in H^2(\mathbb{C}P^n; R)$ generates $H^2(\mathbb{C}P^n; R) \cong R$, and $\gamma(-) = z \smile -$ says $z = x$ in the notation of the Gysin sequence. Moreover, $H^1(\mathbb{C}P^n; R) = 0$, which can be seen in a few ways; for instance, $\mathbb{C}P^n$ has a CW complex decomposition with no one-dimensional cells. (Minor note: McCleary suggests this follows from $\mathbb{C}P^n$ being simply-connected. That doesn't seem to work: the Hurewicz theorem would give this for homology, not cohomology.) Hence $0 = H^1(\mathbb{C}P^n; R) \cong H^3(\mathbb{C}P^n; R)$. Now consider

$$\cdots \to H^{2k+1}(S^{2n+1}; R) \to H^{2k}(\mathbb{C}P^n; R) \xrightarrow{\gamma} H^{2k+2}(\mathbb{C}P^n; R) \to H^{2k+2}(S^{2n+1}; R) \to \cdots.$$

If $n < k$, the first term is 0, and the last term is 0 generally, so γ is an isomorphism. Suppose inductively $H^{2k}(\mathbb{C}P^n; R) \cong R$ is generated by x^k. γ sends a generator to a generator, so $\gamma(x) = x \smile x^k = x^{k+1}$ generates $H^{2k+2}(\mathbb{C}P^n; R)$. Similarly in odd dimensions the cohomology groups are trivial.

On the other hand, if $n = k$, we have

$$\cdots \to H^{2n-1}(\mathbb{C}P^n; R) \to H^{2n+1}(\mathbb{C}P^n; R) \to H^{2n+1}(S^{2n+1}; R)$$
$$\to H^{2n}(\mathbb{C}P^n; R) \xrightarrow{\gamma} H^{2n+2}(\mathbb{C}P^n; R) \to H^{2n+2}(S^{2n+1}; R) \to \cdots$$

which is

$$0 \to 0 \to H^{2n}(\mathbb{C}P^n; R) \to R \xrightarrow{\gamma} 0 \to 0$$

where we've used the fact that $H^i(M; R) = 0$ for a manifold M if $i > \dim M$. It follows that $\gamma(x^k) = x \smile x^k = x^{k+1} = 0$, completing the result.

179 Theorem
Here is a summary of cohomology computations which can also be carried out with the Leray-Serre spectral sequence, taken from McCleary:

(i) Let $\boxed{SU(n)} \subset M_n(\mathbb{C})$ *be the Lie group of unitary matrices of determinant 1, called the* $\boxed{\text{special unitary group}}$. *Then*

$$H^*(SU(n); R) \cong \Lambda(x_3, x_5, \ldots, x_{2n-1}),$$

where $\deg x_i = i$ *(throughout) and* Λ *refers to the exterior algebra (over R). (Recall this is given by formal linear combinations of k-fold tensors of the generators, subject to the relation* $x \otimes x = 0$.*)*

(ii) Let $\boxed{\mathrm{Sp}(n)} \subset M_n(\mathbb{H})$ *be the space of linear transformations which preserve the (quaternionic) inner product, called the* $\boxed{\text{symplectic group}}$. *Then*

$$H^*(\mathrm{Sp}(n); R) \cong \Lambda(x_3, x_7, \ldots, x_{4n-1}).$$

(iii) Let $\boxed{\mathrm{SU}}$ *denote the* $\boxed{\text{infinite special unitary group}}$, *which is the direct limit (union) of special unitary groups* $\mathrm{SU}(2) \subset \mathrm{SU}(3) \subset \cdots$ *(with the natural inclusions). Then*

$$H^*(\mathrm{SU}; R) \cong \Lambda(x_3, x_5, x_7, x_9, \ldots).$$

(iv) Let $\boxed{V_k(\mathbb{C}^n)}$ *denote the space of orthonormal k-frames (ordered bases) in* $\mathbb{C}^n$, *called the* $\boxed{\text{Stiefel manifold}}$. *Then*

$$H^*(V_k(\mathbb{C}^n); R) \cong \Lambda(x_{2(n-k)+1}, x_{2(n-k)+3}, \ldots, x_{2n-1}).$$

