Algebraic Groups Lecture Notes

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Abstract

The following notes were taking during a course on Algebraic Groups at the University of Washington in Fall 2014. Please send any corrections to `jps314@uw.edu`. Thanks!

Contents

September 24th, 2014: Functor of Points, Representable Functors, and Affine (Group) Schemes

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1 Remark

We will largely take the "functor of points" view, which will cut down on a lot of background at the cost of some additional abstraction. The text will be Waterhouse's "Introduction to Affine Group Schemes." We will follow it quite closely, especially in the beginning. There will be perhaps two homework sets.

2 Notation

$\boxed{k}$ will be an arbitrary field throughout this course. Some statements generalize; that's alright.

3 Motivation (Functor of Points)

Algebraic sets are collections of points $X \subset k^n$ which are solutions to a system of polynomial equations; finitely many polynomials suffice, say $\langle f_1, \ldots, f_r \rangle \subset k[x_1, \ldots, x_n]$. To each such algebraic set we have an algebra of functions

$$k[x_1, \ldots, x_n]/\langle f_1, \ldots, f_r \rangle.$$

If $f, g \in k[x_1, \ldots, x_n]$ are in the same coset, then $f = g$ on the algebraic set X. (The converse is not generally true; in classical algebraic geometry over an algebraically closed field, the converse holds if we replace $\langle f_1, \ldots, f_r \rangle$ with its radical.)

Let $\boxed{k\text{-Alg}}$ be the category of commutative k-Algebras (with identity; these are in particular rings). Generalizing the above, if $R \in k\text{-Alg}$, we can solve polynomial equations $\langle f_1, \ldots, f_r \rangle$ over R and find solutions in R^n. Given a particular solution $(a_1, \ldots, a_n) \in R^n$, we can define a map of rings

$$k[x_1, \ldots, x_n]/\langle f_1, \ldots, f_r \rangle \to R$$

given by $x_i \mapsto a_i$. Conversely, given such a map, we can recover a solution by evaluating it at each x_i.

4 Remark

Try solving $x^2 + y^2 = -1$ over $\mathbb{Q}, \mathbb{R}, \mathbb{C}$: solution sets depend on the field. Also try solving $x^2 + y^2 = 1$ over these: over $\mathbb{C}$ they are "isomorphic", while over $\mathbb{Q}$ they very much aren't.

So, let $A := k[x_1, \ldots, x_n]/\langle f_1, \ldots, f_r \rangle$. The set of solutions to $\langle f_1, \ldots, f_r \rangle$ over R corresponds to the set of k-Algebra maps $A \to R$. Letting R vary gives a functor $\mathcal{F} \colon k\text{-Alg} \to \text{Sets}$:

$$\mathcal{F}(R) := \operatorname{Hom}_{k\text{-Alg}}(A, R).$$

This is the "functor of points"; roughly, it encapsulates the solutions of $\langle f_1, \ldots, f_r \rangle$ over all k-algebras.

Definition 5. Let $\mathcal{F} \colon \mathcal{A} \to \text{Sets}$ be a functor. $\mathcal{F}$ is a representable functor if there exists an object $A \in \mathcal{A}$ and an isomorphism of functors $\mathcal{F} \cong \operatorname{Hom}_{\mathcal{A}}(A, -)$. (We will not worry about set-theoretic considerations, i.e. Hom is always a set for us.)

An affine scheme (over k) is a representable functor X from k-Alg to Sets. That is, $X(-) \cong \operatorname{Hom}_{k\text{-Alg}}(A, -)$. We call A the coordinate algebra of X or the algebra of (regular) functions of X; we write $k[X] := A$.

6 Example

1. Affine space $\mathbb{A}^n$ is the functor $R \mapsto R^n$. Here $k[\mathbb{A}^n] = k[x_1, \ldots, x_n]$.

2. The affine scheme of the "punctured affine line" $\mathbb{G}_m(R)$ is the functor sending R to $R^\times$, the set of (multiplicatively) invertible elements. Here $k[\mathbb{G}_m] = k[x, 1/x] \cong k[x, y]/(xy - 1)$. We are currently forgetting about the group structure; we'll add it back in shortly.

7 Lemma (Yoneda Lemma)

There is an anti-equivalence of categories between commutative k-algebras and functors represented by commutative k-algebras, given by

$$A \mapsto \operatorname{Hom}_{k\text{-Alg}}(A, -).$$

In particular, if X is represented by A and Y is represented by B, then

$$\operatorname{Hom}_{k\text{-Alg}}(A, B) \cong \operatorname{Hom}(Y, X).$$

Proof Homework.

8 Remark

If X is a representable functor, what is $k[X]$? Well, by Yoneda,

$$\operatorname{Mor}(X, \mathbb{A}^1) \cong \operatorname{Hom}_{k\text{-Alg}}(k[t], k[X]) \cong k[X].$$

(One must also argue the algebra structure works: for instance, how does one add elements of $\operatorname{Mor}(X, \mathbb{A}^1)$?)

Definition 9. An affine group scheme is a representable functor

$$G : k\text{-Alg} \to \text{Groups}.$$

That is, the induced functor $G \colon k\text{-Alg} \to \text{Sets}$ is isomorphic to some $\operatorname{Hom}_{k\text{-Alg}}(k[G], -) \colon k\text{-Alg} \to \text{Sets}$. As before $k[G] \in k\text{-Alg}$ is the coordinate algebra of G or the algebra of functions of G. We will show that $k[G]$ actually carries a "Hopf algebra" structure which encapsulates the additional structure of the functor G.

10 Example

The following are affine group schemes:

1. $\boxed{\mathbb{G}_a}$ is the functor $R \mapsto R^+$ (that is, the additive group of R). $k[\mathbb{G}_a] = k[x]$ again, but in a sense there is more structure; see next time.

2. $\boxed{\mathbb{G}_m}$ is the functor $R \mapsto R^\times$ (that is, the multiplicative group of units of R). Again $k[\mathbb{G}_m] = k[x, 1/x]$. This differs from the previous version of this example only inasmuch as we no longer forget the group structure of R^x.

3. $\boxed{\mathrm{GL}_n}$ is the functor sending R to the group of invertible $n \times n$ matrices with entries in R. Note that $\mathrm{GL}_1 \cong \mathbb{G}_m$. Here $k[\mathrm{GL}_n] = k[x_{ij}, 1/\det(x_{ij})]$ with $1 \leq i, j \leq n$.

4. $\boxed{\mathrm{SL}_n}$ is the functor sending R to the group of $n \times n$ matrices over R with determinant 1. It turns out that $\mathrm{SL}_n(R) \subset \mathrm{GL}_n(R)$ is a closed subscheme. Here $k[SL_n] = k[x_{ij}]/(\det(x_{ij}) - 1)$.

Continued next time.

September 26th, 2014: Examples of Affine Group Schemes, Hopf Algebras

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Summary Last time there was an important question: what is the algebra structure on $\mathrm{Mor}(X, \mathbb{A}^1)$? Recall this was the key to recovering the coordinate algebra $k[X]$ from an affine scheme. Think about it if you haven't. We also did four examples of affine group schemes, $\mathbb{G}_a, \mathbb{G}_m, \mathrm{GL}_n, \mathrm{SL}_n$.

Outline of today's class:

1. Examples.

2. Hopf algebras.

3. Group schemes vs. Hopf algebras.

11 Example

Continuing examples from the end of last time:

5. $\boxed{\mu_n}$ sends R to $\{a \in R : a^n = 1\}$ and is the group scheme of nth roots of unity. Here $k[\mu_n] = k[x]/(x^n - 1)$ since $\mu_n(R) \cong \mathrm{Hom}(k[x]/(x^n - 1), R)$. It's easy to see $\mu_n \subset \mathbb{G}_m$, that is, there is a natural transformation from μ_n to $\mathbb{G}_m$ which turns out to be an embedding.

6. Suppose $\mathrm{char}\, k = p > 0$. Define $\boxed{\mathbb{G}_{a_{(1)}}}$ as sending R to $\{a \in R : a^p = 0\}$ with addition as the group operation. This is a sub group scheme of $\mathbb{G}_a$ and has coordinate algebra $k[x]/x^p$.

12 Remark

In some sense, nilpotents are a necessary evil. For instance, consider $\mathrm{GL}_n(\overline{k})$ as an algebraic variety; it's reduced and irreducible, so each coordinate ring is a domain, i.e. they contain no nilpotents. Further, we may recover the whole scheme from the single k-algebra $\mathrm{GL}_n(\overline{k})$.

On the other hand, $\mathbb{G}_{a_{(1)}}(\overline{k}) = 0$, and indeed $\mathbb{G}_{a_{(1)}}(K) = 0$ for all K/k, so we can't recover $\mathbb{G}_{a_{(1)}}$ from its value on $\overline{k}$. We need to allow more general rings.

13 Remark

We were able to recover an affine scheme, i.e. a representable functor of sets, using a single k-algebra. How does the equivalent operation occur for affine group schemes, i.e. representable functors of groups? What extra structure do we need to add to the affine scheme's k-algebra to recover the group structure on $\mathrm{Hom}_{k\text{-Alg}}(k[G], -)$?

Definition 14. A is a $\boxed{\text{Hopf } k\text{-algebra}}$ if

1. $A \in k$-Alg; that is, it is a k-vector space with a bilinear, associative, and commutative multiplication together with an identity 1;

2. There exist three k-algebra maps,
   (a) $\boxed{\text{Comultiplication}}$ $\boxed{\Delta}$: $A \to A \otimes A$—tensor products are always over k unless stated otherwise;
   (b) $\boxed{\text{Counit}}$ $\boxed{\epsilon}$: $A \to k$;
   (c) $\boxed{\text{Coinverse}}$ or $\boxed{\text{antipode}}$ $\boxed{\sigma}$: $A \to A$.

3. The following diagrams are commutative; the first is called coassociativity.

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ {\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle 1\otimes\Delta} \\ A \otimes A & \xrightarrow[\Delta\otimes 1]{} & A \otimes A \otimes A \end{array}$$

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ {\scriptstyle\Delta}\downarrow & \overset{\mathrm{id}_A}{\searrow} & \downarrow{\scriptstyle 1\overline{\otimes}\epsilon} \\ A \otimes A & \xrightarrow[\epsilon\overline{\otimes}1]{} & A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ {\scriptstyle\Delta}\downarrow & \overset{\epsilon}{\searrow} & \downarrow{\scriptstyle \mathrm{id}_A\overline{\otimes}\sigma} \\ A \otimes A & \xrightarrow[\sigma\overline{\otimes}\,\mathrm{id}_A]{} & A \end{array}$$

Here $\epsilon\colon A \to A$ is really the composite $A \xrightarrow{\epsilon} k \to A$ given by $a \mapsto \epsilon(a) \mapsto \epsilon(a)1$, and $\overline{\otimes}$ indicates that we first take the tensor product as usual and follow it by "multiplication", either one of the natural isomorphisms $A \otimes k \to A$ or $k \otimes A \to A$ given by multiplication, or the algebra multiplication map $A \otimes A \to A$.

15 Remark

Hopf algebras don't in general need to be commutative, so we will sometimes refer to the above objects as "commutative Hopf algebras". Ours will typically come from coordinate algebras, which are commutative. Another way to define Hopf algebras is by building up algebras and coalgebras, which can be combined into bialgebras, together with an antipode/coinverse, giving a Hopf algebra. This viewpoint is pursued in my Algebraic Combinatorics notes around the April 21st lecture.

Alternatively, one may tweak the above definition to avoid assuming commutativity by simply requiring the antipode to be an antihomomorphism of algebras (rather than a homomorphism of algebras) and leaving the rest unchanged.

16 Example

1. Let π be a group. The group algebra $k\pi$ is a Hopf algebra where $\Delta\colon k\pi \to k\pi \otimes k\pi$ is defined via $g \mapsto g \otimes g$. The antipode $\sigma\colon k\pi \to k\pi$ is given by $g \mapsto g^{-1}$. The counit $\epsilon\colon k\pi \to k$ is the augmentation map, $g \mapsto 1$. (Notational note: $k[\pi]$ and $k\pi$ are very different beasts. $k[\pi]$ is a coordinate algebra of an affine (group) scheme π while $k\pi$ is a group algebra for a group π.)

2. Let V be a k-vector space. Let $T^*(V)$ be the tensor algebra on V, namely $\oplus_{n\geq 0} V^{\otimes n}$ (the $n = 0$ summand is k). Give it a Hopf algebra structure by saying $v \in V$ is sent to $v \mapsto v \otimes 1 + 1 \otimes v$ and extending this to make Δ an algebra map. Use antipode $v \mapsto -v$, extended similarly (note: antipodes are anti-homomorphisms of algebras in general). The counit projects onto the $n = 0$ summand.

3. Let $S^*(V)$ be the symmetric algebra of V, namely the quotient of $T^*(V)$ by the ideal generated by relations $u \otimes v - v \otimes u$. Use the same comultiplication as $T^*(V)$. More concretely, $S^*(V)$ is just a basis-independent construction of the polynomial algebra $k[x_1, \ldots, x_n]$ where $n = \dim(V)$. Hence $\mathbb{A}^n \cong \operatorname{spec} S^*(V)$, i.e. the coordinate algebra of $\mathbb{A}^n$ is $S^*(V)$ where $\dim V = n$.

4. Here we define $\boxed{\mathcal{U}(\mathfrak{g})}$ the $\boxed{\text{universal enveloping algebra}}$ of a $\boxed{\text{Lie algebra}}$ $\mathfrak{g}$. Recall that $\mathfrak{g}$ is a vector space over k with a bracket operation $[-,-]\colon \mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ satisfying certain axioms, for instance the Jacobi identity.

   We define $\mathcal{U}(\mathfrak{g})$ as the quotient of $T^*(\mathfrak{g})$ by the ideal generated by $x\otimes y - y\otimes x - [x,y]$ for $x,y\in\mathfrak{g}$. Hence in $\mathcal{U}(\mathfrak{g})$, the following is true:

$$x\otimes y - y\otimes x = [x,y].$$

   Conveniently, this quotient preserves the Hopf algebra structure above on $T^*(\mathfrak{g})$, so we can give $\mathcal{U}(\mathfrak{g})$ a Hopf algebra structure. In particular, Δ sends x to $x\otimes 1 + 1\otimes x$.

Definition 17. Let A be a Hopf algebra. $a\in A$ is a $\boxed{\text{group-like element}}$ if $\Delta(a)=a\otimes a$ and $\epsilon(a)=1$. It is a $\boxed{\text{primitive element}}$ if $\Delta(a)=1\otimes a + a\otimes 1$. In some sense these are orthogonal cases. "Group-like" corresponds to group algebras while "primitive" corresponds to universal enveloping algebras.

18 Remark
We originally omitted the condition $\epsilon(a)=1$ above, though without it "too many" things (eg. $a=0$) are group-like. See the October 3rd lecture for more.

19 Theorem
There is an antiequivalence of categories between affine group schemes over k and commutative Hopf k-algebras. In particular, if X is an affine scheme, each possible group scheme structure on X corresponds to a k-Hopf algebra structure on its coordinate algebra and vice-versa.

PROOF See next lecture.

20 Remark
While we haven't formally defined morphisms in the category of (commutative) Hopf k-algebras, they are just bialgebra morphisms. Bialgebras are more carefully developed in my Algebraic Combinatorics notes. This theorem is also proved there in some detail, though it uses the usual sheaf-theoretic construction of (affine) group schemes rather than the functor of points.

September 29th, 2014: Fiber and Cartesian Products of Affine Group Schemes; Sweedler Notation; Categorical Equivalence Proof and Practice

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Summary Outline:

1. Fiber products
2. Proof of theorem about group schemes and Hopf algebras
3. Practical usage of theorem
4. Examples

Definition 21. Suppose X and Y are representable functors represented by algebras $A=k[X]$ and $B=k[Y]$. We can form the categorical $\boxed{\text{fiber product}}$ of X and Y

$$\boxed{X\times Y}(R) := X(R)\times Y(R).$$

We claim $X\times Y$ is represented by the algebra $A\otimes B$ and is a direct product of X and Y.

Proof One can check

$$\mathrm{Hom}_{k\text{-Alg}}(A \otimes B, R) = \mathrm{Hom}_{k\text{-Alg}}(A, R) \times \mathrm{Hom}_{k\text{-Alg}}(B, R).$$

Definition 22. Let X, Y, Z be affine schemes represented by $k[X] = A, k[Y] = B, k[Z] = C$. We can form the fiber product $X \times_Z Y$ fitting in to

$$\begin{array}{ccc} X \times_Z Y & \dashrightarrow & Y \\ \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Z \end{array}$$

purely in terms of functors via

$$\boxed{X \times_Z Y}(R) := \{(x, y) \in X(R) \times Y(R)\} : f(R)(x) = g(R)(y)\}.$$

We claim $X \times_Z Y$ is represented by the algebra $A \otimes_C B$ and is a fiber product of X and Y over Z.

Proof The given maps between X, Y, Z correspond to maps of the underlying algebras (via Yoneda)

$$\begin{array}{ccc} A \otimes_C B & \dashleftarrow & B \\ \uparrow & & \uparrow \\ A & \longleftarrow & C \end{array}$$

Verify the remaining details as an exercise.

23 Remark

As an affine scheme, $\mathrm{spec}\, k$ for a field k is a point, call it pt. This is represented by $\mathrm{pt}(R) := \mathrm{Hom}_{k\text{-Alg}}(k, R)$. The direct product $X \times Y$ above is then the same as the fiber product $X \times_{\mathrm{pt}} Y$. This corresponds to the diagrams

$$\begin{array}{ccc} & & Y \\ & & \downarrow \\ X & \longrightarrow & \mathrm{pt} \end{array}$$

$$\begin{array}{ccc} A \otimes_k B & \longleftarrow & B \\ \uparrow & & \uparrow \\ A & \longleftarrow & k \end{array}$$

24 Remark

We next sketch the proof of the antiequivalence of the categories of affine group schemes and commutative Hopf algebras over a field.

Proof (Sketch.) Saying that G is a group scheme is the same as saying G is a "group object" in the category of representable functors.

Definition 25. Let $\mathcal{C}$ be a category with an initial object e. We call $G \in \mathcal{C}$ a group object in $\mathcal{C}$ if there exists maps $G \times G \to G$ ("multiplication"), $e \to G$ ("identity"), and $G \to G$ ("inverse") satisfying the usual associativity, identity, and inverse axioms of a group written in diagrammatic form.

For instance, if $\mathcal{C} = \text{Sets}$, the group objects are precisely the groups and the initial object is the singleton $\{e\}$. In the category of affine schemes over k, the initial object is the point scheme $\text{pt} = \operatorname{spec} k$. The group objects in the category of affine schemes over k, i.e. the category of representable functors over k, are precisely the affine group schemes over k.

Using the Yoneda lemma, to each group object G in the category of representable functors, there are maps

$$\begin{aligned} \Delta &\colon k[G] \to k[G] \otimes k[G] \\ \epsilon &\colon k[G] \to k \\ \sigma &\colon k[G] \to k[G]. \end{aligned}$$

Put the remaining pieces together as an exercise; you'll use the above fact about cartesian products in the category of representable functors.

Definition 26. Here we introduce Sweedler notation for comultiplication in Hopf algebras. To avoid a profusion of subscripts, we write

$$\begin{aligned} \Delta(a) &= \sum a_1 \otimes a_2 \\ &= \sum a' \otimes a''. \end{aligned}$$

There are other variations; see my algebraic combinatorics notes for a more complete discussion.

27 Example

We can write some of the Hopf algebra diagrams in Sweedler notation quite compactly. For instance, letting $\Delta(a) = \sum a_1 \otimes a_2$, the counit diagram is in part

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ & \searrow^{\mathrm{id}_A} & \downarrow 1 \overline{\otimes} \epsilon \\ & & A \end{array} \qquad\qquad \begin{array}{ccc} a & \longmapsto & \sum a_1 \otimes a_2 \\ & \searrow & \downarrow \\ & & a = \sum \epsilon(a_2) a_1. \end{array}$$

Using the other diagram gives $a = \sum \epsilon(a_1) a_2$. Hence the counit diagram is just saying

$$\sum \epsilon(a_2) a_1 = a = \sum \epsilon(a_1) a_2.$$

For the coinverse the condition is

$$\sum a_1 \sigma(a_2) = \epsilon(a) = \sum \sigma(a_1) a_2.$$

(Be careful about commutativity: $\epsilon(a_1) \in k$, so we can write it on either side; $\sigma(a_1) \in A$, which in general might not be commutative, so we shouldn't move it around.)

28 Remark

In practice, how do we go between G and $k[G]$, an affine group scheme over k and its commutative k-Hopf algebra?

- Given $k[G]$, define G as follows:
  (i) Multiplication: $f, g \in G(R) := \operatorname{Hom}(k[G], R)$ are multiplied by using the composite

$$f \times g \colon k[G] \xrightarrow{\Delta} k[G] \otimes k[G] \xrightarrow{f \otimes g} R \otimes R \xrightarrow{m} R.$$

  (Here m is just the algebra multiplication in R. This operation can be generalized by replacing R with an algebra and $k[G]$ with a coalgebra, which is discussed in my algebraic combinatorics notes as a "convolution".)

(ii) Unit: in $G(R) = \mathrm{Hom}(k[G], R)$, the multiplicative unit is the composite

$$e_R\colon A \xrightarrow{\epsilon} k \hookrightarrow R$$

where $k \hookrightarrow R$ is the unit map, sending 1 to 1. An alternate viewpoint is that $k \hookrightarrow R$ corresponds to the map $G(k) \to G(R)$ given by $\mathrm{Hom}(k[G], k) \to \mathrm{Hom}(k[G], R)$ where $\epsilon := e_k \mapsto e_R$.

