Cohomology and support varieties: two points of view

Julia Pevtsova

MSRI Evans Lecture, April 14, 2008

Atiyah-Swan conjecture

D. Quillen, "*The spectrum of an equivariant cohomology ring* I, II, " Ann. Math. 94 (1971)

Atiyah-Swan conjecture

D. Quillen, "*The spectrum of an equivariant cohomology ring* I, II," Ann. Math. 94 (1971)

G - finite group, $k = \mathbb{F}_p^{\mathrm{alg}}$.

What is the Krull dimension of $\mathrm{H}^*(G, k)$?

Atiyah-Swan conjecture

D. Quillen, "*The spectrum of an equivariant cohomology ring* I, II," Ann. Math. 94 (1971)

G - finite group, $k = \mathbb{F}_p^{\mathrm{alg}}$.

What is the Krull dimension of $\mathrm{H}^*(G, k)$?

Conjecture (Atiyah, Swan): Krull dim $\mathrm{H}^*(G, k) = p$ - rank of G

Definition

$$p - \text{rank} = \max_{E \subset G} \text{rk}\, E$$

where $E \simeq \mathbb{Z}/p \times \mathbb{Z}/p \times \cdots \times \mathbb{Z}/p$ runs over all elementary abelian p-subgroups of G.

Finite generation

Back up...

Finite generation

Theorem

The cohomology ring $\mathrm{H}^*(G, k)$ *is a graded commutative* k*-algebra.*

Finite generation

Theorem

The cohomology ring $\mathrm{H}^*(G,k)$ is a graded commutative k-algebra.

$$\mathrm{H}^\bullet(G,k) = \begin{cases} \mathrm{H}^*(G,k), & \text{if } p = 2, \\ \mathrm{H}^{\mathrm{ev}}(G,k) & \text{if } p > 2. \end{cases}$$

Finite generation

Theorem

The cohomology ring $\mathrm{H}^*(G, k)$ is a graded commutative k-algebra.

$$\mathrm{H}^\bullet(G, k) = \begin{cases} \mathrm{H}^*(G, k), & \text{if } p = 2, \\ \mathrm{H}^{\mathrm{ev}}(G, k) & \text{if } p > 2. \end{cases}$$

Theorem (Venkov (1959), Evans (1961))

The cohomology ring $\mathrm{H}^\bullet(G, k)$ of a finite group G is a finitely generated k-algebra.

Extensions

Theorem (Maschke)

Let M be a representation of a finite group G over $\mathbb{C}$. Let $N \subset M$ be a G-invariant subspace. Then N splits off as a direct summand: $M = N \oplus N'$, N' is G-invarient subspace.

Extensions

Theorem (Maschke)

Let M be a representation of a finite group G over $\mathbb{C}$. Let $N \subset M$ be a G-invariant subspace. Then N splits off as a direct summand: $M = N \oplus N'$, N' is G-invarient subspace.

Corollary. Every representation over $\mathbb{C}$ is *completely reducible* - a direct sum of simple modules.

Extensions

Theorem (Maschke)

Let M be a representation of a finite group G over $\mathbb{C}$. Let $N \subset M$ be a G-invariant subspace. Then N splits off as a direct summand: $M = N \oplus N'$, N' is G-invarient subspace.

Corollary. Every representation over $\mathbb{C}$ is *completely reducible* - a direct sum of simple modules.

Modular representation theory: char $k \mid \#G$.
Representations are not completely reducible. Lots of non-split extensions (exact sequences of G-modules).

$$N \hookrightarrow M \twoheadrightarrow M/N$$

and the structure of $\mathrm{H}^*(G, k)$

Cohomology $\mathrm{H}^*(G, k) \quad \longleftrightarrow \quad$ extensions $[k \to \cdots \to k]$.

and the structure of $\mathrm{H}^*(G,k)$

Cohomology $\mathrm{H}^*(G,k) \quad \longleftrightarrow \quad$ extensions $[k \to \cdots \to k]$.

- $\mathrm{H}^i(G,k) = \mathrm{Ext}^i_G(k,k)$, additive group for every n.

and the structure of $\mathrm{H}^*(G, k)$

Cohomology $\mathrm{H}^*(G, k) \quad \longleftrightarrow \quad$ extensions $[k \to \cdots \to k]$.

- $\mathrm{H}^i(G, k) = \mathrm{Ext}^i_G(k, k)$, additive group for every n.
- Yoneda Product: $\mathrm{Ext}^i_G(k, k) \times \mathrm{Ext}^j_G(k, k) \to \mathrm{Ext}^{i+j}_G(k, k)$.

and the structure of $\mathrm{H}^*(G, k)$

Cohomology $\mathrm{H}^*(G, k)$ $\longleftrightarrow$ extensions $[k \to \cdots \to k]$.

- $\mathrm{H}^i(G, k) = \mathrm{Ext}^i_G(k, k)$, additive group for every n.
- Yoneda Product: $\mathrm{Ext}^i_G(k, k) \times \mathrm{Ext}^j_G(k, k) \to \mathrm{Ext}^{i+j}_G(k, k)$.

$$[k \to \cdots \to M_j \to k] \quad \circ \quad [k \to N_1 \to \cdots \to k]$$

and the structure of $\mathrm{H}^*(G,k)$

Cohomology $\mathrm{H}^*(G,k) \quad \longleftrightarrow \quad$ extensions $[k \to \cdots \to k]$.

- $\mathrm{H}^i(G,k) = \mathrm{Ext}^i_G(k,k)$, additive group for every n.
- Yoneda Product: $\mathrm{Ext}^i_G(k,k) \times \mathrm{Ext}^j_G(k,k) \to \mathrm{Ext}^{i+j}_G(k,k)$.

$$[k \to \cdots \to M_j \to k] \quad \circ \quad [k \to N_1 \to \cdots \to k]$$

$$k \to \cdots \to M_j \to \boxed{\mathrm{k} = \mathrm{k}} \to N_1 \to \cdots \to k$$

and the structure of $\mathrm{H}^*(G,k)$

Cohomology $\mathrm{H}^*(G,k) \quad \longleftrightarrow \quad$ extensions $[k \to \cdots \to k]$.

- $\mathrm{H}^i(G,k) = \mathrm{Ext}^i_G(k,k)$, additive group for every n.
- Yoneda Product: $\mathrm{Ext}^i_G(k,k) \times \mathrm{Ext}^j_G(k,k) \to \mathrm{Ext}^{i+j}_G(k,k)$.

$$[k \to \cdots \to M_j \to k] \quad \circ \quad [k \to N_1 \to \cdots \to k]$$

$$k \to \cdots \to M_j \to \boxed{\mathrm{k} = \mathrm{k}} \to N_1 \to \cdots \to k$$

$$k \to \cdots \to M_j \to N_1 \to \cdots \to k$$

and the structure of $H^*(G, k)$

Cohomology $H^*(G,k)$ $\longleftrightarrow$ extensions $[k \to \cdots \to k]$.