(v) Let $\boxed{\mathrm{SO}(n)} \subset M_n(\mathbb{R})$ *denote the space of orthogonal matrices of determinant 1, called the* $\boxed{\text{special orthogonal group}}$. *Then* $H^*(\mathrm{SO}(n); \mathbb{F}_2)$ *has a "simple system of generators" (see below)*

$$\{x_1, x_2, \ldots, x_{n-1}\}, \qquad \deg x_i = i.$$

(vi) Let $V_k(\mathbb{R}^n)$ *denote the space of orthonormal k-frames in* $\mathbb{R}^n$. *Then* $H^*(V_k(\mathbb{R}^n); \mathbb{F}_2)$ *has a simple system of generators*

$$\{x_{n-k}, x_{n-k+1}, \ldots, x_{n-1}\}, \qquad \deg x_i = i.$$

(vii) *Let* $\boxed{K(\mathbb{Z}, n)}$ *denote the* $\boxed{\text{Eilenberg-MacLane spaces}}$*. Then*

$$H^*(K(\mathbb{Z}, n); \mathbb{Q}) \cong \begin{cases} \Lambda(x_n) & n \text{ odd} \\ \mathbb{Q}[x_n] & n \text{ even} \end{cases}$$

(viii) *Let $B\,\mathrm{SO}(n)$ denote the classifying space (see below) of the special orthogonal group, and likewise with other groups we've encountered. Similarly, a lack of "(n)" or an (∞) denotes an "infinite" version. Then*

$$\begin{aligned} H^*(B\,\mathrm{SO}(n); \mathbb{F}_2) &\cong \mathbb{F}_2[w_2, \ldots, w_n] \\ H^*(B\,\mathrm{SO}; \mathbb{F}_2) &\cong \mathbb{F}_2[w_2, w_3, \ldots] \\ H^*(BO(n); \mathbb{F}_2) &\cong \mathbb{F}_2[w_1, \ldots, w_n] \\ H^*(BO; \mathbb{F}_2) &\cong \mathbb{F}_2[w_1, w_2, \ldots] \\ H^*(\mathbb{R}P(\infty); \mathbb{F}_2) &\cong \mathbb{F}_2[w_1]. \end{aligned}$$

Definition 180. Let H^* be a graded-commutative algebra, taken over R. A set $\{y_1, y_2, \ldots\}$ is called a $\boxed{\text{simple system of generators}}$ if the elements 1 and $x_{i_1} x_{i_2} \cdots$ with $i_1 < i_2 < \cdots$ form a basis over R for H^*. (Note: this does not determine the algebra structure fully. For instance, x_i^2 is not determined.)

181 Remark

Next we'll discuss computing the cohomology ring of a flag manifold. This will be a rough overview with many references and little rigor.

Definition 182. The $\boxed{\text{flag manifold}}$ $\boxed{\mathrm{Flag}(n)}$ as a set consists of ordered bases of $\mathbb{C}^n$, or equivalently saturated chains of subspaces in $\mathbb{C}^n$. The (complex) unitary group $U(n)$ acts on $\mathrm{Flag}(n)$ in an obvious way, and has stabilizer $T(n)$, the diagonal (complex) matrices in $U(n)$. Indeed, $\mathrm{Flag}(n) \cong U(n)/T(n)$ is a $\boxed{\text{homogeneous space}}$ and carries a Lie group structure. Note that $U(n)$ is compact, so $\mathrm{Flag}(n)$ is as well. There is an associated fibration

$$T(n) \hookrightarrow U(n) \to U(n)/T(n) \cong \mathrm{Flag}(n).$$

183 Proposition

Associated to a Lie group G is a $\boxed{\text{classifying space}}$ *BG. Indeed, given a closed subgroup $i\colon H \hookrightarrow G$, there is an associated fibration*

$$G/H \hookrightarrow BH \overset{Bi}{\to} BG.$$

In our case, this looks like

$$\mathrm{Flag}(n) \hookrightarrow BT(n) \overset{Bi}{\to} BU(n).$$

184 Theorem (Borel)

Let G be a connected compact Lie group, H a closed connected subgroup of maximal rank, and k a field of characteristic p. Suppose that $p = 0$ or $H^*(G; \mathbb{Z})$ and $H^*(H; \mathbb{Z})$ have no p-torsion. Then

$$H^*(G/H; k) \cong k \otimes_{H^*(BG;k)} H^*(BH; k).$$

PROOF (Statement taken from Frank Neumann's "On the cohomology of homogeneous spaces...", Journal of Pure and Applied Algebra, 1999.) Borel used the Leray-Serre spectral sequence in his Paris thesis in the early 1950's, which is still a classic reference (though it's in French). It can also be proved using the Eilenberg-Moore spectral sequence, which is covered extensively in McCleary.