(iii) Inverse: if $f \in G(R) = \mathrm{Hom}(k[G], R)$, what is $f^{-1} \in G(R)$? It's the composite

$$k[G] \xrightarrow{\sigma} k[G] \xrightarrow{f} R,$$

i.e. $f^{-1} := f \circ \sigma\colon k[G] \to R$.

Let's check this works, at least partly. For instance, let's show $f \times f^{-1} = e_R$. The left side is the composite

$$f \times f^{-1}\colon k[G] \xrightarrow{\Delta} k[G] \otimes k[G] \xrightarrow{f \otimes f^{-1}} R \otimes R \to R$$

which is

$$\begin{aligned} a &\mapsto \sum a_1 \otimes a_2 \mapsto \sum f(a_1) \otimes f^{-1}(a_2) \\ &= \sum f(a_1) \otimes f(\sigma(a_2)) \mapsto \sum f(a_1) f(\sigma(a_2)) \\ &\mapsto f\left(\sum a_1 \sigma(a_2)\right) = f(\epsilon(a)) \\ &= \epsilon(a) f(1) = \epsilon(a). \end{aligned}$$

Hence $f \times f^{-1} = e_R \in \mathrm{Hom}(A, R)$.

- Given G, define $k[G]$ as follows:

29 Aside ("Generic Element" of a Group Scheme)

Suppose G is a group scheme represented by A. Take an element $f \in G(R)$ where we take $G(R) = \mathrm{Hom}_{k\text{-Alg}}(A, R)$. Applying $\mathrm{Hom}_{k\text{-Alg}}(A, -)$, $f\colon A \to R$ induces a map $\eta_f\colon G(A) \to G(R)$, i.e.

$$\eta_f\colon \mathrm{Hom}(A, A) \to \mathrm{Hom}(A, R)$$

which sends id_A to f. In particular, $f = \eta_f(\mathrm{id}_A)$, so every $f \in G(R)$ is in the image of some η_f. We call id_A a generic element since roughly if we can show some property is functorial and holds for id_A, then it holds for all $f \in G(R)$.

This is the functorial version of the generic point of a (say integral) scheme.

More next time.

October 1st, 2014: Hopf Algebras from Group Schemes; Cocommutativity; Closed Subschemes; Closed Subgroup Schemes

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Summary Today's outline:

1. Finish translation from group schemes to Hopf algebras.

2. Example of translation

3. Subgroup schemes: definition and examples

Homework 1 will appear on the web site later today. They should be pretty easy.

30 Remark

Finishing up from last time, given a group scheme G with coordinate algebra $k[G]$, we can recover the Hopf algebra structure on $k[G]$ from the group structure on $G(R)$ as follows.

Note that $\Delta\colon k[G] \to k[G] \times k[G]$ must live in $G(k[G] \otimes k[G]) = \mathrm{Hom}_{k\text{-Alg}}(k[G], k[G] \otimes k[G])$. Let $\iota_1\colon k[G] \to k[G] \otimes k[G]$ be given by $a \mapsto a \otimes 1$ and likewise with ι_2. Define

$$\Delta(a) := (\iota_1 \times \iota_2)(a)$$

where $\times$ indicates the multiplication in $G(k[G] \otimes k[G])$.

To motivate this choice, recall the definition of $\iota_1 \times \iota_2$ given a Hopf algebra structure to begin with:

$$\iota_1 \times \iota_2 \colon k[G] \overset{\Delta}{\to} k[G] \otimes k[G] \overset{\iota_1 \otimes \iota_2}{\to} k[G] \otimes k[G] \otimes k[G] \otimes k[G] \overset{m}{\to} k[G] \otimes k[G].$$

The last two maps together are the identity:

$$x \otimes y \mapsto x \otimes 1 \otimes 1 \otimes y \mapsto x \otimes y.$$

Hence $\iota_1 \times \iota_2 = \Delta$.

Likewise, for the counit, we have an identity $e_k \in G(k) = \mathrm{Hom}_{k\text{-Alg}}(k[G], k)$, which we define to be ϵ, the counit. For the coinverse, we have the identity map $\mathrm{id}_A \in G(A) = \mathrm{Hom}(A, A)$ (*not* in general the identity with respect to the group operation). Since we're in a group, let $\sigma := (\mathrm{id}_A)^{-1}$ with respect to the group operation.

Proof Let's verify one of these statements; the rest are similar. In particular, let's show part of the coinverse diagram holds in the sense that if $\Delta(a) = \sum a' \otimes a''$, then $\sum a'\sigma(a'') = \epsilon(a)$.

Let $A = k[G]$. Consider

$$A \overset{\Delta}{\to} A \otimes A \overset{1 \otimes \sigma}{\to} A \otimes A \overset{m}{\to} A$$

which is

$$a \mapsto \sum a' \otimes a'' \mapsto \sum a' \otimes \sigma(a'') \mapsto a'\sigma(a'').$$

Now $\sigma = (\mathrm{id}_A)^{-1}$ by definition, so this composite is simply computing $\mathrm{id}_A \times \mathrm{id}_A^{-1} \in G(A)$. By definition this is e_A, which is by definition ϵ.

31 Example

Here we translate from the affine group schemes introduced above to the corresponding Hopf algebras.

(1) $\mathbb{G}_a$: recall $k[\mathbb{G}_a] = k[x]$ as an algebra. Comultiplication is the product of ι_1 and ι_2 in $G[k[x]]$. Here $\iota_1\colon k[x] \to k[x] \otimes k[x]$ via $x \mapsto x \otimes 1$ and likewise with ι_2. In the additive group of $k[x] \otimes k[x]$, these are represented by the elements $x \otimes 1$ and $1 \otimes x$, so applying the group operation to these elements gives $x \otimes 1 + 1 \otimes x$. Hence $\Delta(x) = x \otimes 1 + 1 \otimes x$ (and this is extended to be an algebra map).

The counit $\epsilon\colon k[x] \to k$ is the additive identity in $k[x]$, namely 0, i.e. it is the map given by $x \mapsto 0$.

The coinverse is the inverse with respect to addition, so $\sigma\colon k[x] \to k[x]$ is given by $x \mapsto -x$.

(2) GL_n: recall that $A = k[\mathrm{GL}_n] = k[x_{ij}, 1/\det]$ as above. Here $\iota_1\colon A \to A \otimes A$ acts via $x_{ij} \mapsto x_{ij} \otimes 1$ and likewise with ι_2. In $\mathrm{GL}_n(A \otimes A)$, the map ι_1 is given by the matrix $[x_{ij} \otimes 1]_{1 \leq i,j \leq n}$ and ι_2 is given by $[1 \otimes x_{ij}]_{1 \leq i,j \leq n}$. Hence their product in $\mathrm{GL}_n(A \otimes A)$ is given by multiplying these matrices, namely

$$\iota_1 \times \iota_2 = \left[\sum_{\ell=1}^{n} x_{i\ell} \otimes x_{\ell j}\right]_{1 \leq i,j \leq n}.$$

Hence $\iota_1 \times \iota_2 \colon A \to A \otimes A$ (that is, Δ) is given by

$$\Delta(x_{ij}) = \sum_{\ell=1}^{n} x_{i\ell} \otimes x_{\ell j}.$$

Exercise: find ϵ and σ.

(3) $\mathbb{G}_m$: this is just GL_1. From the above formula, comultiplication is given by

$$\begin{aligned} k[x, 1/x] &\to k[x, 1/x] \otimes k[x, 1/x] \\ x &\mapsto x \otimes x. \end{aligned}$$

(4) $\mathrm{SL}_n \subset \mathrm{GL}_n$: there is a surjection $k[GL_n] \twoheadrightarrow k[\mathrm{SL}_n]$ and the coproduct will be induced by this surjection. "Comultiplication is inherited by subgroups": see below.

(5) $\mu_n \subset \mathbb{G}_m$.

(6) $\mathbb{G}_{a_{(1)}} \subset \mathbb{G}_a$.

Definition 32. A Hopf algebra A is cocommutative if the following diagram commutes:

$$\begin{array}{ccccc} & & A & & \\ & \swarrow^{\Delta} & & \searrow^{\Delta} & \\ A \otimes A & & \xrightarrow[\tau]{\text{"Twist"}} & & A \otimes A \end{array}$$

(The twist map simply sends $a \otimes b$ to $b \otimes a$.)

Indeed, the equivalence of categories above gives a correspondence between commutative cocommutative k-Hopf algebras and commutative affine group schemes over k.

33 Example

Cocommutativite Hopf algebras in action:

1. $\mathbb{G}_a$, $\mathbb{G}_m$ are both commutative group schemes. Subgroup schemes like $\mathbb{G}_{a_{(1)}}$, μ_n inherit this property. Hence the corresponding Hopf algebras are all cocommutative. For instance, for $\mathbb{G}_m$ this is plain from the formula: $\Delta(x) = x \otimes x$.

2. If π is a group, then we can give $k\pi$ its usual k-Hopf algebra structure. If π is not commutative, then $k\pi$ is not commutative. However, it is cocommutative: $g \mapsto g \otimes g$. Since group algebras in general do not have commutative multiplication, they are not in general affine group schemes. For finite groups we can dualize; more on this later in the course.

   Technical note: we have only defined commutative Hopf algebras. The general definition is in my algebraic combinatorics notes; the only differences are that multiplication is no longer necessarily commutative and the antipode is an antihomomorphism of algebras.

3. $k[\mathrm{GL}_n]$ is not cocommutative, so GL_n is not a commutative group scheme.

Definition 34. Let X be an affine scheme with coordinate algebra $k[X]$. $Y \subset X$ is classically defined to be a closed subscheme "if it is defined by an ideal $I \subset k[X]$". For our purposes, we define a closed subscheme $Y \subset X$ to be one which is represented by a quotient $k[X]/I$.

We next define an equivalent notion for affine group schemes.

Definition 35. Let A be a (commutative, for simplicity) Hopf algebra. An ideal $I \subset A$ is a Hopf ideal if:

1. $\Delta(I) \subset I \otimes A + A \otimes I$;

2. $\epsilon(I) = 0$;

3. $\sigma(I) \subset I$.

Equivalently, A/I has a natural, well-defined Hopf algebra structure inherited from that of A via the projection map $A \to A/I$.

Definition 36. Let H, G be affine group schemes. Then $\boxed{H \subset G}$ is a $\boxed{\text{closed subgroup scheme}}$ if:

1. $H \subset G$ is a subgroup functor (i.e. there is a natural transformation $H \hookrightarrow G$ such that for any R, $H(R) \hookrightarrow G(R)$ is a subgroup).

2. H is a closed subscheme of G; that is, $k[H] = k[G]/I$ for some ideal $I \subset k[G]$.

Note: (2) uses only the scheme structure of G and not its group structure.

37 Proposition

Given group schemes H, G, a closed subscheme $H \subset G$ is a closed subgroup scheme if and only if $I := \ker(k[G] \to k[H])$ is a Hopf ideal.

Proof Exercise; see next lecture for one direction; see the following lecture for the other.

38 Remark

The second condition in the definition of a closed subgroup scheme is redundant by the following:

39 Fact

If $H \subset G$ is a subgroup functor, then it is a closed subscheme.

This is a relatively deep fact about Hopf algebras. We will not have need of it.

October 3rd, 2014: Kernels and Character Groups of Group Schemes

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Summary Today's outline: injectivity, surjectivity, and ker for group schemes

40 Remark

Last time we mentioned that a closed subscheme $H \subset G$ for group schemes H and G is a subgroup scheme if and only if $I := \ker(k[G] \to k[H])$ is a Hopf ideal. We'll sketch a proof of one direction now.

Proof Suppose I is a Hopf ideal. We need to show that $H(R)$ is a subgroup of $G(R)$. For instance, let's show that if $f, g \in H(R) \subset G(R)$ then $f \times g \in H(R)$. On the level of Hopf algebras, we have

$$f \times g \colon k[G] \overset{\Delta}{\to} k[G] \otimes k[G] \overset{f \otimes g}{\to} R \otimes R \overset{m}{\to} R.$$

For this to descend to the quotient $k[H] = k[G]/I$, if $f|_I, g|_I = 0$, we need $(f \times g)|_I = 0$. If $a \in I$ with $\Delta(a) = \sum a' \otimes a''$, then $(f \times g)(a) = \sum f(a')g(a'')$. By assumption, $\Delta(I) \subset I \otimes A + A \otimes I$, so each term $f(a')g(a'')$ is in IA or AI, hence is 0, as required.

Show the other direction as an exercise. By the equivalence of categories, it suffices to show that the kernel of a Hopf algebra map is a Hopf ideal.

Definition 41. Let $f \colon G \to H$ be a morphism of affine group schemes. (That is, a natural transformation of functors.) Then define

$$\boxed{\ker f}(R) := \ker(f(R) \colon G(R) \to H(R)).$$

Claim: $\ker f$ is an affine group scheme.

PROOF Recall that a kernel can be defined categorically as a pullback

$$\begin{array}{ccc} G \times_H \mathrm{pt} = \ker f & \longrightarrow & \mathrm{pt} \\ \downarrow & & \downarrow \\ G & \longrightarrow & H \end{array}$$

We already decided how fiber products behave; the above definition is a special case.

More concretely, $k[\ker f] = k[G] \otimes_{k[H]} k$. Sometimes the right-hand side is written as

$$k[G]//k[H] \cong k[G]/f^*(I_H)k[G]$$

where $\boxed{I_H} = \ker(\epsilon\colon k[H] \to k)$, which is called the $\boxed{\text{augmentation ideal}}$ of H.

42 Example

Here are some maps of affine group schemes together with their kernels:

1. We have $\mathrm{SL}_n \hookrightarrow \mathrm{GL}_n \stackrel{\det}{\to} \mathbb{G}_m$. In particular, SL_n is the kernel of det, so is a subgroup scheme (indeed, a normal subgroup scheme).

2. $\mu_n \hookrightarrow \mathbb{G}_m \stackrel{-^n}{\to} \mathbb{G}_m$ similarly.

3. If $\operatorname{char} k = p$, then $\mathbb{G}_{a_{(1)}} \hookrightarrow \mathbb{G}_a \stackrel{-^p}{\to} \mathbb{G}_a$.

4. If $\operatorname{char} k = p$, then there is a map $F\colon \mathrm{GL}_n \to \mathrm{GL}_n$ called the $\boxed{\text{Frobenius map}}$ given by $(a_{ij}) \mapsto (a_{ij}^p)$. The kernel $\boxed{\mathrm{GL}_{n_{(r)}}} \hookrightarrow \mathrm{GL}_n \stackrel{F^r}{\to} \mathrm{GL}_n$ is called the $\boxed{r\text{th Frobenius kernel of } \mathrm{GL}_n}$.

43 Example

Let's consider the Hopf algebra structures associated to the kernel $\mu \hookrightarrow \mathbb{G}_m \stackrel{-^n}{\to} \mathbb{G}_m$. The corresponding sequence of Hopf algebras is

$$k[x]/(x^n - 1) \twoheadleftarrow k[x, 1/x] \leftarrow k[x, 1/x].$$

Let's verify $I = (x^n - 1)$ is a Hopf ideal in $k[x, 1/x]$:

$$\begin{aligned} \Delta(x^n - 1) &= \Delta(x)^n - \Delta(1) \\ &= (x \otimes x)^n - 1 \otimes 1 \\ &= x^n \otimes x^n - 1 \otimes 1 \\ &= x^n \otimes (x^n - 1) + (x^n - 1) \otimes 1. \end{aligned}$$

Hence indeed $\Delta(I) \subset I \otimes A + A \otimes I$. We won't take the time to check the conditions on ϵ or σ.

44 Remark

Correction: we defined a group-like element in a Hopf algebra A as one for which $\Delta(g) = g \otimes g$. This is a bit too liberal–for instance, 0 is group-like. The "correct" definition adds the condition that the counit is 1 on g, $\epsilon(g) = 1$.

45 Proposition

If A is a Hopf algebra and $g \in A$ is group-like, then $\sigma(g) = g^{-1}$, so g is invertible. Indeed, group-like elements form a subgroup in $A^\times$.

PROOF If $\Delta(g) = \sum g' \otimes g''$, then since $m \circ \mathrm{id} \otimes \sigma \circ \Delta = \epsilon \circ u$ (where $u\colon k \to A$ is the unit map sending 1 to 1), we have $\epsilon(g) = \sum g' \sigma(g'')$. Here $\epsilon(g) = 1$ and $\sum g' \otimes g'' = g \otimes g$, whence $1 = g\sigma(g)$; etc.

Definition 46. Let G be an affine group scheme. We define $\boxed{G^\vee} := \mathrm{Hom}(G, \mathbb{G}_m)$ as the $\boxed{\text{character group}}$ of G. Another notation for this is $\boxed{X(G)}$. Here Hom takes place in the category of group schemes.

47 Proposition

There is an isomorphism (of groups) between $G^\vee$ and the subgroup of $k[G]^\times$ consisting of group-like elements.

PROOF Let $\mu \in \mathrm{Hom}(G, \mathbb{G}_m)$ be a "character". In terms of Hopf algebras, it corresponds to a map $\mu^* \colon k[x, 1/x] \to k[G]$. Since x is group-like in $k[x, 1/x]$, $\mu^*(x)$ is group-like in $k[G]$; etc.

Note: One may define the group structure on $G^\vee$ by transfer of structure from the preceding proposition or using the structure of $\mathbb{G}_m$.

October 6th, 2014: Injective and Surjective maps; Images; Finite Groups and Group Schemes

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Summary Today's outline:

1) Surjective and injective maps of group schemes
2) Constant group schemes
3) Restriction and corestriction

48 Remark

We continue the proof which was continued at the start of last lecture:

PROOF Given $\Delta \colon A \to A \otimes A$, if $\Delta(a) = 0$ in $A/I \otimes A/I$, we need that $\Delta(a) \in I \otimes A + A \otimes I$.

The main ingredient is the following fact:

$$A/I \otimes A/I \cong A \otimes A/(I \otimes A + A \otimes I).$$

The two maps involved are as follows. First, the natural map $(x, y) \to \overline{x \otimes y}$ corresponds to a bilinear map $\phi \colon A/I \otimes A/I \to A \otimes A/(I \otimes A + A \otimes I)$. Second, we may take $A \otimes A \to A/I \otimes A/I$ via $x \otimes y \mapsto \overline{x} \otimes \overline{y}$; this evidently annihilates $I \otimes A + A \otimes I$. You can check the resulting induced map is the inverse of ϕ.

(One can also check that $\epsilon(I) = 0$ and $\sigma(I) \subset I$.)

Definition 49. A map $f \colon H \to G$ is $\boxed{\text{injective}}$ (in the category of affine group schemes) if $f^* \colon k[G] \to k[H]$ is surjective. $f \colon G \to H$ is $\boxed{\text{surjective}}$ (in the category of affine group schemes) if $f^* \colon k[H] \to k[G]$ is injective.

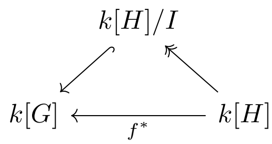
Injectivity is equivalent to $f(R) \colon H(R) \to G(R)$ being injective for all R, or in other words to f being a subgroup functor. Note that from the equivalence of categories, these are the correct categorical definitions of injective and surjective maps.

Gotcha: f surjective does *not* imply that $f(R) \colon G(R) \to H(R)$ is surjective for all R. (It "almost never" happens.)

50 Example

Consider $\mathbb{G}_m \xrightarrow{-^n} \mathbb{G}_m$ which takes an element to its nth power. The corresponding map $k[x, 1/x] \leftarrow k[x, 1/x]$ given by sending x to x^n is injective. Hence $\mathbb{G}_m \xrightarrow{-^n} \mathbb{G}_m$ is surjective. On the other hand, $R^\times \xrightarrow{-^n} R^\times$ is not always surjective. (For instance, let R be a non-perfect field extension of k.)

Definition 51. A map $f\colon G \to H$ of affine group schemes with corresponding map $f^*\colon k[G] \leftarrow k[H]$ of Hopf algebras with kernel I factors through $k[H]/I$:



Define the image (in the category of affine group schemes) $\operatorname{im} f$ as the group scheme represented by the Hopf algebra $k[H]/I$. This is the categorical image of the morphism by the equivalence of categories as before.

52 Remark

Let π be a finite group. The association $R \mapsto \pi$ is not a group scheme in general. We will next "do our best" to construct a constant group scheme.

Definition 53. Given a finite group π, set

$$A := k \times \cdots \times k = k^{\#\pi}.$$

Let e_g be the gth standard basis element of A ($g \in \pi$). We give A a Hopf algebra structure as follows.

- Define $\Delta\colon A \to A \otimes A$ by $e_g \mapsto \sum_{ht=g} e_h \otimes e_t$.
- Define $\sigma\colon A \to A$ by $e_g \mapsto e_{g^{-1}}$.
- Define $\epsilon\colon A \to k$ by $e_{\mathrm{id}} \mapsto 1$, $e_g \mapsto 0$ for $g \neq \mathrm{id}$.

Define the finite group scheme $\underline{\pi}$ as the group scheme represented by A.

54 Remark

Let $-^\#$ denote the linear dual (of a vector space, for now). Then $k[\underline{\pi}]^\# := \operatorname{Hom}_k(k[\underline{\pi}], k) \cong k\pi$ as algebras. (Here Hom_k refers to k-linear maps, as opposed to k-algebra maps.)

In general, let A be any finite dimensional Hopf algebra. Then $A^\sharp$ is also a Hopf algebra. (If A is infinite dimesional, we no longer have $(A \otimes A)^\sharp \cong A^\sharp \otimes A^\sharp$, which fundamentally breaks the naive construction of the dual Hopf algebra.) For more details, see my algebraic combinatorics notes under "Aside: Dual Algebras, Coalgebras, Bialgebras, and Hopf Algebras."