- $H^i(G,k) = \mathrm{Ext}^i_G(k,k)$, additive group for every n.
- Yoneda Product: $\mathrm{Ext}^i_G(k,k) \times \mathrm{Ext}^j_G(k,k) \to \mathrm{Ext}^{i+j}_G(k,k)$.

$$[k \to \cdots \to M_j \to k] \quad \circ \quad [k \to N_1 \to \cdots \to k]$$

$$k \to \cdots \to M_j \to \boxed{\mathrm{k} = \mathrm{k}} \to N_1 \to \cdots \to k$$

$$k \to \cdots \to M_j \to N_1 \to \cdots \to k$$

- $H^*(G,k) = \mathrm{Ext}^*_G(k,k) = \bigoplus_{i \geq 0} \mathrm{Ext}^i_G(k,k)$

and the structure of $\mathrm{H}^*(G,k)$

Cohomology $\mathrm{H}^*(G,k) \quad \longleftrightarrow \quad$ extensions $[k \to \cdots \to k]$.

- $\mathrm{H}^i(G,k) = \mathrm{Ext}^i_G(k,k)$, additive group for every n.
- Yoneda Product: $\mathrm{Ext}^i_G(k,k) \times \mathrm{Ext}^j_G(k,k) \to \mathrm{Ext}^{i+j}_G(k,k)$.

$$[k \to \cdots \to M_j \to k] \quad \circ \quad [k \to N_1 \to \cdots \to k]$$

$$k \to \cdots \to M_j \to \boxed{\mathrm{k} = \mathrm{k}} \to N_1 \to \cdots \to k$$

$$k \to \cdots \to M_j \to N_1 \to \cdots \to k$$

- $\mathrm{H}^*(G,k) = \mathrm{Ext}^*_G(k,k) = \bigoplus_{i \geq 0} \mathrm{Ext}^i_G(k,k)$

Remark. This gives the same cohomology ring as the one defined in Dave Benson's talk two weeks ago (in terms of projective resolutions and cup product).

Support variety

Example [Cohomology of elementary abelian p-groups].

- $E = (\mathbb{Z}/p)^{\times r}$, $\operatorname{rk} E = r$

Support variety

Example [Cohomology of elementary abelian *p*-groups].

- $E = (\mathbb{Z}/p)^{\times r}$, $\operatorname{rk} E = r$
- $\mathrm{H}^*(E, k) = k[x_1, \ldots, x_r] \otimes \underbrace{\Lambda^*(y_1, \ldots, y_n)}$

Support variety

Example [Cohomology of elementary abelian p-groups].

- $E = (\mathbb{Z}/p)^{\times r}$, $\operatorname{rk} E = r$
- $\mathrm{H}^*(E, k) = k[x_1, \dots, x_r] \otimes \underbrace{\Lambda^*(y_1, \dots, y_n)}_{\text{nilpotents}}$

Support variety

Example [Cohomology of elementary abelian p-groups].

- $E = (\mathbb{Z}/p)^{\times r}$, $\operatorname{rk} E = r$
- $\mathrm{H}^*(E, k) = k[x_1, \dots, x_r] \otimes \underbrace{\Lambda^*(y_1, \dots, y_n)}_{\text{nilpotents}}$
- $\mathrm{H}^*(E, k)_{\text{red}} = k[x_1, \dots, x_r]$

Support variety

Example [Cohomology of elementary abelian p-groups].

- $E = (\mathbb{Z}/p)^{\times r}$, $\operatorname{rk} E = r$
- $\mathrm{H}^*(E,k) = k[x_1,\dots,x_r] \otimes \underbrace{\Lambda^*(y_1,\dots,y_n)}_{\text{nilpotents}}$
- $\mathrm{H}^*(E,k)_{\mathrm{red}} = k[x_1,\dots,x_r]$
- $\operatorname{Spec} k[x_1,\dots,x_r] \simeq \mathbb{A}^r$

Example [Cohomology of elementary abelian p-groups].

- $E = (\mathbb{Z}/p)^{\times r}$, $\operatorname{rk} E = r$
- $\mathrm{H}^*(E, k) = k[x_1, \dots, x_r] \otimes \underbrace{\Lambda^*(y_1, \dots, y_n)}_{\text{nilpotents}}$
- $\mathrm{H}^*(E, k)_{\mathrm{red}} = k[x_1, \dots, x_r]$
- $\operatorname{Spec} k[x_1, \dots, x_r] \simeq \mathbb{A}^r$
  $\underbrace{(x_1 - \lambda_1, \dots, x_r - \lambda_r)}_{\text{max ideal}} \leftrightarrow \underbrace{(\lambda_1, \dots \lambda_r)}_{\text{point on } \mathbb{A}^r}.$

Example [Cohomology of elementary abelian *p*-groups].

- $E = (\mathbb{Z}/p)^{\times r}$, $\operatorname{rk} E = r$
- $\mathrm{H}^*(E,k) = k[x_1,\dots,x_r] \otimes \underbrace{\Lambda^*(y_1,\dots,y_n)}_{\text{nilpotents}}$
- $\mathrm{H}^*(E,k)_{\mathrm{red}} = k[x_1,\dots,x_r]$
- $\operatorname{Spec} k[x_1,\dots,x_r] \simeq \mathbb{A}^r$
  $\underbrace{(x_1-\lambda_1,\dots,x_r-\lambda_r)}_{\text{max ideal}} \leftrightarrow \underbrace{(\lambda_1,\dots\lambda_r)}_{\text{point on } \mathbb{A}^r}.$

Definition (Support variety)

$$|G| = \operatorname{Spec} \mathrm{H}^\bullet(G,k),$$

the **support variety** of G (set of prime ideals with Zariski topology).

Support variety

Example [Cohomology of elementary abelian p-groups].

- $E = (\mathbb{Z}/p)^{\times r}$, $\operatorname{rk} E = r$
- $\mathrm{H}^*(E,k) = k[x_1,\ldots,x_r] \otimes \underbrace{\Lambda^*(y_1,\ldots,y_n)}_{\text{nilpotents}}$
- $\mathrm{H}^*(E,k)_{\mathrm{red}} = k[x_1,\ldots,x_r]$
- $\operatorname{Spec} k[x_1,\ldots,x_r] \simeq \mathbb{A}^r$
  $\underbrace{(x_1-\lambda_1,\ldots,x_r-\lambda_r)}_{\text{max ideal}} \leftrightarrow \underbrace{(\lambda_1,\ldots\lambda_r)}_{\text{point on } \mathbb{A}^r}.$

Definition (Support variety)

$$|G| = \operatorname{Spec} \mathrm{H}^\bullet(G,k),$$

the **support variety** of G (set of prime ideals with Zariski topology).

Example. $|E| \simeq \mathbb{A}^r$, $\dim |E| = r$.

Quillen stratification theorem

Roughly: $|G|$ is "determined" by $|E| \subset |G|$, where $E \subset G$ runs over all elementary abelian p-subgroups of G.

Quillen stratification theorem

Roughly: $|G|$ is "determined" by $|E| \subset |G|$, where $E \subset G$ runs over all elementary abelian p-subgroups of G.

$E \subset G$

Quillen stratification theorem

Roughly: $|G|$ is "determined" by $|E| \subset |G|$, where $E \subset G$ runs over all elementary abelian p-subgroups of G.