185 Example

For the flag manifold, this gives

$$H^*(\mathrm{Flag}(n); \mathbb{Q}) \cong \mathbb{Q} \otimes_{H^*(BU(n);\mathbb{Q})} H^*(BT(n); \mathbb{Q}).$$

One can argue that

$$H^*(BT(n);\mathbb{Q}) \cong \mathbb{Q}[y_1, \ldots, y_n].$$

We discuss two methods for doing so. First, one can use a variation on Theorem 6.38 in McCleary, which in turn relies on another classic result of Borel, Theorem 3.27. The advantage of this method is that we may relate $H^*(BU(n);\mathbb{Q})$ and $H^*(BT(n);\mathbb{Q})$ explicitly, which we must do to apply Borel's theorem. (Theorem 6.38 doesn't quite apply without change, since for instance the field is of prime characteristic there.)

Second, we can note that $T(n)$ is $(S^1)^n$: each diagonal entry must have norm 1. Since $BS^1 = \mathbb{C}P^\infty$, we computed $H^*(BT(1);\mathbb{Q}) \cong \mathbb{Q}[y_1]$ (with $\deg y_1 = 2$) above. Now use the fact that $B(S^1)^n = (BS^1)^n$ to see

$$H^*(BT(n);\mathbb{Q}) = H^*((BS^1)^n;\mathbb{Q}) = H^*(BS^1;\mathbb{Q})^{\otimes n} = \mathbb{Q}[y_1, \ldots, y_n],$$

where we have used the Künneth theorem and the fact that our coefficients are in a field to break the product into a tensor product.

In any case, another general result (Definition 8.4) gives

$$H^*(BU(m);\mathbb{Q}) \cong \mathbb{Q}[y_1, \ldots, y_n]^{S_n},$$

where here S_n is the corresponding $\boxed{\text{Weyl group}}$, with the usual action. Putting it all together,

$$\begin{aligned} H^*(\mathrm{Flag}(n)) &\cong \mathbb{Q} \otimes_{\mathbb{Q}[y_1,\ldots,y_n]^{S_n}} \mathbb{Q}[y_1, \ldots, y_n] \\ &\cong \frac{\mathbb{Q}[y_1, \ldots, y_n]}{(e_1, \ldots, e_n)} \end{aligned}$$

where $y_1, \ldots, y_n$ act on $\mathbb{Q}$ by 0 and e_i is the degree i elementary symmetric polynomial on n variables. That is, we quotient by the ideal of non-constant symmetric polynomials.

186 Example

In algebraic combinatorics, the cohomology ring of both Grassmannians and flag manifolds figure prominently, with bases given by Schur and Schubert polynomials, respectively, both of which have been studied extensively. For instance, for the Grassmannian $\boxed{\mathrm{Gr}(k,n)}$ of k-planes in $\mathbb{C}^n$, we have

$$H^*(\mathrm{Gr}(k,n);\mathbb{Z}) \cong \frac{\mathbb{Z}[x_1, \ldots, x_k]^{S_k}}{\mathcal{I}_{k,n}}$$

where $\mathcal{I}_{k,n}$ is the ideal generated by the Schur polynomials whose diagram does not fit in a box with k rows and $n-k$ columns. The $\boxed{\text{Schur polynomials}}$ s_λ are indexed by integer partitions λ. More precisely,

$$\boxed{s_\lambda} := \sum_{T \in \mathrm{SSYT}(\lambda)} x^T,$$

where $\boxed{\mathrm{SSYT}(\lambda)}$ is the set of all $\boxed{\text{semi-standard Young tableaux}}$ of shape λ. That is, we form a certain diagram out of λ and label boxes with numbers from $1, \ldots, k$ so that rows weakly increase and columns strictly increase. $x^T := x_1^{a_1} \cdots x_k^{a_k}$ where a_i is the number of times the label i appears in T. We frequently forget about the underlying topological interpretation of these polynomials, but it's nice to see where they come from.

June 3rd, 2014: $K(\pi, n)$'s; Principal and Universal G-Bundles

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Summary Riley and Becca are going to prove group cohomology for any finite group is finitely generated. Riley is up first. A summary of the overall talk, which will take several lectures:

a) Discuss Eilenberg-MacLane spaces, in particular $K(\pi, 1)$-spaces.

