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Infinite dimensional Modules for Infinitesimal Group Schemes

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Abstract

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We prove that the projectivity of an arbitrary (possibly infinite dimensional) module for an infinitesimal group scheme can be detected by restrictions to one-parameter subgroups. Building upon this result, we introduce the support cone of such a module, extending the construction of support variety for a finite dimensional module, and show that such support cones satisfy most of the familiar properties of support varieties. We also verify that our representation-theoretic definition of support cones admits an interpretation in terms of Rickard idempotent modules associated to thick subcategories of the stable category of finite dimensional modules. As a necessary step towards the proof of the projectivity detection theorem we investigate properties of the induction functor in the context of infinitesimal group schemes.

Acknowledgements

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Introduction

CHAPTER 1

Background

In this chapter we recall some necessary definitions and set some notation which will be used throughout. Since the material of this section is widely known, we will try to get through it quickly and painlessly (skipping all the details), most of which can be found in [**Jan**].

1.1. Group schemes and representations

Let k be an algebraically closed field of characteristic $p > 0$. An affine scheme X over k is a representable k-functor (represented by a k-algebra R) from the category of commutative k-algebras to the category of sets:

$$X(A) = \mathrm{Hom}_{k-alg}(R, A)$$

for any commutative k-algebra A. The algebra R is the *coordinate algebra* of the affine scheme X. It will be denoted $k[X]$. The simplest example of an affine scheme is an affine line $\mathbb{A}^1$, which, as a k-functor, is given as:

$$\mathbb{A}^1(A) = A.$$

The coordinate algebra of $\mathbb{A}^1$ is $k[\mathbb{A}^1] = k[x]$, polynomial ring of one variable over k.

Recall the *Yoneda lemma*: for any two affine schemes X and Y we have

$$\mathrm{Hom}_{k-alg}(k[Y], k[X]) = \mathrm{Mor}_{Sh/k}(X, Y).$$

Applying the Yoneda lemma to $Y = \mathbb{A}^1$, we get

$$k[X] = \mathrm{Mor}_{Sh/k}(X, \mathbb{A}^1).$$

An affine k-group scheme G is a representable functor from the category of commutative k-algebras to the category of groups. If the coordinate algebra $k[G]$ is finitely generated over k, then G is an *algebraic* group scheme. In this case algebra $k[G]$ has an extra structure of a Hopf algebra, i.e. there is a coproduct map

$$\Delta : k[G] \to k[G] \otimes k[G],$$

a counit map

$$\epsilon : k[G] \to k$$

and a coinverse map

$$s : k[G] \to k[G],$$

which are homomorphisms of k-algebras and make standard diagrams commutative. In particular, the coproduct is coassociative. When the coproduct is also cocommuative, the Hopf algebra is called cocommutative. The *augmentation ideal* of a Hopf algebra is the kernel of the counit map.

Next we mention two basic examples of algebraic group schemes and their coordinate algebras, which will be referred to throughout the text.

- *General linear group.* As a k-group functor, GL_n is defined as

$$GL_n(A) = \{n \times n \text{ matrices with entries in } A \text{ and non-zero determinant}\},$$

where multiplication is the usual matrix multiplication. Furthermore,

$$k[GL_n] = k[T_{ij}, 1/det(T_{ij})]_{1 \leq i,j \leq n},$$

and coalgebra structure is given as follows:

$$\Delta(T_{ij}) = \sum_{k=1}^{k=n} T_{ik} \otimes T_{kj},\ \epsilon(T_{ij}) = \delta_{ij}.$$

- *Additive group* $\mathbb{G}_a$. Again, as a functor,

$$\mathbb{G}_a(A) = A^+,$$

the additive group of the ring A. The coordinate algebra $k[\mathbb{G}_a] = k[T]$, a polynomial ring of 1 variable, and coproduct is given by $T \rightarrow 1 \otimes T + T \otimes 1$.

An *algebraic group* is a reduced algebraic group scheme, where we say that an affine scheme is reduced if its coordinate algebra is reduced. Although we would be taking a functorial point of view on algebraic groups, we would like to mention here that they are affine algebraic groups in the "standard" sense: the set of k-rational points of an algebraic group is an affine variety over k with group operations given by regular functions.

Over an algebraically closed field k, a reduced affine group scheme G is automatically smooth, hence $k[G]$ is a regular ring (cf. [**Har**]). Any connected algebraic group can be embedded into some GL_n for an appropriate n.

Representations. We will briefly describe three different (equivalent) ways of giving a G-representation.

A module (or a representation) of an affine algebraic group scheme G is a k-vector space M such that the corresponding affine scheme M_a ($M_a(A) = M \otimes_k A$) is endowed with an action of G:

$$G(A) \times M \otimes A \to M \otimes A,$$

and the action is functorial on A. For each A we thereby get a group homomorphism $G(A) \to Aut_A(M \otimes A)$ which leads to a group scheme homomorphism

$$G \to GL(M).$$

If we take $A = k[G]$, then the identity homomorphism $id_{k[G]} \in G(k[G]) = Hom_{k-alg}(k[G], k[G])$ acts on $M \otimes k[G]$ and we define

$$\Delta_M : M \to M \otimes k[G]$$

via $\Delta_M(m) = id_{k[G]}(m \otimes 1) \in M \otimes k[G]$. The map Δ_M gives a *comodule* structure on M. One has an equivalence of categories

$$\{G \text{ - modules}\} \longleftrightarrow \{k[G] \text{ - comodules}\}.$$

The action of G on $k[G]$ defined via

$$g \circ f(\bullet) \stackrel{def}{=} f(g^{-1}\bullet),$$

$g \in G, f \in k[G]$, yields the *left regular representation* of G. By setting

$$g \circ f(\bullet) \stackrel{def}{=} f(\bullet g)$$

one gets the *right regular representation.* As $k[G]$-comodules, the right regular representation is given by the coproduct map $\Delta : k[G] \to k[G] \otimes k[G]$, whereas the comodule structure for the left regular representation is given by the the formula $T_{12} \circ (s \otimes id) \circ \Delta : k[G] \to k[G] \otimes k[G]$, where s is the coinverse map and T_{12} permutes the factors in the tensor product.

Frobenius kernels. An affine algebraic group scheme is called finite if its coordinate algebra is finite-dimensional over the ground field k. In this case the linear dual to the coordinate algebra is again a finite-dimensional Hopf algebra. We will use # for the linear dual. For a finite k-group scheme G we can extend the equivalence of categories mentioned above:

$$\{G \text{ - modules}\} \longleftrightarrow \{k[G] \text{ - comodules}\} \longleftrightarrow \{k[G]^{\#} \text{ - modules}\}.$$

Finite group scheme is called *infinitesimal* is its coordinate algebra is local. This is equivalent to the augmentation ideal being nilpotent. The most important (for us) example of an infinitesimal group scheme is provided by Frobenius kernels which we define below.

Let $f : k \to k$ be the Frobenius automorphism of the field k: $f(a) = a^p$. Let M be any k-module. Define $M^{(1)}$, the Frobenius twist of M, as

$$M^{(1)} = M \otimes_f k,$$

i.e. $\lambda m \otimes 1 = m \otimes \lambda^p$ for $\lambda \in k$. Another way of saying this is that M coincides with $M^{(1)}$ as an abelian group and $\lambda \in k$ acts on $M^{(1)}$ as λ^{-p} acts on M.

For a commutative k-algebra A we get a map of k-algebras:

$$F^* : A^{(1)} \to A; a \otimes \lambda \to \lambda a^p$$

Note that if A is a Hopf algebra, then so is $A^{(1)}$, and the map above is a map of Hopf algebras.

Let G be an affine k-group scheme. Then $k[G]^{(1)}$ defines a new k-group scheme, which we denote $G^{(1)}$. The map $F^* : k[G]^{(1)} \to k[G]$ induces a map of group schemes, which we call Frobenius map:

$$F : G \to G^{(1)}.$$

$G^{(1)}$ is the first *Frobenius twist* of G. By iterating F we get higher Frobenius twists $G^{(r)}$. Next, we define the *r-th Frobenius kernel* of a k-group scheme G, denoted $G_{(r)}$, to be the scheme-theoretic kernel of the map $F^r : G \to G^{(r)}$.

If the group scheme G is defined over the prime field $\mathbb{F}_p$, then there is a natural isomorphism $G^{(1)} \cong G$. Indeed, in this case $k[G] = \mathbb{F}_p[G] \otimes_{\mathbb{F}_p} k$. Thus, $k[G]^{(1)} = k[G] \otimes_f k \simeq \mathbb{F}_p[G] \otimes_{\mathbb{F}_p} k \otimes_f k \simeq \mathbb{F}_p[G] \otimes_{\mathbb{F}_p} k = k[G]$. Therefore, for group schemes defined over $\mathbb{F}_p$, Frobenius map can be written as

$$F : G \to G.$$

If M is a representation of G, then $M^{(1)}$ has a natural structure of a G-module obtained via twisting the structure on M with the Frobenius map:

$$G \to GL(M) \xrightarrow{F} GL(M)^{(1)} = GL(M^{(1)}).$$

One can also apply Frobenius twist to the comodule map on M to obtain the same G-structure on $M^{(1)}$:

$$M^{(1)} \xrightarrow{\Delta^{(1)}} (M \otimes k[G])^{(1)} = M^{(1)} \otimes k[G]^{(1)} \xrightarrow{id_{M^{(1)}} \otimes F^*} M^{(1)} \otimes k[G]$$

In short, the category of G - modules is closed under taking Frobenius twists.

- $GL_{n(r)}$. The Frobenius map on GL_n is given by taking each matrix entry to the p-th power. Precisely,

$$F^r(A) : GL_n(A) \to GL_n(A) : F^r((a_{ij})) = (a_{ij}^{p^r}).$$

As a k-group functor, the r-th Frobenius kernel of GL_n can be described as

$$GL_{n,(r)}(A) = KerF^r(A) = \{(a_{ij}) \in M_n(A) : a_{ij}^{p^r} = \delta_{ij}\}.$$

We also describe the coordinate algebra of $GL_{n,(r)}$:

$$k[GL_{n,(r)}] = k[T_{ij}]/(T_{ij}^{p^r} - \delta_{ij}), 1 \leq i, j \leq n.$$

- $\mathbb{G}_{a(r)}$. $F : \mathbb{G}_a \to \mathbb{G}_a$ is simply given by $F(a) = a^p$. We immediately get

$$\mathbb{G}_{a(r)}(A) = \{a \in A : a^{p^r} = 0\},$$

$$k[\mathbb{G}_{a(r)}] = k[T]/T^{p^r}.$$

We will fix notations for the dual algebra $k[\mathbb{G}_{a(r)}]^{\#}$ which will be used later in the text. Let $v_0, \dots v_{p^r-1}$ be the basis of $k[\mathbb{G}_{a(r)}]^{\#} = (k[T]/T^{p^r})^{\#}$ dual to the standard basis of $k[T]/T^{p^r}$. Denote v_{p^i} by u_i. Then

$$k[\mathbb{G}_{a(r)}]^{\#} = k[u_0, \dots, u_{r-1}]/(u_0^p, \dots, u_{r-1}^p).$$

Let G be an infinitesimal finite k-group scheme and I be the maximal ideal of $k[G]$. Since G is infinitesimal, I is nilpotent. The *height* of G, denoted ht$G$, is the minimal integer r suth that for any $x \in I$, $x^{p^r} = 0$. Clearly, if G is an algebraic group, then ht$G_{(r)} = r$. Any infinitesimal group scheme of height r can be embedded into the r-th Frobenius kernel of GL_n for an appropriate n.