$E \subset G$

$$\rightsquigarrow \quad \mathrm{H}^\bullet(G, k) \to \mathrm{H}^\bullet(E, k)$$

Quillen stratification theorem

Roughly: $|G|$ is "determined" by $|E| \subset |G|$, where $E \subset G$ runs over all elementary abelian p-subgroups of G.

$E \subset G$

$$\rightsquigarrow \quad H^\bullet(G,k) \to H^\bullet(E,k)$$

$$\rightsquigarrow \quad \mathrm{res}_{G,E} : |E| \to |G|$$

Quillen stratification theorem

Roughly: $|G|$ is "determined" by $|E| \subset |G|$, where $E \subset G$ runs over all elementary abelian p-subgroups of G.

$E \subset G$

$$\rightsquigarrow \quad H^\bullet(G,k) \to H^\bullet(E,k)$$

$$\rightsquigarrow \quad \mathrm{res}_{G,E} : |E| \to |G| \quad \text{finite map}$$

Quillen stratification theorem

Roughly: $|G|$ is "determined" by $|E| \subset |G|$, where $E \subset G$ runs over all elementary abelian p-subgroups of G.

$E \subset G$

$$\rightsquigarrow \quad \mathrm{H}^\bullet(G,k) \to \mathrm{H}^\bullet(E,k)$$

$$\rightsquigarrow \quad \mathrm{res}_{G,E} : |E| \to |G| \quad \text{finite map}$$

$\mathrm{res}_{G,E}\,|E| \simeq |E|/W_E$, where $W_E = N_G(E)/E$

Quillen stratification theorem

Roughly: $|G|$ is "determined" by $|E| \subset |G|$, where $E \subset G$ runs over all elementary abelian p-subgroups of G.

$E \subset G$

$$\rightsquigarrow \quad H^\bullet(G,k) \to H^\bullet(E,k)$$

$$\rightsquigarrow \quad \mathrm{res}_{G,E} : |E| \to |G| \quad \text{finite map}$$

$\mathrm{res}_{G,E}\,|E| \simeq |E|/W_E$, where $W_E = N_G(E)/E$

Theorem (Quillen (weak form))

$$|G| = \bigcup_{E \subset G} \mathrm{res}_{G,E}\,|E|$$

Consequences

Theorem (Quillen (weak form))

$$|G| = \bigcup_{E \subset G} \mathrm{res}_{G,E} |E|$$

Consequences

Theorem (Quillen (weak form))

$$|G| = \bigcup_{E \subset G} \mathrm{res}_{G,E} \, |E|$$

Corollary (Atiyah-Swan conjecture)

$\mathrm{Krull\,dim}\, H^\bullet(G,k) = \dim \mathrm{Spec}\, H^\bullet(G,k) = \dim |G| = \max_{E \subset G} \dim |E| = \max_{E \subset G} \dim \mathbb{A}^{\mathrm{rk}\, E} = \max_{E \subset G} \mathrm{rk}\, E$

Consequences

Theorem (Quillen (weak form))

$$|G| = \bigcup_{E \subset G} \mathrm{res}_{G,E} |E|$$

Corollary (Atiyah-Swan conjecture)

$\mathrm{Krull\,dim}\, H^\bullet(G,k) = \dim \mathrm{Spec}\, H^\bullet(G,k) = \dim |G| = \max_{E \subset G} \dim |E| = \max_{E \subset G} \dim \mathbb{A}^{\mathrm{rk}\, E} = \max_{E \subset G} \mathrm{rk}\, E$

Corollary

Irreducible components of $|G|$ $\leftrightarrow$ *conjugacy classes of maximal elementary abelian subgroups*

Example: D_8

$$D_8 = \langle \sigma, \tau \mid \sigma^4 = \tau^2 = 1, \tau\sigma\tau = \sigma^{-1} \rangle$$

Example: D_8

$$D_8 = \langle \sigma, \tau \mid \sigma^4 = \tau^2 = 1, \tau\sigma\tau = \sigma^{-1} \rangle$$

$$\langle \tau, \sigma^2\tau \rangle = (\mathbb{Z}/2)^2 \qquad\qquad \langle \sigma\tau, \sigma^3\tau \rangle = (\mathbb{Z}/2)^2$$

$$\langle \sigma^2 \rangle = \mathbb{Z}/2$$

Example: D_8

$$D_8 = \langle \sigma, \tau \,|\, \sigma^4 = \tau^2 = 1, \tau\sigma\tau = \sigma^{-1} \rangle$$

$$\langle \tau, \sigma^2\tau \rangle = (\mathbb{Z}/2)^2 \longleftarrow \langle \sigma^2 \rangle = \mathbb{Z}/2 \longrightarrow \langle \sigma\tau, \sigma^3\tau \rangle = (\mathbb{Z}/2)^2$$

$$|D_8| \qquad \mathbb{A}^2 \longleftarrow \mathbb{A}^1 \longrightarrow \mathbb{A}^2$$

Example: D_8

$$D_8 = \langle \sigma, \tau \mid \sigma^4 = \tau^2 = 1, \tau\sigma\tau = \sigma^{-1} \rangle$$

$\langle \tau, \sigma^2\tau \rangle = (\mathbb{Z}/2)^2$ $\quad\longleftarrow\quad$ $\langle \sigma^2 \rangle = \mathbb{Z}/2$ $\quad\longrightarrow\quad$ $\langle \sigma\tau, \sigma^3\tau \rangle = (\mathbb{Z}/2)^2$

$|D_8|$

$\mathbb{A}^2$ $\quad\longleftarrow\quad$ $\mathbb{A}^1$ $\quad\longrightarrow\quad$ $\mathbb{A}^2$

$$|D_8| \simeq \mathbb{A}^2 \times_{\mathbb{A}^1} \mathbb{A}^2$$

Example: D_8

$$D_8 = \langle \sigma, \tau \mid \sigma^4 = \tau^2 = 1, \tau\sigma\tau = \sigma^{-1} \rangle$$

$$\langle \tau, \sigma^2\tau \rangle = (\mathbb{Z}/2)^2 \longleftarrow \langle \sigma^2 \rangle = \mathbb{Z}/2 \longrightarrow \langle \sigma\tau, \sigma^3\tau \rangle = (\mathbb{Z}/2)^2$$

$|D_8|$

$$\mathbb{A}^2 \longleftarrow \mathbb{A}^1 \longrightarrow \mathbb{A}^2$$

$$|D_8| \simeq \mathbb{A}^2 \times_{\mathbb{A}^1} \mathbb{A}^2$$

Can check the answer because ...

Example: D_8

$$D_8 = \langle \sigma, \tau \mid \sigma^4 = \tau^2 = 1, \tau\sigma\tau = \sigma^{-1} \rangle$$

$$\begin{array}{ccc} \langle \tau, \sigma^2\tau \rangle = (\mathbb{Z}/2)^2 & & \langle \sigma\tau, \sigma^3\tau \rangle = (\mathbb{Z}/2)^2 \\ & \nwarrow \quad \nearrow & \\ & \langle \sigma^2 \rangle = \mathbb{Z}/2 & \end{array}$$

$$\begin{array}{cccc} |D_8| & \mathbb{A}^2 & & \mathbb{A}^2 \\ & & \nwarrow \quad \nearrow & \\ & & \mathbb{A}^1 & \end{array}$$

$$|D_8| \simeq \mathbb{A}^2 \times_{\mathbb{A}^1} \mathbb{A}^2$$

Can check the answer because ...
$\mathrm{H}^*(D_8, k) = k[x_1, x_2, z]/(x_1x_2)$

Varieties for modules

Alperin – Evens, Carlson, Avrunin – Scott.