Note that $\operatorname{Hom}_k(k\pi, k)$ has a linear basis "e_g" defined via $e_g(h) = \delta_{g,h}$. The elements $\{e_g\}$ in $k[\underline{\pi}]$ are orthogonal idempotents. One can check that $\underline{\pi}(R) = \pi$ if R does not have any idempotents, and more generally that $\underline{\pi}(R) = \pi^c$ where c is the number of connected components of $\operatorname{spec} R$, i.e. $c = \#\pi_0(R)$. (If R is a field, c should be 1, but $\underline{\pi}(k)$ should always be trivial. What?)

Definition 55. An affine group scheme G is a finite group scheme if $k[G]$ is a finite dimensional k-algebra. (This unfortunately conflicts with what we've denoted $\underline{\pi}$ and called "finite group schemes". The difference is roughly "(finite group) scheme" vs. "finite (group scheme)".)

56 Example

For any finite group π, $\underline{\pi}$ is finite.

$k[\mu_n] = k[x]/(x^n - 1)$ is also finite (of dimension n), so μ_n is a finite group scheme. Is μ_n of the form $\underline{\pi}$? If we could find two fields K, K' such that $\mu_n(K) \ncong \mu_n(K')$, then no: if μ_n were of the form $\underline{\pi}$, these would all be isomorphic since the underlying topological spaces are. Over a field of characteristic 0, we will see that we can extend scalars and get $k[\mu_n] \cong \underline{\pi}$ for some finite group π.

$\mathbb{G}_{a_{(1)}} = k[x]/x^p$ (or the more general version from homework): these are finite group schemes, though they are not of the form $\underline{\pi}$.

October 8th, 2014: Extension of Scalars, Galois Descent, and Weil Restriction

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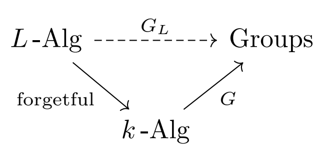
Summary Outline: restriction and corestriction; Galois descent.

57 Remark

First an algebraic detour. Fix a field extension L/k. We may extend scalars of (Hopf) algebras as usual: if H is a k-Hopf algebra, then $H \otimes_k L$ is an L-Hopf algebra. This is functorial. On the other hand, we may view the situation in the category of group schemes.

Definition 58. If L/k is a field extension, there is a "forgetful" functor L-Alg $\to$ k-Alg. Given a group scheme G over L, we form a functor G_L via



The assignment $G \mapsto G_L$ is a functor from the category of group schemes over k to the category of group schemes over L. Depending on your perspective, this functor is called restriction or extension of scalars.

59 Proposition

If L/k is a field extension and G is a group scheme over k, then $L[G_L] = k[G] \otimes_k L$.

Proof Let R be an L-algebra. Then

$$G_L(R) = G(R) \cong \mathrm{Hom}_{k\text{-Alg}}(k[G], R{\downarrow_k}) \cong \mathrm{Hom}_{L\text{-Alg}}(k[G] \otimes_k L, R).$$

60 Remark

Now that we've defined a functor k-GrSch $\to$ L-GrSch, our next target is to define a functor

$$\mathrm{Res}_{L/k} \colon L\text{-GrSch} \to k\text{-GrSch}$$

in the opposite direction called either Weil restriction or corestriction. The existence of this functor requires some assumptions on the field extension L/k; we will restrict to finite separable L/k. First, however, a digression on Galois descents.

Definition 61. We briefly review some standard facts about separable closures. Let k^{sep} be the separable closure of k; k^{sep}/k is Galois (though typically infinite). The absolute Galois group $\Gamma := \mathrm{Gal}(k^{\mathrm{sep}}/k)$

is the inverse limit of $\mathrm{Gal}(L/k)$ where L ranges over the finite separable extensions L/k. Γ is thus a profinite group (by definition). The fundamental theorem of Galois theory generalizes in the infinite case by means of adding a topology to the underlying Galois group called the Krull topology. Equivalently, we give the finite groups $\mathrm{Gal}(L/k)$ the discrete topology; the product of all of these groups has its usual topology; the inverse limit can be thought of as a subset of the product; we can define the topology on Γ as the corresponding subspace topology. The usual Galois correspondence in this case gives a bijection between intermediate fields $k^{\mathrm{sep}}/L/k$ and closed subgroups of Γ. Finite intermediate extensions $k^{\mathrm{sep}}/L/k$ correspond to open subgroups of Γ.

62 Fact

Let V be a k-vector space on which the absolute Galois group Γ above acts (possibly semi-linearly). Then Γ acts on V continuously if

$$V = \bigcup_{\substack{\text{some } H \subset \Gamma \\ H \text{ open}}} V^H.$$

Here V^H denotes the H-invariants of V. Note that U is open if it is closed of finite index.

63 Proposition

Let V be a vector space over k. Set $V_{k^{\mathrm{sep}}} := V \otimes_k k^{\mathrm{sep}}$. Then Γ acts on $V_{k^{\mathrm{sep}}}$ continuously via semi-linear (or Γ-equivariant) automorphisms. In particular, given $\gamma \in \Gamma$, $v \otimes x \in V_{k^{\mathrm{sep}}}$, then

$$\gamma \cdot (v \otimes x) := v \otimes \gamma x.$$

(Semi-linear transformations of a vector space are linear transformations except the scalars are "twisted" by an automorphism. Here, if $\gamma \in \Gamma$, $y \in k^{sep}$ and $\widetilde{v} \in V_{k^{\mathrm{sep}}}$, then $\gamma \cdot (y\widetilde{v}) = \gamma(y)\gamma(\widetilde{v})$, so the twisting automorphism is γ itself.)

Proof For semi-linearity, we let $\widetilde{v} = \sum v_i \otimes x_i$ and compute:

$$\gamma(y\widetilde{v}) = \gamma\left(y \sum v_i \otimes x_i\right) = \gamma\left(\sum v_i \otimes yx_i\right) = \sum v_i \otimes \gamma(yx_i) = \sum v \otimes \gamma(y)\gamma(x_i) = \gamma(y)\gamma(\widetilde{v}).$$

To show continuity, we show that $V_{k^{\mathrm{sep}}}$ is the union of fixed points for subgroups of finite index. Let $\widetilde{v} \in V_{k^{\mathrm{sep}}}$. For $\widetilde{v}$ above, there is some finite separable extension L/k which contains all the x_i (since there are finitely many of them). Then $\mathrm{Gal}(k^{\mathrm{sep}}/L)$ acts trivially on $\widetilde{v}$, so $\widetilde{v} \in V_{k^{\mathrm{sep}}}^{\mathrm{Gal}(k^{\mathrm{sep}}/L)}$, and this subgroup is closed of finite index.

64 Lemma (Galois Descent)

Let V be a vector space over k^{sep} such that Γ acts on V continuously via semi-linear automorphisms. Then we can reconstruct V in the following way: there is a k^{sep}-linear isomorphism

$$V^\Gamma \otimes_k k^{sep} \cong V$$

where $V^\Gamma := \{v \in V : \gamma \cdot v = v\}$.

Proof Exercise.

Definition 65. Let L/k be a finite separable field extension. Let G be a group scheme over L. Define

$$\boxed{\mathrm{Res}_{L/k}} : L\text{-GrSch} \to k\text{-GrSch}$$

by

$$(\mathrm{Res}_{L/k} G)(R) := G(R \otimes_k L).$$

This gives a well-defined functor k-Alg $\to$ Groups, though it's not at all clear that this is affine.

66 Example

Let $G = \mathbb{A}^1$ over L. Say L/k is separable of degree n. Then

$$(\mathrm{Res}_{L/k} \mathbb{A}^1)(R) = \mathbb{A}^1(R \otimes_k L) = (R \otimes_k L)^+.$$

If we pick a basis of L/k, we have $(R \otimes_k L)^+ \cong (R^{\oplus n})^+ = \mathbb{A}^n_k(R)$. Hence $\mathrm{Res}_{L/k}(\mathbb{A}^1) = \mathbb{A}^{[L:k]}$.

67 Example

$\mathbb{S} := \operatorname{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$ (the "torus group"; this is a "non-split" torus over $\mathbb{R}$; we'll get there). We have

$$\mathbb{S}(R) = \mathbb{G}_m(R \otimes_{\mathbb{R}} \mathbb{C}) = \mathbb{G}_m(R \oplus Ri).$$

The invertible elements are those of the form $a + bi$ for which $a^2 + b^2$ is a unit. If $R = \mathbb{R}$, we get $\mathbb{C}^\times$, a torus.

68 Proposition

$\operatorname{Hom}_L(H_L, G) \cong \operatorname{Hom}_k(H, \operatorname{Res}_{L/k} G)$, *i.e. Weil restriction is right adjoint to extension of scalars.*

Proof Exercise.

69 Theorem

The Weil restriction functor maps into the subcategory of affine group schemes. That is, $\operatorname{Res}_{L/k} G$ *is representable when* L/k *is separable.*

Proof Next time.

October 10th, 2014: Weil Restriction is Affine, Separable Algebras

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70 Remark

Today we'll begin by proving the theorem from the end of last time, that Weil restriction for a separable extension L/k is representable.

Proof Let $X := \{\tau : L \to k^{\text{sep}}\}$ be the set of embeddings of L in the separable closure (fixing k); this is finite. Let $\Gamma := \operatorname{Gal}(k^{\text{sep}}/k)$; Γ acts on X: $\tau \in X, \gamma \in \Gamma$, $x \in L$ gives $(\gamma \cdot \tau)(x) := \gamma(\tau(x))$. Let $A = L[G]$, $\tau \in X$. Then $A^\tau := A \otimes_\tau k^{\text{sep}}$. (This is $A \otimes_L k^{\text{sep}}$ using the embedding $\tau : L \to k^{\text{sep}}$. Explicitly, $sa \otimes x = a \otimes \tau(s)x$ for all $s \in L, a \in A, x \in k^{\text{sep}}$.)

Take $\widetilde{B} = \otimes_{\tau \in X} A^\tau$. This is a k^{sep}-algebra and we may define an action of $\gamma \in \Gamma$ by

$$\widetilde{\gamma}_\tau : A^\tau \to A^{\gamma\tau}$$

via

$$a \otimes x \mapsto a \otimes \gamma(x).$$

This is a k^{sep}-algebra homomorphism and we may "glue" them together to get a k^{sep}-algebra automorphism of $\widetilde{B}$. More precisely, $\gamma : \widetilde{B} \to \widetilde{B}$ is given by

$$\otimes a_\tau \mapsto \otimes a'_\tau$$

where $a'_{\gamma\tau} := \widetilde{\gamma}_\tau(a_\tau)$. Define B to be the Γ-invariants of $\widetilde{B}$, $B := \widetilde{B}^\Gamma$; this is a k-algebra. It also inherits the Hopf algebra structure of A (exercise).

We will show B represents $\operatorname{Res}_{L/k}(G)$. Recall

$$(\operatorname{Res}_{L/k} G)(R) = G(R \otimes_k L) = \operatorname{Hom}_L(A, R \otimes_k L).$$

We must show the right-hand term is given by $\operatorname{Hom}_k(B, R)$ (functorially). Claim: there is a natural isomorphism

$$\operatorname{Hom}_k(B, k) \cong \operatorname{Hom}_{k^{\text{sep}}}(B \otimes_k k^{\text{sep}}, R \otimes_k k^{\text{sep}})^\Gamma.$$

Proof: exercise. By Galois descent (from last time), $B \otimes_k k^{\text{sep}} \cong \widetilde{B}$ since $B = \widetilde{B}^\Gamma$. (One must check the algebra structure is also preserved by Galois descent.) So, we want to show that

$$\operatorname{Hom}_{k^{\text{sep}}}(\widetilde{B}, R \otimes_k k^{\text{sep}})^\Gamma \cong \operatorname{Hom}_L(A, R \otimes_k L).$$

Let $f : \widetilde{B} \to R \otimes_k k^{\text{sep}}$ be a Γ-invariant k^{sep}-algebra map. Since $\widetilde{B} = \otimes_\tau A^\tau$, this is equivalent to a collection of maps $f_\tau : A^\tau \to R \otimes_k k^{\text{sep}}$. The following commutes:

$$\begin{array}{ccc} A^\tau & \xrightarrow{f_\tau} & R \otimes_k k^{\mathrm{sep}} \\ \big\downarrow{\scriptstyle \widetilde{\gamma}_\tau} & & \big\downarrow{\scriptstyle 1\otimes\gamma} \\ A^{\gamma\tau} & \xrightarrow{f_{\gamma\tau}} & R \otimes_k k^{\mathrm{sep}} \end{array}$$

Now define $g_\tau := f_\tau|_A \colon A \to R \otimes_k k^{\mathrm{sep}}$ where $A \subset A^\tau$ is given by $A \otimes 1 \subset A \otimes_\tau k^{\mathrm{sep}}$. If $\gamma\tau = \tau$, then

$$(1 \otimes \gamma)g)\tau(a) = g_{\gamma\tau}(a) = g_\tau(a).$$

We have a tower of extensions $k^{\mathrm{sep}}/\tau L/k$. If $\gamma \in \mathrm{Gal}(k^{\mathrm{sep}}/\tau L)$, then $1 \otimes \gamma$ fixes $\mathrm{im}\, g_\tau \in R \otimes_k k^{\mathrm{sep}}$. One can check this implies $\mathrm{im}\, g_\tau \in R \otimes_k \tau L$. Finally define $g := (1 \otimes \tau^{-1})g_\tau$. Claim: g does not depend on τ and is L-linear. Proof: exercise.

Conclusion: we have a map $\mathrm{Hom}_{k^{\mathrm{sep}}}(\widetilde{B}, R \otimes_k k^{\mathrm{sep}})^\Gamma \to \mathrm{Hom}_k(A, R_L)$ given by sending f to $(1 \otimes \tau^{-1}) f_\tau|_A$. There is a map in the other direction given by sending $g \colon A \to R_L$ to $f = \otimes f_\tau$ where $f_\tau(a \otimes x) := [(1 \otimes \tau)(g(a)][1 \otimes x]$. Exercise: these are mutually inverse maps. One must also check they are functorial. This completes the proof.

Question: the above gives us a recipe for computing coordinate algebras of Weil restrictions. Using the example from last time, what is the coordinate algebra of $\mathrm{Res}_{\mathbb{R}}\, \mathbb{G}_m$ with $\mathbb{G}_m$ viewed over $\mathbb{C}$? Hao may tell us next time.

71 Example

Here are some special classes of group schemes; they appear in the structure theory we'll get to.

1. Constant finite group schemes $\underline{\pi}$.
2. Étale group schemes.
3. Diagonalizable group schemes.
4. Group schemes of multiplicative type.
5. (Cartier duality.)

72 Proposition

Let A be a finite dimensional commutative k-algebra. The following are equivalent:

(1) For all L/k, $A_L := A \otimes_k L$ is reduced. (That is, $\mathrm{nil}(A_L) = 0$*, i.e. there are no nonzero nilpotents in A_L.)*

(2) $A \cong L_1 \times \cdots \times L_n$ where each L_i/k is a finite separable field extension.

(3) $A \otimes_k k^{sep} \cong k^{sep} \times \cdots \times k^{sep}$.

Such an algebra is called a separable algebra *or an* étale algebra*. (Bourbaki give around seven equivalent conditions in chapter five; (3) above was probably the original. Milne's book on étale cohomology is another good source of information.)*

Proof Exercise.

October 13th, 2014: Étale Algebras and Finite Γ-Sets

Summary Today we will discuss étale algbras and étale group schemes. Recall that étale (or "separable") algebras were defined at the end of last lecture.

73 Remark

Geometrically, what is an étale algebra? "Étale" roughly means smooth of dimension 0. One usually encounters étale morphisms first, where the fibers are étale. Our condition is equivalent to the morphism $\operatorname{spec} A \to \operatorname{spec} k$ being étale.

Can we add another condition to the list from last lecture, namely "A is reduced"? Equivalently, can we have a reduced algebra which under scalar extension gets nilpotents? If k is a perfect field, then we can indeed add this condition.

As a counterexample for non-perfect fields, let $A = L$ be a non-separable extension of k, for instance use $k = \mathbb{F}_p(t)$ and $L = k[x]/(x^p - t)$. Now L is reduced, though if we adjoin a pth root of t, we get nilpotents:

$$L \otimes_k \overline{k} \cong \overline{k}[x]/(x - \sqrt[p]{t})^p.$$

74 Proposition

Étale algebras are closed under subalgebras, quotients, tensors, and products.

Proof Let A be étale. If $B \subset A$, then by condition (1), B is étale. If $B = A/I$ use (2) and look at all possible ideals. If B is also étale, use (2) for $A \times B$ and use (3) for $A \otimes_k B$.

75 Corollary

If L/k is a field extension and A is a k-algebra, then A is étale (over k) if and only if $A_L := A \otimes_k L$ is étale (over L).

Proof ($\Leftarrow$) If A_L is étale, choose $\overline{k} \subset \overline{L}$. Then $A_{\overline{L}}$ is reduced, so $A_{\overline{k}}$ is reduced since

$$\mathrm{nil}(A_{\overline{k}}) \otimes_{\overline{k}} \overline{L} \subset \mathrm{nil}(A_{\overline{k}}) \otimes_{\overline{k}} \overline{L}.$$

Then we claim that condition (1) is equivalent to the condition that $A \otimes_k \overline{k}$ is reduced (exercise). (Roughly, nilpotents are not transcendental.) Hence A is étale.

($\Rightarrow$) If A is étale, then $A \otimes_k k^{\mathrm{sep}} = k^{\mathrm{sep}} \times \cdots \times k^{\mathrm{sep}}$. Choose $k^{\mathrm{sep}} \subset L^{\mathrm{sep}}$; check that this implies $A_L \otimes_L L^{\mathrm{sep}} = L^{\mathrm{sep}} \times \cdots \times L^{\mathrm{sep}}$. (In general, the "splitting condition" implies a similar splitting condition for all larger extensions of scalars.)

Definition 76. Let $\Gamma := \mathrm{Gal}(k^{\mathrm{sep}}/k)$. Then Γ acts on a set X continuously iff $X = \cup X_i$ such that for all i there exists a finite Galois extension L_i such that the action of Γ on X_i factors through $\mathrm{Gal}(L_i/k)$.

We had a previous version of this definition a few lectures ago, which required letting X be the union of X^U over open subgroups U of Γ. These should be equivalent.

77 Theorem

There is an anti-equivalence of categories

$$\{\text{étale } k\text{-algebras}\} \leftrightarrow \{\text{finite } \Gamma\text{-sets}\}$$

where X is a finite Γ-set if $|X| < \infty$ and Γ acts (continuously) on X.

Proof The functor is as follows. Given an étale algebra A, consider $\mathrm{Hom}_{k\text{-Alg}}(A, k^{\mathrm{sep}})$. Γ acts on the second factor and turns this into a (finite) Γ-set. On the other hand, given a finite Γ-set X, consider $\mathrm{Hom}_{\mathrm{Sets}}(X, k^{\mathrm{sep}})$. Γ acts on $\mathrm{Hom}_{\mathrm{Sets}}(X, k^{\mathrm{sep}})$: if $\alpha\colon X \to k^{\mathrm{sep}}$ and $\gamma \in \Gamma$, then

$$(\gamma \cdot \alpha)(-) := \gamma\alpha(\gamma^{-1}-).$$

$\mathrm{Hom}_{\mathrm{Sets}}(X, k^{\mathrm{sep}})$ also inherits an algebra structure from k^{sep}, and this is compatible with the action above. We need to show the result is a separable algebra. To be continued.

October 15th, 2014: Étale Algebras and Finite Γ-Sets Continued

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78 Remark

There will be a couple of homework problems on Weil restriction: proving the proposition above that Weil restriction is right adjoint to extension of scalars, in addition to the following:

$$(\operatorname{Res}_{L/k} G)_{k^{\text{sep}}} = \prod_{[L:k]} G_{k^{\text{sep}}}.$$

79 Remark

At the end of last lecture, we stated an anti-equivalence of categories. We continue the proof now.

Proof Recall the functors suggested last time: $A \mapsto \operatorname{Hom}_{k\text{-Alg}}(A, k^{\text{sep}})$ with Γ acting on the second factor for the forward direction; for the backwards direction, send a finite Γ-set X to $\operatorname{Hom}_{\text{Sets}}(X, k^{\text{sep}})^\Gamma$, which is given a k-algebra structure.

Step 1: $\operatorname{Hom}_k(\prod_i L_i, k^{\text{sep}}) \cong \coprod_i \operatorname{Hom}_k(L_i, k^{\text{sep}})$. This occurs since the kernel of any map $\prod_i L_i \to k^{\text{sep}}$ with L_i/k each finite separable has kernel a maximal ideal (the image being a subfield of k^{sep}, which forces the kernel to kill all but one of the factors L_i. Similarly, for finite Γ-sets X, Y,

$$\operatorname{Hom}_{\text{Sets}}(X \coprod Y, k^{\text{sep}})^\Gamma = \operatorname{Hom}_{\text{Sets}}(X, k^{\text{sep}})^\Gamma \times \operatorname{Hom}_{\text{Sets}}(Y, k^{\text{sep}})^\Gamma.$$

Hence the functors interchange finite products and coproducts.