A *1-parameter subgroup* of height r of an infinitesimal group scheme G is a homomorphism $\mathbb{G}_{a(r)} \to G$. We say that a 1-parameter subgroup is injective if this homomorphism is a closed embedding of group schemes.

1.2. Overview of projectivity detection results

Since the key result of this work concerns detection of projectivity of modules, we will briefly state the known results in this direction.

As always, historically (chronologically) finite groups come first. An elementary abelian p-group is a group isomorphic to a direct product of $\mathbb{Z}/p$.

Theorem 1.2.1. *(Chouinard* [**C**]*, 1976). Let G be a finite group, and M be a kG-module. Then M is projective if and only if it is projective as a kE - module for every elementary abelian p-subgroup $E \subset G$.*

Let us make a note here that the theorem of Chouinard does not require module M to be finite dimensional.

Once we reduce to the case of elementary abelian p-group, we can apply "Dade's lemma" to reduce to even smaller objects. We define these smaller objects first.

Let $E_r = \mathbb{Z}/p^n$ be an elementary abelian p-group, and denote generators by $\sigma_1, \ldots, \sigma_n$. A *cyclic shifted subgroup* of kE is a subalgebra generated by a p-unipotent element $1 + \alpha_1(\sigma_1 - 1) + \cdots + \alpha_n(\sigma_n - 1)$, where $\alpha_1, \ldots \alpha_n \in k$. Note that this subalgebra is isomorphic to $k\mathbb{Z}/p$ but does not necessarily come from a subgroup of E - here is the name "cyclic shifted subgroup".

Theorem 1.2.2. *(Dade* [**D**]*, 1978). Let E be an elementary abelian p-group and M be a finite dimensional kE-module. M is projective if and only if it is projective upon restriction to every non-trivial cyclic shifted subgroup of kE.*

Dade's lemma was generalized to infinite dimensional modules in the work of Benson, Carlson and Rickard. In particular, they observed that in its original form Dade's lemma does not hold in the infinite dimensional case: one can produce a non-projective module (which will be done in the context of infinitesimal group schemes in Section 4.4), whose restrictions to all cyclic shifted subgroups defined over k are projective.

Theorem 1.2.3. *(Benson, Carlson, Rickard* [**BCR2**]*, 1997). Let E be an elementary abelian p-group, and M be any kE-module. M is projective if and only if the* *restriction of $M \otimes K$ to every non-trivial cyclic shifted subgroup of KE is projective for any field extension K/k.*

So, in order to detect projectivity of an infinite dimensional module, we have to pass to field extensions. This fact has an easy geometric interpretation which will be explained in the next section, once we introduce the notion of support varieties.

Representation theories of E_r and $\mathbb{G}_{a(r)}$ are equivalent thanks to the fact that $kE_r \cong k[\mathbb{G}_{a(r)}]^{\#}$. In order to apply the "generalized Dade's lemma" of Benson-Carlson-Rickard to the representations of $\mathbb{G}_{a(r)}$, we reformulate the theorem in a more suitable for our purposes way.

Theorem 1.2.4. *Let $A = k[u_0, \dots, u_{r-1}]/(u_0^p, \dots, u_{r-1}^p)$ and M be an A-module. M is projective if and only if for any field extension K/k and any element $z = c_0u_0 + \cdots + c_{r-1}u_{r-1}$, where $c_0, \dots, c_{r-1} \in K$, the restriction of $M \otimes K$ to $K[z]/(z^p)$ is projective.*

In representation theory of restricted Lie algebras (which is equivalent to the representation theory of a suitable infinitesimal group scheme of height 1), the projectivity detection result is "one-step" and follows from works of Jantzen [**Jan1**] and Friedlander and Parshall:

Theorem 1.2.5. *(Friedlander-Parshall,* [**FP3**]*, 1986) Let g be a finite dimensional restricted Lie algebra and let M be a finite-dimensional restricted g-module. Then*

M is projective if and only if M is projective as a restricted $< X >$-module for every non-zero p-nilpotent element X in g.

Next, we turn to the case of an arbitrary infinitesimal group scheme. In [**SFB1**], [**SFB2**] Suslin, Friedlander and Bendel develop a theory of support varieties for infinitesimal group schemes. As an application, they generalize the theorem of Friedlnader-Parshall and prove the following projectivity detection result:

Theorem 1.2.6. *(Suslin-Friedlander-Bendel,* [**SFB2**]*, 1997). Let G be an infinitesimal group scheme over an algerbaically closed field k. Let further M be a finite dimensional G-module. Then M is projective if and only if for any subgroup scheme $H \subset G$ isomorphic to $\mathbb{G}_{a(r)}$ the restriction of M to H is a projective H-module.*

Bendel in [**B1**] further generalized this result to infinite dimensional modules for unipotent infinitesimal group schemes. A finite group scheme is called *unipotent* if it admits a filtration with quotients isomorphic to $\mathbb{G}_{a(1)}$. Or, equivalently, if it can be embedded in a Frobenius kernel of the algebraic group of upper triangular matrices (= unipotent radical of GL_n). Note that once we turn to the case of infinite dimensional modules, we have to take into account field extensions of k.

Theorem 1.2.7. *(Bendel,* [**B1**]*, 2001) Let G be an infinitesimal unipotent group scheme over k. Let further M be any G-module. Then M is projective if and only if for any field extension K/k and any subgroup scheme $H \otimes K \subset G \otimes K$ isomorphic to $\mathbb{G}_{a(r)} \otimes K$, the restriction of $M \otimes K$ to $H \otimes K$ is a projective $H \otimes K$-module.*

As a closing point of this discussion we can now state the main theorem of chapter two (Theorem 2.3.4), which proves projectivity detection property for any infinitesimal group scheme:

Theorem. *Let G be an infinitesimal group scheme over k. Let further M be any G-module. Then M is projective if and only if for any field extension K/k and any subgroup scheme $H \otimes K \subset G \otimes K$ isomorphic to $\mathbb{G}_{a(r)} \otimes K$, the restriction of $M \otimes K$ to $H \otimes K$ is a projective $H \otimes K$-module.*

1.3. Support varieties

Since one of the goals of this work is to generalize the notion of support variety to infinite dimensional modules, we briefly go over their definition and basic properties in the finite dimensional case. The following general constructions can be done both for finite groups (as it originally happened) and infinitesimal group schemes. Let G be either of them until specified further. A good reference for finite groups is [**Ben**]. A literature for infinitesimal group schemes is more scattered: see [**FP1**], [**FP2**], [**FP3**], [**Jan1**] for the case of a restricted Lie algebra (= infinitesimal group scheme of height one), and [**SFB1**], [**SFB2**] [**B2**] for the general case.

The even part of the cohomology algebra of G, $H^{ev}(G, k)$ is commutative Noetherian algebra over k ([**E**] for finite groups, [**FS**] for arbitrary finite group schemes). Thus, one can consider an affine scheme which is the prime ideal spectrum of this algebra. For $p = 2$ we look at the entire cohomology ring which is commutative in this case. To simplify notation assume that $p > 2$ for the rest of the section. For $p = 2$ everything is the same once we substitute $H^*(G,k)$ for $H^{ev}(G,k)$.

Define the *cohomological support scheme* of G to be $\operatorname{Spec} H^{ev}(G,k)$. The *cohomological support variety* of G is the variety of k-rational points of this scheme, or, equivalenty, the maximal ideal spectrum of $H^{ev}(G,k)$. Denote this variety by $|G|$. For a finite dimensional G - module M, let the *cohomological support variety* of M, $|G|_M$, be the closed homogenious subvariety of $|G|$ defined by the graded ideal $\operatorname{Ann}_{H^{ev}(G,k)}(\operatorname{Ext}^*_G(M,M))$, i.e.

$$|G|_M = V_k(\operatorname{Ann}_{H^{ev}(G,k)}(\operatorname{Ext}^*_G(M,M)).$$

Of course, $|G|_M$ is the variety of k-rational points of the closed conical affine subscheme of $\operatorname{Spec} H^{ev}(G,k)$ defined by the same ideal.

Support varieties capture substantial amount of information about G-modules. Here is a list of their basic properties:

(a) $|G|_M = 0$ if and only if M is a projective G-module.

(b) Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of G-modules. Then for any permutation (ijk) of (123) we have $|G|_{M_i} \subset |G|_{M_j} \cup |G|_{M_k}$.
A special case of this property is $|G|_{M\oplus N} = |G|_M \cap |G|_N$.

(c) "Tensor product theorem". $|G|_{M\otimes N} = |G|_M \cap |G|_N$ for any two G-modules M and N.

The most elusive property if we take the cohomological approach described above is the tensor product theorem.

?Naturality?

In both finite group and infinitesimal group cases the solution to the problem comes from developing a different, representation-theoretic approach.

For finite groups one defines *rank variety* of a module for an elementary abelian p-group ([**?**]). Let E be an elementary abelian p-group of rank r, and let J be the augmentation ideal of kE. The rank variety of E, V_E, is the affine r-space J/J^2. The rank variety of a finite-dimensional E-module M is a closed homogeneous subset of V_E defined by

$$V_E(M) = \{\bar{v} \in V_E : M \downarrow_{1+v} \text{ is not free}\} \cup 0.$$

(one shows this is weel-defined). With this definition the tensor product theorem is straightforward. For instance, the proof of Theorem 3.3.2 immediately applies here. By a thereom of Avrunin and Scott, $V_E(M) \simeq |E|_M$. The tensor product theorem for an arbitrary finite group G now follows from the weak version of the Quillen's stratification theorem:

$$|G|_M = \bigcup_{E \subset G \mathrm{ab}} (\mathrm{res}^G_E)^*(|E|_M).$$

cohom defn, list of properties. rank varieties. nullcone.

CHAPTER 2

Local Projectivity Test

2.1. Some algebraic lemmas

Recall that a finite dimensional Artin algebra A is called *Frobenius* if it admits a non-degenerate bilinear form and is called *quasi-Frobenius* if it is self-injective, i.e. $A \cong A^{\#}$ as an A-module. By a theorem of Faith-Walker ([**FW**]), an algebra is quasi-Frobenius if and only if any projective A-module is injective and vice versa.