A G-module M $\longrightarrow$ a subvariety $|G|_M \subset |G|$.

Varieties for modules

Alperin – Evens, Carlson, Avrunin – Scott.

A G-module $M \longrightarrow$ a subvariety $|G|_M \subset |G|$.

- $\mathrm{Ext}^*_G(M,M)$ is a ring (operations as for $\mathrm{Ext}^*_G(k,k)$).

Varieties for modules

Alperin – Evens, Carlson, Avrunin – Scott.

A G-module M $\longrightarrow$ a subvariety $|G|_M \subset |G|$.

- $\mathrm{Ext}^*_G(M,M)$ is a ring (operations as for $\mathrm{Ext}^*_G(k,k)$).
- $\mathrm{H}^\bullet(G,k) = \mathrm{Ext}^\bullet_G(k,k) \xrightarrow{\otimes M} \mathrm{Ext}^*_G(M,M)$

Varieties for modules

Alperin – Evens, Carlson, Avrunin – Scott.

A G-module M $\longrightarrow$ a subvariety $|G|_M \subset |G|$.

- $\mathrm{Ext}^*_G(M, M)$ is a ring (operations as for $\mathrm{Ext}^*_G(k, k)$).
- $\mathrm{H}^\bullet(G, k) = \mathrm{Ext}^\bullet_G(k, k) \xrightarrow{\otimes M} \mathrm{Ext}^*_G(M, M)$

$$k \to \cdots \to k \quad \mapsto \quad k \otimes M \longrightarrow \cdots \longrightarrow k \otimes M$$

Varieties for modules

Alperin – Evens, Carlson, Avrunin – Scott.

A G-module $M \longrightarrow$ a subvariety $|G|_M \subset |G|$.

- $\mathrm{Ext}^*_G(M, M)$ is a ring (operations as for $\mathrm{Ext}^*_G(k, k)$).
- $\mathrm{H}^\bullet(G, k) = \mathrm{Ext}^\bullet_G(k, k) \xrightarrow{\otimes M} \mathrm{Ext}^*_G(M, M)$

$$k \to \cdots \to k \quad \mapsto \quad k \otimes M \longrightarrow \cdots \longrightarrow k \otimes M$$

- $\mathcal{I}_M = \mathrm{Ker}\{\mathrm{H}^\bullet(G, k) \to \mathrm{Ext}^*_G(M, M)\}$

Varieties for modules

Alperin – Evens, Carlson, Avrunin – Scott.

A G-module M $\longrightarrow$ a subvariety $|G|_M \subset |G|$.

- $\mathrm{Ext}^*_G(M, M)$ is a ring (operations as for $\mathrm{Ext}^*_G(k, k)$).
- $\mathrm{H}^\bullet(G, k) = \mathrm{Ext}^\bullet_G(k, k) \xrightarrow{\otimes M} \mathrm{Ext}^*_G(M, M)$

$$k \to \cdots \to k \quad \mapsto \quad k \otimes M \longrightarrow \cdots \longrightarrow k \otimes M$$

- $\mathcal{I}_M = \mathrm{Ker}\{\mathrm{H}^\bullet(G, k) \to \mathrm{Ext}^*_G(M, M)\}$

Definition

The support variety of a G-module M

$$|G|_M = Z(\mathcal{I}_M) \subset |G|,$$

where $Z(\mathcal{I}_M) = \langle \wp \,|\, \mathcal{I}_M \subset \wp \rangle$

Properties

- $|G|_{M \oplus N} = |G|_M \cup |G|_N$.
- If $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence, then $|G|_{M_i} \subset |G|_{M_{i+1}} \cup |G|_{M_{i+2}}$.
- $|G|_{\Omega M} = |G|_M$.
- (Tensor product property) $|G|_{M \otimes N} = |G|_M \cap |G|_N$
- (Restriction) Let $H \subset G$, M - a G-module. Then $\mathrm{res}_{G,H}(|H|_M) = |G|_M$.
- $\dim |G|_M =$ complexity of M ($=$ rate of growth of the minimal projective resolution).
- $|G|_{L_\zeta} = \langle \zeta \rangle$ - a hypersurface in $|G|$ defined by $\zeta = 0$, $\zeta \in \mathrm{H}^\bullet(G, k)$.

Froms groups to algebras

- Group algebra kG
  basis as k-vector space: $\{e_g\}_{g \in G}$
  multiplication: $e_g \cdot e_h = e_{gh}$

Froms groups to algebras

- Group algebra kG
  basis as k-vector space: $\{e_g\}_{g\in G}$
  multiplication: $e_g \cdot e_h = e_{gh}$
- kG is a finite-dimensional Hopf algebra
  (has a coproduct: $kG \to kG \otimes kG$, $e_g \mapsto e_g \otimes e_g$)

Froms groups to algebras

- Group algebra kG
  basis as k-vector space: $\{e_g\}_{g \in G}$
  multiplication: $e_g \cdot e_h = e_{gh}$
- kG is a finite-dimensional Hopf algebra
  (has a coproduct: $kG \to kG \otimes kG$, $e_g \mapsto e_g \otimes e_g$)
- Representations of $G \stackrel{\sim}{\longleftrightarrow} kG$-modules

Froms groups to algebras

- Group algebra kG
  basis as k-vector space: $\{e_g\}_{g \in G}$
  multiplication: $e_g \cdot e_h = e_{gh}$
- kG is a finite-dimensional Hopf algebra
  (has a coproduct: $kG \to kG \otimes kG$, $e_g \mapsto e_g \otimes e_g$)
- Representations of $G \overset{\sim}{\longleftrightarrow}$ kG-modules
- Cohomology of $G \overset{\sim}{\longleftrightarrow}$ cohomology of kG

Other structures

Other algebraic structures that correspond to fin. dim-l Hopf algebras (and have theories of support vareities)

Other structures

Other algebraic structures that correspond to fin. dim-l Hopf algebras (and have theories of support vareities)

- Lie algebras in char p

Other structures

Other algebraic structures that correspond to fin. dim-l Hopf algebras (and have theories of support vareities)

- Lie algebras in char p
- Finite group schemes (e.g., inifinitesimal subgroups of algebraic groups, such as GL_n)

Other structures

Other algebraic structures that correspond to fin. dim-l Hopf algebras (and have theories of support vareities)

- Lie algebras in char p
- Finite group schemes (e.g., inifinitesimal subgroups of algebraic groups, such as GL_n)
- Small quantum groups

Other structures

Other algebraic structures that correspond to fin. dim-l Hopf algebras (and have theories of support vareities)

- Lie algebras in char p
- Finite group schemes (e.g., inifinitesimal subgroups of algebraic groups, such as GL_n)
- Small quantum groups
- Lie superalgebras (actually, no Hopf algebra here)

Other structures

Other algebraic structures that correspond to fin. dim-l Hopf algebras (and have theories of support vareities)