- Prove existence of $K(\pi, 1)$ for any group π.
- Show $K(\pi, 1)$ is unique up to homotopy.
- Relate cellular cohomology of this to the cohomology of π.

b) Define principal G-bundles and the universal G-bundle whose base space BG is a $K(G, 1)$-space which we call the classifying space. (Again, these are unique up to homotopy.)

c) Define equivariant cohomology $H^i_G(X; R)$ and use it to achieve our goal.

Main reference: Benson's *Representations and Cohomology.*

Definition 187. A space X is of type (π, n) if $\pi_i(X) = \delta_{i,n}\pi$. (Here "0" is the terminal object of the appropriate category, so we require $\pi_i(X)$ to be π if $i = n$, the 0 group if $0 < i \neq n$, and a singleton if $i = 0$). A (π, n)-space is an Eilenberg-MacLane space if it has the homotopy type of a CW complex.

188 Notation
$K(\pi, n)$ refers to an Eilenberg-MacLane space of type (π, n).

189 Example
We'll focus on $K(\pi, 1)$, but our statements will usually be true for $K(\pi, n)$ when π is abelian. (Recall the higher homotopy groups are in general abelian.)

We will frequently use the following equivalent condition: being a $K(\pi, 1)$ space is the same as being homotopic to a CW complex with $\pi_1 = \pi$ and a contractible universal cover. (Proof "by topology.")

190 Example
$\mathbb{R}P^\infty$ (see two lectures ago when James introduced $\mathbb{C}P^\infty$) is a $K(\mathbb{Z}/2, 1)$-space. Indeed, S^∞, which is $\lim_n S^n$, is the universal covering space of $\mathbb{R}P^\infty$. One can argue $\pi_1(\mathbb{R}P^\infty) = \mathbb{Z}/2$; for instance, $S^\infty \to \mathbb{R}P^\infty$ is a double cover.

191 Example
$K(\mathbb{Z}, 1) = S^1$. Here $\mathbb{Z}$ is essentially given the discrete topology, though one can work more generally with (non-discrete, infinite, etc.) topological groups. We will not.

192 Theorem
For any group π, there exists a $K(\pi, 1)$-space.

Proof In their lecture notes; see Becca; also very standard.

193 Theorem
$K(\pi, 1)$-spaces are unique up to homotopy equivalence.

Proof If X is a $K(\pi, 1)$ space, there is a map $[Y, X] \to \mathrm{Hom}(\pi_1(Y), \pi)$ induced by $[f : Y \to X] \mapsto f^*$. This is a bijection in general if X has the homotopy type of a CW complex. (Recall the notation $[Y, X]$ indicates the space of continuous maps $Y \to X$ up to homotopy equivalence, given the compact-open topology.)

194 Theorem
Let π be a group, R a commutative ring with identity. Then

- $H_i(\pi, R) \cong H_i^{cellular}(K(\pi, 1); R)$
- $H^i(\pi, R) \cong H^i_{cellular}(K(\pi, 1); R)$.

(Here R on the left has trivial π-action.)

Proof Let X be a CW complex and also a $(\pi, 1)$-space. Let $\tilde{X}$ be its universal cover, which can be taken to be a CW complex. (In particular, $\tilde{X}$ is contractible, as above.) π acts freely on $\tilde{X}$ by deck transformations, which permute the cells of $\tilde{X}$. Thus cellular chains $C_*(\tilde{X}; R)$ on X are $R\pi$-modules, and $\tilde{H}_i(\tilde{X}; R) = 0$ since $\tilde{X}$ is contractible. Hence we have a free resolution of $R\pi$-modules

$$\cdots \to C_1(\tilde{X}; R) \to C_0(\tilde{X}; R) \xrightarrow{\epsilon} R \to 0.$$

Since $\tilde{X}/\pi = X$, quotienting $C_i(\tilde{X}; R)$ by the action of π (as a set, this consists of orbits) is $C_i(X; R)$, i.e. $C_i(X; R) \cong C_i(\tilde{X}; R) \otimes_{R\pi} R$. But then

$$H_i(X; R) \cong \mathrm{Tor}_i^{R\pi}(R, R) = H_i(\pi, R).$$

In the same way, we get

$$C^i(X, R) := \mathrm{Hom}_R(C_i(X; R), R) = \mathrm{Hom}_{R\pi}(C_i(\tilde{X}; R), R),$$

and it follows that

$$H^i(X; R) \cong \mathrm{Ext}^i_{R\pi}(R, R) = H^i(\pi, R).$$

195 Remark

Becca is up next.