By a result of Larson and Sweedler [**LS**], any finite dimensional cocommutative Hopf algebra is Frobenius. Since we only need to know that projectives coincide with injectives, it is sufficient for our purposes to prove that for a finite group scheme H, $k[H]^{\#}$ is quasi-Frobenius. We will follow [**Jan**] in the proof of the following

Lemma 2.1.1. *Let H be a finite group scheme. Then $k[H]^{\#}$ is a quasi-Frobenius algebra.*

Proof. Let $n = \dim_k k[H]$. By applying tensor identity (cf. Section 2.2), we get the following isomorphism:

$$k[H] \otimes k[H]^{\#} = \mathrm{Ind}_1^H(k) \otimes k[H]^{\#} = \mathrm{Ind}_1^H(k \otimes \mathrm{Res}_1^H(k[H]\#)) = \mathrm{Ind}_1^H(k^n) = k[H]^n.$$

On the other hand, $k[H] \otimes k[H]^{\#}$ is self-injective, so we conclude that $k[H]^n \cong (k[H]^{\#})^n$. By Krull-Shmidt theorem, $k[H] \cong k[H]^{\#}$. $\square$

The lemma implies that in the category of $k[H]^{\#}$ - modules projectives and injectives are the same. Since this category is equivalent to the category of H-modules, we conclude that projective H-modules coincide with injective ones.

Lemma 2.1.2. *Let A be a quasi-Frobenius algebra and M be an A-module. If M admits a finite injective resolution, then M is injective.*

Proof. Assume that M is not injective and let

$$M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \ldots \longrightarrow I^n \longrightarrow 0$$

be an injective resolution of M of minimal length. By our assumption $n > 0$.

Since A is quasi-Frobenius and I^n is injective, it is also projective. Then the last map $\delta^n : I^{n-1} \to I^n$ in the injective resolution above splits and $I^{n-1} = J^{n-1} \bigoplus I^n$ for some injective module J^{n-1}. Then

$$M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow \ldots \longrightarrow I^{n-2} \longrightarrow J^{n-1} \longrightarrow 0$$

is an injective resolution of M of smaller length than the original one. Thus, M is injective.

$\square$

We shall denote by $\Omega^{-n}M$ the $-n$-th Heller operator of M. Precisely, if $M \to I^0 \to I^1 \to \ldots$ is the minimal injective resolution of M, then $\Omega^{-n}M = coker(I^{n-2} \to I^{n-1})$ for $n > 1$ and $\Omega^{-1}M = coker(M \to I^0)$.

Lemma 2.1.3. *Let A be a quasi-Frobenius algebra and M be an A-module. If there exists an integer n_0 such that $Ext^n_A(S, M) = 0$ for all $n > n_0$ and any simple A-module S, then M is projective.*

Proof. Let

$$M \to I^0 \to I^1 \to \cdots \to I^n \to \dots$$

be the minimal injective resolution of M. Let further S be a simple A-module. Minimality of $I^\bullet$ implies that the complex $\mathrm{Hom}_A(S, I^\bullet)$ has zero differentials. Indeed, suppose $\delta : I^{n-1} \to I_n$ induces a non-zero differential $\bar{\delta} : \mathrm{Hom}(S, I^{n-1}) \to \mathrm{Hom}(S, I^n)$. Then there is an embedding $S \hookrightarrow I^{n-1}$ such that composing with δ on the right, we still get am embedding $S \hookrightarrow I^{n-1}\delta I^n$. If $I(S)$ denotes the injective hull of S, then embeddings of S into I_{n-1} and I_n extend to embeddings of $I(S)$. Since projective=injective, $I(S)$ splits off as a direct summand of both I_{n-1}, on which δ is an identity. This contradicts the minimality of $I^\bullet$. Thus, differentials are zero as claimed and $\mathrm{Ext}^n_A(S, M) = \mathrm{Hom}_A(S, I^n) = \mathrm{Hom}_A(S, \Omega^{-n}M)$, where the last equality holds because I^n is the injective hull of $\Omega^{-n}M$). Our assumption now implies that $\mathrm{Hom}_A(S, \Omega^{-n}M) = 0$ for all $n > n_0$. Therefore, $\Omega^{-n}M = 0$ for all $n > n_0$ (since any non-trivial module has a simple submodule). This implies that the minimal injective resolution of M is finite. The statement now follows from Lemma 2.1.2. □

We will need the following algebraic lemma to finish the proof of Proposition 2.2.2.

Lemma 2.1.4. *Let A be a regular ring of finite Krull dimension d and $J^\bullet$ be a cochain complex of flat A-modules such that $J^\bullet \otimes_A k(\mu)$ is acyclic in positive degrees for any prime ideal $\mu \subset A$. Then $H^n(J^\bullet) = 0$ for all $n > d$.*

Proof. We proceed by induction on $d = \dim A$.

First note that $J^\bullet$ has zero cohomology in degrees greater than m if and only if $J^\bullet_\mu$ has zero cohomology in degrees greater than m for all prime ideals μ. Indeed, the only if part follows from the exactness of localization. To prove the opposite direction assume that $J^\bullet$ is not acyclic. Let $[\alpha] \in H^n(J^\bullet)$ be a non-zero cycle. Since $J^\bullet$ is a complex of A-modules, $H^n(J^\bullet)$ also has a structure of an A-module. Let μ be a prime ideal in A containing $\mathrm{Ann}_A[\alpha]$. Then $[\alpha]_\mu = [\alpha_\mu]$ is a non-zero cycle in $H^n(J^\bullet)_\mu = H^n(J^\bullet_\mu)$ or, equivalently, $H^n(J^\bullet_\mu) \neq 0$.

In view of the preceding remark it suffices to prove the assertion of the lemma for local rings.

Let $d = 1$.

In this case A is a discrete valuation ring. Denote by π a generator of the maximal ideal of A, and by K the fraction field of A. Consider the short exact sequence

$$0 \to A \to A \to A/\pi A \to 0$$

and tensor it with $J^\bullet$ over A. Since $J^\bullet$ is flat we get an exact sequence of cochain complexes:

$$0 \to J^\bullet \to J^\bullet \to J^\bullet/\pi J^\bullet \to 0$$

and, therefore, a long exact sequence in cohomology:

$$\cdots \to H^{n-1}(J^\bullet/\pi J^\bullet) \to H^n(J^\bullet) \to H^n(J^\bullet) \to H^n(J^\bullet/\pi J^\bullet) \to \ldots .$$

Note that $J^\bullet/\pi J^\bullet = J^\bullet \otimes_A A/\pi A$ is acyclic in degrees higher than 0 by the assumption of the lemma. Therefore, multiplication by π induces an isomorphism on $H^n(J^\bullet)$ for all $n > 1$, which implies that the action of A on $H^n(J^\bullet)$ extends to an action of $K = Frac(A)$. Thus, $H^n(J^\bullet) = H^n(J^\bullet) \otimes_A K = H^n(J^\bullet \otimes_A K) = 0$.

$d - 1 \Rightarrow d$

Denote by $\mathcal{M}$ the maximal ideal of A. Let $t \in \mathcal{M}$ but $t \notin \mathcal{M}^2$. To apply the induction hypothesis to $J^\bullet/tJ^\bullet$ as a module over A/tA we have to check:

(i) $J^\bullet/tJ^\bullet$ *is flat.*

Let $M \to N$ be an injective map of A/tA modules. Then $M \otimes_{A/tA} J^n/tJ^n \cong M \otimes_{A/tA} J^n \otimes_A A/tA \cong M \otimes_A J^n$ and in the same way $N \otimes_{A/tA} J^n/tJ^n \cong N \otimes_A J^n$. To complete the argument we notice that $M \times_A J^n \to N \otimes_A J^n$ is injective since J^n is flat.

(ii) *"local acyclicity".*

Let $\mu \in \mathrm{Spec}\ (A/tA)$. Denote by π^* the map induced on spectra $\mathrm{Spec}\, A/tA \to \mathrm{Spec}\, A$ and let $\nu = \pi^*(\mu)$. We have

$$J^\bullet/tJ^\bullet \otimes_{A/tA} k(\mu) = J^\bullet \otimes_A A/tA \otimes_{A/tA} k(\mu) = J^\bullet \otimes_A k(\nu)$$

which implies that $J^\bullet/tJ^\bullet$ is acyclic in positive degrees.

(iii) $\dim A/tA \leq \dim A - 1$.

This allows us to conclude that $H^n(J^\bullet/tJ^\bullet) = 0$ for $n > d-1$. Combining this observation with a long exact sequence in cohomology:

$$\cdots \to H^{n-1}(J^\bullet/tJ^\bullet) \to H^n(J^\bullet) \to H^n(J^\bullet) \to H^n(J^\bullet/tJ^\bullet) \to \ldots,$$

we get that multiplication by t induces an isomorphism on $H^n(J^\bullet)$ for $n > d$.

Let $S = \{t \in A : \text{multiplication by } t \text{ induces an isomorphism on } H^n(J^\bullet) \text{ for } n > d\}$. Then S is a multiplicative system in A which contains $\mathcal{M}\backslash\mathcal{M}^2$. Therefore, $\dim S^{-1}A < \dim A$ and we can apply induction hypothesis to $S^{-1}A$.

Let $[a] \in H^n(J^\bullet)$, $n > d$. $S^{-1}[a] \in S^{-1}H^n(J^\bullet) = H^n(S^{-1}J^\bullet) = 0$. So there exists $t \in S$ such that $t[a] = 0$. Since multiplication by any element in S induces an isomorphism on cohomology we conclude that $[a] = 0$ and, therefore, $H^n(J^\bullet) = 0$ for $n > d$ $\square$

2.2. Induction

We start by recalling the definition and basic properties of induction. Details can be found in [**Jan**].

Let G be any affine group scheme and H be a subgroup scheme of G. We define the functor

$$\mathrm{Ind}_H^G : \{H \text{ - mod}\} \to \{G \text{ - mod}\}$$

by setting $\mathrm{Ind}_H^G(M) = (k[G] \otimes M)^H$, where the action of H is as given on M and via the right regular representation on $k[G]$, and the structure of G-module is given via the left regular action of G on $k[G]$.

Definition 2.2.1. *Let H be an infinitesimal group scheme and M be an H-module. We will say that M is locally projective if for any field extension K/k and any injective 1-parameter subgroup $\mathbb{G}_{a(r)} \otimes K \to H \otimes K$, the restriction of $M \otimes K$ to $\mathbb{G}_{a(r)} \otimes K$ is projective.*

Proposition 2.2.2. *Let G be a connected smooth algebraic group and $G_{(r)}$ be the r-th Frobenius kernel of G. Let further M be a locally projective $G_{(r)}$-module. Then $Ind^G_{G_{(r)}}(M)$ is locally projective as a $G_{(r)}$-module.*

Proof. We shall follow closely the proof of Theorem 4.1 of [**SFB2**].