- Lie algebras in char p
- Finite group schemes (e.g., inifinitesimal subgroups of algebraic groups, such as GL_n)
- Small quantum groups
- Lie superalgebras (actually, no Hopf algebra here)

Theorem (Friedlander-Suslin, (1997))

Let A be a finite-dimensional co-commutative Hopf algebra over a field k of positive characteristic. Then the cohomology algebra $\mathrm{H}^\bullet(A, k)$ is finitely generated.

p-Lie algebras

Let G be an algebraic group defined over k, $g = \text{Lie}(G)$.

$$g \leftrightarrow u(g),$$

the *restricted enveloping algebra* of g.

p-Lie algebras

Let G be an algebraic group defined over k, $g = \text{Lie}(G)$.

$$g \leftrightarrow u(g),$$

the *restricted enveloping algebra* of g. This is a Hopf algebra, which is a finite dimensional quotient of the universal enveloping algebra of g.

p-Lie algebras

Let G be an algebraic group defined over k, $g = \mathrm{Lie}(G)$.

$$g \leftrightarrow u(g),$$

the *restricted enveloping algebra* of g. This is a Hopf algebra, which is a finite dimensional quotient of the universal enveloping algebra of g.

Assume $p = \mathrm{char}\ \mathrm{k}$ is "big enough" $(p > h)$.

Theorem (Friedlander-Parshall, Andersen-Jantzen (1983-84))

$$|g| = \mathcal{N}(g),$$

where $\mathcal{N}(g)$ is the nullcone of g, the variety of all nilpotent elements of g.

p-Lie algebras

Let G be an algebraic group defined over k, $g = \text{Lie}(G)$.

$$g \leftrightarrow u(g),$$

the *restricted enveloping algebra* of g. This is a Hopf algebra, which is a finite dimensional quotient of the universal enveloping algebra of g.
Assume $p = \mathrm{char}\, \mathrm{k}$ is "big enough" $(p > h)$.

Theorem (Friedlander-Parshall, Andersen-Jantzen (1983-84))

$$|g| = \mathcal{N}(g),$$

where $\mathcal{N}(g)$ is the nullcone of g, the variety of all nilpotent elements of g.

Very different from finite groups! In particular, $\mathcal{N}$ is irreducible.

p-Lie algebras

Let G be an algebraic group defined over k, $g = \mathrm{Lie}(G)$.

$$g \leftrightarrow u(g),$$

the *restricted enveloping algebra* of g. This is a Hopf algebra, which is a finite dimensional quotient of the universal enveloping algebra of g.
Assume $p = \mathrm{char}\ \mathrm{k}$ is "big enough" $(p > h)$.

Theorem (Friedlander-Parshall, Andersen-Jantzen (1983-84))

$$|g| = \mathcal{N}(g),$$

where $\mathcal{N}(g)$ is the nullcone of g, the variety of all nilpotent elements of g.

Very different from finite groups! In particular, $\mathcal{N}$ is irreducible.
Support varieties for modules $\leftrightarrow$ theory of nilpotent orbits.

Representation theory of the cyclic group $\mathbb{Z}/p$

Representation theory of a finite group is usually "wild" - we cannot classify indecomposable modules.

Representation theory of the cyclic group $\mathbb{Z}/p$

Representation theory of a finite group is usually "wild" - we cannot classify indecomposable modules.

Exception: $\mathbb{Z}/p = \langle \sigma \rangle$.

$$k\mathbb{Z}/p = \frac{k[\sigma]}{(\sigma^p - 1)} = \frac{k[\sigma]}{(\sigma - 1)^p} = \frac{k[t]}{t^p},$$

Representation theory of the cyclic group $\mathbb{Z}/p$

Representation theory of a finite group is usually "wild" - we cannot classify indecomposable modules.

Exception: $\mathbb{Z}/p = \langle \sigma \rangle$.

$$k\mathbb{Z}/p = \frac{k[\sigma]}{(\sigma^p - 1)} = \frac{k[\sigma]}{(\sigma - 1)^p} = \frac{k[t]}{t^p},$$

Complete description of representation theory:

Representation theory of the cyclic group $\mathbb{Z}/p$

Representation theory of a finite group is usually "wild" - we cannot classify indecomposable modules.

Exception: $\mathbb{Z}/p = \langle \sigma \rangle$.

$$k\mathbb{Z}/p = \frac{k[\sigma]}{(\sigma^p - 1)} = \frac{k[\sigma]}{(\sigma - 1)^p} = \frac{k[t]}{t^p},$$

Complete description of representation theory:

- k - simple module

Representation theory of the cyclic group $\mathbb{Z}/p$

Representation theory of a finite group is usually "wild" - we cannot classify indecomposable modules.

Exception: $\mathbb{Z}/p = \langle \sigma \rangle$.

$$k\mathbb{Z}/p = \frac{k[\sigma]}{(\sigma^p - 1)} = \frac{k[\sigma]}{(\sigma - 1)^p} = \frac{k[t]}{t^p},$$

Complete description of representation theory:

- k - simple module
- $k,\ k[t]/t^2, \ldots, k[t]/t^p$ – p indecomposable modules.

Representation theory of the cyclic group $\mathbb{Z}/p$

Representation theory of a finite group is usually "wild" - we cannot classify indecomposable modules.

Exception: $\mathbb{Z}/p = \langle \sigma \rangle$.

$$k\mathbb{Z}/p = \frac{k[\sigma]}{(\sigma^p - 1)} = \frac{k[\sigma]}{(\sigma - 1)^p} = \frac{k[t]}{t^p},$$

Complete description of representation theory:

- k - simple module
- $k,\ k[t]/t^2, \dots, k[t]/t^p \quad - \quad p$ indecomposable modules.

$\mathbb{Z}/p$–module M

Representation theory of the cyclic group $\mathbb{Z}/p$

Representation theory of a finite group is usually "wild" - we cannot classify indecomposable modules.

Exception: $\mathbb{Z}/p = \langle \sigma \rangle$.

$$k\mathbb{Z}/p = \frac{k[\sigma]}{(\sigma^p - 1)} = \frac{k[\sigma]}{(\sigma - 1)^p} = \frac{k[t]}{t^p},$$

Complete description of representation theory:

- k - simple module
- $k,\ k[t]/t^2, \ldots, k[t]/t^p$ – p indecomposable modules.

$\mathbb{Z}/p$–module $M \leftrightarrow$ Jordan canonical form of σ as an operator on M

Representation theory of the cyclic group $\mathbb{Z}/p$

Representation theory of a finite group is usually "wild" - we cannot classify indecomposable modules.

Exception: $\mathbb{Z}/p = \langle \sigma \rangle$.

$$k\mathbb{Z}/p = \frac{k[\sigma]}{(\sigma^p - 1)} = \frac{k[\sigma]}{(\sigma - 1)^p} = \frac{k[t]}{t^p},$$

Complete description of representation theory:

- k - simple module
- $k,\ k[t]/t^2, \ldots, k[t]/t^p$ – p indecomposable modules.