Step 2: $A \mapsto X_A \mapsto \operatorname{Hom}_{\text{Sets}}(X_A, k^{\text{sep}})$ is isomorphic to the identity functor. By Step 1, it suffices to show this for $A = L$ finite separable over k. Consider:

$$L \mapsto \operatorname{Hom}_{k\text{-Alg}}(L, k^{\text{sep}}) \mapsto \operatorname{Hom}_{\text{Sets}}(\operatorname{Hom}_{k\text{-Alg}}(L, k^{\text{sep}}), k^{\text{sep}})^\Gamma.$$

We must show this is naturally isomorphic to the identity functor. Observation: before we take Γ-invariants of the right-hand side, we get $L \otimes_k k^{\text{sep}}$ with Γ acting on k^{sep}, as follows. Given $a \otimes x \in L \otimes_k k^{\text{sep}}$, we send it to a map $\alpha\colon \operatorname{Hom}_{k\text{-Alg}}(L, k^{\text{sep}}) \to k^{\text{sep}}$ given by $\alpha(f) := f(a)x$. One may check this yields an isomorphism. Now take Γ-invariants and apply Galois descent. One must also check naturality.

Step 3: $A \mapsto X_A$ is surjective. Observe that $\operatorname{Hom}_{k\text{-Alg}}(L, k^{\text{sep}}) \cong \Gamma/\Gamma_L$ where $\Gamma_L = \operatorname{Stab}_\Gamma(L)$. Hence we are more or less sending L to its Galois group, or at least the corresponding quotient. Write X as the disjoint union of its orbits Γx_i. Each Γx_i is given by $\Gamma/\operatorname{Stab}_\Gamma(x_i)$ as Γ-sets. By Galois theory there is a corresponding extension $k \subset L_i \subset k^{\text{sep}}$ with $[L_i : k] = |\Gamma x_i|$ such that $\operatorname{Stab}_\Gamma(L_i) \cong \operatorname{Stab}_{L_i}(x_i)$. Hence

$$\operatorname{Hom}_{k\text{-Alg}}(L_i, k^{\text{sep}}) \cong \Gamma/\operatorname{Stab}_\Gamma(L) \cong \Gamma x_i.$$

(The continuity hypothesis is hiding in the "by Galois theory" comment.)

Julia originally thought we would be able to finish by a general abstract nonsense argument, but it seems we must prove a little more; see next lecture.

October 17th, 2014: Étale Algebras and Finite Γ-Sets Concluded; Equivalences of Categories

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80 Remark

We had hoped to give an abstract nonsense argument to finish the proof from last time, though it doesn't seem to quite work. Nonetheless, here are some useful basic properties related to (anti-)equivalences of categories.

We take the definition of an equivalence of categories between $\mathcal{A}$ and $\mathcal{B}$ to be the existence of functors $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{A}$ such that there is a natural isomorphism $\epsilon\colon \mathrm{id}_{\mathcal{A}} \to GF$ and a natural isomorphism $\eta\colon FG \to \mathrm{id}_{\mathcal{B}}$. Here ϵ is called the unit and η is called the counit.

81 Lemma

A single functor $F\colon \mathcal{A} \to \mathcal{B}$ determines an equivalence of categories if and only if:

1) *F is fully faithful–that is, the induced map $\mathrm{Hom}_A(X, Y) \to \mathrm{Hom}_B(F(X), F(Y))$ is a bijection;*

2) *F is essentially surjective–that is, for each $B \in \mathcal{B}$, there is a an object in the image of F isomorphic to B.*

This requires some form of the axiom of choice, which we ignore.

82 Lemma

Let F, G be an adjoint pair, i.e.

$$\mathrm{Hom}_{\mathcal{B}}(F(X), Y) \cong \mathrm{Hom}_{\mathcal{A}}(X, G(Y))$$

is a functorial bijection. Define $\epsilon\colon \mathrm{id}_A \to GF$ by sending $\mathrm{id}_{F(X)}$ in $\mathrm{Hom}_{\mathcal{B}}(F(X), F(X))$ to its image in $\mathrm{Hom}_{\mathcal{A}}(X, GF(X))$, and likewise define $\eta_{\mathcal{B}}\colon FG \to \mathrm{id}_{\mathcal{B}}$.

1) *$\epsilon\colon \mathrm{id}_A \to GF$ is an isomorphism if and only if F is fully faithful.*

2) *$\eta_{\mathcal{B}}\colon FG \to \mathrm{id}_{\mathcal{B}}$ is a natural isomorphism if and only if G is fully faithful.*

3) *(F, G) is an equivalence of categories if and only if F, G are fully faithful.*

83 Remark

We next continue the proof from the end of last lecture.

Proof Recall we were establishing an anti-equivalence between étale k-algebras and finite Γ-sets with continuous action. We had $F(A) := \mathrm{Hom}_{k\text{-Alg}}(A, k^{\mathrm{sep}})$ and $G(X) := \mathrm{Hom}_{\mathrm{Sets}}(X, k^{\mathrm{sep}})^{\Gamma}$. Here the functors are contravariant (which is confusing). We had shown that $GF(A) = \mathrm{Hom}_{\mathrm{Sets}}(F(A), k^{\mathrm{sep}})^{\Gamma}$ corresponds to A via $a \mapsto f_a$ where $f_a\colon \mathrm{Hom}_{k\text{-Alg}}(A, k^{\mathrm{sep}}) \to k^{\mathrm{sep}}$ via $f_a(\tau) := \tau(a)$. We also note that (1) $\dim_k A = |F(A)|$ and (2) $\dim_k G(X) = |X|$. For (1), we can reduce to A finite separable over k; then by the definition of separability, there are exactly $[A : k]$ embeddings of A into k^{sep}. For (2), by Galois descent,

$$\mathrm{Hom}_{\mathrm{Sets}}(X, k^{\mathrm{sep}})^{\Gamma} \otimes_k k^{\mathrm{sep}} \cong \mathrm{Hom}_{\mathrm{Sets}}(X, k^{\mathrm{sep}}).$$

The dimension over k^{sep} is $|X|$. To check that $A \to GF(A)$ is an isomorphism, it hence suffices to check that it is injective. If $f_a = 0$, then $f_a(\tau) = \tau(a) = 0$ for any $\tau \in \mathrm{Hom}_{k\text{-Alg}}(A, k^{\mathrm{sep}})$; it is now straightforward to show that $a = 0$.

In the other direction, let $\eta_X\colon X \to FG(X)$ by

$$\eta_X\colon X \to \mathrm{Hom}_{k\text{-Alg}}(G(X), k^{\mathrm{sep}})$$

where if $x \in X$, then $\eta_X(x)\colon \mathrm{Hom}_{\mathrm{Sets}}(X, k^{\mathrm{sep}})^{\Gamma} \to k^{\mathrm{sep}}$ by $\eta_X(x)(f) = f(x)$. As before, showing η_X is an isomorphism reduces to showing it is injective. If $x, y \in X$ and if $f(x) = f(y)$ for all $f \in \mathrm{Hom}_{\mathrm{Sets}}(X, k^{\mathrm{sep}})^{\Gamma}$, we must show $x = y$. Consider

$$f \otimes 1 \in \mathrm{Hom}_{Sets}(X, k^{\mathrm{sep}})^{\Gamma} \otimes_k k^{\mathrm{sep}}) = \mathrm{Hom}_{Sets}(X, k^{\mathrm{sep}}).$$

One can check from here that $x = y$. Let's call this good enough.

Definition 84. A group scheme G over k is an étale group scheme if $k[G]$ is étale. In particular, any étale group scheme is finite.

85 Corollary

There is an anti-equivalence of categories between étale group schemes over k and finite groups with continuous Γ-action.

Proof Group-like objects go to group-like objects in an equivalence of categories.

86 Example

If π is a finite group, the finite group scheme $\underline{\pi}$ is étale, since the algebra structure is just a finite product of copies of k. Indeed, the finite group schemes correspond to finite groups with trivial Γ-action under the above equivalence of categories.

By a constant group scheme, we mean a group scheme G with $k[G] = k^{\times n}$.) Let G be an étale group scheme. Then $G_{k^{\mathrm{sep}}}$ is a constant group scheme. One sometimes captures this idea by saying "étale group schemes are forms of constant group schemes". A group scheme is a form of a type of group scheme if the original group scheme is of the suggested type after some extension of scalars.

Consider μ_3 over $k = \mathbb{R}$. The coordinate algebra is $\mathbb{R}[\mu_3] = \mathbb{R}[x]/(x^3 - 1)$, which splits as a product of a degree 1 and a degree 2 extension of $\mathbb{R}$; in particular, it is not of the form $\underline{\pi}$. However, $\mathbb{C}[\mu_3] = \mathbb{C} \times \mathbb{C} \times \mathbb{C}$. Since there is just one group of order three, $(\mu_3)_{\mathbb{C}} \cong \underline{\mathbb{Z}/3}$. More generally, μ_n is a "form" of what finite group scheme?

October 20th, 2014: Diagonalizable Group Schemes; Multiplicative Type; Tori; Cartier Duality

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87 Fact

If k is of characteristic 0, then any finite group scheme is étale.

88 Remark

We next discuss two more classes of groups, namely the diagonalizable groups and the groups of multiplicative type.

Definition 89. Let Λ be an abelian group (not necessarily finite). Several weeks ago we defined a natural Hopf algebra structure on $k\Lambda$; the only obstruction to it being a commutative Hopf algebra was Λ being abelian. (Recall $g \in \Lambda$ was defined to be group-like, so $\Delta(g) := g \otimes g$ and $\epsilon(g) = 1$; the antipode was given by $g \mapsto g^{-1}$.)

This construction yields a functor $Ab \to k$-GrSch given by $\Lambda \mapsto \Lambda_{\mathrm{diag}}$ where Λ_{diag} is defined to be the affine group scheme corresponding to the Hopf algebra $k\Lambda$; for instance, $k[\Lambda_{\mathrm{diag}}] := k\Lambda$. A group scheme of the form Λ_{diag} is called diagonalizable.

Warning: $\Lambda_{\mathrm{diag}} \neq \underline{\Lambda}$ in general. For instance, the coordinate algebra of the former is $k\Lambda$, while the coordinate algebra of the latter is $k^{|\Lambda|}$ (when Λ is finite).

Here

$$\Lambda_{\mathrm{diag}}(R) = \mathrm{Hom}_{k\text{-Alg}}(k\Lambda, R) \cong \mathrm{Hom}_{\mathrm{Groups}}(\Lambda, R^\times).$$

90 Example

We see:

(i) $\mathbb{Z}_{\mathrm{diag}} = \mathbb{G}_m$, essentially because $\mathbb{Z}_{\mathrm{diag}}(R) = \mathrm{Hom}_{\mathrm{Groups}}(\mathbb{Z}, R) = R^\times$

(ii) $(\mathbb{Z}/n)_{\mathrm{diag}} \cong \mu_n$.

91 Theorem

Let G be a diagonalizable group scheme of finite type (that is, $k[G]$ is finitely generated as an algebra). Then

$$G \cong \mathbb{G}_m^{\times s} \times \mu_{n_1} \times \cdots \times \mu_{n_t}.$$

(This is just a reformulation of the structure theorem for finitely generated $\mathbb{Z}$-modules.)

92 Theorem (Criterion of Diagonalizability)

G is diagonalizable if and only if $k[G]$ is spanned by group-like elements.

Proof First, a lemma:

93 Lemma

Group-like elements in any Hopf algebra H are linearly independent.

Proof Let $\{a_1, \ldots, a_n\} \subset H$ be a minimal linearly dependent collection of distinct group-like elements. By renumbering, we can write a_1 as a linear combination of the rest, $a_1 = \sum_{i>1} \lambda_i a_i$. Applying the counit $\epsilon \colon H \to k$ gives

$$1 = \epsilon(a_1) = \sum_{i>1} \lambda_i \epsilon(a_i) = \sum_{i>1} \lambda_i.$$

Also apply the coproduct $\Delta \colon H \to H \otimes H$:

$$a_1 \otimes a_1 = \Delta(a_1) = \sum_{i>1} \lambda_i a_i \otimes a_i.$$

However, the left-hand side is just $\sum_{i,j>1} \lambda_i \lambda_j a_i \otimes a_j$. Since $\{a_2, \ldots, a_n\}$ are linearly independent by minimality, it follows that $\lambda_i \lambda_j = \delta_{ij} \lambda_i$. Since $\sum_{i>1} \lambda_i = 1$, some $\lambda_i \neq 0$. But then $\lambda_i \lambda_j = 0$ for $j \neq i$ forces $\lambda_j = 0$ for all $j \neq i$. That is, $a_1 = \lambda_i a_i = a_i$, contradicting distinctness.

Given the lemma, the collection of group-like elements essentially gives us the coordinate algebra, and they form a group, which allows us to reconstruct Λ. The details are left as an exercise.

Definition 94. G is a group scheme of **multiplicative type** if $G_{k^{\text{sep}}}$ is diagonalizable.

(If at any point it stops making sense to consider infinitely generated coordinate algebras, just restrict to finitely generated ones.)

95 Proposition

There is an anti-equivalence of categories

$$\{\text{group schemes of multiplicative type over } k\} \leftrightarrow \{\text{abelian groups with continuous } \Gamma\text{-action}\}.$$

Here $\Gamma := \mathrm{Gal}(k^{sep}/k)$. The functors are roughly given by

$$G \mapsto G_{k^{\text{sep}}}^{\vee}$$

and

$$\Lambda_{mult} \leftarrow\!\shortmid \Lambda$$

where $k[\Lambda_{mult}] := (k\Lambda \otimes_k k^{sep})^{\Gamma}$. On the level of functors, $\Lambda_{mult}(R) := \mathrm{Hom}_{\Gamma}(\Lambda, R_{k^{\text{sep}}}^{\times})$. (Here Hom_{Γ} denotes Γ-invariant maps of groups.)

Proof Exercise.

Definition 96. A group scheme T is called a **torus** if it is a group scheme such that

$$T_{k^{\text{sep}}} = \mathbb{G}_m \times \cdots \times \mathbb{G}_m.$$

(In particular, T is of multiplicative type.)

97 Example

Let $k = \mathbb{R}$ and define

$$T_a(R) := \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a^2 + b^2 = 1 \right\}.$$

As a Lie group, this is essentially SO_2. If we consider $(T_a)_{\mathbb{C}}$, we get $\mathrm{SO}(2, \mathbb{C}) = \mathbb{G}_m$, hence T_a is a torus. The coordinate algebra is

$$\mathbb{R}[T_a] = \frac{\mathbb{R}[x, y]}{(x^2 + y^2 - 1)}.$$

98 Fact

$\mathrm{Hom}_{\mathbb{R}\text{-GrSch}}(\mathrm{SO}_2, \mathbb{G}_m) = 1$. This is a homework problem.

Definition 99. A torus with no non-trivial maps to $\mathbb{G}_m$ is anisotropic. A torus $T \cong \mathbb{G}_m \times \cdots \times \mathbb{G}_m$ is called split. Hence any torus becomes split after we extend scalars to k^{sep}, though before we get there all sorts of things can happen, and this is roughly encoded in the action of the Galois group on the group-like elements.

Definition 100 (Cartier Duality). Let G be a finite abelian group scheme. Equivalently, the corresponding Hopf algebra $k[G]$ is commutative, cocommutative, and finite dimensional. The Cartier dual G^D is the group scheme corresponding to the Hopf algebra $k[G]^\sharp$. (Here $k[G]^\sharp$ refers to the dual Hopf algebra structure inherited from $k[G]$ mentioned above and discussed in greater detail in my algebraic combinatorics notes. For the naive construction, it is essential that $k[G]$ be finite dimensional.)

Claim: $G \mapsto G^D$ is a duality. For instance, we get a duality between étale abelian finite group schemes and finite abelian group schemes of multiplicative type. Both of these are (anti-)equivalent to finite abelian groups with continuous Γ-action.

October 22nd, 2014: Examples of Cartier Duality; Rational Representations and Comodules

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101 Remark

No lecture on Friday; there will be a guest lecturer on Monday.

102 Example

Here are some examples of Cartier duality.

- The Cartier dual of $\underline{\mathbb{Z}/n}$ is μ_n.
- In characteristic $p > 0$, $\mathbb{G}_{a_{(1)}}$, the first Frobenius kernel of $\mathbb{G}_a$, has coordinate algebra $k[x]/x^p$, and in fact is self-dual.

What about $\mathbb{G}_{a_{(2)}} := \{a \in R : a^{p^2} = 0\}$? Its coordinate algebra is $k[x]/x^{p^2}$, which is finite dimensional, so we can dualize it. One can check that the pth power of the dual of x will vanish, suggesting it is not self-dual. In fact,

$$k[\mathbb{G}_{a_{(2)}}]^\sharp \cong k\left[\frac{\partial}{\partial x}, \frac{\partial^p}{\partial x^p}\right] \Big/ \left(\left(\frac{\partial}{\partial x}\right)^p = \left(\frac{\partial^p}{\partial x^p}\right)^p = 0 \right).$$

Indeed, given $k[u_1, u_2]/(u_1^p, u_2^p)$, we have

$$\Delta u_2 = 1 \otimes u_2 + u_2 \otimes 1 + \sum \frac{1}{p} \binom{p}{i} u_1^i \otimes u_1^{p-i}$$

where the right-hand side is a bit of an abuse of notation: we compute $\frac{1}{p}\binom{p}{i} \in \mathbb{Z}$ and take the result mod p in k. In any case, u_2 is not primitive.

The ring of Witt vectors $W_n(R)$ has an addition formula similar to the above coproduct. We can define W_n as an abelian group scheme $R \mapsto W_n(R)^+$. It happens that $\mathbb{G}_{a_{(2)}} = W_{2_{(1)}}$, where $_{(1)}$ denotes the first Frobenius kernel, which has a natural interpretation in this context. For instance,

$$W_2(R) := \{(x_1, x_2) : x_1, x_2 \in R\}$$

but the group operation is given by

$$(x_1, x_2) + (y_1, y_2) := (x_1 + y_1, x_2 + y_2 + \frac{(x_1 + y_1)^p - x_1^p - y_1^p}{p}).$$

(Again, dividing by p is an abuse of notation; the right-most term here means the same thing as the binomial formula sum above.) There is a duality which gives $W_{n_{(r)}} \cong W_{r_{(n)}}$. Since $W_1 = \mathbb{G}_a$, the $n = 1, r = 2$ case of the claim is $\mathbb{G}_{a_{(2)}} \cong W_{1_{(2)}} \cong W_{2_{(1)}}$.

Definition 103. We next define representations of affine group schemes G over k. There are three equivalent notions. Let V be a k-vector space.

(1) We can associated to V the group scheme $\mathbb{V}$ defined by $\mathbb{V}(R) := V \otimes_k R =: V_R$ (additively). If $\dim_k V = n$, then $\mathbb{V} \cong \mathbb{A}^n$. We say V is a **rational representation** of G if for all R we have a group action $G(R) \times V_R \to V_R$ which is functorial in R. Equivalently, this is an action of schemes $G \times \mathbb{V} \to \mathbb{V}$.

104 Remark

The term "rational" indicates that we want a family of actions functorial in R rather than just a single action for a particular R. Not every action of a particular R extends to a functorial action. For instance, $\mathrm{GL}_n(\mathbb{F}_p)$-mod $\neq \mathrm{GL}_n$-mod. Indeed, the right-hand category has no projective modules, and the left-hand category has free, hence projective, modules. GL_n-mod doesn't really have an analogue of the group algebra, though it does have enough injectives. On the other hand, injectives and projectives coincide in $\mathrm{GL}_n(\mathbb{F}_p)$-mod.

(2) V is a rational representation of G if we have a map of group schemes $G \to \mathrm{GL}(V) \cong \mathrm{Aut}(V)$. If $\dim_k V = n$, this is the same as a map $G \to \mathrm{GL}_n$.

(3) Let $A := k[G]$. Given a rational representation from (1), in particular we have a map $G(A) \times (M \otimes A) \to M \otimes A$. Hence $\xi \in G(A) = \mathrm{Hom}_{k\text{-Alg}}(A, A)$ associated to the identity map id_A acts on $M \otimes A$, and we can define the composite

$$\Delta_V := V \to V \otimes A \xrightarrow{\xi} V \otimes A$$

given by $v \mapsto v \otimes 1 \mapsto \xi \cdot (v \otimes 1)$.

A has a coalgebra structure given by the coproduct Δ and the counit ϵ. We call Δ_V a **comodule map** since it fits in the following diagrams:

$$\begin{array}{ccc} V & \xrightarrow{\Delta_V} & V \otimes A \\ {\scriptstyle \Delta_V}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_V \otimes \Delta} \\ V \otimes A & \xrightarrow[\Delta_V \otimes \mathrm{id}_A]{} & V \otimes A \otimes A. \end{array}$$

and

$$\begin{array}{ccc} V & \xrightarrow{\Delta_V} & V \otimes A \\ & {\scriptstyle \sim}\searrow & \downarrow{\scriptstyle 1 \overline{\otimes} \epsilon} \\ & & V \end{array}$$

Definition 105. For any coalgebra A over k and k-vector space V with a k-linear map $\Delta_V : V \to V \otimes A$ satisfying the two diagrams above, V is called an $\boxed{A\text{-comodule}}$. Next time we'll discuss how to translate between these three notions.

October 27th, 2014: Basic Comodule Theory

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106 Remark

Jim Stark is subbing today.

Let $G = \mathrm{Hom}_{k\text{-Alg}}(k[G], -)$ be a group scheme over k, M a k-module. We saw three ways to make M into a representations of G last time:

1) A group scheme homomorphism $G \to \mathrm{GL}_M$;
2) A functorial sequence of maps $G \times (M \otimes -) \to (M \otimes -)$;
3) A k-linear map $\rho\colon M \to M \otimes k[G]$ such that

$$(\rho \otimes \mathrm{id})\rho = (\mathrm{id} \otimes \Delta)\rho \qquad\qquad (\mathrm{id} \overline{\otimes} \epsilon)\rho = \mathrm{id}\,.$$

(M is a "comodule" for the Hopf algebra.)