Fix a Borel subgroup $B \subset G$ and let T and U be the corresponding torus and unipotent subgroup respectively. Denote by $B^{(r)}$ and $G^{(r)}$ the r-th Frobenius twists of corresponding groups. Furthermore, let $B_{(r)} = B \cap G_{(r)}$, $U_{(r)} = U \cap G_{(r)}$, and $T_{(r)} = T \cap G_{(r)}$.

Let $H \otimes K \to G_{(r)} \otimes K$ be an injective one-parameter subgroup. We need to show that $\mathrm{Ind}^G_{G_{(r)}}(M) \otimes K$ restricted to $H \otimes K$ is projective. By extending scalars from k to K and by taking further the image of H in G we can assume that H is a k-subgroup scheme of G.

All invariants throughout the proof will be taken with respect to the action via the left regular representation of various subgroup schemes of G on $k[G]$ unless specified otherwise. To distinguish between right and left regular representations we shall use subscripts "l" or "r". Normality of $G_{(r)}$ in G implies that $k[G]_r^{G_{(r)}} = k[G]_l^{G_{(r)}}$, so in this particular case we will just write $k[G]^{G_{(r)}}$.

Let $M \to I^\bullet$ be the standard $G_{(r)}$-injective resolution of M: $I^m = M \otimes k[G_{(r)}]^{\otimes m+1}$, where $G_{(r)}$ acts on I^m via the right regular representation on the last tensor factor. Then $\mathrm{Ind}_{G_{(r)}}^G(M) \to \mathrm{Ind}_{G_{(r)}}^G(I^\bullet)$ is an injective resolution of $\mathrm{Ind}_{G_{(r)}}^G(M)$ as an H-module. ($\mathrm{Ind}_{G_{(r)}}^G$ is exact since $G_{(r)}$ is a finite group scheme (cf. [**Jan**, I.5.13b)]) and Res_H^G takes injectives to injectives because any injective G-module is a direct summand of $k[G]\otimes$ <trivial G-module> and injectivity of $k[G]\downarrow_H$ itself is equivalent to the exactness of Ind_H^G (cf. [**Jan**, I.4.12]).)

If we set $J^\bullet = (\mathrm{Ind}_{G_{(r)}}^G(I^\bullet))^H$, then $H^*(J^\bullet) = H^*(H, \mathrm{Ind}_{G_{(r)}}^G(M))$. Note that $J^\bullet$ has a natural structure of a complex of $k[G^{(r)}] = k[G/G_{(r)}]$-modules. Indeed, for any map $M_1 \otimes M_2 \to M_3$ of $G_{(r)}$-modules, we get a G-module map $\mathrm{Ind}_{G_{(r)}}^G(M_1) \otimes \mathrm{Ind}_{G_{(r)}}^G(M_2) \to \mathrm{Ind}_{G_{(r)}}^G(M_3)$. By taking $M_1 = k$ and $M_2 = M_3 = I^n$, we get a natural structure of an $\mathrm{Ind}_{G_{(r)}}^G k = k[G/G_{(r)}]$-module on $\mathrm{Ind}_{G_{(r)}}^G I^n$ compatible with the action of G. Since $k[G/G_{(r)}] \cong k[G]^{G_{(r)}}$ is H-invariant, $J^\bullet$ is a $k[G/G_{(r)}]$-subcomplex of $\mathrm{Ind}_{G_{(r)}}^G(I^\bullet)$.

We point out next that all J^n are flat $k[G/G_{(r)}]$-modules. Indeed,

$$J^n = (\mathrm{Ind}_{G_{(r)}}^G(I^n))^H = \mathrm{Ind}_{G_{(r)}}^G(Q \otimes k[G_{(r)}]))^H = Q \otimes (\mathrm{Ind}_{G_{(r)}}^G(k[G_{(r)}]))^H \cong Q \otimes k[G]_l^H$$

where $Q = M \otimes k[G_{(r)}]^{\otimes n}$ is a vector space with trivial $G_{(r)}$-action. We have an extension of rings $k[G/G_{(r)}] \cong k[G_{(r)}\backslash G] \to k[H\backslash G] \to k[G]$ where the composition and the second extension are faithfully flat since they correspond to a quotient by a finite group scheme acting freely (cf. [**Jan**, I.5.7]). Consequently, the first extension

$k[G]_l^{G_{(r)}} \cong k[G_{(r)}\backslash G] \to k[H\backslash G] \cong k[G]_l^H$ is flat, which implies that $J^n = Q \otimes k[G]^H$ is flat over $k[G]^{G_{(r)}}$.

For any point $g \in G$ we are going to establish the following isomorphism:

$$J^\bullet \otimes_{k[G]^{G_{(r)}}} k(g) \cong (I^\bullet \otimes k(g))^{g^{-1}(H \otimes k(g))g}. \tag{$*$}$$

First note that there is a natural isomorphism $(\mathrm{Ind}_{G_{(r)}\otimes k(g)}^{G\otimes k(g)}(N \otimes k(g)))^{H\otimes k(g)} \cong (\mathrm{Ind}_{G_{(r)}}^G(N))^H \otimes k(g)$. Furthermore, $J^\bullet \otimes k(g) \otimes_{k(g)[G/G_{(r)}]} k(g) \cong J^\bullet \otimes_{k[G/G_{(r)}]} k(g)$. Thus, it suffices to prove $(*)$ for a k-rational point g and then proceed by extension of scalars.

For a k-rational point $g \in G$ denote by $\overline{g}$ its image under the projection $G \to G/G_{(r)}$. For any $G_{(r)}$-module N we have a natural homomorphism

$$\epsilon_g : \mathrm{Ind}_{G_{(r)}}^G(N) \to N$$

given by evaluation at g, i.e. $\epsilon_g(n \otimes f) = f(g)n$. The restriction of ϵ_g to $(\mathrm{Ind}_{G_{(r)}}^G(N))^H$ lands in $N^{g^{-1}Hg}$. As it was noted above, $(\mathrm{Ind}_{G_{(r)}}^G(N))^H$ has a natural structure of a $k[G]^{G_{(r)}}$- module. If we make N into a $k[G]^{G_{(r)}}$-module via evaluation at $\overline{g}$, then ϵ_g becomes a homomorphism of $k[G]^{G_{(r)}}$-modules. Tensoring the left hand side with k over $k[G]^{G_{(r)}}$, we get a natural map of k-vector spaces:

$$\epsilon_g : (\mathrm{Ind}_{G_{(r)}}^G(N))^H \otimes_{k[G]^{G_{(r)}}} k \to N^{g^{-1}Hg}.$$

When $N = k[G_{(r)}]$ this is an isomorphism as one sees from the following Cartesian square:

$$\begin{tikzcd}
g^{-1}Hg\backslash G_{(r)} \arrow[r] \arrow[d, "x\to gx"'] & Spec\, k \arrow[d, "\overline{g}"'] \\
H\backslash G \arrow[r] & G_{(r)}\backslash G
\end{tikzcd}$$

Hence, ϵ_g is an isomorphism for any injective $G_{(r)}$-module. This implies the isomorphism of complexes $(*)$.

Computing cohomology of both sides of $(*)$ we get that $H^*(J^\bullet \otimes_{k[G]^{G_{(r)}}} k(g)) = H^*(g^{-1}(H\otimes k(g))g, M\otimes k(g))$ and the latter is trivial for $* > 0$, since $g^{-1}(H\otimes k(g))g$ is again a one-parameter subgroup of $G_{(r)} \otimes k(g)$ and M is locally projective. We conclude that $J^\bullet \otimes_{k[G]^{G_{(r)}}} k(g)$ is acyclic in positive degrees for any point $g \in G$.

We have $J^\bullet \otimes_{k[G]^{G_{(r)}}} k(g) = J^\bullet \otimes_{k[G]^{G_{(r)}}} k(\overline{g}) \otimes_{k(\overline{g})} k(g)$ and the extension of scalars $k(\overline{g}) \to k(g)$ gives an injective map on cohomology. Therefore, $J^\bullet \otimes_{k[G]^{G_{(r)}}} k(\overline{g})$ is also acyclic in positive degrees. Since the projection $G \to G/G_{(r)}$ is a bijection on points, we get that for any point $x \in G/G_{(r)}$ the complex $J^\bullet \otimes_{k[G]^{G_{(r)}}} k(x)$ is acyclic in positive degrees. Since $G_{(r)}$ is a closed normal subgroup of G, $G/G_{(r)}$ is a smooth affine scheme and hence $k[G/G_{(r)}]$ is a regular ring. Lemma 2.1.4 now implies that $J^\bullet$ is acyclic in all sufficiently large degrees. Hence, $H^*(H, \mathrm{Ind}_{G_{(r)}}^G(M)) = 0$ in all sufficiently large degrees. Since k is the only simple H-module, we get that $\mathrm{Ind}_{G_{(r)}}^G(M)$ is injective by applying Lemma 2.1.3. $\square$

We will announce the following proposition here, postponing its proof until the end of section 3.4. The strategy of the proof will be the following: using Proposition 2.2.2, we prove Local Criterion for Projectivity for Frobenius kernels (Theorem 2.3.2). Building upon this result, we develop the theory of support varieties for Frobenius kernels in chapter 3. We then employ it to prove Proposition 2.2.3, derive Local Criterion for Projectivity for an arbitrary infinitesimal group scheme and finally claim that the entire chapter 3 goes through for any infinitesimal group scheme, since the only thing we really use there is Local Criterion for Projectivity.

Proposition 2.2.3. *Let G be an infinitesimal group scheme and M be a locally projective G-module. Let $G \hookrightarrow G'$ be a closed embedding of G into some Frobenius kernel of the same height as G. Then $\mathrm{Ind}_G^{G'}(M)$ is locally projective as a G'-module.*

Remark 2.2.4. In fact, we will prove the following statement: if $M \otimes K$ is projective restricted to an injective 1-parameter subgroup $H \otimes K \hookrightarrow G \otimes K$ for some field extension K/k, then the restriction of $\mathrm{Ind}_G^{G'}(M)$ to $H \otimes K$ is also projective.

2.3. Suslin-Friedlander-Bendel spectral sequence

To prove Theorem 2.3.4 we are going to exploit one more construction introduced in [**SFB2**, §3] which we briefly discuss below.