Definition 196. A $\boxed{\text{principal } G\text{-bundle}}$ over a G-space B is a bundle $\rho\colon E \to B$ in the category of G-spaces which factors uniquely through the orbit space E/B. That is, the fibers are G (in the guise of orbits in E), and there is an open cover of B by $\{U_i\}$ and isomorphisms such that the following is a commutative diagram of G-spaces (i.e. all the arrows are G-maps):

$$\begin{array}{ccc} U_i \times G & \overset{\cong}{\longleftrightarrow} & \rho^{-1}U_i \\ \downarrow & & \downarrow \\ U_i & \hookrightarrow & B \end{array}$$

A morphism between principal G-bundles B, B' is just a G-map $B \to B'$. Let $\boxed{\mathrm{Prin}_G(B)}$ denote the set of all principal G-bundles over B.

197 Remark

$f\colon B' \to B$ induces $f^*\colon \mathrm{Prin}_G(B) \to \mathrm{Prin}_G(B')$ via

$$\begin{array}{ccc} B' \times_B E & \longrightarrow & E \\ {\scriptstyle f^*(\xi)}\downarrow & & \downarrow{\scriptstyle \xi} \\ B' & \xrightarrow{f} & B \end{array}$$

Claim: if B' is paracompact and $f, g\colon B' \to B$ are homotopic, then $f^*(\xi)$ and $g^*(\xi)$ are isomorphic as G-bundles. (Here "homotopy" refers to G-equivariant homotopy, meaning the homotopy is G-equivariant, where G acts trivially on the unit interval.)

Claim: there is a map $[B', B] \to \mathrm{Prin}_G(B')$ given as follows. Pick $\xi \in \mathrm{Prin}_G(B)$ and send $[f\colon B' \to B]$ to $f^*(\xi)$.

Definition 198. A universal G-bundle is a principal G-bundle ξ such that for all paracompact B', the map

$$[B', B] \xrightarrow{\sim} \mathrm{Prin}_G(B')$$

as above is a bijection. (Roughly, one can consider this as viewing $[-, B]$ as a representable functor, which suggests this definition has intrinsic interest.)

199 Proposition

There is a unique universal G-bundle with paracompact base (up to homotopy equivalence).

Proof We show part of uniqueness. Given principal G-bundles ξ, ξ', from the bijection we get f, f' such that $f^*(\xi') = \xi$ and $(f')^*(\xi) = \xi'$. But then $(ff')^*(\xi') = (f')^*(f^*(\xi')) = \xi' = \mathrm{id}_{B'}^*(\xi')$. Since ξ' is universal, $ff' \sim \mathrm{id}_{B'}$. Similarly $f'f \sim \mathrm{id}_B$, so B and B' are homotopy equivalent.

Some remarks on existence: Benson uses $EG := G * G * \cdots$ with as many factors of G as $|G|$, where $*$ denotes the join of topological spaces. Equivalently (as a set), EG consists of sequences $(t_1 g_1, t_2 g_2, \ldots)$ for $t_i \geq 0, g_i \in G$, with only finitely many non-zero terms and where $\sum t_i = 1$. These sequences are taken modulo $0g = 0g'$ for all $g, g' \in G$. (This is essentially a topological version of the space of linear interpolations between finitely many elements of G.) G acts diagonally on the right on such sequences. Then the classifying space of G is $BG := EG/G$, with associated universal G-bundle $\xi\colon EG \to BG$ via projection. There are certainly many details to check.

June 4th, 2014—G-Equivariant Cohomology and Finite Generation

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200 Remark

Becca is continuing from last time.

Definition 201. Let $\xi = EG \to BG$ be "the" (see previous proposition) universal G-bundle with paracompact base. Recall there is a bijection $[B', B] \xrightarrow{\sim} \mathrm{Prin}_G(B')$ where $[f\colon B' \to B] \mapsto f^*(\xi)$.

202 Remark

For G discrete, BG is $K(G, 1)$. (In general, it appears EG is contractible.)

203 Remark

We now take our groups to be finite and suppose our ring R is commutative and Noetherian. We turn back to our goal of proving the finite generation of finite group cohomology. The idea is to use $H^i(G, R) \cong H^i(K(G, 1); R)$, mentioned last time.