How does one translate between them?

- (1) $\leftrightarrow$ (2): given $\Phi\colon G \to \mathrm{GL}_M$, define $\Psi\colon G \times (M \otimes -) \to (M \otimes -)$ via $\Psi(g, m) := \Phi(g)(m)$, and conversely given Ψ define $\Phi(g) = \Psi(g, -)$.
- (1), (2) $\to$ (3): consider the linear map $M \otimes k[G] \to M \otimes k[G]$ given by $\Phi(\mathrm{id}_{k[G]})$; precompose with the natural map $M \to M \otimes k[G]$ given by $m \mapsto m \otimes 1$. Alternatively, replace $\Phi(\mathrm{id}_{k[G]})$ by $\Psi(\mathrm{id}_{k[G]}, -)$.
- (3) $\to$ (1), (2): given ρ, define $\Psi(g, m \otimes 1) := (\mathrm{id} \otimes g)\rho(m)$; extend linearly to $M \otimes R$.

We next define basic representation-theoretic operations (eg. tensor products of representations) in terms of comodules.

Definition 107. A k-submodule $N \subset M$ is a $\boxed{\text{subcomodule}}$ if $\rho(N) \subset N \otimes k[G]$. A quotient comodule is given by

$$\begin{array}{ccc} M & \xrightarrow{\rho} & M \otimes k[G] \\ \downarrow & & \downarrow \\ M/N & \overset{\exists!}{\dashrightarrow} & M/N \otimes k[G] \end{array}$$

We write $\rho_{M/N}$ for the dashed arrow at the bottom; this is a quotient map.

For direct sums, use the composite

$$M \oplus N \overset{\rho_M, \rho_N}{\to} (M \otimes k[G]) \oplus (N \otimes k[G]) \cong (M \oplus N) \otimes k[G].$$

For tensor products, use the composite

$$M \otimes N \overset{\rho_M \otimes \rho_N}{\to} M \otimes k[G] \otimes N \otimes k[G] \to M \otimes N \otimes k[G],$$

where the second map sends $m \otimes a \otimes n \otimes b$ to $m \otimes n \otimes ab$.

In fact, the category of A-comodules is an abelian category.

108 Example

We always have certain representations given a Hopf algebra A with coproduct $\Delta\colon A \to A \otimes_k A$ and unit $\eta\colon k \to A$:

1. Take $\rho := \Delta$ and $M := k[G]$. This is called the regular representation.

2. $\rho := \eta\colon k \to k[G] \cong k \otimes k[G]$; this is the trivial representation.

3. The standard representation of GL_n, namely the GL_n-module k^n with basis $\{e_i\}$ and $\rho\colon k^n \to k^n \otimes k[x_{ij}, 1/\det]$ given by $\rho(e_j) := \sum_i e_i \otimes x_{ij}$.

   For instance, given the matrix $g = [1, 2; 0, 3] \in \mathrm{GL}_2(k)$, this is represented by the homomorphism $k[\mathrm{GL}_2] \to k$ given by $x_{11} \mapsto 1, x_{12} \mapsto 2, x_{21} \mapsto 0, x_{22} \mapsto 3$. Then $g \cdot [0; 1]$ is given by $\rho(e_2) = e_1 \otimes x_{12} + e_2 \otimes x_{22}$, which is $e_1 \cdot 2 + e_2 \cdot 3$, which is $[2; 3]$.

109 Lemma

Let M be a finite-dimensional comodule for a Hopf algebra A. Suppose $\{m_i\}$ is a basis for M and write $\rho(m_j) =: \sum_i m_i \otimes a_{ij}$. Then $\Delta(a_{ij}) = \sum_k a_{ik} \otimes a_{kj}$. (Hence, for these elements, comultiplication looks like matrix comultiplication.)

Proof Having chosen a basis, we have a map $\Phi\colon G \to \mathrm{GL}_n$ with induced map of Hopf algebras Φ^* fitting into the diagram

$$\begin{array}{ccc} k[\mathrm{GL}_n] & \xrightarrow{\Phi^*} & k[G] \\ \downarrow{\scriptstyle \Delta_{\mathrm{GL}_n}} & & \downarrow{\scriptstyle \Delta_G} \\ k[\mathrm{GL}_n] \otimes k[\mathrm{GL}_n] & \xrightarrow{\Phi^* \otimes \Phi^*} & k[G] \otimes k[G] \end{array}$$

Run x_{ij} through the diagram; from the left-hand arrow, we get $\sum_k x_{ik} \otimes x_{kj}$; from the top arrow, we get find that $\Phi(\mathrm{id}_{k[G]})$ is given by the matrix $[a_{ij}]$, so the morphism that represents that is given by sending x_{ij} to a_{ij}, so unwinding Yoneda's lemma, $\Phi^*(x_{ij}) = a_{ij}$. The result follows.

Definition 110. Modules for affine group scheme are locally finite, meaning the following theorem is true:

111 Theorem

Let M be a comodule for a Hopf algebra A. For any $m \in M$, there is a finite dimensional submodule $N \subset M$ which contains m.

Proof Let $\{a_i\}$ be a basis for $k[G]$. Write $\rho(m) =: \sum_i m_i \otimes a_i$ and $\Delta(a_i) =: \sum_{j,k} c_{ijk} a_j \otimes a_k$. Note that only finitely many m_i are nonzero. So, $N := \mathrm{Span}(\{m\} \cup \{m_i\}_i)$ is finite-dimensional. We need to show N is a comodule, i.e. $\rho(N) \subset N \otimes k[G]$. By definition this holds for m; we must show it holds for the m_i. Plug m into $(\rho \otimes \mathrm{id})\rho = (\mathrm{id} \otimes \Delta)\rho$, yielding

$$\sum_i \rho(m_i) \otimes a_i = \sum_i m_i \otimes \Delta(a_i) = \sum_{i,j,k} c_{ijk} m_i \otimes a_j \otimes a_k.$$

Since we're tensoring over a field, grouping terms of the form $- \otimes a_i$ gives $\rho(m_i) = \sum_{i,j} c_{iji} m_i \otimes a_j \in N \otimes k[G]$.

112 Remark

Is there a more memorable way to phrase the comodule axiom used above? It says the following two composites are the same:

$$M \xrightarrow{\rho} M \otimes k[G] \underset{\text{id} \otimes \Delta}{\overset{\rho \otimes \text{id}}{\rightrightarrows}} M \otimes k[G] \otimes k[G].$$

But this is just a reversed version of saying that we can either act twice or multiply and act once, hence the term "comodule".

113 Fact

Hopf algebras are "locally" finitely generated. That is, for any $a \in k[G]$, there exists a sub-Hopf algebra $A \subset k[G]$ which is finitely generated (as an algebra) and contains a.

PROOF Look at the regular representation; use local finiteness; throw in the images under the antipode to get closure.

114 Remark

By taking (internal) direct sums, we can extend the previous two results to finite sets of elements m or a.

Definition 115. G is an algebraic group scheme if $k[G]$ is a finitely generated k-algebra. Next time, we will show that any algebraic G can be embedded in GL_n for some n large enough.

October 29th, 2014: Affine Algebraic Groups are Linear; Left Regular, Right Regular, and Adjoint Representations

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116 Remark

Recall that an affine group scheme G is algebraic if $k[G]$ is a finitely generated k-algebra.

117 Theorem

Any affine algebraic group scheme G is a closed subgroup of GL_n (up to isomorphism), i.e. there exists a closed embedding of group schemes $G \hookrightarrow \mathrm{GL}_n$.

118 Remark

An algebraic group is an algebraic group scheme which is smooth (reduced), so dropping the word "scheme" gives a much stronger condition than including it.

PROOF Let $A := k[G]$, $f_1, \ldots, f_m$ algebraic generators of A. By the local finiteness theorem from last time, there exists a finite dimensional subcomodule $V \subset A$ which contains all the algebraic generators. (Here A is viewed as a comodule using the regular representation $\rho := \Delta$.) That is, $\Delta|_V : V \to V \otimes A$. Let $\{v_i\}$ be a linear basis of V; write $\Delta(v_i) =: \sum_j v_j \otimes a_{ij}$. The corresponding Hopf algebra map $k[\mathrm{GL}_n] \to k[G]$, i.e. $k[x_{ij}, 1/\det] \to A$, is determined by $x_{ij} \mapsto a_{ij}$. Claim: this map is surjective.

By the counit axiom, $v_i = \sum_j \epsilon(v_j) a_{ij}$, so each v_i is in the image, so each f_i is in the image, and since this is a map of algebras, surjectivity follows. Hence the corresponding map of groups $G \to \mathrm{GL}_n$ is a closed embedding.

(A "closed embedding" here means a closed embedding of schemes, which happens to coincide with surjectivity of the corresponding map of Hopf algebras.)

119 Remark

Let G be a discrete group, X a set on which G acts. Set $k[X] := \mathrm{Mor}(X, k)$, which can be given the structure of a k-algebra by acting in the second coordinate. Define an action $G \times k[X] \to k[X]$ by $(g, f)(-) \mapsto f(g^{-1}-)$; this is called the left regular action on $k[G]$. Indeed, X has compatible left and right actions: $(gf)(-) := f(-g)$ also works, giving the right regular action. Finally, we have the adjoint action $(gf)(-) := f(g^{-1} - g)$. (Confusingly, these are each left actions.)

120 Example

$X = G$ has compatible left and right actions, so we may use any of the three actions on $k[G]$ above.

A Hopf algebra $k[G]$ is a $k[G]$-comodule using its own coproduct. Indeed, we can give three different $k[G]$-comodule structures to $k[G]$. Letting $R \in k\text{-Alg}$, we define (functorial) maps $G(R) \times k[G]_R \to k[G]_R$ as follows. Take $k[G] := \mathrm{Mor}(G, \mathbb{A}^1)$, $k[G]_R := \mathrm{Mor}_R(G_R, \mathbb{A}^1_R)$. For $f \in k[G]_R, g \in G(R)$, define the following for all $x \in G_R(R')$ where $R' \in R\text{-Alg}$:

$$(g \cdot f)(x) := \begin{cases} f(g^{-1}x) & \boxed{\text{left regular representation}} \\ f(xg) & \boxed{\text{right regular representation}} \\ f(g^{-1}xg) & \boxed{\text{adjoint representation}} \end{cases}$$

(Here xg for instance means we send g through the map $G(R) \to G(R')$ induced by $R \to R'$ and act on $x \in G_R(R')$ by the result.)

Now Δ corresponds to the right regular representation. The left regular representation corresponds to $\tau \circ (\sigma \otimes \mathrm{id}) \circ \Delta$ where σ is the coinverse and τ is the usual twist map. Explicitly, $f \mapsto \sum f_2 \otimes \sigma(f_1)$ where $\Delta(f) = \sum f_1 \otimes f_2$. The adjoint representation corresponds to $f \mapsto \sum f_2 \otimes \sigma(f_1) f_3$ where $(\mathrm{id} \otimes \Delta) \circ \Delta(f) = \sum f_1 \otimes f_2 \otimes f_3$.

Exercise: prove this. (It's computed in Janzten's book, I2.8.)

October 31st, 2014: G-modules and $G(k)$-modules; Fixed Point Functor

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Summary Today's outline:

- "Universality" of regular representation
- Fixed point functor
- Representations of diagonalizeable groups

121 Remark

We will se the phrase "G-modules" to refer to representations of G. There is a difference between G-modules and $G(k)$-modules. For instance, let G be a group scheme over $\mathbb{F}_p$. If G is a "matrix group", then $G(\mathbb{F}_p)$ is a finite group. Such groups are ubiquitious in the classification of finite simple groups. In any case, $G(\mathbb{F}_p)\text{-mod} \neq G\text{-mod}$. For instance, there are enough projectives on the left and no projectives on the right.

122 Remark

Let G be an affine group scheme over $\mathbb{F}_p$. You can restrict G-modules to $G(\mathbb{F}_p)$-modules, which is essentially a "forgetful" functor. However, how one lifts $G(\mathbb{F}_p)$-modules back up to G-modules is not clear. For Lie groups, the representation theory of G is very similar to the representation theory of $\mathrm{Lie}\, G$, so the corresponding functor is significantly nicer than the forgetful functor. There is no known nice way to go from $\mathrm{Lie}\, G$-modules to $G(\mathbb{F}_p)$-modules. The lifting (or lack thereof) of $\mathrm{Lie}\, G$-modules or $G(\mathbb{F}_p)$-modules is an open question. For reasonably nice G, you can describe irreducible representations for $G(\mathbb{F}_p)$ and for $\mathrm{Lie}\, G$, and the lists happen to be identical, and they all lift to the same G-modules. The same is true for the projective covers of irreducible modules, though the same is not known for general projective modules.

123 Remark

If $\operatorname{char} k = p > 0$, $\operatorname{Lie} G$ in a certain sense is the same as the kernel $G_{(1)}$ of the Frobenius map $F\colon G \to G$. In particular, the category of representations of $\operatorname{Lie} G$ (which we will define formally later) is equivalent to the category of representations of $G_{(1)}$. $\operatorname{Lie} G$ will be a purely algebraic version of the tangent space at the identity. Indeed, one can consider $G_{(2)}$ or $G_{(3)}$, etc., which grow. In this sense, the Lie algebra is only the first kernel, whereas the remaining kernels contain other information, so it's not terribly surprising the $\operatorname{Lie} G$-representations don't fully capture the G-representations.

124 Remark

Take $k = \overline{k}$. Assume $k[G]$ is reduced, i.e. contains no nilpotents. In this case the scheme is recovered by is $\overline{k}$-valued points. In this case G-mod is equivalent to (rational) $G(k)$-mod. Rational modules are group objects in the category of $G(k)$-mod. The easiest way to show this is to go from rational $G(k)$-modules to $k[G(k)]$-comodules, where $k[G(k)] = k[G]$.

(In characteristic 0, any affine group scheme is automatically reduced.)

125 Theorem

Any finite dimensional G-module can be embedded in $\oplus_n k[G]$, where $k[G]$ denotes the right regular representation of G.

Proof Let V be a finite dimensional G-module. Let $\Delta_V\colon V \to V \otimes_k k[G]$ be the corresponding comodule map. Set $M := V \otimes_k k[G]$; M is also a G-module via $M \overset{\mathrm{id} \otimes \Delta}{\to} M \otimes_k k[G]$ where Δ is comultiplication in $k[G]$. We showed last time that $k[G] \overset{\Delta}{\to} k[G] \otimes k[G]$ is the right regular representation of G. Hence $M \cong k[G]^{\oplus \dim_k V}$. We claim Δ_V is an injective map of G-modules:

Definition 126. If V and W are comodules with $f\colon V \to W$, f is a comodule map if

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \downarrow{\scriptstyle \Delta_V} & & \downarrow{\scriptstyle \Delta_W} \\ V \otimes A & \xrightarrow{f \otimes 1} & W \otimes A \end{array}$$

commutes.

That Δ_V is a comodule map is essentially a tautology here. That it's injective follows from the second axiom, namely

$$\begin{array}{ccc} V & \xrightarrow{\Delta_V} & V \otimes A \\ & \searrow^{\sim} & \downarrow{\scriptstyle \mathrm{id}\,\overline{\otimes}\epsilon} \\ & & V \end{array}$$

This essentially tells you that Δ_V is split. More concretely, $(\mathrm{id}\,\overline{\otimes}\epsilon)(\Delta_V(v)) = v$, so Δ_V is injective. The same proof seems to work without finite dimensionality; we just embed our G-module in a sum of $k[G]$'s indexed by $\dim_k V$.

Definition 127. Let G be an affine group scheme, M a G-module. The invariants under the action of G are defined as

$$\boxed{M^G} := \{m \in M : g(m \otimes 1) = m \otimes 1\}.$$

Here $g \in G(R)$ for $R \in k$-Alg and $m \otimes 1 \in M \otimes R$, and the above condition must hold functorially as R varies.

Claim: $M^G = \{m \in M : \Delta_M(m) = m \otimes 1\}$ when we view M as a comodule. Indeed, for $R \in k$-Alg, $g \in G(R)$, the composite

$$M^G \xrightarrow{\Delta_m} M \otimes A \xrightarrow{1\otimes g} M \otimes R$$

is $m \mapsto m \otimes 1 \mapsto m \otimes 1$.

128 Proposition

Let G be an affine group scheme.

(1) $-^G\colon G\text{-mod} \to k\text{-mod}$ given by $M \mapsto M^G$ is a functor.

(2) $-^G$ is left exact.

November 3rd, 2014: Characters, Restriction, and Induction

Summary Today's outline:

(1) Representations of diagonalizable groups

(2) Induction

(3) Injective modules

129 Remark

Suppose G is any affine group scheme. Given a character $\lambda \in G^\vee$ (so $\lambda\colon G \to \mathbb{G}_m$) and a G-module M, we can define

$$M_\lambda := \{m \in M : g(m \otimes 1) = m \otimes \lambda(g)\}$$

where $g \in G(R)$, $m \otimes 1 \in M \otimes R$, and $m \otimes \lambda(g) \in M \otimes R^\times$ for a particular $R \in k\text{-Alg}$.

This is a slight generalization of the definition of invariants from before, where then λ was the trivial character sending everything to 1.

Claim:

$$M_\lambda = \{m \in M : \Delta_M(m) = m \otimes \lambda\}.$$

where $m \otimes \lambda \in M \otimes k[G]$, viewing a character equivalently as a group-like element of the corresponding Hopf algebra. Also, $\sum_{\lambda \in G^\vee} M_\lambda \subset M$ is a direct sum (i.e. the pairwise intersections are trivial). Fundamentally this comes from the linear independence of characters. The inclusion may in general be strict.

130 Proposition

Let $G := \Lambda_{\text{diag}}$ be a diagonalizable group scheme with Λ an abelian group. Suppose M is a G-module with comodule map $\Delta_M\colon M \to M \otimes_k k\Lambda$ given by $m \mapsto \sum_{\lambda\in\Lambda} m_\lambda \otimes \lambda$. This allows us to define projection maps $\rho_\lambda\colon M \to M$ by $m \mapsto m_\lambda$.

We have $\rho_\lambda\rho_{\lambda'} = 0$ if $\lambda \neq \lambda'$, $\sum_{\lambda\in\Lambda} \rho_\lambda = 1$, $\rho_\lambda^2 = \rho_\lambda$, so they are an orthogonal system of idempotents. It follows that $M = \oplus_\lambda \rho_\lambda(M)$ and $\rho_\lambda(M) = M_\lambda$, so that M is just a direct sum of its M_λ's,

$$M = \bigoplus_{\lambda\in\Lambda} M_\lambda.$$

Definition 131. Continuing the notation of the previous proposition, let $\{e^\lambda\}$ be a $\mathbb{Z}$-basis for $\mathbb{Z}\Lambda$ for which $e^{\lambda+\mu} = e^\lambda e^\mu$. (Julia seems to intend us to simply use the natural basis Λ for $\mathbb{Z}\Lambda$ here, just with a different notation.) We define the $\boxed{\text{character of } M}$ as

$$\boxed{\operatorname{ch} M} := \sum_\lambda (\dim M_\lambda) e_\lambda \in \mathbb{Z}\Lambda.$$

If M is finite dimensional, this sum has finitely many non-zero terms. One must use more care in the infinite-dimensional case, which we will not discuss further. The character behaves well with respect to many constructions: exact sequences, scalar extension, etc.

Definition 132. Let H be a subgroup scheme of a group scheme G. Define a restriction functor (or "forgetful functor")

$$\operatorname{Res}_H^G : G\text{-mod} \to H\text{-mod}.$$

On the level of comodules, we use the composite

$$\Delta_M^H := M \overset{\Delta_M^G}{\to} M \otimes k[G] \to M \otimes k[H]$$

as the new comodule map.

133 Remark

Claim: Res_H^G has a right adjoint called the induction functor,

$$\operatorname{Ind}_H^G : H\text{-mod} \to G\text{-mod},$$

i.e. there is a natural isomorphism

$$\operatorname{Hom}_{H\text{-GrSch}}(\operatorname{Res}_H^G N, M) \cong \operatorname{Hom}_{G\text{-GrSch}}(N, \operatorname{Ind}_H^G M).$$

Warning: The terminology is confused in that our "induction" is called "coinduction" for finite groups. Be careful you know which is meant. In our context, there is also sometimes another adjoint called coinduction, and likewise for finite groups, though "induction" and "coinduction" are switched in the two contexts.

Definition 134. Let H be a subgroup scheme of a group scheme G. Define the induction functor

$$\operatorname{Ind}_H^G : H\text{-mod} \to G\text{-mod},$$

by

$$\operatorname{Ind}_H^G(M) := (M \otimes k[G])^H.$$

Here $G \times H$ acts on $M \otimes k[G]$ where:

(1) H acts as given on M and via the right regular representation on $k[G]$,

$$h \cdot (m \otimes f(-)) := (h \cdot m) \otimes f(-h).$$

(This notation is as usual rather informal since it must be extended functorially to all $R \in k\text{-Alg}$.)

(2) G acts identically on M and via the left regular representation on $k[G]$,

$$g \cdot (m \otimes f(-)) := m \otimes f(g^{-1}-).$$

One must check the actions of G and H commute. In any case, G acts on $(M \otimes k[G])^H$.

135 Notation

We will use Hom for "algebraic" morphisms and Mor for "geometric" ones.

136 Remark

We can view $k[G]$ as $\operatorname{Mor}(G, \mathbb{A}^1)$, so we can view $M \otimes k[G]$ as $\operatorname{Mor}(G, \mathbb{M})$ where $\mathbb{M} \cong \mathbb{A}^{\dim M}$ as before.