Let H be an affine k-group scheme, H' be a closed subgroup scheme, and X be the quotient scheme H/H' with the quotient map $p : H \to X$. There is an equivalence of categories between the category of quasi-coherent sheaves $\mathcal{M}$ on X

and the category of rational H'-modules M provided with the structure of a left $k[H]$-module such that the multiplication $k[H] \otimes M \to M$ is a homomorphism of rational H'-modules, where H' acts on $k[H]$ via the right regular representation, given by the functor $\mathcal{M} \mapsto \Gamma(H, p^*\mathcal{M})$. Moreover, the sheaf cohomology $H^*_{Zar}(X, \mathcal{M})$ is naturally isomorphic to the rational cohomology $H^*(H', \Gamma(H, p^*\mathcal{M}))$.

Let G_r be an infinitesimal group scheme which is a normal closed subgroup of a smooth connected algebraic group G. Fix a Borel subgroup B and unpotent radical U in G. Let further $B_r = G_r \cap B$, $G' = G/G_r$ and $B' = B/B_r$. Since B' is a Borel subgroup of G', G'/B' is a projective variety. We are going to show that cohomology groups $H^n(B_r, \mathrm{Ind}_{G_r}^G(M))$ belong to the aforementioned category of rational B'-modules with the compatible structure of a left $k[G']$-module. Once this is done, we can associate to $H^n(B_r, \mathrm{Ind}_{G_r}^G(M))$ a quasi-coherent sheaf on X, denoted $\mathcal{H}^q(\mathcal{B}_r, M)$, with the property

$$H^p(B/B_r, H^q(B_r, \mathrm{Ind}_{G_r}^G(M))) \cong H^p(X, \mathcal{H}^q(\mathcal{B}_r, M)). \qquad (**)$$

Lemma 2.3.1. *For any G_r-module M and any $n \geq 0$, the cohomology group $H^n(B_r, Ind_{G_r}^G(M))$ has the natural structures of a rational B/B_r-module and a left $k[G/G_r]$-module such that the action of $k[G/G_r]$ on M is a B/B_r-homomorphism.*

Proof. Let $M \to I^\bullet$ be the standard G_r-injective resolution of M. The cohomology groups $H^n(B_r, \mathrm{Ind}_{G_r}^G(M))$ can be computed via the complex $J^\bullet = (\mathrm{Ind}_{G_r}^G(I^\bullet))^{B_r}$, which has the natural structures of B/B_r and $k[G/G_r]$-modules. The action of $k[G/G_r]$ is given explicitly via

$$k[G/G_r] \otimes (\mathrm{Ind}_{G_r}^G(I^\bullet))^{B_r} \to (\mathrm{Ind}_{G_r}^G(I^\bullet))^{B_r}$$

$$\phi \otimes (f \otimes s) \longrightarrow \phi f \otimes s$$

which one easily checks to be a homomorphism of B/B_r-modules, where B/B_r acts on $k[G/G_r]$ via the *left* regular representation (since this is how the standard G-action on $\mathrm{Ind}_{G_r}^G(N) = (k[G] \otimes N)^{G_r}$ is defined).

To get the compatibility with B/B_r acting on $k[G/G_r]$ via the *right* regular representation we have to change the structure of $k[G/G_r]$ on $J^\bullet$ via the automorphism of G/G_r: $G/G_r \xrightarrow{\sigma} G/G_r$, $\sigma(x) = x^{-1}$. $\square$

Next, we prove Local Criterion for Projectivity. We treat the case of a Frobenius kernel first.

Theorem 2.3.2. *Let $G_{(r)}$ be the r-th Frobenius kernel of a smooth connected algebraic group G and M be a $G_{(r)}$-module such that for any field extension K/k and any injective one-parameter subgroup $\mathbb{G}_{a(s)} \otimes K \to G_{(r)} \otimes K$ the restriction of $M \otimes K$ to $\mathbb{G}_{a(s)} \otimes K$ is projective. Then M is projective as a $G_{(r)}$-module.*

Proof. Let X be the quotient scheme $G^{(r)}/B^{(r)}$. Consider the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(B^{(r)}, H^q(B_{(r)}, \mathrm{Ind}_{G_{(r)}}^G(M))) \Longrightarrow H^{p+q}(B, \mathrm{Ind}_{G_{(r)}}^G(M)).$$

By Theorem 3.6 in [**SFB2**], which is an extension to not necessarily reductive algebraic groups of a fundamental theorem of [**CPSvdK**],

$$H^n(B, \mathrm{Ind}_{G_{(r)}}^G(M)) \cong H^n(G, \mathrm{Ind}_{G_{(r)}}^G(M)),$$

and by Shapiro's lemma

$$H^n(G, \mathrm{Ind}_{G_{(r)}}^G(M)) \cong H^n(G_{(r)}, M).$$

Since M is locally projective, Proposition 2.2.2 implies that $\mathrm{Ind}_{G_{(r)}}^G(M)$ is also locally projective as a $G_{(r)}$-module and thus as a $U_{(r)}$-module. Now, by a theorem of Bendel (Theorem 1.2.7), which applies to unipotent infinitesimal group schemes, $\mathrm{Ind}_{G_{(r)}}^G(M)$ is projective as a $U_{(r)}$-module. We have a short exact sequence of group schemes: $1 \to U_{(r)} \to B_{(r)} \to T_{(r)} \to 1$, where $T_{(r)} = T \cap G_{(r)}$ is diagonalizable and hence cohomologically trivial. Applying the Serre spectral sequence, we get an isomorphism: $H^*(B_{(r)}, \mathrm{Ind}_{G_{(r)}}^G(M)) \cong H^*(U_{(r)}, \mathrm{Ind}_{G_{(r)}}^G(M))^{T_{(r)}}$ and the latter is 0 in positive degrees, since $\mathrm{Ind}_{G_{(r)}}^G(M)$ is a projective $U_{(r)}$-module. Thus, $H^q(B_{(r)}, \mathrm{Ind}_{G_{(r)}}^G(M)) = 0$ for $q > 0$, so that the Hochschild-Serre spectral sequence above collapses and we get an isomorphism

$$H^p(B^{(r)}, H^0(B_{(r)}, \mathrm{Ind}_{G_{(r)}}^G(M))) \cong H^p(G_{(r)}, M).$$

Combining this with the isomorphism $(**)$ one gets:

$$H^p(X, \mathcal{H}^0(\mathcal{B}_r, M)) \cong H^p(G_{(r)}, M).$$

Let $x = \dim X$. Since X is a projective variety, its cohomology groups with coefficients in any quasi-coherent sheaf are trivial in degrees higher than x (cf. [**Har**, III.2.7]). Thus, $H^p(G_{(r)}, M) = 0$ for $p > x$. Applying the same argument to $M \otimes N^{\#}$, we get $\mathrm{Ext}^p_{G_{(r)}}(N, M) = 0$ for all $p > x$ and all finite-dimensional modules N. By Lemma 2.1.3, M is projective.

$\square$

Remark 2.3.3. Let G be a semi-simple simply connected algebraic group. Assume all the hypotheses of Corollary 2.3.2 and also assume that the $G_{(r)}$-structure on M comes from a structure of a rational G-module. In this case we do not need to consider induced modules and can significantly simplify the proof of our local criterion for projectivity. Indeed, for a rational G-module M, we have the following spectral sequence ([**AJ**]):

$$H^p(G^{(r)}/B^{(r)}, \mathcal{L}(H^q(B_{(r)}, M))) \Longrightarrow H^{p+q}(G_{(r)}, M),$$

where $\mathcal{L}(H^q(B_{(r)}, M))$ is the sheaf on $G^{(r)}/B^{(r)}$ associated to $H^q(B_r, M)$ considered as a $B^{(r)}$-module (cf. [**Jan**, I.5]).

Local projectivity of M implies that M is a projective $B_{(r)}$-module which makes the spectral sequence collapse. Thus, we get an isomorphism:

$$H^p(G^{(r)}/B^{(r)}, \mathcal{L}(H^0(B_{(r)}, M))) \cong H^p(G_{(r)}, M).$$

Since $G^{(r)}/B^{(r)}$ is a projective variety, $H^p(G^{(r)}/B^{(r)}, \mathcal{L}(H^0(B_{(r)}, M))) = 0$ for $p > \dim G^{(r)}/B^{(r)}$. Thus, $H^p(G_{(r)}, M) = 0$ for $p > \dim G^{(r)}/B^{(r)}$. Applying the same argument to $M \otimes S^{\#}$, we get that

$$\mathrm{Ext}^p_{G_{(r)}}(S, M) = 0$$

for any simple G-module and any $p > \dim G^{(r)}/B^{(r)}$. Due to the assumptions made on G we know that all simple $G_{(r)}$-modules come from restricting simple G-modules corresponding to restricted dominant weights (cf. [**Jan**, II.3]). Thus, we have vanishing of Ext-groups in all sufficiently large degrees for all simple $G_{(r)}$-modules. By Lemma 2.1.3, M is projective as a $G_{(r)}$-module.

Finally, we derive Local Projectivity Test for any infinitesimal group scheme.

Theorem 2.3.4. *Let G be an infinitesimal k-group scheme of height r. Let M be a G-module such that for any field extension K/k and any injective one-parameter subgroup $\mathbb{G}_{a(s)} \otimes K \to G \otimes K$ the restriction of $M \otimes K$ to $\mathbb{G}_{a(s)} \otimes K$ is projective. Then M is projective as a G-module.*

Proof. Embed G into some Frobenius kernel $G_{(r)}$. In view of the Prop. 2.2.3, local projectivity of G-module M implies local projectvity of $G_{(r)}$-module $\mathrm{Ind}_G^{G_{(r)}}(M)$. The Local Criterion for Projectivity for Frobenius kernels enables us to conclude that $\mathrm{Ind}_G^{G_{(r)}}(M)$ is projective as a $G_{(r)}$-module. Therefore, $H^*(G, M) = H^*(G_{(r)}, \mathrm{Ind}_G^{G_{(r)}}(M)) = 0$ for $* > 0$. Applying the same argument to all modules of the form $M \otimes S^{\#}$ for all simple G-modules S, we get that

$$\mathrm{Ext}^*_G(S, M) = 0 \text{ for} * > 0$$

Hence, M is projective.

$\square$

CHAPTER 3

Support Cones for Infinitesimal Group Schemes

In this chapter G will denote an arbitrary infinitesimal k-group scheme of height r. We assume that $p > 2$ simply for notational convenience: everything still holds for $p = 2$ if we change $H^{ev}(G, k)$ to $H^*(G, k)$.

3.1. Definition of Support Cones

The purpose of this section is to give a suitable definition of a "support" of an infinite dimensional G-module. To start, we recall some necessary constructions and results from [**SFB1**], [**SFB2**].