$\mathbb{Z}/p$–module $M \leftrightarrow$ Jordan canonical form of σ as an operator on M $\leftrightarrow$ partition $(1^{a_1} 2^{a_2} \ldots p^{a_p}) \vdash \dim M$

Representation theory of the cyclic group $\mathbb{Z}/p$

Representation theory of a finite group is usually "wild" - we cannot classify indecomposable modules.

Exception: $\mathbb{Z}/p = \langle \sigma \rangle$.

$$k\mathbb{Z}/p = \frac{k[\sigma]}{(\sigma^p - 1)} = \frac{k[\sigma]}{(\sigma - 1)^p} = \frac{k[t]}{t^p},$$

Complete description of representation theory:

- k - simple module
- $k,\ k[t]/t^2, \ldots, k[t]/t^p$ – p indecomposable modules.

$\mathbb{Z}/p$–module $M \leftrightarrow$ Jordan canonical form of σ as an operator on M $\leftrightarrow$ partition $(1^{a_1}2^{a_2}\ldots p^{a_p}) \vdash \dim M$

Write additively:
$k[t]/t^p$ - module $M \longleftrightarrow$ Jordan type $a_1[1] + a_2[2] + \cdots + a_p[p]$

Representation theory of the cyclic group $\mathbb{Z}/p$

Representation theory of a finite group is usually "wild" - we cannot classify indecomposable modules.

Exception: $\mathbb{Z}/p = \langle \sigma \rangle$.

$$k\mathbb{Z}/p = \frac{k[\sigma]}{(\sigma^p - 1)} = \frac{k[\sigma]}{(\sigma - 1)^p} = \frac{k[t]}{t^p},$$

Complete description of representation theory:

- k - simple module
- $k,\ k[t]/t^2, \dots, k[t]/t^p$ – p indecomposable modules.

$\mathbb{Z}/p$–module $M \leftrightarrow$ Jordan canonical form of σ as an operator on M $\leftrightarrow$ partition $(1^{a_1}2^{a_2}\dots p^{a_p}) \vdash \dim M$

Write additively:

$k[t]/t^p$ - module $M \longleftrightarrow$ Jordan type $a_1[1] + a_2[2] + \cdots + a_p[p]$

Free $k[t]/t^p$-module $= \bigoplus_a k[t]/t^p \longleftrightarrow$ Jordan type $a[p]$.

Cyclic Shifted Subgroups

$E = (\mathbb{Z}/p)^{\times r}$. Choose generators $g_1, \dots, g_r$. Let $t_1 = g_1 - 1, \dots, t_r = g_r - 1$.

Cyclic Shifted Subgroups

$E = (\mathbb{Z}/p)^{\times r}$. Choose generators $g_1, \ldots, g_r$. Let $t_1 = g_1 - 1, \ldots, t_r = g_r - 1$.

$$kE = k[g_1, \ldots, g_r]/(g_i^p - 1) = k[t_1, \ldots, t_r]/(t_i^p).$$

Cyclic Shifted Subgroups

$E = (\mathbb{Z}/p)^{\times r}$. Choose generators $g_1, \ldots, g_r$. Let $t_1 = g_1 - 1, \ldots, t_r = g_r - 1$.

$$kE = k[g_1, \ldots, g_r]/(g_i^p - 1) = k[t_1, \ldots, t_r]/(t_i^p).$$

Definition

Let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{A}^r$. A shifted cyclic subgroup $< \underline{\alpha} >$ of E corresponding to $\underline{\alpha}$ is a cyclic subgroup of kE generated by a p-unipotent element $\alpha_1 t_1 + \cdots + \alpha_r t_r + 1$.

Cyclic shifted sub-s are parametrized by the affine space $\mathbb{A}^n_k$.

Cyclic Shifted Subgroups

$E = (\mathbb{Z}/p)^{\times r}$. Choose generators $g_1, \ldots, g_r$. Let $t_1 = g_1 - 1, \ldots, t_r = g_r - 1$.

$$kE = k[g_1, \ldots, g_r]/(g_i^p - 1) = k[t_1, \ldots, t_r]/(t_i^p).$$

Definition

Let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{A}^r$. A shifted cyclic subgroup $< \underline{\alpha} >$ of E corresponding to $\underline{\alpha}$ is a cyclic subgroup of kE generated by a p-unipotent element $\alpha_1 t_1 + \cdots + \alpha_r t_r + 1$.

Cyclic shifted sub-s are parametrized by the affine space $\mathbb{A}^n_k$.

$V_E =$ variety of cyclic shifted subgroups.

Cyclic Shifted Subgroups

$E = (\mathbb{Z}/p)^{\times r}$. Choose generators $g_1, \dots, g_r$. Let $t_1 = g_1 - 1, \dots, t_r = g_r - 1$.

$$kE = k[g_1, \dots, g_r]/(g_i^p - 1) = k[t_1, \dots, t_r]/(t_i^p).$$

Definition

Let $\underline{\alpha} = (\alpha_1, \dots, \alpha_r) \in \mathbb{A}^r$. A shifted cyclic subgroup $< \underline{\alpha} >$ of E corresponding to $\underline{\alpha}$ is a cyclic subgroup of kE generated by a p-unipotent element $\alpha_1 t_1 + \dots + \alpha_r t_r + 1$.

Cyclic shifted sub-s are parametrized by the affine space $\mathbb{A}^n_k$.

V_E = variety of cyclic shifted subgroups.

There is a natural isomorphism $V_E \simeq |E|$.

Rank variety

Definition (Carlson)

$$V_E(M) = \{\underline{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{A}^r \mid \langle \alpha_1 t_1 + \ldots + \alpha_r t_r + 1 \rangle$$

$$\text{does not act freely on } M\}$$

- the **rank variety** of M.

Rank variety

Definition (Carlson)

$$V_E(M) = \{\underline{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{A}^r \mid \langle \alpha_1 t_1 + \ldots + \alpha_r t_r + 1 \rangle \text{ does not act freely on } M\}$$

- the **rank variety** of M.

Theorem (Avrunin-Scott, (1982))

Let M be a finite-dimensional kE-module. The isomorphism $V_E \simeq |E|$ restricts to

$$V_E(M) \simeq |E|_M.$$

Rank variety

Definition (Carlson)

$$V_E(M) = \{\underline{\alpha} = (\alpha_1, \ldots, \alpha_r) \in \mathbb{A}^r \mid \langle \alpha_1 t_1 + \ldots + \alpha_r t_r + 1 \rangle \text{ does not act freely on } M\}$$

- the **rank variety** of M.

Theorem (Avrunin-Scott, (1982))

Let M be a finite-dimensional kE-module. The isomorphism $V_E \simeq |E|$ restricts to

$$V_E(M) \simeq |E|_M.$$

Cyclic shifted subgroup is NOT a subgroup of E. It is a subgroup of kE.