$G \times H$ acts on $\operatorname{Mor}(G, \mathbb{M})$ as follows. If $f : G \to \mathbb{M}$, then

$$(g,h)f(x) := hf(g^{-1}xh).$$

$\operatorname{Ind}_H^G M := \{f \in \operatorname{Mor}(G, \mathbb{M}) : f(x) = hf(xh)\}$ for $x \in G(R)$, $h \in H(R')$ and $R \to R'$ is a morphism in k-Alg. Sometimes it's more convenient to write this as

$$\{f \in \operatorname{Mor}(G, \mathbb{M}) : h^{-1} f(x) = f(xh)\}.$$

$G(R) \otimes H(R)$ acts on

$$M \otimes k[G] \otimes R \cong (M \otimes_k R) \otimes_R (k[G] \otimes_k R) \cong \operatorname{Mor}_{R\text{-GrSch}}(G_R, (\mathbb{M} \otimes R))$$

Exercise: take the definition of coinduction for finite groups and verify it agrees with the above definition of induction when it can.

137 Remark
For any M, we have a map $\epsilon_M \colon \operatorname{Ind}_H^G M \overset{1 \overline{\otimes} \epsilon}{\to} M$ given by $m \otimes f \mapsto \epsilon(f) \otimes m$. This is a map of H-modules called the evaluation map.

November 5th, 2014: Induction and the Tensor Identity

Summary Today's outline:

(1) Induction

(2) Injective modules

138 Remark
Recall we gave two definitions of the induction functor. That is, if $H \leq G$, then $\operatorname{Ind}_H^G M := (M \otimes k[G])^H$ with certain actions described explicitly above. Equivalently, we could look at H-equivariant morphisms G to $\mathbb{M}$ where $\mathbb{M}(R) := M \otimes_k R$, that is, $h(f(-h) = f(-)$.

The comodule map associated with $(M \otimes k[G])^H$ is induced by

$$M \otimes k[G] \overset{\mathrm{id} \otimes \Delta_\ell}{\to} M \otimes k[G] \otimes k[G]$$

where Δ_ℓ is the comodule map associated to the left regular representation. We must argue this descends to a map

$$(M \otimes k[G])^H \to (M \otimes k[G])^H \otimes k[G].$$

To translate between the two definitions, take H-invariants of

$$M \otimes k[G] = \mathbb{M}(k[G]) = \operatorname{Mor}(G, \mathbb{M})$$

where the second equality comes from Yoneda's lemma.

139 Remark
Given $H \leq G$, last time we also defined the evaluation map $\epsilon_M \colon \operatorname{Ind}_H^G M \overset{1 \overline{\otimes} \epsilon}{\to} M$ given by $m \otimes f \mapsto \epsilon(f) m$. Defining $\operatorname{Ind}_H^G M$ in terms of natural transformations, this map corresponds precisely to $f \mapsto f(1)$, hence the name.

140 Proposition
Let $H \leq G$. Properties of Ind_H^G:

(1) $\epsilon_M \colon \operatorname{Ind}_H^G M \to M$ is a map of H-modules.

(2) *Frobenius reciprocity,*

$$\mathrm{Hom}_H(\mathrm{Res}^G_H N, M) \cong \mathrm{Hom}_G(N, \mathrm{Ind}^G_H M).$$

Equivalently, $\mathrm{Res}^G_H, \mathrm{Ind}^G_H$ *form an adjoint pair. (*Res^G_H *is more or less trivially exact.)*

(3) Ind^G_H *takes injectives to injectives.*

(4) Ind^G_H *preserves* $\oplus$.

(5) Ind^G_H *is left exact.*

(6) $\mathrm{Ind}^G_H \circ \mathrm{Ind}^H_K = \mathrm{Ind}^G_K$.

(7) Ind^G_H *commutes with extension of scalars.*

Note: $R^i \mathrm{Ind}^G_H$ *are very important in the theory of algebraic groups.*

141 Proposition (Tensor Identity)
If $H \leq G$ *,* M *is a* G*-module,* N *is an* H*-module, then, as* G*-modules,*

$$\mathrm{Ind}^G_H(\mathrm{Res}^G_H M \otimes N) \cong M \otimes \mathrm{Ind}^G_H N.$$

PROOF (Sketch.) Note that the LHS and the RHS can be embedded in $\mathrm{Mor}(G, M \otimes N \otimes -)$ as follows. The left-hand side consists of H-invariant morphisms $f \colon G \to M \otimes N \otimes -$, namely $(h \otimes h) f(-h) \cong f(-)$. The right-hand side can be treated similarly: it is

$$M \otimes (N \otimes k[G])^H \subset M \otimes N \otimes k[G] \cong \mathrm{Mor}(G, M \otimes N \otimes -).$$

Hence the right-hand side is the set of all functions $f \colon G \to M \otimes N \otimes -$ such that $(1 \otimes h) f(-h) = f(-)$. Hence the left-hand and right-hand sides are different though similar subsets of $\mathrm{Mor}(G, M \otimes N \otimes -)$. We will construct endomorphisms of $\mathrm{Mor}(G, M \otimes N \otimes -)$ sending the left-hand side to the right-hand side and vice-versa. Precisely, we define mutual inverses α, β by

$$(\alpha f)(x) := (x^{-1} \otimes 1) f(x) \qquad (\beta f)(x) := (x \otimes 1) f(x).$$

Claim: $\alpha(\mathrm{LHS}) \subset \mathrm{RHS}$; $\beta(\mathrm{RHS}) \subset \mathrm{LHS}$; α and β are G-equivariant.

On the left-hand side, the action is $(gf)(-) = f(g^{-1}-)$. On the right-hand side, the action is $(gf)(-) = (g \otimes 1) f(g^{-1}-)$.

142 Corollary
Suppose M *is a* G*-module. There is a natural isomorphism*

$$M \otimes k[G] \cong M_{\mathrm{triv}} \otimes k[G],$$

where the left-hand side's $k[G]$ *uses the right regular representation (true for left regular as well) and* M_{triv} *denotes the trivial action on* M*.*

PROOF We see

$$M_{\mathrm{triv}} \otimes k[G] = \mathrm{Ind}^G_1 M{\downarrow}^G_1 \cong \mathrm{Ind}^G_1 M{\downarrow}^G_1 \otimes_k k \cong M \otimes \mathrm{Ind}^G_1 k = M \otimes k[G]$$

where we've used the fact that $k[G] = \mathrm{Ind}^G_1 k$, which incidentally gives us our first example of induction.

143 Proposition
Important properties of injective modules:

(1) *Any* G*-module embeds in an injective* G*-module.*

(2) A G-module M is injective if and only if M is a direct summand of $V \otimes k[G]$ where V is a k-vector space.

Note: $k[G] = \operatorname{Ind}_1^G k$ is injective, since k is injective over trivial modules trivially, and induction takes injectives to injectives.

Proof (1) Recall we have an embedding $\Delta_M \colon M \to M_{\text{triv}} \otimes k[G]$, using the trivial action on M on the right and the right regular representation of $k[G]$, where Δ_M is an injective G-module map. Now $M_{\text{triv}} \otimes k[G] \cong \operatorname{Ind}_1^G M$. This is injective module using the same argument as for $k[G]$.

November 7th, 2014: Injective and Projective G-Modules

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144 Remark

We were talking about induction and injective modules. We had a proposition: (1) all G-modules can be embedded in an injective module; (2) I is injective if and only if I is a direct summand of $V \otimes k[G]$ for a k-vector space V with trivial action. We now add (3) a finite sum of injectives is injective when the sum exists, and (4) if I is injective, and M is any G-module, then $M \otimes I$ is injective.

Proof (1) $\Delta_M \colon M \to M_{\text{triv}} \otimes k[G]$ as last time. (2) $V \otimes k[G] \cong \operatorname{Ind}_1^G V$ is injective since induction takes injectives to injectives. Now if I is injective, $\Delta_I \colon I \hookrightarrow I_{\text{triv}} \otimes k[G]$ splits since I is injective. For (3), a more general statement is true. Julia will tell us later what that is and what its proof is. (4) I is a direct summand of $V \otimes k[G]$, so $M \otimes I$ is a direct summand of $M \otimes V \otimes k[G]$, which is isomorphic to $M_{\text{triv}} \otimes V \otimes k[G]$, so is injective.

145 Proposition

We have the following:

(1) Any projective G-module is injective.

(2) If there exists a non-zero projective G-module, then any injective module is projective.

(This should remain true for comodules over a general, not necessarily commutative, Hopf algebra.)

Proof (1) Let P be a projective G-module. We must show $\operatorname{Hom}_{G\text{-mod}}(-, P)$ is exact. Suppose $0 \to V_1 \to V_2 \to V_3 \to 0$ is exact. Assume for now the V_i are finite dimensional. We have

$$\begin{array}{ccccccc}
0 \longrightarrow & \operatorname{Hom}_G(V_3, P) & \longrightarrow & \operatorname{Hom}_G(V_2, P) & \longrightarrow & \operatorname{Hom}_G(V_1, P) \\
& \| & & \| & & \| \\
0 \longrightarrow & \operatorname{Hom}_k(V_3, P)^G & \longrightarrow & \operatorname{Hom}_k(V_2, P)^G & \longrightarrow & \operatorname{Hom}_k(V_1, P)^G \\
& \| & & \| & & \| \\
0 \longrightarrow & (V_3^\sharp \otimes P)^G & \longrightarrow & (V_2^\sharp \otimes P)^G & \longrightarrow & (V_1^\sharp \otimes P)^G
\end{array}$$

Our finiteness assumption gives $0 \to V_3^\sharp \to V_2^\sharp \to V_1^\sharp \to 0$ exact, so applying $- \otimes P$ yields an exact sequence of projective G-modules, since tensoring a finite-dimensional module with a projective module is projective, as follows. P is projective iff $\operatorname{Hom}_G(P, -)$ is exact. We want to show $V \otimes P$ is projective, so we must show $\operatorname{Hom}_G(V \otimes P, -)$ is exact. But by adjointness

$$\operatorname{Hom}_G(V \otimes P, -) \cong \operatorname{Hom}_G(P, V^\sharp \otimes_k -),$$

and $V^\sharp \otimes -$ is exact since tensoring over k is exact, so our functor is the composite of two exact functors, so is exact. Now the sequence above splits since short exact sequences of projective modules spit. This implies that taking invariants gives a split exact sequence. To show that for any two modules $N \hookrightarrow M$ the map $\mathrm{Hom}_G(M, P) \to \mathrm{Hom}_G(N, P)$ is surjective, use local finiteness. (Local finiteness here meaning our modules are each colimits of finite-dimensional modules.)

(2) First, a digression.

146 Fact

Here are some basic facts about G-module representation theory:

1. All simple modules are finite dimensional (by local finiteness).

2. Let M be a G-module. The socle functor soc exists and gives the maximal semisimple submodule of M. (A semisimple submodule is just a direct sum of simple submodules.) There is a one-to-one correspondence between simple G-modules and indecomposable injective G-modules. Precisely, given a module S, there is an injective hull I_S of S, which in an appropriate sense is a minimal injective module containing S. The universal property of I_S is that it is a unique (up to non-canonical isomorphism) injective module such that $\mathrm{soc}\, I_S = S$. On the other hand, given an indecomposable injective G-module I, $\mathrm{soc}\, I$ is simple.

3. All injective modules are a direct sum of indecomposable injective modules.

4. An injective module is indecomposable if and only if its socle is simple.

To be continued next time.

November 10th, 2014: Connected Components of Linear Algebraic Groups

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147 Remark

There were some unfinished things from last time. We had claimed that arbitrary direct sums of injectives are injective. This is not true for general categories, though in our category, injectives are of a very particular form, namely we can form $(\oplus I_i) \oplus (\oplus I_i') \cong \oplus k[G]$, more or less using the usual proof of a direct sum of projectives being projective.

We left off in the middle of proving (2) of the proposition immediately above, namely if there exists a non-zero projective G-module, then any injective module is projective.

Proof Let P be a non-zero projective G-module. Julia claims there exists a simple argument that P has a finite-dimensional projective submodule, so assume P is finite dimensional. We can embed $k \hookrightarrow \mathrm{End}_k(P)$ by $1 \mapsto \mathrm{id}$. Since P is finite dimensional, $\mathrm{End}_k(P) \cong P^\sharp \otimes P$. Tensoring with a simple module S gives $S \hookrightarrow S \otimes P^\sharp \otimes P$. S also embeds into the injective hull of S. Since P is injective, $S \otimes P^\sharp \otimes P$ is injective, so there exists a commutative diagram

$$\begin{array}{ccc} S & \hookrightarrow & S \otimes P^\sharp \otimes P \\ \downarrow & \nearrow & \\ \mathrm{Inj}(S) & & \end{array}$$

Indeed, the dashed arrow splits since $\mathrm{Inj}(S)$ is an injective hull. Hence $\mathrm{Inj}(S)$ is a direct summand of a projective modlue, so is projective.

Note: Julia is not terribly happy with this proof and will update it.

148 Remark

We next discuss more geometric questions concerned with the connected components of algebra group schemes. We've been viewing group schemes as functors, though one could also take the classical approach (over an algebraically closed field $\overline{k}$, say) where we have algebraic maps giving the group operations $S \times S \to S$, $\sigma\colon S \to S$, $e\colon e \to S$. This suggests going from an affine group scheme G over k to $G(\overline{k})$. From the variety perspective, supposing S is defined over k, we can go from the variety S to the ring of regular functions $k[S]$, so that $G_S(-) := \mathrm{Hom}_{k\text{-Alg}}(k[S], -)$. (Here we pick $k[S]$ without nilpotents, i.e. reduced.) If $k = \overline{k}$ and $\mathrm{char}\, k = 0$, this is a one-to-one correspondence. Waterhouse calls affine varieties over $\overline{k}$ which we think of as embedded in $\mathrm{GL}_n(\overline{k})$ where the group operation is matrix multiplication $\boxed{\text{algebraic matrix groups}}$. A more common name is $\boxed{\text{linear algebraic groups}}$.

Recall (1) S is irreducible if and only if $k[S]$ is an integral domain. In other words, reducibility is detected by zero-divisors, eg. $k[x, y]/(xy)$ is reducible, being the union of two lines. (2) Connectedness in the Zariski topology occurs if and only if there are no non-trivial idempotents.

149 Proposition (Connected Components of Linear Algebraic Groups)

Let S be a linear algebraic group and let S^0 be the connected component of S at the identity e. Then:

(1) S^0 is a normal subgroup of S of finite index;

(2) S^0 is irreducible;

(3) S is a union of finitely many connected components which are cosets of S^0.

Moral: For S, "connected" and "irreducible" coincide.

Proof Since S is noetherian, write $S = \cup_{i=1}^n X_i$ as the union of finitely many irreducible components so that no X_i is covered by the rest of them. Let $x_i \in X_1 - \cup_{i=2}^n X_i$. Take $g \in S$ and consider the map $S \to S$ given by $y \mapsto gx^{-1}y$. This is continuous and sends x to g. Hence we can send any point to any other point by a homeomorphism. Hence this map takes irreducible components to irreducible components. In particular it takes the irreducible component of x to the irreducible component of g, from which it follows that the irreducible components are pairwise disjoint, i.e. $\cup_{i=1}^n X_i = \coprod_{i=1}^n X_i$. Let $x = e$, $X_1 = S^0$. Then multiplication by g sends S^0 to some X_i. There are only finitely many possible X_i, so there are only finitely many cosets X/S^0. That is, $S = \coprod_{g \in S/S^0} gS^0$. (One can check S^0 is indeed closed under multiplication using similar reasoning.)

Definition 150. For any algebra A, we denote by $\boxed{\pi_0 A}$ the maximal separable subalgebra of A. This turns out to be functorial.

Definition 151. If G is an affine group scheme, we define $\boxed{\pi_0 G}$ as the group scheme with coordinate algebra $\pi_0(k[G])$. (Really $\pi_0(k[G])$ is a Hopf subalgebra, giving the group structure on $\pi_0 G$.)

November 12th, 2014: Connected Components of Group Schemes

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152 Remark

Julia was not terribly happy with the proof at the beginning of last lecture. We had made a claim that any projective module has a finite dimensional simple projective submodule, which she has begun to doubt, so we won't use this.

Proof Suppose there is a non-zero projective module P. We were showing any injective module is projective. Let $M \subset P$ be a finite dimensional submodule. Consider $k \to \mathrm{End}_k(M) \cong M^\sharp \otimes M \subset M^\sharp \otimes P$ by $1 \mapsto \mathrm{id}$. If P is projective, then $N \otimes P$ is projective for all N, since

$$\mathrm{Hom}_G(N \otimes P, -) \cong \mathrm{Hom}_G(P, \mathrm{Hom}_k(N, -)),$$

the right-hand side is exact (since k is a field and P is projective), so $N \otimes P$ is projective. Hence $M^\sharp \otimes P$ is projective. If S is irreducible, then tensor $k \to M^\sharp \otimes P$ with S to get $S \hookrightarrow S \otimes M^\sharp \otimes P$ where the right-hand side remains projective. On the other hand, S can be embedded in its injective hull $\mathrm{Inj}(S)$. We showed in the first part of the proposition that any projective module is injective, so $S \otimes M^\sharp \otimes P$ is also injective. But then we have

$$\begin{array}{ccc} S & \hookrightarrow & S \otimes M^\sharp \otimes P \\ \downarrow & \swarrow & \\ \mathrm{Inj}(S) & & \end{array}$$

where the dashed line is a splitting map. It follows that $\mathrm{Inj}(S)$ is a direct summand of a projective module, hence is projective. But then any indecomposable injective is projective, and any injective is a direct sum of such, so any injective module is projective.

We now return to our discussion of connected components.

153 Remark

Let A be a finitely generated k-algebra. Recall we had defined $\pi_0 A$ to be the largest separable subalgebra of A.

Why does $\pi_0 A$ exist? Suppose $B \subset A$ is separable. Then we can extend scalars to $\overline{k}/k$, yielding $B_{\overline{k}} \subset A_{\overline{k}}$. Now $\dim_{\overline{k}} B_{\overline{k}}$ is $\leq$ the number of connected components of $\operatorname{spec} A_{\overline{k}}$. This is finite since there are finitely many irreducible components, hence finitely many connected components. Given $B_1, B_2 \subset A$ separable, $B_1 B_2 \subset A$ is again a separable subalgebra. This is because there is a map $B_1 \otimes B_2 \to B_1 B_2 \subset A$ and the tensor product of separable algebras is separable. Existence now follows.

154 Proposition

π_0 has nice properties:

(1) $\pi_0 A$ is a Hopf subalgebra when A is a Hopf algebra.

(2) If L/k is a field extension, then $\pi_0(A_L) \cong \pi_0(A)_L$

(3) $\pi_0(A \otimes B) \cong \pi_0(A) \otimes \pi_0(B)$.

In particular, $\pi_0(A_{\overline{k}}) \cong \overline{k} \times \cdots \times \overline{k}$ where the right-hand side has as many factors as connected components of $A_{\overline{k}}$.

Proof See Waterhouse, 6.5.

Definition 155. Let G be an affine group scheme over k. Let $\pi_0 G$ be the group functor represented by $\pi_0(k[G])$, where here we give $k[G]$ its Hopf algebra structure. As schemes, $\pi_0 G = \operatorname{spec} \pi_0(k[G])$. We have a map $G \to \pi_0 G$. $\pi_0 G$ is an étale group scheme and is sometimes called the group of connected components of G.

We define the connected component at the identity via

$$G^\circ := \ker G \to \pi_0 G.$$

We say that G is $\boxed{\text{connected}}$ if and only if $\pi_0 G$ is trivial.

156 Proposition

Let G be an algebraic affine group scheme, $A := k[G]$. The following are equivalent:

(1) $\pi_0 G$ is trivial.

(2) $\operatorname{spec} A$ is connected.

(3) $\operatorname{spec} A$ is irreducible.

(4) $A/\operatorname{Nil}(A)$ is an integral domain.

Proof (2) $\Leftrightarrow$ (4) was mentioned last time and is standard commutative algebra. (3) $\Rightarrow$ (2) trivially. We do (2) $\Rightarrow$ (1) and (1) $\Rightarrow$ (4). For (2) $\Rightarrow$(1), suppose $\operatorname{spec} A$ is connected. Claim: in general, $\pi_0 A$ is then a field. Indeed, suppose $\pi_0 A = L_1 \times \cdots \times L_m$ for $m > 1$. Let e be a non-trivial idempotent corresponding to L_1. Then $\operatorname{spec} A = V(e) \cup V(1-e)$ where $V(-)$ indicates the set of all prime ideals containing an element $-$, since $e(1-e) = 0$. Since e and $1-e$ are non-invertible, $V(e)$ and $V(1-e)$ are non-empty. Indeed, this union is disjoint since any ideal containing e and $1-e$ contains 1, so is not prime. This contradicts the fact that $\operatorname{spec} A$ is connected, giving the claim. So, let $L = \pi_0 A$ be a field. Composing with the counit ϵ yields

$$\begin{array}{ccc} \epsilon\colon A & \longrightarrow & k \\ \uparrow & \nearrow & \\ L = \pi_0 A & & \end{array}$$

so $\pi_0 A = k$.

For (1) $\Rightarrow$ (4), $\pi_0 G$ is trivial if and only if $\pi_0 A = k$, $\Rightarrow \pi_0 A_{\overline{k}} = \overline{k}$, $\Rightarrow G_{\overline{k}}$ is connected, $\Rightarrow G_{\overline{k}}$ is irreducible (as a matrix algebraic group), $\Rightarrow \overline{k}[G_{\overline{k}}]/\operatorname{Nil} = A_{\overline{k}}/\operatorname{Nil}$ is an integral domain, which implies we have an embedding $A/\operatorname{Nil} \hookrightarrow A_{\overline{k}}/\operatorname{Nil}$.