Define the functor

$$V(G) : (\text{comm } k\text{-alg}) \to (\text{sets})$$

by setting

$$V(G)(A) = \mathrm{Hom}_{Gr/A}(\mathbb{G}_{a(r)} \otimes_k A, G \otimes_k A).$$

This functor is representable by an affine scheme of finite type over k, which we will still denote $V(G)$. Indeed, the following statement, which is Theorem 1.5 in [**SFB1**], holds:

Proposition 3.1.1. *The functor $V(G)$ is represented by an affine scheme of finite type over k. Moreover, $G \to V(G)$ is a covariant functor from the category of affine* *group schemes over k of height $\leq r$ to the category of affine schemes of finite type over k, which takes closed embeddings to closed embeddings.*

We shall specify further the correspondence between one-parameter subgroups of G (i.e. group scheme homomorphisms $\mathbb{G}_{a(r)} \otimes K \to G \otimes K$) and points of $V(G)$. Let $s \in V_r(G)$ be a point. This point defines a canonical $k(s)$-point of $V(G)$ and hence an associated group scheme homomorphism over $k(s)$:

$$\nu_s : \mathbb{G}_{a(r)} \otimes_k k(s) \to G \otimes_k k(s).$$

Note that if K/k is a field extension and $\nu : \mathbb{G}_{a(r)} \otimes_k K \to G \otimes_k K$ is a group scheme homomorphism, then this data defines a point $s \in V_r(G)$ and a field embedding $k(s) \hookrightarrow K$ such that ν is obtained from ν_s by extending scalars from $k(s)$ to K.

As it will be discussed in more details in the next section, the affine scheme $V(G)$ is a cone, or, which amounts to the same thing, its coordinate algebra is graded connected. We have that $V(G)$ is homeomorphic to the cohomological support of G, $\operatorname{Spec} H^{ev}(G, k)$, which we denoted by $|G|$. Again, we quote a result from [**SFB2**].

Theorem 3.1.2. *There is a natural homomorphism of graded commutative k-algebras*

$$\psi : H^{ev}(G, k) \to k[V(G)]$$

which induces a finite universal homeomorphism of schemes

$$\Psi : V(G) \to |G|$$

([**SFB1**, 1.14]; [**SFB2**, 5.2]). Furthermore, restricted to $V(G)_M$, the "representation-theoretic" support variety of a *finite* dimensional G-module M, defined as in 3.1.3 below, Ψ is a homeomorphism onto $|G|_M$ ([**SFB2**, 6.8]).

Looking for a good definition of a "support" for an infinite dimensional module it seems natural to establish the following criteria:

1. Restricted to the finite dimensional case our new construction should give the standard support variety for finite dimensional modules.

2. Standard properties of support varieties for finite dimensional modules should remain valid as properties of "supports" for all G-modules.

The natural extension of the cohomological definition of support variety does not satisfy the "tensor product property" for infinite dimensional modules. We will give an example of this failure as we look at Rickard idempotent modules in the next chapter. On the other hand, our extension of the representation-theoretic construction is not necessarily a closed subset of $V(G)$. This particular feature, though, shows that, extended to infinite dimensional modules, $V(G)_M$ gives a "finer" invariant than $|G|_M$. As it will be shown later (cf. Corollary 4.3.6), any conical subset of $V(G)$ can be realized as $V(G)_M$ for some G-module M.

Getting more sets as support cones also emphasizes the difference between finite and infinite dimensional case. The category of all modules is "richer" with respect to this invariant than the category of finite dimensional modules.

For these reasons we choose as our definition of "support" of an arbitrary G-module module M the representation-theoretic construction appearing below.

We recall a notation introduced in the Chapter . Let $v_0, \dots v_{p^r-1}$ be the basis of $k[\mathbb{G}_{a(r)}]^\# = (k[T]/T^{p^r})^\#$ dual to the standard basis of $k[T]/T^{p^r}$. Denote v_{p^i} by u_i. Then the algebra $k[\mathbb{G}_{a(r)}]^\#$ coincides with $k[u_0, \dots, u_{r-1}]/(u_0^p, \dots, u_{r-1}^p)$. We will give a special name ϵ to the map $k[\mathbb{G}_{a(1)}] = k[u]/u^p \hookrightarrow k[u_0, \dots, u_{r-1}]/(u_0^p, \dots, u_{r-1}^p) = k[\mathbb{G}_{a(r)}]^\#$ defined by sending $u \to u_{r-1}$. This is just a map of algebras, not a map of group schemes.

Definition 3.1.3. *Let G be an infinitesimal k-group scheme of height r and let M be a rational G-module. The support cone of M is the following subset of $V(G)$:*

$$V(G)_M = \{s \in V(G) : M \otimes_k k(s) \textit{ is not projective as a module for the subalgebra}$$

$$k(s)[u_{r-1}]/(u_{r-1}^p) \subset k(s)[u_0, \dots, u_{r-1}]/(u_0^p, \dots, u_{r-1}^p) = k(s)[\mathbb{G}_{a(r)}]^\#\}.$$

We remark that by a "subset" of an affine scheme $X = \operatorname{Spec} A$ we would mean simply a set of prime ideals in A. We shall often use the same notation for a point in X and the corresponding prime ideal in A.

Let E_r be an elementary abelian p-group of rank r (i.e. $E_r = (\mathbb{Z}/p)^r$). If we view E_r as a commutative Lie algebra with trivial restriction, then its representation theory is equivalent to the representation theory of the infinitesimal group scheme $\mathbb{G}_{a(1)}^{\times r}$. Taking this point of view on elementary abelian groups, it is easy to see that our definition of support cone in the special case of $\mathbb{G}_{a(1)}^{\times r}$ agrees with the extension to infinite dimensional E_r-modules of the notion of rank variety given in [**BCR2**].

Note that any group scheme homomorphism $\mathbb{G}_{a(s)} \to G$, $s \leq r$, can be extended canonically to a one-parameter subgroup of height r, $\mathbb{G}_{a(r)} \to G$, via the projection $p_{r,s} : \mathbb{G}_{a(r)} \to \mathbb{G}_{a(s)}$ given by the natural embedding of coordinate algebras

$$k[\mathbb{G}_{a(s)}] = k[T_1]/(T_1^{p^s}) \xrightarrow{T_1 \to T^{p^{r-s}}} k[T]/(T^{p^r}) = k[\mathbb{G}_{a(r)}].$$

The corresponding map of dual algebras is

$$k[\mathbb{G}_{a(r)}]^{\#} = k[u_0, \dots, u_{r-1}]/(u_0^p, \dots, u_{r-1}^p) \to$$

$$k[u_{r-s}, \dots, u_{r-1}]/((u_{r-s}^p, \dots, u_{r-1}^p) \cong k[\mathbb{G}_{a(s)}]^{\#}.$$

As we immediately see from the formula above, extending a 1-parameter subgroup this way does not change the image of the map ϵ.

Conversely, any one-parameter subgroup $\mathbb{G}_{a(r)} \to G$ can be decomposed as

$$\mathbb{G}_{a(r)} \xrightarrow{p_{r,s}} \mathbb{G}_{a(s)} \hookrightarrow G$$

for some $s \leq r$.

As an example, which will also be used later in the text, we compute $V_{\mathbb{G}_{a(r)}}$ (see also [**SFB1**, 1.10]).

Lemma 3.1.4. $V_{\mathbb{G}_{a(r)}} = \mathbb{A}^r$

Proof. By definition, $V_{\mathbb{G}_{a(r)}}(A) = \mathrm{Hom}_{Gr/A}(\mathbb{G}_{a(r)} \otimes A, \mathbb{G}_{a(r)} \otimes A)$. A map of group schemes $\mathbb{G}_{a(r)} \otimes A \to \mathbb{G}_{a(r)} \otimes A$ is given by a map of Hopf algebras $A[T]/T^{p^r} \to A[T]/T^{p^r}$. Since the generator T is primitive ($\Delta(T) = 1 \otimes T + T \otimes 1$), to give such a map is equivalent to giving an additive polynomial on T, i.e. $P(T) = a_0 T + a_1 T^p +$

$\ldots a_{r-1}T^{p^{r-1}}$. Thus, 1-parameter subgroup corresponds to an r-tuple $(a_0, \ldots, a_{r-1})$.

□

We apply the Generalized Dade's lemma (Theorem 1.2.4) to show that support cones satisfy "projectivity detection" property for representations of $\mathbb{G}_{a(r)}$.

Lemma 3.1.5. *Let M be a $\mathbb{G}_{a(r)}$-module. M is projective if and only if $V(\mathbb{G}_{a(r)})_M = 0$.*

Proof. The category of $\mathbb{G}_{a(r)}$-modules is equivalent to the category of $k[\mathbb{G}_{a(r)}]^{\#} = k[u_0, \ldots, u_{r-1}]/(u_0^p, \ldots, u_{r-1}^p)$-modules. To apply Theorem 1.2.4 we have to show that $V(\mathbb{G}_{a(r)})_M = 0$ is equivalent to the assumption of the theorem. Let $z = c_0u_0 + \cdots + c_{r-1}u_{r-1}$, where $c_0, \ldots, c_{r-1} \in K$, K is an extension of k, which we assume to be perfect (we can always extend scalars further). Consider an endomorphism α of $\mathbb{G}_{a(r)} \otimes K$ defined on the level of coordinate algebras via the formula:

$$\begin{tikzcd}
K[\mathbb{G}_{a(r)}] \arrow[r] \arrow[d, equal] & K[\mathbb{G}_{a(r)}] \arrow[d, equal] \\
K[T]/(T^{p^r}) \arrow[r, "T \to \sum_0^{r-1} c_i^{p^{-i}} T^{p^{r-1-i}}"] & K[T]/(T^{p^r})
\end{tikzcd}$$

Dual to this map is an endomorphism of $K[\mathbb{G}_{a(r)}]^{\#} = K[u_0, \ldots, u_{r-1}]/(u_0^p, \ldots, u_{r-1}^p)$, which takes u_{r-1} to $c_0u_0 + \cdots + c_{r-1}u_{r-1}$. By definition of $V(\mathbb{G}_{a(r)})$, α corresponds to a point there defined over K.

Since $V(\mathbb{G}_{a(r)})_M$ is assumed to be 0, the restriction of $M\otimes K$ to $K[u_{r-1}]/(u_{r-1}^p) \subset K[u_0,\ldots,u_{r-1}]/(u_0^p,\ldots,u_{r-1}^p) = K[\mathbb{G}_{a(r)}]^\#$ is projective, where $M \otimes K$ is considered as a $K[\mathbb{G}_{a(r)}]^\#$-module via the pull-back of α. By the construction of α this is equivalent to $M \otimes K$ being projective when restricted to the subalgebra of $K[\mathbb{G}_{a(r)}]^\# = K[u_0,\ldots,u_{r-1}]/(u_0^p,\ldots,u_{r-1}^p)$ generated by $z = c_0u_0 + \cdots + c_{r-1}u_{r-1}$. Thus we proved that for any z as above $M \otimes K$ is projective when restricted to $K[z]/(z^p) \subset K[\mathbb{G}_{a(r)}]^\#$. Now we can apply Theorem 1.2.4 to conclude that M is projective.