π-points

Cyclic shifted subgroup $\langle \underline{\alpha} \rangle = \langle \alpha_1 t_1 + \ldots + \alpha_r t_r + 1 \rangle$ of E determines a map of algebras

$$k[t]/t^p \xrightarrow{t \mapsto \alpha_1 t_1 + \ldots + \alpha_r t_r} kE$$

π-points

Cyclic shifted subgroup $\langle \underline{\alpha} \rangle = \langle \alpha_1 t_1 + \ldots + \alpha_r t_r + 1 \rangle$ of E determines a map of algebras

$$k[t]/t^p \xrightarrow{t \mapsto \alpha_1 t_1 + \ldots + \alpha_r t_r} kE$$

Definition (π-point)

A π-**point** α of a finite group G is a map of algebras

$$\begin{array}{ccc} k[t]/t^p & \overset{\alpha}{\dashrightarrow} & kG \\ & \searrow \quad \nearrow & \\ & kA & \end{array}$$

which factors through some abelian p-subgroup $A \subset G$.

The map $kA \to kG$ is induced by a subgroup, the other two are just maps of algebras.

π-points

Cyclic shifted subgroup $\langle \underline{\alpha} \rangle = \langle \alpha_1 t_1 + \ldots + \alpha_r t_r + 1 \rangle$ of E determines a map of algebras

$$k[t]/t^p \xrightarrow{t \mapsto \alpha_1 t_1 + \ldots + \alpha_r t_r} kE$$

Definition (π-point)

A π-**point** α of a finite group G is a map of algebras

$$\begin{array}{ccc} k[t]/t^p & \overset{\alpha}{\dashrightarrow} & kG \\ & \searrow \qquad \nearrow & \\ & kA & \end{array}$$

which factors through some abelian p-subgroup $A \subset G$.

The map $kA \to kG$ is induced by a subgroup, the other two are just maps of algebras.

From π-points to cohomology

- A π-point $k[t]/t^p \longrightarrow kG \quad \rightsquigarrow$

From π-points to cohomology

- A π-point $k[t]/t^p \longrightarrow kG \quad \rightsquigarrow$
- $\mathrm{H}^\bullet(G, k) \to \mathrm{H}^\bullet(k[t]/t^p, k) \simeq k[x] \quad \rightsquigarrow$

From π-points to cohomology

- A π-point $k[t]/t^p \to kG \quad \rightsquigarrow$
- $\mathrm{H}^\bullet(G,k) \to \mathrm{H}^\bullet(k[t]/t^p, k) \simeq k[x] \quad \rightsquigarrow$
- $\operatorname{Spec} k[x] \to \operatorname{Spec} \mathrm{H}^\bullet(G,k) = |G|$

From π-points to cohomology

- A π-point $k[t]/t^p \to kG \quad \rightsquigarrow$
- $\mathrm{H}^\bullet(G,k) \to \mathrm{H}^\bullet(k[t]/t^p,k) \simeq k[x] \quad \rightsquigarrow$
- $\mathbb{A}^1 = \operatorname{Spec} k[x] \to \operatorname{Spec} \mathrm{H}^\bullet(G,k) = |G|$

From π-points to cohomology

- A π-point $k[t]/t^p \to kG \quad \rightsquigarrow$
- $\mathrm{H}^\bullet(G,k) \to \mathrm{H}^\bullet(k[t]/t^p, k) \simeq k[x] \quad \rightsquigarrow$
- $\mathbb{A}^1 = \operatorname{Spec} k[x] \to \operatorname{Spec} \mathrm{H}^\bullet(G,k) = |G|$
  Projectivize (factor out the scalar action of k^*):

From π-points to cohomology

- A π-point $k[t]/t^p \to kG \quad \rightsquigarrow$
- $\mathrm{H}^\bullet(G,k) \to \mathrm{H}^\bullet(k[t]/t^p,k) \simeq k[x] \quad \rightsquigarrow$
- $\mathbb{A}^1 = \operatorname{Spec} k[x] \to \operatorname{Spec} \mathrm{H}^\bullet(G,k) = |G|$
  Projectivize (factor out the scalar action of k^*):
- $\mathrm{pt} \in \operatorname{Proj} |\mathrm{G}|$.

From π-points to cohomology

- A π-point $k[t]/t^p \to kG \quad \rightsquigarrow$
- $H^\bullet(G,k) \to H^\bullet(k[t]/t^p, k) \simeq k[x] \quad \rightsquigarrow$
- $\mathbb{A}^1 = \text{Spec}\, k[x] \to \text{Spec}\, H^\bullet(G,k) = |G|$
  Projectivize (factor out the scalar action of k^*):
- $\mathrm{pt} \in \text{Proj}\, |\mathrm{G}|$.

Some π - points $\quad \mapsto \quad$ same point on $\text{Proj}\, H^\bullet(G,k)$

From π-points to cohomology

- A π-point $k[t]/t^p \to kG \quad \rightsquigarrow$
- $\mathrm{H}^\bullet(G,k) \to \mathrm{H}^\bullet(k[t]/t^p, k) \simeq k[x] \quad \rightsquigarrow$
- $\mathbb{A}^1 = \operatorname{Spec} k[x] \to \operatorname{Spec} \mathrm{H}^\bullet(G,k) = |G|$
  Projectivize (factor out the scalar action of k^*):
- $\mathrm{pt} \in \operatorname{Proj} |\mathrm{G}|$.

Some π - points $\quad \mapsto \quad$ same point on $\operatorname{Proj} \mathrm{H}^\bullet(G,k)$

Example. $E = \mathbb{Z}/p \times \mathbb{Z}/p$, $kE = k[t_1,t_2]/(t_1^p, t_2^p)$.
$t \mapsto \alpha_1 t_1 + \alpha_2 t_2 \qquad t \mapsto \alpha_1 t_1 + \alpha_2 t_2 + t_1^2$

From π-points to cohomology

- A π-point $k[t]/t^p \to kG \quad \rightsquigarrow$
- $\mathrm{H}^\bullet(G,k) \to \mathrm{H}^\bullet(k[t]/t^p, k) \simeq k[x] \quad \rightsquigarrow$
- $\mathbb{A}^1 = \operatorname{Spec} k[x] \to \operatorname{Spec} \mathrm{H}^\bullet(G,k) = |G|$
  Projectivize (factor out the scalar action of k^*):
- $\mathrm{pt} \in \operatorname{Proj} |\mathrm{G}|$.

Some π - points $\quad \mapsto \quad$ same point on $\operatorname{Proj} \mathrm{H}^\bullet(G,k)$

Example. $E = \mathbb{Z}/p \times \mathbb{Z}/p$, $kE = k[t_1, t_2]/(t_1^p, t_2^p)$.
$t \mapsto \alpha_1 t_1 + \alpha_2 t_2 \qquad t \mapsto \alpha_1 t_1 + \alpha_2 t_2 + t_1^2$

Equivalence relation on π–points

$$\alpha \sim \beta$$

From π-points to cohomology

- A π-point $k[t]/t^p \to kG \quad \rightsquigarrow$
- $H^\bullet(G,k) \to H^\bullet(k[t]/t^p, k) \simeq k[x] \quad \rightsquigarrow$
- $\mathbb{A}^1 = \operatorname{Spec} k[x] \to \operatorname{Spec} H^\bullet(G,k) = |G|$
  Projectivize (factor out the scalar action of k^*):
- $\mathrm{pt} \in \operatorname{Proj} |\mathrm{G}|$.