157 Example

Consider μ_3 over $\mathbb{R}$. We have $\mathbb{R}[\mu_3] \cong \mathbb{R}[x]/(x^3-1) = \mathbb{R} \oplus \mathbb{R}[x]/(x^2+x+1)$. It is its own largest separable subalgebra. The number of connected components is 2 corresponding to the two summand fields. Hence μ_3 is not connected. However, if we extend scalars to $\mathbb{C}$, the coordinate algebra becomes $\mathbb{C}^{\oplus 3}$. The number of connected components changed, though the property of not being connected didn't.

158 Remark

We have the sequence $G^0 \hookrightarrow G \stackrel{\pi}{\to} \pi_0 G$. G^0 is connected and $\pi_0 G$ is étale. Very often this extension is split—it is enough to assume your field is perfect. Recall the coordinate algebra of a kernel, which gives

$$k[G^\circ] = k[G] \otimes_{k[\pi_0 G]} k \cong k[G]//k[\pi_0 G] := k[G]/Ik[\pi_0 G]$$

where $I := \pi^*(\ker \epsilon)$ where $\pi^*\colon k[\pi_0 G] \to k[\pi G]$. Here k is viewed as a $k[\pi_0 G]$-module through the counit map.

November 14th, 2014: Introducing Kähler Differentials

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159 Remark

Let $A = k[G]$ be the group algebra of an algebraic affine group scheme. $\pi_0 A$ is separable, so splits as a product of a bunch of fields. Each factor yields an idempotent, and all of them together form an orthogonal system of such. The counit $\epsilon\colon k[A] \to k$ restricts to a counit $\epsilon\colon \pi_0(A) \to k$. It follows that (without loss of generality) some idempotent e_0 is sent to 1 under ϵ. In fact, $Ae_0 = k[G^\circ]$. In this sense, G° is "detected" by ϵ.

160 Example
Connectedness in action:

(1) $\mathrm{GL}_n, \mathrm{SL}_n$ are connected. Note: $\mathrm{GL}_n(\mathbb{R})$ is not connected in the usual topology, though we are essentially using the Zariski topology.

(2) $(G^\circ)_L = (G_L)^\circ$ and $(G_1 \times G_2)^\circ = G_1^\circ \times G_2^\circ$. In particular, the product of connected groups is connected.

(3) Suppose $G = \Lambda_{\text{diag}}$. Hence $k[G] = k\Lambda$ for the abelian group Λ. G is connected iff Λ does not have p'-torsion where p' is relatively prime to the characteristic of k. Here every positive integer is considered relatively prime to 0. In general, $G^\circ = (\Lambda/\Lambda')_{\text{diag}}$ where Λ' is the p'-torsion.

161 Remark
Our next main topic will be the infinitesimal theory: derivations, differentials, and Lie algebras.

Definition 162. Let A be a k-algebra, M an A-module. Then $D\colon A \to M$ is a k-linear $\boxed{\text{derivation}}$ if it satisfies the Leibniz rule,

$$D(ab) = aD(b) + bD(a),$$

and $D(k) = 0$ or equivalently $D(\alpha a) = \alpha D(a)$ for all $\alpha \in k$. We write $\boxed{\mathrm{Der}_k(A, M)}$ for the set of k-linear derivations $A \to M$.

We next define the module of Kähler differentials. A standard reference for such differentials is Matsumura, Chapter 2.

163 Example
Let $A = k[x_1, \ldots, x_n]$. Let Ω_A be the free A-module on $dx_1, \ldots, dx_n$ using the usual partial derivative operators. Let $d\colon A \to \Omega_A$ be given by $x_i \mapsto dx_i$. Given any derivation $D\colon A \to M$, the map $\phi\colon \Omega_A \to M$ given by $dx_i \mapsto Dx_i$ satisfies $D = \phi d$.

164 Theorem
Suppose A is a noetherian k-algebra. Then there exists a unique A-module $\boxed{\Omega_A}$ together with a k-linear derivation $d\colon A \to \Omega_A$ such that any derivation $D\colon A \to M$ factors uniquely through d as

$$\begin{array}{ccc} A & \xrightarrow{D} & M \\ \downarrow{\scriptstyle d} & \nearrow{\scriptstyle \exists!} & \\ \Omega_A & & \end{array}$$

That is, $\mathrm{Der}_k(A, M) \cong \mathrm{Hom}_A(\Omega_A, M)$. *We call Ω_A the* $\boxed{\textit{module of Kähler differentials}}$

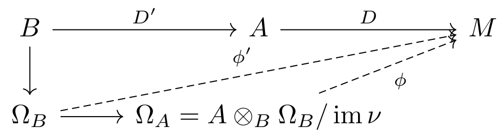
Proof We may write $A = B/I$ for some $B := k[x_1, \ldots, x_n]$ and $I \subset B$ an ideal. Let Ω_B be the B-module from the preceding example together with the differential $d\colon B \to \Omega_B$. Let $\nu\colon I \to A \otimes_B \Omega_B$ be given by $f \mapsto 1 \otimes df$. This is a B-module homomorphism, as follows. Pick $h \in B$ and compute

$$\nu(hf) = 1 \otimes d(hf) = h \otimes df + f \otimes dh = h \cdot (1 \otimes df) + 0 = h\nu(f).$$

Now set $\Omega_A := A \otimes_B \Omega_B / \operatorname{im} \nu$. We have

$$\begin{aligned} B &\xrightarrow{d} \Omega_B \to \Omega_A \\ f &\mapsto df \mapsto 1 \otimes df. \end{aligned}$$

This map factors through $A \to \Omega_A$, yielding $d_A\colon A \to \Omega_A$. We claim (Ω_A, d_A) satisfies the suggested properties. Let $D\colon A \to M$ be a derivation. Now



Define $\phi\colon \Omega_A \to M$ by $1 \otimes df \mapsto \phi'(df)$. Check the remaining details yourself, namely that this is well-defined and unique.

165 Remark

In the notation of the theorem and proof, suppose $I = (f_1, \ldots, f_m)$. If we unwind the proof, then

$$\Omega_A = \frac{\langle dx_1, \ldots, dx_n \rangle}{\{\sum_{i=1}^n \frac{\partial f_j}{\partial x_i} dx_i : 1 \leq j \leq m\}}$$

166 Example

Let $A = k[x, y]/(x^2 + y^2 - 1)$. Let $f = x^2 + y^2 - 1$. Here $df = 2x\,dx + 2y\,dy$. Hence

$$\Omega_A = \frac{\langle dx, dy \rangle}{(2x\,dx + 2y\,dx)}.$$

Say $\operatorname{char} k \neq 2$. Geometrically, this should be a free module of rank 1, since the circle has one-dimensional cotangent space. Algebraically, this is true. Let $dt := y\,dx - x\,dy \in \Omega_A$. Notice that

$$\begin{aligned} y\,dt &= y^2\,dx - yx\,dy \\ &= (1 - x^2)\,dx - yx\,dy \\ &= dx - x(x\,dx + y\,dy) \\ &= dx \\ -x\,dt &= dy. \end{aligned}$$

It follows that dt generates Ω_A. One should check this implies Ω_A is a rank 1 free A-module with generator dt.

In characteristic 2, the relation is trivial, so Ω_A is free of rank 2. Using the standard notion of dimension, the Krull dimension of A is 1. In this case, A is not reduced (with nilpotent $(x+y-1)^2 = 0$), so the fact that Ω_A has rank $2 > 1$ is less surprising since A is then not smooth.

167 Fact

If $A = k[G]$ where G is an algebraic affine group scheme, then Ω_A is always a free A-module. Given this, to check that G is smooth, we must check the dimension of the tangent space at each point is the Krull dimension of A. Since G is a group, the dimension is constant. It then suffices to check whether or not the rank of Ω_A is of the same rank as the Krull dimension of A.

November 17th, 2014: Properties of Kähler Differentials and Hopf Algebras

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168 Proposition

Properties of Ω_A:

1) It commutes with extension of scalars: $\Omega_{A \otimes k'} \cong \Omega_A \otimes k'$.

2) It commutes with products: $\Omega_{A \times B} \cong \Omega_A \times \Omega_B$.

3) It commutes with localizations: if S is a multiplicatively closed subset of A, then $\Omega_{S^{-1}A} \cong S^{-1}\Omega_A := S^{-1}A \otimes_A \Omega_A$.

4) A is étale (separable) if and only if $\Omega_A = 0$.

Proof $\Rightarrow$ in (4) follows since $\Omega_k = 0$ trivially and we can apply (1) and (2) in this case.

169 Lemma

Let $f\colon A \to k$, $I := \ker f$. Then there is a canonical isomorphism

$$\Omega_A \otimes_A k \cong I/I^2.$$

The left-hand side is $\Omega_A/I\Omega_A$. (It is the formal definition of the specialization *at k. It roughly tells us what's happening at the point $\operatorname{spec} k \hookrightarrow \operatorname{spec} A$. It is the fiber at $\operatorname{spec} A$.)*

Proof $\Omega_A \otimes_A k$ is the universal module of differentials for all derivations $D\colon A \to N$ which factor through f. (This may be referred to as f-linear derivations.) More precisely, we require

$$D(ab) = f(a)D(b) + f(b)D(a).$$

Hence we must show I/I^2 is also the universal module of f-linear derivations. That is, we need natural isomorphisms $\mathrm{Der}_f(A, N) \cong \mathrm{Hom}_k(I/I^2, N)$. We need a "universal" derivation $\pi\colon A \to I/I^2$.

170 Example

Let $A = k[x_1, \ldots, x_n]$, $g(x_1, \ldots, x_n) = a_0 + a_1x_1 + \cdots a_nx_n + \cdots$. We want to single out just the linear terms when constructing $\pi(g)$. Here our evaluation map is $f\colon A \to k$ given by evaluating at the origin. To pick off the linear terms, we simply take $\pi(g) = g - f(g) \bmod I^2$.

In analogy with the example, we define $\pi(g) := g - f(g) \bmod I^2$. We check this is a derivation:

$$\begin{aligned}\pi(ab) &= ab - f(a)f(b)\\ &= f(b)(a - f(a)) + f(a)(b - f(b)) + (a - f(a))(b - f(b)).\end{aligned}$$

The right-hand term is indeed in I^2, and the remaining two terms are $f(a)\pi(b) + f(b)\pi(a)$, as required. Now we need to show universality. Suppose $D\colon A \to N$ is an f-derivation. As usual, $D(k) = 0$, and indeed $D(I^2) = 0$. Hence D factors through I/I^2 via $\phi(\overline{a}) := D(a)$:

$$\begin{array}{ccc} A & & \\ \downarrow{\scriptstyle \pi} & \searrow^{D} & \\ I/I^2 & \xrightarrow{\phi} & N \end{array}$$

171 Theorem

Let A be a (commutative) Hopf algebra. Let $I := \ker \epsilon$ where $\epsilon\colon A \to k$ is the counit map. Let $\pi\colon A \to I/I^2$ by $a \mapsto a - \epsilon(a) \bmod I^2$. Then

(1) $\Omega_A \cong A \otimes_k I/I^2$

(2) The universal derivation $d\colon A \to \Omega_A \cong A \otimes_k I/I^2$ is given by $a \mapsto \sum a_i \otimes \pi(a_i')$ where $\Delta(a) = \sum a_i \otimes a_i'$.

Proof We begin with a pair of lemmas:

172 Lemma

Suppose B is an algebra, N is a B-module. Then $C := B \oplus N$ can be given an algebra structure as follows. $(b, n) \cdot (b', n') := (bb', bn' + b'n)$. Further,

$$\mathrm{Hom}_{\mathrm{Alg}}(A, B \oplus N) = \{(\phi, D) : \phi \in \mathrm{Hom}_{\mathrm{Alg}}(A, B), D \in \mathrm{Der}_k(A, N)\}.$$

Here D is a ϕ-derivation.

PROOF Exercise.

173 Lemma

Under the assumptions of the preceding lemma, suppose also that A is a Hopf algebra. Let G be the group scheme with $k[G] = A$. Then the group structure of $G(C) = \mathrm{Hom}_{k\text{-Alg}}(A, B \oplus N)$ is given as follows. $(\phi, D) \cdot (\phi', D') = (\phi\phi', \phi D' + \phi' D)$ where $\phi' D$ is given by the composite

$$\phi' D \colon A \overset{\Delta}{\to} A \otimes A \overset{\phi \otimes D}{\to} B \otimes N \overset{mult}{\to} N,$$

so $\phi' D(a) = \sum \phi'(a_i) D(a_i')$.

PROOF Exercise.

To be continued.

November 19th, 2014: Commutative Hopf Algebras are Reduced over Characteristic 0

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174 Remark

We continue the proof from last time. We had a commutative Hopf algebra A, $I := \ker \epsilon$. We were showing $\Omega_A \cong A \otimes_k I/I^2$, $d \colon A \to \Omega_A$ by $A \overset{\Delta}{\to} A \otimes A \overset{\mathrm{id} \otimes \pi}{\to} A \otimes I/I^2$.

PROOF Recall the two lemmas stated last time; we will not repeat them here. Let N be an A-module and $A \oplus N$ be the algebra as in the second lemma. Then $p \colon A \to A \oplus N$ splits, i.e. we have $\mathrm{id} \colon A \overset{p}{\to} A \oplus N \to A$. It follows that $G(A \oplus N) \overset{p}{\to} G(A)$ splits, that is, we have

$$\mathrm{id} \colon \mathrm{Hom}(A, A) \to \mathrm{Hom}(A, A \oplus N) \overset{p}{\to} \mathrm{Hom}(A, A).$$

Now $\ker p = \{(\phi, D) : p(\phi, D) = \phi = e \in G(A)\}$, so that $\ker p = \{(\epsilon, D) \in \mathrm{Hom}(A, A \oplus N)\}$. Since $G(A \oplus N) \to G(A)$ splits, for all $(\phi, \delta) \in G(A \oplus N)$ we can write $(\phi, \delta) = (\epsilon, D) \cdot (\phi, 0)$. (Here ϵ really refers to the composite of the counit map $A \to k$ and the unit map $k \to A$.) Hence if $\delta \colon A \to N$ is a derivation, there is a corresponding derivation $D \colon A \to N$ which is ϵ-linear. Indeed, we have

$$\mathrm{Der}_k(A, N) \cong \mathrm{Der}_\epsilon(A, N) = \mathrm{Hom}_k(I/I^2, N) \cong \mathrm{Hom}_A(A \otimes_k I/I^2, N) = \mathrm{Hom}_A(\Omega_A, N).$$

Unwinding these maps, given a derivation $d \colon A \to N$, form the pair (id_A, d)–note that d is id_A-linear trivially. Write $\xi := \mathrm{id}_A$ to avoid confusing it with the identity of the group $G(A)$. (It happens that ξ is the inverse of the coinverse.) Use the above splitting to write $(\xi, d) = (\epsilon, D) \cdot (\xi, 0)$ for some ϵ-linear derivation D. By the formula for ϕD from the second lemma, $d = \xi \cdot D$. Last time we defined the universal derivation π. Set $D \colon A \overset{\pi}{\to} I/I^2 \to A \otimes I/I^2$, so $D(a) = 1 \otimes \pi(a)$. Using the above recipe, we set

$$d := \xi \cdot D \colon A \overset{\Delta}{\to} A \otimes A \overset{\xi \otimes D}{\to} A \otimes \Omega_A \to \Omega_A$$

which works via

$$a \mapsto \sum a_i \otimes a_i' \mapsto \sum a_i \otimes 1 \otimes \pi(a_i') \mapsto \sum a_i \otimes \pi(a_i').$$

175 Corollary

If A is a Hopf algebra, then Ω_A is a free A-module of rank $\dim_k I/I^2$.

176 Theorem (Cartier)

Let A be a commutative Hopf algebra of characteristic 0. Then A is reduced.

Proof Reduce to the algebraic case by writing an arbitrary such algebra as a limit of algebraic ones.

177 Lemma

Let A be a commutative, finitely generated Hopf algebra over a field k of characteristic 0. Let I be the augmentation ideal $\ker \epsilon$. Suppose $\{x_1, \ldots, x_n\}$ form a k-basis for I/I^2.Then $\{x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}\}$ where $i_1 + \cdots + i_n = m$ form a basis of I^m/I^{m+1}.

Proof Consider the dual basis of $\{x_1, \ldots, x_n\}$ in I/I^2, namely $d_i \colon I/I^2 \to k$ with $d_i(x_j) := \delta_{i,j}$. Construct $\mathrm{id}_A \otimes d_i \colon A \otimes I/I^2 \to A$ with $a \otimes \overline{f} \mapsto a d_i(\overline{f})$. We have $\mathrm{id}_A \otimes d_i \in \mathrm{Hom}_A(\Omega_A, A) \cong \mathrm{Der}(A, A)$ by the previous theorem. Precisely, let $D_i \colon A \to A$ be the corresponding derivation, namely with

$$\begin{array}{ccc} \Omega_A & \xrightarrow{\mathrm{id}_A \otimes d_i} & A \\ {\scriptstyle d}\uparrow & \overset{D_i}{\nearrow} & \\ A & & \end{array}$$

or more explicitly

$$D_i \colon A \overset{\Delta}{\to} A \otimes A \overset{\mathrm{id} \otimes \pi}{\to} A \otimes I/I^2 \overset{\mathrm{id}_A \overline{\otimes} d_i}{\to} A \otimes k \to A$$

with

$$a \mapsto \sum a_j \otimes a'_j \mapsto \sum a_j \otimes \pi(a'_j) \mapsto \cdots \mapsto \sum d_i \pi(a'_j) a_j.$$

Now consider $\epsilon \circ D_i \colon A \to k$, which gives

$$\epsilon \circ D_i(a) = \sum \epsilon(a_j) d_i \pi(a'_j) = d_i \pi(\sum \epsilon(a_j) a'_j) = d_i \pi(a).$$

In particular, $\epsilon \circ D_i(x_j) = \delta_{ij}$.

To be continued.

November 21st, 2014: Smooth Algebraic Affine Group Schemes

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178 Remark

We were proving Cartier's theorem last time, which says that a Hopf algebra in characteristic 0 is reduced. We continue the theorem's proof now.

Proof We were proving a lemma, namely if $\{x_1, \ldots, x_n\}$ is a basis for I/I^2 where I is the augmentation ideal $\ker \epsilon \colon k[G] \to k$, then $\{x_1^{i_1} \cdots x_n^{i_n}\}_{i_1 + \cdots + i_n = m}$ is a basis for I^m/I^{m+1}. We now continue the lemma's proof.

Proof We had gotten to the point of having derivations $D_i \colon A \to A$ such that $(\epsilon \circ D_i)(x_j) = \delta_{ij}$, so that $D_i(x_j) = \delta_{ij} \bmod I$. Hence $D_1^{j_1} D_2^{j_2} \cdots D_n^{j_n}(x_1^{i_1} \cdots x_n^{i_n}) = i_1! i_2! \cdots i_n! \bmod I$ if each $j_k = i_k$ and 0 otherwise. Since ϵ is a k-algebra map, it cannot annihilate any element of $\mathbb{Z} \subset k$. Linear independence mod I^{m+1} now follows.

We now prove the theorem. We may extend scalars and assume $k = \overline{k}$ since $\mathrm{Nil}(A) \subset \mathrm{Nil}(A_{\overline{k}})$. Suppose to the contrary $\mathrm{Nil}(A) \neq 0$. It is enough to show that $y^2 = 0$ implies $y = 0$ (by taking an element of minimal nilpotence degree).

Claim: if $y^2 = 0$, then $y \in \cap_{n\geq 0} I^n$. Note that A is noetherian, so if A were local, we would be done by Krull's intersection theorem. We will have to work a little more and use the group structure. For the claim, suppose not, so there is some m such that $y \in I^m$ and $y \notin I^{m+1}$. Let $\{x_1, \ldots, x_n\}$ form a basis of I/I^2. From the lemma, $\{x_1^{i_1} \cdots x_n^{i_n}\}_{i_1+\cdots+i_n=m}$ is a basis of I^m/I^{m+1}, so we can write $y = P(x_1, \ldots, x_n)$ where P is a homogeneous polynomial over k of degree m. Now $y \neq 0 \bmod I^{m+1}$. It follows that $y^2 = P(x_1, \ldots, x_n)^2 \bmod I^{2m+1}$ is non-zero since $P \neq 0$ as a polynomial, so $P^2 \neq 0$ as well. This is a contradiction since $y^2 = 0$ by assumption, giving the claim.

Now let $\mathfrak{m}$ be a maximal ideal in A. Since k is algebraically closed, all maximal ideals are of the trivial form, namely the kernel of a map $g\colon A \to k$. Now $g \in G(k)$, so there is a translation-by-g map

$$T_g\colon A \overset{\Delta}{\to} A \otimes A \overset{\mathrm{id}\otimes g}{\to} A \otimes k \overset{\mathrm{mult}}{\to} A.$$

T_g is an algebra isomorphism (with inverse $T_{g^{-1}}$). It is called a translation map. One can check $T_g(\mathfrak{m}) = I$ (or perhaps $T_g(I) = \mathfrak{m}$—one or the other), which is essentially clear geometrically: translating takes one point to another. Also, $T_g(\mathrm{Nil}(A)) = \mathrm{Nil}(A)$. So, if $y \in \cap_{i\geq 0} I^i$, then $y \in \cap_{i\geq 0}\mathfrak{m}^i$ for all $\mathfrak{m}$. By Krull's intersection theorem, $y \in \cap_{m\geq 0}\mathfrak{m}^m = 0$, so $y = 0$.

179 Corollary

In characteristic 0, any finite group scheme is étale. (The book of involutions should have more examples of finite group schemes in characteristic 0.)

180 Corollary

In characteristic 0, any affine group scheme comes from a linear algebraic group.