To prove the "only if" part it suffices to show that for any one-parameter subgroup $\mathbb{G}_{a(r)} \xrightarrow{\alpha} \mathbb{G}_{a(r)}$, $k[\mathbb{G}_{a(r)}]^\#$ is projective over $k[u_{r-1}]/(u_{r-1}^p)$, where the module structure on $k[\mathbb{G}_{a(r)}]^\#$ is given via the composition $k[u_{r-1}]/(u_{r-1}^p) \subset k[\mathbb{G}_{a(r)}]^\# \xrightarrow{\alpha_*} k[\mathbb{G}_{a(r)}]^\#$. Decompose α as $\mathbb{G}_{a(r)} \xrightarrow{p_{r,s}} \mathbb{G}_{a(s)} \hookrightarrow \mathbb{G}_{a(r)}$. Since $\mathbb{G}_{a(s)}$ is a finite group scheme, $k[\mathbb{G}_{a(r)}]^\#$ is injective (and hence projective) as a $\mathbb{G}_{a(s)}$-module (cf. [**Jan**, I.5.13b)]). The composition $k[u_{r-1}]/(u_{r-1}^p) \subset k[\mathbb{G}_{a(r)}]^\# \xrightarrow{(p_{r,s})_*} k[\mathbb{G}_{a(s)}]^\# = k[u_0,\ldots,u_{s-1}]/(u_0^p,\ldots,u_{s-1}^p)$ takes u_{r-1} to u_{s-1}, which clearly implies that $k[\mathbb{G}_{a(s)}]^\#$ (and, therefore, $k[\mathbb{G}_{a(r)}]^\#$) is free as a $k[u_{r-1}]/(u_{r-1}^p)$-module.

$\square$

3.2. Conical Sets

We shall call an affine k-scheme $X = \operatorname{Spec} A$ *conical* if A is a graded connected k-algebra. The data of a (non-negative) grading on A is equivalent to a right monoid action of $\mathbb{A}^1$ on X, where the monoid structure on $\mathbb{A}^1$ is just the usual multiplication.

(The correspondence is given in the following way: the canonical k-algebra homomorphism $A \to A[T]$ defined by the grading on A induces a morphism of schemes $X \times \mathbb{A}^1 \to X$ which defines a monoid action of $\mathbb{A}^1$. Conversely, given an action we get a homomorphism $A \to A[T]$ which defines a non-negative grading on A).

Definition 3.2.1. *(conical subset) Let $X = \mathit{Spec}\, A$ be a conical affine scheme, where the conical structure is given by the map $\rho : X \times \mathbb{A}^1 \to X$. Denote by $\pi_X : X \times \mathbb{A}^1 \to X$ the canonical projection onto X. A subset W of X is said to be conical if it is stable under the action of $\mathbb{A}^1$ on X and if for any point $s \in X$ we have $\pi_X(\rho^{-1}(s)) \subset X$.*

Note that if W is a closed subset, then it is conical if and only if it is defined by a graded ideal, or, equivalently, if it corresponds to a homogeneous subvariety. In fact, in this familiar case or even in the more general case of a subset closed under specialization, the second condition is redundant and implied by the first.

Next we give an example of a conical set which we find to be more illuminating. Since A is connected we can give a precise meaning to the 0-point: this is the point corresponding to the augmentation ideal in A and it belongs to any conical subset.

Example 3.2.2. *Let $s \in X$ be a point corresponding to a graded prime ideal $\mu_s \subset A$. Denote $\pi_X(\rho^{-1}(s)) \subset X$ by $L(s)$. Then $L(s) \cup 0$ is the minimal conical subset containing s: by our definition of "conical", $s \in W$ implies $L(s) \subset W$ for any* *conical subset W. We give a description of $L(s)$ in terms of prime ideals:*

$$L(s) = \{\mu \in \mathit{Spec}\, A : \ \mu \text{ is not homogeneous, } \mu_s \subset \mu \text{ and } \mathrm{ht}(\mu) = \mathrm{ht}(\mu_{\mathrm{s}}) + 1\} \cup \{\mathrm{s}\}$$

To justify this claim we make three simple observations. Denote the action of $\mathbb{A}^1$ on X by $\bullet$.

First, the action of $\mathbb{A}^1$ cannot increase the height of the ideal and can lower it at most by one.

Second, since any set of the form $\{$homogeneous ideal$\} \cup 0$ is stable under the action, $L(s)$ does not contain any homogeneous ideals other than μ_s.

Third, let p be any point in X, c be the generic point of $\mathbb{A}^1$, and s be the point corresponding to the maximal homogeneous ideal contained in the ideal μ_p. Assume also that μ_s is strictly contained in μ_p (i.e. μ_p is not homogeneous), in which case $ht(\mu_s) = ht(\mu_p) - 1$. Then $p \bullet c = s$ which implies that $p \in L(s)$.

To see that $p \bullet c = s$ we note that if μ_p is the kernel of the map $A \to k(p)$, then the kernel of the induced map

$$A \xrightarrow{\sum_0^n a_i \to \sum_0^n a_i T^i} A[T] \longrightarrow k(p)(T)$$

is the maximal homogeneous ideal contained in μ_p, i.e. μ_s.

Next we describe how to give an action of $\mathbb{A}^1$ on $V(G)$ and, therefore, define a grading on $k[V(G)]$.

We have a natural morphism of schemes defined by taking composition of morphisms

$$V(G) \times V(\mathbb{G}_{a(r)}) \to V(G).$$

Namely, if $\nu_A : \mathbb{G}_{a(r)} \otimes A \to \mathbb{G}_{a(r)} \otimes A$ and $\mu_A : \mathbb{G}_{a(r)} \otimes A \to G \otimes A$ are one-parameter subgroups, then we send $\mu_A \times \nu_A$ to $\mu_A \circ \nu_A \mathbb{G}_{a(r)} \otimes A\ to G \otimes A$.

Taking G to be $\mathbb{G}_{a(r)}$ we see that $V(\mathbb{G}_{a(r)})$ has a natural structure of a monoid scheme over k. Restricting the action to a submonoid of $V(\mathbb{G}_{a(r)})$ consisting of homomorphisms of $\mathbb{G}_{a(r)}$ given by linear maps of coordinate algebras, we get a right monoid action of $\mathbb{A}^1$ on $V(G)$, which, consequently, defines a grading on $k[V(G)]$. Moreover, $k[V(G)]$ becomes a graded *connected* ([**SFB1**, 1.12]) k-algebra with respect to this grading which makes $V(G)$ into a conical k-scheme.

3.3. Properties of Support Cones

The following theorem establishes the list of properties satisfied by support cones. The most difficult one is 3.3.2.3, the detection of projectivity "on" support cones, which follows from the Local Criterion of Projectivity of section 2.3 and Corollary 3.1.5.

We will need an obvious little lemma about $\mathbb{G}_{a(1)}$-modules (equivalently, $\mathbb{Z}/p$-modules), which seems to be easier to prove than to find a reference for.

Lemma 3.3.1. *Let M, N be finite dimensional $\mathbb{G}_{a(1)}$-modules and assume further that $M \otimes N$ is projective. Then either M or N is projective.*

Proof. Since modules are finite dimensional, we can apply the theory of support varieties. Suppose neither M, nor N is projective. Then varieties of both M and N, being non-zero conical subsets of $\mathbb{A}^1 = V_{\mathbb{G}_{a(1)}}$, must coincide with $\mathbb{A}^1$. Thus, $V_{\mathbb{G}_{a(1)}}(M) \cap V_{\mathbb{G}_{a(1)}}(N) = \mathbb{A}^1$. On the other hand, projectivity of $M \otimes N$ together with tensor product property implies that $V_{\mathbb{G}_{a(1)}}(M) \cap V_{\mathbb{G}_{a(1)}}(N) = V_{\mathbb{G}_{a(1)}}(M \otimes N) = 0$. We get a contradiction, so the statement follows. $\square$

Theorem 3.3.2. *Let G be an infinitesimal k-group scheme of height r which is a closed normal subgroup of a smooth algebraic group and let M and N be G-modules. Support cones satisfy the following properties:*

0. For a finite dimensional module M, $V(G)_M \cong |G|_M$.

1. $V(G)_M$ is a conical subset of $V(G)$.

2. "Naturality." Let $f : H \to G$ be a homomorphism of infinitesimal group schemes of height $\leq r$. Denote by $f_* : V(H) \to V(G)$ the associated morphism of schemes. Then

$$f_*^{-1}(V(G)_M) = V(H)_M,$$

where M is considered as an H-module via f.

3. $V(G)_M = 0$ if and only if M is projective.

4. "Tensor product property." $V(G)_{(M \otimes N)} = V(G)_M \cap V(G)_N$.

5. $V(G)_{(M \bigoplus N)} = V(G)_M \cup V(G)_N$.

6. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ *be a short exact sequence of* G*-modules. Then for any permutation* (ijk) *of* (123) *we have*

$$V(G)_{M_i} \subset V(G)_{M_j} \cup V(G)_{M_k}.$$

Proof. Note that over the algebra $K[u]/(u^p)$ projective=free which we shall use without mention throughout the argument.

0. This is proved in [**SFB2**], Cor.6.8.

1. The proof for finite dimensional modules given in [**SFB2**], Prop.6.1, generalizes immediately to our case but we shall include it here for the completeness of the argument. Denote the action of $\mathbb{A}^1$ on $V(G)$, $V(G) \times \mathbb{A}^1 \to V(G)$, by $\bullet$.