204 Lemma

If G, G' are finite groups, $G \to G'$ is a group homomorphism, X is a G-space, and $EG \to BG$ and $EG' \to BG'$ are universal bundles, then

$$EG \times_G X \sim EG' \times_G X$$

(that is, they are homotopy equivalent).

(The notation $X \times_G Y$ means $(X \times Y)/(xg, y) \sim (x, gy)$. Compare with tensor products. Here EG' is a G-space via pullback through $G \to G'$.)

205 Remark

$H^*(EG \times_G X; R)$ is then independent of the choice of EG.

Definition 206. If X is a G-space, we define the $\boxed{G\text{-equivariant cohomology}}$ with coefficients in R to be

$$\boxed{H^*_G(X;R)} := H^*(EG \times_G X; R).$$

207 Remark

$EG \times_G X \to EG \times_G * = BG$ is a Serre fibration with fiber X (indeed, a fiber bundle).

208 Lemma

There exists a spectral sequence

$$E_2^{pq} = H^p(BG; H^q(X;R)) \Rightarrow H^{p+q}_G(X;R).$$

(This is just the Serre spectral sequence applied to the preceding fibration.)

209 Lemma

$H^*(BU(n);R) \cong R[c_1,\ldots,c_n]$ *where* $\deg c_i = 2i$.

(This was used to compute the cohomology of the flag manifold in Josh's second lecture above.)

210 Theorem

Let G be a finite group, R a commutative noetherian ring, and $G \hookrightarrow U(n)$ an embedding (of groups) into a complex unitary group. Suppose X is a G-space with $H^*(X;R)$ a finitely generated R-module. Then $H^*_G(X;R)$ is a finitely generated $H^*(BU(n);R)$-module.

211 Corollary

$H^*(G,R)$ *is a finitely generated R-algebra.*

Proof Take $X = *$, a point. Then $EG \times_G X = BG$. By the theorem, $H^*(BG;R)$ is a finitely generated module over $R[c_1,\ldots,c_n]$, so it is a finitely generated R-algebra.

Note that $G \hookrightarrow U(|G|)$ is always possible by representing the elements of G via permutation matrices.

June 6th, 2014—Finite Group Cohomology is Finitely Generated

Summary Riley and Becca are finishing their proof today. See the previous two lectures for background.

212 Theorem

Suppose G is a finite group, R is a commutative Noetherian ring, and $G \hookrightarrow U(n)$ is an embedding into the complex unitary group. Suppose also X is a G-space where $H^*(X;R)$ is a finitely generated R-module. Then $H^*_G(X;R)$ is a finitely generated module over $H^*(BU(n);R)$.

Proof Start with the Serre fibration

$$X \to U(n) \times_G X \to U(n)/G.$$

A similar fibration was mentioned last time, $EG \times_G X \to BG = EG/G$, however $U(n)$ is not (weakly) contractible (it has fundamental group $\mathbb{Z}$, for instance), so $U(n) \to U(n)/G$ is not a universal G-bundle. In any case, the associated Serre spectral sequence is

$$E_2^{pq} = H^p(U(n)/G; H^q(X;R)) \Rightarrow H^{p+q}(U(n) \times_G X; R).$$

Also, $U(n)/G$ is a finite CW complex, so the entries of the E_2 page are finitely generated R-modules. (This step is potentially non-trivial. Need the CW complex property to "descend to the quotient". For one source, ask Steve Mitchell.) This property is preserved under quotients since R is Noetherian, so the entires of the E_∞ page are also finitely generated R-modules. Hence $H^{p+q}(U(n) \times_G X; R)$ has a filtration by finitely generated R-modules. Since $H^*(X; R)$ is by assumption a finitely generated R-module, $H^m(X; R) = 0$ for m large, so the filtration is finite, so $H^*(U(n) \times_G X; R)$ is a finitely generated R-module (roughly, it comes from a finite rectangle).

Next we remark that $EU(n) \times_G X \sim EG \times_G X$. In particular, we have projection maps

$$\begin{array}{ccccc} & & (EU(n) \times EG) \times_G X & & \\ & \swarrow^{\sim} & & \searrow^{\sim} & \\ EU(n) \times_G X & & \overset{\sim}{\longleftarrow\cdots\longrightarrow} & & EG \times_G X \end{array}$$

which turn out to be homotopy equivalences.