Some π - points $\quad \mapsto \quad$ same point on $\operatorname{Proj} H^\bullet(G,k)$

Example. $E = \mathbb{Z}/p \times \mathbb{Z}/p$, $kE = k[t_1, t_2]/(t_1^p, t_2^p)$.
$t \mapsto \alpha_1 t_1 + \alpha_2 t_2 \qquad t \mapsto \alpha_1 t_1 + \alpha_2 t_2 + t_1^2$

Equivalence relation on π–points
$\alpha \sim \beta$

Solely in terms of local properties of representations

Π-space

Definition (Π-space)

$$\Pi(G) = \frac{\langle \pi - \text{points} \quad \alpha : k[t]/t^p \to kG \rangle}{\sim}$$

This is a topological space.

Π-space

Definition (Π-space)

$$\Pi(G) = \frac{\langle \pi - \text{points} \quad \alpha : k[t]/t^p \to kG \rangle}{\sim}$$

This is a topological space.

Definition

Let M be a G-module.

$$\Pi(G)_M = < [\alpha] : k[t]/t^p \to kG : \alpha^* M \text{ is not free} >$$

$\alpha^* M$ is a $k[t]/t^p$ – module where t acts via $\alpha(t) \in kG$.

M - finite dimensional $\mapsto \Pi(G)_M$ are precisely the closed sets of $\Pi(G)$.

Carlson's conjecture holds for Π-spaces:

Theorem (Friedlander-P.)

$$\Pi(G) \simeq \operatorname{Proj}|G|$$

$$\underbrace{\Pi(G)_M}_{\textit{local prop}} \simeq \underbrace{\operatorname{Proj}|G|_M}_{\textit{cohomology}}$$

Carlson's conjecture holds for Π-spaces:

Theorem (Friedlander-P.)

$$\Pi(G) \simeq \operatorname{Proj} |G|$$

$$\underbrace{\Pi(G)_M}_{\textit{local prop}} \simeq \underbrace{\operatorname{Proj} |G|_M}_{\textit{cohomology}}$$

Theorem (Detection of projectivity)

M is projective $\Leftrightarrow$ $\Pi(G)_M = \emptyset$ $\Leftrightarrow$ M is free when restricted to any subalgebra $k[t]/t^p \to kG$.

Carlson's conjecture holds for Π-spaces:

Theorem (Friedlander-P.)

$$\Pi(G) \simeq \operatorname{Proj}|G|$$

$$\underbrace{\Pi(G)_M}_{\textit{local prop}} \simeq \underbrace{\operatorname{Proj}|G|_M}_{\textit{cohomology}}$$

Theorem (Detection of projectivity)

M is projective $\Leftrightarrow$ $\Pi(G)_M = \emptyset$ $\Leftrightarrow$ M is free when restricted to any subalgebra $k[t]/t^p \to kG$.

- Projectivity can be detected locally on π-points.

Carlson's conjecture holds for Π-spaces:

Theorem (Friedlander-P.)

$$\Pi(G) \simeq \operatorname{Proj}|G|$$

$$\underbrace{\Pi(G)_M}_{\textit{local prop}} \simeq \underbrace{\operatorname{Proj}|G|_M}_{\textit{cohomology}}$$

Theorem (Detection of projectivity)

M is projective $\Leftrightarrow$ $\Pi(G)_M = \emptyset$ $\Leftrightarrow$ M is free when restricted to any subalgebra $k[t]/t^p \to kG$.

- Projectivity can be detected locally on π-points.
- Can replace kG by any finite dimensional co-commutative Hopf algebra

Modules of Constant Jordan type

- M is projective $\Leftrightarrow$ at every π-point $\alpha : k[t]/t^p \to kG$ the Jordan type of M is $a[p]$.

Modules of Constant Jordan type

- M is projective $\Leftrightarrow$ at every π-point $\alpha : k[t]/t^p \to kG$ the Jordan type of M is $a[p]$.
- In particular, the Jordan type is the *same* at every π-point.

Modules of Constant Jordan type

- M is projective $\Leftrightarrow$ at every π-point $\alpha : k[t]/t^p \to kG$ the Jordan type of M is $a[p]$.
- In particular, the Jordan type is the *same* at every π-point.
- Are there other modules with this property?

Modules of Constant Jordan type

- M is projective $\Leftrightarrow$ at every π-point $\alpha : k[t]/t^p \to kG$ the Jordan type of M is $a[p]$.
- In particular, the Jordan type is the *same* at every π-point.
- Are there other modules with this property?

Definition

M is a module of *constant Jordan type* if the Jordan type of M at every π-point $\alpha : k[t]/t^p \to kG$ is the same (the operator $\alpha(t)$ on M has the same Jordan canonical form for all α).

A very interesting and elusive class of modules!

Realizibility

- If M is of constant Jordan type, then M is determined by this unique type.

Realizibility

- If M is of constant Jordan type, then M is determined by this unique type.
- Which Jordan types can occur?

Realizibility

- If M is of constant Jordan type, then M is determined by this unique type.
- Which Jordan types can occur?
- Some constructible examples:

Realizibility

- If M is of constant Jordan type, then M is determined by this unique type.
- Which Jordan types can occur?
- Some constructible examples:
  - indecomposable modules of type $n[1] + m[p]$ for any n (Auslander-Reiten theory),

Realizibility

- If M is of constant Jordan type, then M is determined by this unique type.
- Which Jordan types can occur?
- Some constructible examples:
  - indecomposable modules of type $n[1] + m[p]$ for any n (Auslander-Reiten theory),
  - $n[p-1] + m[p]$,

Realizibility

- If M is of constant Jordan type, then M is determined by this unique type.
- Which Jordan types can occur?
- Some constructible examples:
  - indecomposable modules of type $n[1] + m[p]$ for any n (Auslander-Reiten theory),
  - $n[p-1] + m[p]$,
  - $[1] + n[2]$

Realizibility

- If M is of constant Jordan type, then M is determined by this unique type.
- Which Jordan types can occur?
- Some constructible examples:
  - indecomposable modules of type $n[1] + m[p]$ for any n (Auslander-Reiten theory),
  - $n[p-1] + m[p]$,
  - $[1] + n[2]$
- Quite limited!

Realizibility

- If M is of constant Jordan type, then M is determined by this unique type.
- Which Jordan types can occur?
- Some constructible examples:
  - indecomposable modules of type $n[1] + m[p]$ for any n (Auslander-Reiten theory),
  - $n[p-1] + m[p]$,
  - $[1] + n[2]$
- Quite limited!

Conjecture. [Carlson-Friedlander-P.] Let $p > 3$, $\dim |G| > 1$. The type $[2] + n[p]$ does not occur.

Realizibility

- If M is of constant Jordan type, then M is determined by this unique type.
- Which Jordan types can occur?
- Some constructible examples:
  - indecomposable modules of type $n[1] + m[p]$ for any n (Auslander-Reiten theory),
  - $n[p-1] + m[p]$,
  - $[1] + n[2]$
- Quite limited!

Conjecture. [Carlson-Friedlander-P.] Let $p > 3$, $\dim |G| > 1$. The type $[2] + n[p]$ does not occur.

Theorem (D. Benson, March 2008)

Let G be a finite group, $\dim |G| > 1$. There does not exist a G-module of constant Jordan type $[\mathbf{a}] + \mathbf{n}[\mathbf{p}]$ for $2 \le a \le p-2$.