181 Remark

We next discuss smoothness of group schemes.

Definition 182. An affine group scheme G is smooth if $\dim G = \operatorname{rank} \Omega_{k[G]}$. Since $\Omega_{k[G]} = I/I^2$ is the cotangent space at the identity, and G is a group so whatever holds at the identity happens everywhere, this definition agrees with the usual one for schemes.

183 Fact

Let k be a perfect field. Suppose L/k is a finitely generated field extension.

1) $k \subset k(x_1, \ldots, x_n) \subset L$ where the second inclusion is a separable algebraic extension and the first inclusion is a purely transcendental extension of degree n.

2) $\dim_L \Omega_{L/k} = \operatorname{trdeg} L/k$.

3) If L is an arbitrary extension of k, then $\dim_L \Omega_{L/k} = \operatorname{trdeg} L/k$. If $\langle dx_1, \ldots, dx_n\rangle$ is a basis of $\Omega_{L/k}$, then $k \subset k(x_1, \ldots, x_n) \subset L$ as in (1).

(Waterhouse proves this in Chapter 11.)

184 Theorem

Let k be any field. If G is an algebraic affine group scheme, then G is smooth if and only if $\overline{k}[G]$ is reduced.

185 Remark

Note: the original statement of this result used k instead of $\overline{k}$; it was corrected in the next lecture. Julia will try to find an example where $k[G]$ is reduced but $\overline{k}[G]$ is not. It is easy to do so for algebras, but less clear for Hopf algebras.

186 Corollary

In characteristic 0, all algebraic group schemes are smooth.

Proof We'll prove $\Leftarrow$. The other direction is more involved, and we might prove it later, or we might skip it. To be continued next time.

November 24th, 2014: Lie Algebras of Affine Group Schemes

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187 Remark

Today we'll prove the assertion from last time, that if $\overline{k}[G]$ is reduced, then G is smooth. We'll then define Lie algebras.

Proof Recall $\dim G = \operatorname{rank} \Omega_{k[G]}$. Since $\dim G$, $\operatorname{rank} \Omega_{k[G]}$ are preserved by field extensions, we may assume $k = \overline{k}$. If G is connected, then G is irreducible, so $k[G]/\operatorname{Nil}(k[G])$ is an integral domain from classical algebraic geometry. But $\operatorname{Nil}(k[G]) = 0$ by assumption, so $k[G]$ is an integral domain. Let $K = \operatorname{Frac} k[G]$. Then $\dim G$ is the Krull dimension of $k[G]$, which from general dimension theory is the transcendence degree of K/k. From the fact mentioned last time, this transcendence degree is $\dim_K \Omega_K$, which is $\dim_K \Omega_{k[G]} \otimes_{k[G]} K$ since Ω behaves well with respect to localization. But this is just $\operatorname{rank}_{k[G]} \Omega_{k[G]}$.

For the other direction, see Chapter 14 of Waterhouse.

Definition 188. $\mathcal{L}$ is a Lie algebra over k if $\mathcal{L}$ is a k-vector space equipped with a map $[-,-]\colon \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ such that

(1) $[-,-]$ is k-bilinear;

(2) $[-,-]$ is anticommutative

(3) $[-,-]$ satisfies the Jacobi identity, $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$.

The Jacobi identity can be restated in terms of derivations. Precisely, we want $\star\colon \mathcal{L} \times \mathcal{L} \to \mathcal{L}$ and $d\colon \mathcal{L} \to \mathcal{L}$ to satisfy $d(a \star b) = (da) \star b + a \star (db)$. Here if $a \star b := [a,b]$ and $d := [x,-]$ for some fixed $x \in \mathcal{L}$, one may check the desired property is equivalent to the Jacobi identity (after assuming linearity appropriately).

Definition 189. Let A be a k-Hopf algebra. $\operatorname{Der}_k A$ denotes the space of derivations of A as before. A derivation $D\colon A \to A$ is a left invariant derivation if $(\operatorname{id} \otimes D) \circ \Delta = \Delta \circ D$, that is,

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ \downarrow{\scriptstyle D} & & \downarrow{\scriptstyle \operatorname{id} \otimes D} \\ A & \xrightarrow{\Delta} & A \otimes A \end{array}$$

190 Lemma

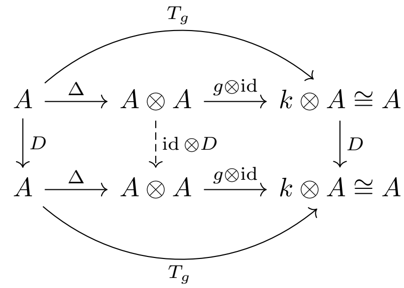
Let $D_1, D_2 \in \operatorname{Der}_k A$ be left invariant. Then $[D_1, D_2] := D_1 \circ D_2 - D_2 \circ D_1$ is a left invariant derivation of A.

Proof Exercise.

Definition 191. Given an affine group scheme G, define the Lie algebra of G, $\operatorname{Lie} G$, as the space of left invariant derivations in $\operatorname{Der}_k(A)$.

192 Lemma

An equivalent description of left invariants is the following. If D is a left invariant derivation on a Hopf algebra $A = k[G]$, then for all $g \in G(k)\colon A \to k$, the following commutes:



That is, $T_g \circ D = D \circ T_g$. (The dashed arrow is not technically part of this diagram. Recall that $\xi = \mathrm{id} \otimes D$ is the "general element".)

193 Remark

Thinking of $k[G] = \mathrm{Mor}(G, \mathbb{A}^1)$, $T_g \colon f(-) \mapsto f(g-)$ is honest left translation. Then the preceding condition just says $D(f(g-)) = (Df)(g-)$.

194 Lemma

$\mathrm{Lie}\, G \cong \mathrm{Der}_k(k[G], k)$ *where the right-hand side denotes derivations which factor through k, or equivalently ϵ-linear derivations.*

Proof If D is a left invariant derivation, then $\epsilon \circ D \colon k[G] \to k$, which one can check is indeed a derivation. On the other hand, given a derivation $d \colon k[G] \to k$, then $(\mathrm{id} \otimes d) \circ \Delta$ given by $A \overset{\Delta}{\to} A \otimes A \overset{\mathrm{id} \otimes d}{\to} A \otimes k \cong A$ is left invariant. One must check they are inverses.

Indeed, $\epsilon \circ D$ is an ϵ-linear derivation since ϵ is a ring homomorphism and D is a derivation. Now

$$(\mathrm{id} \overline{\otimes} \epsilon) \circ (\mathrm{id} \otimes D) \circ \Delta = \Delta \circ D$$
$$(\mathrm{id} \overline{\otimes} \epsilon \circ D) \circ \Delta = (\mathrm{id} \overline{\otimes} \epsilon) \circ \Delta \circ D$$
$$(\mathrm{id} \overline{\otimes} d) \circ \Delta = \mathrm{id} \circ D = D.$$

Hence the map is injective. Is it surjective? We compute

$$(\mathrm{id} \otimes D) \circ \Delta = \Delta \circ D$$
$$(\mathrm{id} \otimes ((\mathrm{id} \otimes d) \circ \Delta)) \circ \Delta = \Delta \circ (\mathrm{id} \otimes d) \circ \Delta,$$

and the second equality follows from coassociativity.

November 26th, 2014: Draft

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195 Remark

To be added.

December 1st, 2014: Commutators and p-Restricted Lie Algebras

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196 Remark

Let G be an affine group scheme. How do we actually compute $[-, -]$ in $\mathrm{Lie}(G)$? Let $R := k[u, v]/(u^2, v^2)$. There are two natural maps $k[t]/t^2 \to k[u, v]/(u^2, v^2)$, namely we may send t to u or v. Suppose $d_1, d_2 \in \ker(G(k[t]/t^2) \to G(K))$. Set $\phi_1 := \epsilon + ud_1 \in G(R)$, $\phi_2 := \epsilon + vd_2 \in G(R)$. One may check (exercise) $\phi_1 \phi_2 = (\epsilon + uv[d_1, d_2])\phi_2 \phi_1$. That is, $\phi_1 \phi_2 \phi_1^{-1} \phi_2^{-1} = \epsilon + uv[d_1, d_2]$.

197 Example

Let $G := \mathrm{GL}_n$. What is $[-,-]$ in $\mathrm{Lie}(\mathrm{GL}_n) =: \mathfrak{gl}_n$? Given $A, B \in \mathfrak{gl}_n$, $\phi_1 = I + uA$ and $\phi_2 = I + vB$. Hence $\phi_1\phi_2\phi_1^{-1}\phi_2^{-1}$ is

$$(I + uA)(I + vB)(I - uA)(I - vB) = \cdots = I + uv(AB - BA),$$

so $[A, B] = AB - BA$.

198 Example

Let $O_n(R) := \langle g \in \mathrm{GL}_n(R) : gg^T = I\rangle$.

199 Aside

This is a very familiar definition of O_n, though it's not choice-free. An alternative is to pick a symmetric non-degenerate matrix S and define O_n as consisting of matrices g which preserve S, namely $gSg^T = S$. This of course depends on S in general and is also not choice-free. A nicer version of this construction is to start with a symmetric non-degenerate bilinear form $B\colon V \times V \to k$ on a vector space V. Then define O_n as consisting of linear transformations $V \to V$ which leave the form invariant, i.e. $B(gv, gv) = B(v, v)$. This yields $O(V, B)$; one may make it functorial by extending scalars. To get back to the condition $gg^T = I$, take $B(x, y) = \sum x_i y_i$. This construction happens to give "most" algebraic groups.

What is $\mathrm{Lie}(O_n)$? Given $A \in \mathrm{Lie}(O_n)$, take $I + tA \in \ker(O_n(k[t]/t^2) \to O_n(k))$. Since $(I + tA)(I + tA)^T = I + t(A + A^T) = I$, we must have $A + A^T = 0$. Hence $\mathrm{Lie}(O_n) = \{A \in \mathfrak{gl}_n : A = -A^T\}$, i.e. it consists of the skew-linear transformations. Is O_n connected? Checking this directly from the definition in terms of maximal étale subalgebras is rather daunting. However, the determinant map $O_n \to \mathbb{Z}/2 = \{\pm 1\}$ is surjective and $\mathbb{Z}/2$ is not connected, so O_n is as well. In terms of the underlying coordinate algebras, this map is $k \times k = k[\mathbb{Z}/2] \hookrightarrow k[O_n]$. Indeed, O_n has two connected components. We set $\mathrm{SO}_n := O_n^0 = \{g : gg^T = I, \det g = 1\}$.

Is O_n smooth? From the definition, we'll have to compute the dimension of the Lie algebra and the coordinate algebra. It turns out that $\dim O_n = n(n-1)/2$. This is done in Borel 23.6; Julia thinks there must be a better source. In any case, $\mathrm{Lie}(O_n)$ has dimension $n(n-1)/2$ if $\mathrm{char}\, k \neq 2$, but $n(n+1)/2$ if $\mathrm{char}\, k = 2$. Hence O_n is smooth if and only if $\mathrm{char}\, k \neq 2$.

200 Aside

The Book of Involutions handles the characteristic 2 case. Julia says it took the authors twice as long to not ignore the characteristic 2 case, which is apparently customary.

Definition 201. Let $\mathrm{char}\, k = p > 0$. Suppose $\mathfrak{g}$ is a Lie algebra. $\mathfrak{g}$ is a p-restricted Lie algebra if it comes with a map $[p]\colon \mathfrak{g} \to \mathfrak{g}$ such that

(1) $(\alpha x)^{[p]} = \alpha^p x^{[p]}$

(2) $[x^{[p]}, y] = [x, [x, [\ldots, [x, y] \ldots]]]$ (where there are p brackets here). (That is, $\mathrm{ad}^p\, x = \mathrm{ad}\, x^{[p]}$.)

(3) $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{s=1}^{p-1} s_i(x, y)$ where $\iota \cdot s_i(x, y)$ is a polynomial given as the coefficient by t^{i-1} in $[tx + y, [tx + y, [\ldots, [tx + y, x] \ldots]]]$ (where there are $p - 1$ brackets).

202 Example

Let $\mathrm{char}\, k = p > 0$.

(1) If $\mathfrak{g} = \mathfrak{gl}_n$, then $A^{[p]} := A^p$ yields a p-restricted Lie algebra.

(2) For any p-restricted Lie algebra $\mathfrak{g}$, there exists an embedding $\mathfrak{g} \hookrightarrow \mathfrak{gl}_n$ as a restricted Lie algebra, so $[p]$ is sent to the pth power map.

(3) If G is an affine group scheme over k, we defined $\mathrm{Lie}(G)$ as the left-invariant derivations on $k[G]$. An exercise is to show that if D is a left invariant derivation on $k[G]$, then $D^p = D \circ \cdots \circ D$ is again left invariant. One should embed $\mathrm{Lie}(G)$ in $\mathfrak{gl}_n$. $\mathrm{Lie}(G)$ is said to be naturally a restricted Lie algebra.

203 Remark

If $G \xrightarrow{f} G'$ is a map of affine group schemes in characteristic $p > 0$, then $\mathrm{Lie}(G) \xrightarrow{df} \mathrm{Lie}(G')$ is a map of restricted Lie algebras.

Definition 204. Let $\mathfrak{g}$ be a restricted Lie algebra. Suppose M is a representation of $\mathfrak{g}$ (that is, a representation which turns the bracket operation into matrix commutation). We call M a restricted representation of $\mathfrak{g}$ if

(1) $g^{[p]} \circ m = g \circ \cdots \circ g \circ m$ (using p g's)

(2) $[g_1, g_2] \circ m = g_1 \circ g_2 \circ m - g_2 \circ g_1 \circ m$.

205 Remark

If M is a representation of G, then M is a representation of $\mathrm{Lie}(G)$. In particular, if $\rho_M \colon G \to \mathrm{GL}(M)$, then $d\rho_M \colon \mathrm{Lie}(G) \to \mathfrak{gl}(M)$.

Definition 206. Let $\mathfrak{g}$ be a restricted Lie algebra. The restricted enveloping algebra $\mathfrak{u}(\mathfrak{g})$ is the quotient of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ by the ideal generated by $x^p - x^{[p]}$ for all $x \in \mathfrak{g}$.

We can give $\mathfrak{U}(\mathfrak{g})$ a Hopf algebra structure by defining the elements of the Lie algebra to be primitive, i.e. $\Delta(x) := 1 \otimes x + x \otimes 1$. It turns out that $\mathfrak{u}(\mathfrak{g})$ is a finite dimensional (cocommutative) Hopf algebra. Indeed, if $\dim \mathfrak{g} = n$, then $\dim \mathfrak{u}(\mathfrak{g}) = p^n$.

207 Proposition

There is a one-to-one correspondence between p-restricted representations of $\mathfrak{g}$ *and* $\mathfrak{u}(\mathfrak{g})$*-modules. Compare with the fact that there is a one-to-one correspondence between representations of* $\mathfrak{g}$ *and* $\mathfrak{U}(\mathfrak{g})$*-modules.*

December 3rd, 2014: Algebraic Groups, Varieties, and Fixed Points

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208 Example

Let $\mathfrak{g}_a = \mathrm{Lie}(\mathbb{G}_a)$ defined over a field of characteristic $p > 0$. $\mathfrak{g}_a$ is one dimensional, say with generator x. Then $x^{[p]} = 0$. Another way to put it: we can embed $\mathbb{G}_a \hookrightarrow \mathrm{GL}_2$ in the upper right corner, which induces $\mathfrak{g}_a \to \mathfrak{gl}_2$ also by embedding in the upper right corner, but squaring such an element gives zero, and in general the pth power will annihilate the image. In this case, $\mathfrak{u}(\mathfrak{g}_a) = k[x]/x^p \cong k(\mathbb{Z}/p)$. (This is an isomorphism of algebras but not of Hopf algebras: the left-hand side has primitive generators whereas the right-hand side has group-like ones. Heuristically, primitive generators seem nicer from a representation-theory perspective.)

For another example, $\mathfrak{gl}_1 = \mathrm{Lie}(\mathrm{GL}_1)$. Let x be a generator of $\mathfrak{gl}_1$. Then $x^{[p]} = x$ (really, this is twisted by a constant, but we can set it equal to 1 if we wish). Now $\mathfrak{gl}_1 \ncong \mathfrak{g}_a$ as restricted Lie algebras. For instance, $\mathfrak{u}(\mathfrak{gl}_1) \cong k[x]/(x^p - x)$.

209 Remark

We next compare Frobenius kernels and restricted Lie algebras. Let G be an affine group scheme over characteristic $p > 0$. As usual, the Frobenius map $G \xrightarrow{F} G$ has a kernel denoted $G_{(1)}$ with corresponding Hopf algebra $k[G_{(1)}]$. Letting $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{u}(\mathfrak{g})$ is a cocommutative Hopf algebra. If we take the Cartier dual (so there must be some finiteness assumption) we have $k[G_{(1)}]^\sharp \cong \mathfrak{u}(\mathfrak{g})$. It follows that representations of $G_{(1)}$ are the same as comodules of $k[G_{(1)}]$ which are the same as modules $k[G_{(1)}]^\sharp$ which are the same as restricted $\mathfrak{g}$-modules.

210 Remark

The plan for the rest of the course is to give a crash course in the structure theory of semisimple algebraic groups. We'll skip over most of the root system and Dynkin diagram material, which will be covered in the algebra course in the Spring. (Note: Unfortunately, Julia was sick and was unable to give the final lecture.)

Definition 211. G is an algebraic group if it is a smooth algebraic group scheme. Smoothness here means the coordinate algebra is reduced.

212 Remark

Recall that a group of multiplicative type was a group which becomes diagonalizable upon extension to the separable closure. For algebraic groups, they decompose according to the usual structure theory for finite abelian groups. Recall also that a group is a torus if extending to the separable closure it is isomorphic to a product of (finitely many) $\mathbb{G}_m$'s. A split torus is a group which is isomorphic to a product of $\mathbb{G}_m$'s over its base field. Note that a split torus is a diagonalizable group scheme. If V is a representation of a split torus T, then $V \cong \oplus_{\alpha \in T^\vee} V_\alpha$ (where $T^\vee = \mathrm{Hom}(T, \mathbb{G}_m)$). (There is a section in Waterhouse covering this from the Hopf algebra perspective. Here $V_\alpha = \{v \in V : \Delta_V(v) = v \otimes m_\alpha\}$ using $V \to V \otimes k[T]$.)

Definition 213. We say that a variety X is a complete variety if for any other variety Y, the projection $X \times Y \to Y$ is closed. Equivalently, the map $X \to \mathrm{pt}$ is "universally closed", meaning it is preserved under pullbacks.

214 Example

A projective variety is complete. If X is affine and complete over an algebraically closed field, then X is finite. A classical example: take $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ by projecting the hyperbola $xy = 1$ down to the x-axis, the image misses 0, so the image is not closed. This correctly suggests that completeness is analogous to compactness.

215 Proposition

We have the following:

(1) A closed subvariety of a complete variety is complete.

(2) If $f\colon X \to Y$ and X is complete, then $f(X)$ is closed and complete.

(3) The product of complete varieties is complete.

Definition 216. Let G be an affine group scheme. We say G is solvable if its derived series terminates at the trivial group after finitely many steps. Here the derived series $\mathcal{D}^i G = [\mathcal{D}^{i-1}G, \mathcal{D}^{i-1}G]$ with $\mathcal{D}^0 G := G$ where $[G, G]$ is defined in Waterhouse 10.1 functorially.

217 Fact

G is solvable if and only if $G(\overline{k})$ is solvable as an abstract group.

218 Theorem (Borel's Fixed Point Theorem)

If G is a connected solvable algebraic group acting on a complete variety $X \neq \varnothing$, then G has a fixed point on X.

Proof Next time, if there's time. It's in Borel's Linear Algebraic Groups.

219 Corollary (Lie-Kolchin)

Let G be a connected solvable algebraic group. Let V be a representation of G, $\rho\colon G \to \mathrm{GL}(V)$. Then $\rho(G)$ stabilizes a flag in V, where a flag is a nested collection of subspaces $0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$ with $\dim_i V_i = i$.

Algebraically but equivalently, this says there is always a basis for V in which $\rho(G)$ is upper triangular.

PROOF (Sketch.) Let $\mathcal{F}(V)$ be the flag variety for V, which turns out to be a homogeneous space, namely GL_n modulo the upper triangular matrices. This is in fact a projective variety, so a complete variety. The result then follows immediately from Borel's fixed point theorem.

Definition 220. A Borel subgroup $B \subset G$ is a maximal connected solvable subgroup of G.

221 Example

B_n consisting of upper triangular $n \times n$ matrices is a Borel subgroup of GL_n.

222 Theorem

Let G be an algebraic group.

(1) Any two Borel subgroups are conjugate.

(2) If B is any Borel subgroup of G, then G/B is a projective variety parameterizing all Borel subgroups.

(3) If $H \subset G$ is a closed subgroup, then G/H is complete if and only if there is a Borel subgroup B of G contained in H.

(Quotients G/H in general are complicated and require a rather delicate theory. They are discussed in Janzten, who references Demazure-Gabriel.)

List of Symbols

μ_n Group Scheme of nth Roots of Unity, page 4

$\pi_0 A$ Maximal Separable Subalgebra, page 38

$\pi_0 G$ Group of Connected Components of G, page 38

σ Coinverse or Antipode, page 5

soc Socle Functor, page 37

$\underline{\pi}$ Finite Group Scheme of π, page 15

k Arbitrary field, page 2

$k[G]$ Coordinate Algebra of Affine Group Scheme G, page 3

$k[X]$ Coordinate Algebra of Affine Scheme X, page 3

k-Alg Category of Commutative k-Algebras, page 2

k^{sep} Separable Closure of k, page 16

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