213 Remark

Given a fibration $F \to E \to B$, there is a long exact sequence

$$\cdots \to \pi_2(F) \to \pi_2(E) \to \pi_2(B) \to \pi_1(F) \to \cdots .$$

Furthermore, $EU(n) \times_G X = EU(n) \times_{U(n)} (U(n) \times_G X)$, so that $H^*_{U(n)}(U(n) \times_G X; R) \cong H^*_G(X; R)$; here we use the fact that $EU(n) \times_G X \sim EG \times_G X$. Now consider the spectral sequence from the Borel construction

$$\widetilde{E}_2^{pq} = H^p(BU(n); H^q(U(n) \times_G X; R)) \Rightarrow H^{p+q}_{U(n)}(U(n) \times_G X; R) \cong H^{p+q}_G(X; R).$$

(This is just the Serre spectral sequence coming from $U(n) \times_G X \to EU(n) \times_G X \to BU(n)$.) The constant map $U(n) \times_G X \to *$ induces a map on cohomology

$$H^*(*; R) \to H^*(U(n) \times_G X; R)$$

and thereby a morphism of spectral sequences

$$\begin{array}{ccc} H^p(BU(n); H^q(*; R)) & \xrightarrow{\Rightarrow} & H^{p+q}(BU(n); R) \\ \downarrow & & \downarrow \\ H^p(BU(n); H^q(U(n) \times_G X; R)) & \xrightarrow{\Rightarrow} & H^{p+q}_G(X; R). \end{array}$$

Here the top line comes from the (trivial) fibration $\star \to BU(n) \to BU(n)$, the rightmost arrow is induced by

$$EU(n) \times_G U(n) \times_G X \to EU(n) \times_G * \sim BU(n),$$

and these maps are evidently compatible. Hence $H^*_G(X; R)$ has an $H^*(BU(n); R)$-module structure. Since $H^*(BU(n); H^*(*; R))$ is essentially just $H^*(BU(n); R)$, we find

$$H^*(BU(n); H^*(U(n) \times_G X; R))$$

has an $H^*(BU(n); R)$-module structure. By the lemma computing $H^*(BU(n); -)$ from last time, we have

$$H^*(BU(n); H^q(U(n) \times_G X; R)) \cong H^*(U(n) \times_G X; R)[c_1, \ldots, c_n].$$

Since $H^*(U(n) \times_G X; R)$ is a finitely generated R-module, we have $H^*(U(n) \times_G X; R)[c_1, \ldots, c_n]$ is a finitely generated $R[c_1, \ldots, c_n]$-module. Hence we can apply the argument from before to say that the abutment $H_G(^{p+q}(X; R)$ is also a finitely generated $R[c_1, \ldots, c_n]$-module; note the fact that $R[c_1, \ldots, c_n]$ is Noetherian is used in this step. Hence the abutment is a finitely generated R-algebra, as required.

June 9th, 2014—Čech Cohomology and Sheaf Cohomology Frequently Agree

Summary Hao will define Čech cohomology and sheaf cohomology, and show they are isomorphic in nice cases.

Definition 214. Let X be a quasicompact, separated scheme. Equivalently, X has a finite affine open cover $\{U_i\}_{i=1}^n$ whose pairwise intersections are affine. For $I \subset [n]$, let $U_I = \cap_{i \in I} U_i$.

Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. There is a complex called the Čech complex, denoted $C^*(\mathcal{F})$:

$$0 \to \bigoplus_{|I|=1} \mathcal{F}(U_I) \to \bigoplus_{|I|=2} \mathcal{F}(U_I) \to \cdots \to \bigoplus_{|I|=n} \mathcal{F}(U_I) \to 0,$$

where the differentials $d_{I,J} \colon \mathcal{F}(U_I) \to \mathcal{F}(U_J)$ when $|J| = |I| + 1$ are

$$d_{I,J} = \begin{cases} 0 & \text{if } J \neq I \cup \{*\} \\ (-1)^{k+1} \operatorname{Res}_{U_I}^{U_J} & \text{if } J = I \cup \{t\} \text{ and } t \text{ is the } k\text{th element in } J \end{cases}$$

215 Example
Suppose $n = 3$. The complex is

$$\begin{aligned} 0 &\to \mathcal{F}(U_1) \oplus \mathcal{F}(U_2) \oplus \mathcal{F}(U_3) \to \mathcal{F}(U_{12}) \oplus \mathcal{F}(U_{13}) \oplus \mathcal{F}(U_{23}) \\ &\to \mathcal{F}(U_{123}) \to 0 \end{aligned}$$

where the first differential is

$$(f_1, f_2, f_3) \mapsto (f_1 - f_2, f_1 - f_3, f_2 - f_3)$$

and the second differential is

$$(g_{12}, g_{13}, g_{23}) \mapsto (g_{12} - g_{13} + g_{23}).$$

Definition 216. The Čech cohomology of $\mathcal{F}$ is

$$\widetilde{H}^i(X, \mathcal{F}) := H^i(C^*(\mathcal{F})).$$

217 Remark
$\widetilde{H}^0(X, \mathcal{F}) = \mathcal{F}(X)$; this is essentially a restatement of the sheaf axiom.

218 Remark
Recall the global sections functor $\Gamma(X, -)$ is left exact:

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

becomes

$$0 \to \mathcal{F}(X) \to \mathcal{G}(X) \to \mathcal{H}(X).$$

(It is not generally right exact.)

Definition 219. Define sheaf cohomology as

$$H^i(X, \mathcal{F}) := (R^i\Gamma(X, -))(\mathcal{F}).$$

220 Remark

Fact: the category of $\mathcal{O}_X$–mod is an abelian category with enough injectives (not obvious), so right derived functors in fact exist.

Certainly $H^0(X, \mathcal{F}) = \widetilde{H}^0(X, \mathcal{F})$.

221 Theorem

If X is a quasicompact, separated scheme, and $\mathcal{F}$ is a quasicoherent sheaf of $\mathcal{O}_X$-modules, then $H^i(X, \mathcal{F}) \cong \check{H}^i(X, \mathcal{F})$ for $i \geq 0$.

(That is, there is an open cover where $U = \operatorname{spec} A \subset X$ open implies $\mathcal{F}_U = \widetilde{M}$ for some $M \in A$–mod.)

222 Remark

This implies that Čech cohomology is independent of the open cover chosen: it's always just isomorphic to sheaf cohomology, under the above assumptions.

223 Lemma

The following are necessary for the theorem to possibly be true, and they are true:

(1) If $\mathcal{I}$ is an injective $\mathcal{O}_X$-module, then $\check{H}^i(X, \mathcal{I}) = 0$ for all $i \geq 1$. (Moreover, restricting an injective resolution of $\mathcal{O}_X$-modules to an open subset gives us another injective resolution.)

(2) If $X = \operatorname{spec} A$ is affine, then $H^i(X, \mathcal{F}) = 0$ for all $\mathcal{F}$ quasicoherent on X and $i \geq 1$.

Proof of Theorem. Let $\{U_i\}_{i=1}^n$ be an open cover of X by affines with affine pairwise intersections, and take an injective resolution of $\mathcal{O}_X$-modules

$$0 \to \mathcal{F} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \cdots .$$

We have a first quadrant double complex

$$\begin{array}{ccccc}
 & \cdots & & \cdots & \\
 & \uparrow & & \uparrow & \\
0 \to & \oplus_{|I|=1}\mathcal{I}^0(U_I) & \longrightarrow & \oplus_{|I|=2}\mathcal{I}^0(U_I) & \longrightarrow \cdots \\
 & \uparrow & & \uparrow & \\
0 \to & \oplus_{|I|=1}\mathcal{F}(U_I) & \longrightarrow & \oplus_{|I|=2}\mathcal{F}(U_I) & \longrightarrow \cdots \\
 & \uparrow & & \uparrow & \\
 & 0 & & 0 &
\end{array}$$

Delete the bottom row and take row homology. They are almost entirely zero, except the zeroth Čech cohomology of $\mathcal{I}^i$ is just the global sections. Hence the left column is just the complex involved in computing sheaf cohomology before taking homology. Now take column homology to get sheaf cohomology in the first column and zeros everywhere else; the sequence stabilizes.

On the other hand, delete the bottom row of the above diagram and take column cohomology. Each factor $\mathcal{I}^i(U_I)$ turns into $R^i\Gamma(U_I, \mathcal{F})$, which is $H^i(U_I, \mathcal{F})$. Again using the lemma, the higher derived cohomology here is zero, so the only nonzero terms in the E^1 page are in the first row. That is, we get

$$0 \to \oplus_{|I|=1}\mathcal{F}(U_I) \to \oplus_{|I|=2}\mathcal{F}(U_I) \to \cdots .$$

But this is just the Čech complex! Taking row cohomology gives Čech cohomology. The result follows since the spectral sequence converges.

List of Symbols

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