Let $s \in V(G)$ and let $\nu_s : \mathbb{G}_{a(r)} \otimes k(s) \to G \otimes k(s)$ be the one-parameter subgroup determined by s. By the definition of $V(G)_M$, $s \in V(G)_M$ if and only if the restriction of $M \otimes k(s)$ to $k(s)[u_{r-1}]/u_{r-1}^p \subset k(s)[\mathbb{G}_{a(r)}]^\#$ via ν_s is not projective. Let c be a point in $\mathbb{A}^1$. We can extend the scalars to a field K/k such that both s and c are defined over K. Let $\nu_{s,K} : \mathbb{G}_{a(r)} \otimes K \to G \otimes K$ be the one-parameter subgroup which is obtained from ν_s by extending scalars from $k(s)$ to K. If $c = 0$, then the corresponding one-parameter subgroup is trivial and the restriction of the pull-back of M via the trivial subgroup to $K[u_{r-1}]/u_{r-1}^p$ is never projective. So, in this case $c \bullet s \in V(G)_M$. Assume $c \neq 0$. To prove that $V(G)_M$ is conical we have to show that $s \in V(G)_M$ if and only if $c \bullet s \in V(G)_M$. Considered as a point in $V(\mathbb{G}_{a(r)})$ defined over K, c determines a group scheme homomorphism $\nu_{c,K} : \mathbb{G}_{a(r)} \otimes K \to \mathbb{G}_{a(r)} \otimes K$, given by the multiplication by c^{-1} on the coordinate algebra $K[\mathbb{G}_{a(r)}]$. By definition of the action of $\mathbb{A}^1$ on $V(G)$, the group scheme homomorphism $\nu_{c \bullet s,K} : \mathbb{G}_{a(r)} \otimes K \to G \otimes K$ is defined via the composition

$$\mathbb{G}_{a(r)} \otimes K \xrightarrow{\nu_{c,K}} \mathbb{G}_{a(r)} \otimes K \xrightarrow{\nu_{s,K}} G \otimes K.$$

The homomorphism $(\nu_{c,K})_* : K[\mathbb{G}_{a(r)}]^\# \to K[\mathbb{G}_{a(r)}]^\#$ restricted to $K[u_{r-1}]/u_{r-1}^p$ is given by

$$K[u_{r-1}]/u_{r-1}^p \xrightarrow{u_{r-1} \to c^{p^{r-1}} u_{r-1}} K[u_{r-1}]/u_{r-1}^p$$

which is clearly a ring isomorphism. Consequently, M is not projective as a module over the right hand side of the above isomorphism if and only if M is not projective when restricted to the left hand side. The statement follows.

2. Follows immediately from the definition of $V(G)_M$.

3. Note that $V(G)_M = 0$ implies $V(G \otimes K)_{M \otimes K} = 0$ for any field extension K/k. Let $\mathbb{G}_{a(r)} \otimes K \to G \otimes K$ be any non-trivial one-parameter subgroup. By naturality $V_{\mathbb{G}_{a(r)} \otimes K}(M \otimes K) = 0$, which is equivalent, in view of Cor. 3.1.5, to the fact that the restriction of $M \otimes K$ to $\mathbb{G}_{a(r)} \otimes K$ is projective. Applying Theorem 2.3.4, we conclude that M is projective.

Now suppose that M is a projective G-module. Then M is a direct summand of $k[G] \otimes <$ trivial module $>$, and the support variety of $k[G]$ is trivial, since it is injective as a G-module.

4. The inclusion $V(G)_{M \otimes N} \subset V(G)_M \cap V(G)_N$ follows from the fact that tensor product of a projective module with anything is projective. Indeed, let $s \in$

$V(G)_{M\otimes N}$. By the definition of support cone, $M\otimes N\otimes k(s)$ is not projective when restricted to $k(s)[u_{r-1}]/u_{r-1}^p \subset k(s)[u_0,\ldots,u_{r-1}]/(u_0^p,\ldots,u_{r-1}^p) = k(s)[\mathbb{G}_{a(r)}]^\#$, where $\mathbb{G}_{a(r)} \otimes k(s) \to G \otimes k(s)$ is the one-parameter subgroup of $G\otimes k(s)$ corresponding to the point $s \in V(G)$. In view of the remark above, neither $M \otimes k(s)$ nor $N\otimes k(s)$ is projective and, therefore, $s \in V(G)_M \cap V(G)_N$.

To prove the other inclusion we have to show that if both $M\otimes k(s)$ and $N\otimes k(s)$ are not free as modules over $k(s)[u_{r-1}]/(u_{r-1}^p)$, then $M\otimes N\otimes k(s)$ is not free. Denote $k(s)[u_{r-1}]/(u_{r-1}^p)$ by A. Note that

$$M \otimes N \otimes k(s) \cong (M \otimes k(s)) \otimes_{k(s)} (N \otimes k(s)).$$

Since any A-module is a direct sum of finite dimensional indecomposables (cf. [**FW**]), we can write $M \otimes k(s) = \bigoplus_I M_i$ and $N \otimes k(s) = \bigoplus_J N_j$ for some finite dimensional A-modules M_i and N_j. Consequently,

$$(M \otimes k(s)) \otimes_{k(s)} (N \otimes k(s)) = \bigoplus_{I,J} (M_i \otimes_{k(s)} N_j).$$

If both $M\otimes k(s)$ and $N\otimes k(s)$ are not free, then there exist i and j such that M_i and N_j are not free A-modules. The tensor product of two finite dimensional A-modules is free if and only if at least one of them is free, which implies that $M_i \otimes N_j$ is not free. Since over A projective=free, we get that $M\otimes N\otimes k(s)$ has a direct summand, namely $M_i \otimes N_j$, which is not projective. Therefore, $M \otimes N \otimes k(s)$ is not free.

5. The restriction of $(M\oplus N)\otimes k(s) = (M\otimes k(s))\oplus(N\otimes k(s))$ to $k(s)[u_{r-1}]/(u_{r-1}^p)$ is not free if and only if the restriction of either $(M \otimes k(s))$ or $(N \otimes k(s))$ is not.

6. This follows immediately from the fact that when two $k(s)[u_{r-1}]/(u_{r-1}^p)$-modules out of three in a short exact sequence are free, then the third module has to be free.

□

3.4. Weak cohomological properties

Unlike the situation with finite dimensional modules, the support cone $V(G)_M$ for an infinite dimensional G-module M is typically not a closed subset of $V(G)$ and thus is not homeomorphic to $V(\mathrm{Ann}_{H^{ev}(G,k)}(\mathrm{Ext}_G^*(M,M)))$. As the following proposition shows, a much weaker relationship does hold.

In what follows we identify $V(G)$ and $\mathrm{Spec}\, H^{ev}(G,k)$ via the homeomorphism Ψ of Theorem 3.1.2.

Proposition 3.4.1. *Let G be an infinitesimal k-group scheme of height r satisfying the hypotheses of Theorem 2.3.4 and M be a G-module. Then*

$$V(G)_M \subset V(Ann_{H^{ev}(G,k)}(Ext_G^*(M,M))).$$

Proof. Let $s \in V(G)_M$. Since both $V(G)_M$ and $V(\mathrm{Ann}_{H^{ev}(G,k)}(\mathrm{Ext}_G^*(M,M)))$ are conical (the latter corresponds to an annihilator of a graded module, i.e. is defined by a graded ideal), they are completely determined by their homogeneous ideals, so we can assume that the point s corresponds to a homogeneous prime ideal. To simplify notation denote $k(s)$ by K and $M \otimes K$ by M_K. Then s corresponds to a one-parameter subgroup $\nu_s : \mathbb{G}_{a(r)} \otimes K \to G \otimes K$ such that M_K

restricted to $K[u_{r-1}]/u_{r-1}^p \subset K[\mathbb{G}_{a(r)}]^\#$ via ν_s is not projective. We have an equivalence of categories between the category of H-modules and $K[H]^\#$-modules for any finite group scheme H. Hence, the composition of algebra homomorphisms $K[u_{r-1}]/u_{r-1}^p \subset K[\mathbb{G}_{a(r)}]^\# \to K[G]^\#$ induces a map on cohomology which, by some abuse of notation, we denote ν_s^*: $H^{ev}(G \otimes K, K) \to H^{ev}(\mathbb{G}_{a(1)} \otimes K, K)$. (Note that this map does not correspond to any "real" map of group schemes, but only to a map of coalgebras on the level of coordinate algebras.)

$\mathbb{G}_{a(1)}$ has representation theory equivalent to that of $\mathbb{Z}/p$. Recall that $H^{ev}(\mathbb{G}_{a(1)} \otimes K, K) \cong K[x]$ where x is a generator in degree 2. Note that $H^{ev}(\mathbb{G}_{a(1)} \otimes K, M_K) = \mathrm{Ext}^*_{\mathbb{G}_{a(1)} \otimes K}(K, M_K)$ is naturally a left module for the algebra $\mathrm{Ext}^*_{\mathbb{G}_{a(1)} \otimes K}(M_K, M_K)$ via Yoneda composition. Furthermore, the action of $H^{ev}(\mathbb{G}_{a(1)} \otimes K, K)$ on $\mathrm{Ext}^*_{\mathbb{G}_{a(1)} \otimes K}(K, M_K)$ factors through the action of $\mathrm{Ext}^*_{\mathbb{G}_{a(1)} \otimes K}(M_K, M_K)$ via the natural map of algebras $H^{ev}(\mathbb{G}_{a(1)} \otimes K, K) \overset{\otimes M_K}{\to} \mathrm{Ext}^*_{\mathbb{G}_{a(1)} \otimes K}(M_K, M_K)$. Since x induces a "periodicity" isomorphism on $H^*(\mathbb{G}_{a(1)} \otimes K, M_K)$ (cf. [**Ben**, v.1,3.5]) and the latter is non-trivial in positive degrees due to the fact that M_K is not projective restricted to $K[u_{r-1}]/u_{r-1}^p$, we conclude that the map $H^{ev}(\mathbb{G}_{a(1)} \otimes K, K) \to \mathrm{Ext}^*_{\mathbb{G}_{a(1)} \otimes K}(M_K, M_K)$ is injective.

Thinking of Ext-groups in terms of extensions one sees easily that the following diagram of algebra homomorphisms is commutative:

$$\begin{array}{ccccc} H^{ev}(G,k) & \xrightarrow{\otimes K} & H^{ev}(G\otimes K, K) & \xrightarrow{\nu_s^*} & H^{ev}(\mathbb{G}_{a(1)}\otimes K, K) \\ \downarrow{\scriptstyle \otimes M} & & \downarrow{\scriptstyle \otimes M_K} & & \downarrow{\scriptstyle \otimes M_K} \\ \mathrm{Ext}^*_G(M,M) & \xrightarrow{\otimes K} & \mathrm{Ext}^*_{G\otimes K}(M_K, M_K) & \longrightarrow & \mathrm{Ext}^*_{\mathbb{G}_{a(1)}\otimes K}(M_K, M_K) \end{array}$$

where the right lower map is again restriction via ν_s. By the construction of the homeomorphism $\Psi : V(G) \to \operatorname{Spec} H^{ev}(G,k)$ (cf. [**SFB1**]), the point $s \in V(G)$ corresponds to the homogeneous prime ideal $\mu_s \subset H^{ev}(G,k)$ which is the kernel of the map $H^{ev}(G,k) \to H^{ev}(\mathbb{G}_{a(1)} \otimes K, K)$ appearing as the top row of the commutative diagram above. Now, the commutativity of the diagram together with the injectivity of the right vertical arrow imply that

$$\operatorname{Ker}(H^{ev}(G,k) \to \mathrm{Ext}^*_G(M,M) \to \mathrm{Ext}^*_{\mathbb{G}_{a(1)}\otimes K}(M_K, M_K)) = \operatorname{Ker}(H^{ev}(G,k) \to H^{ev}(\mathbb{G}_{a(1)} \otimes K, K)).$$

Since $\operatorname{Ker}(H^{ev}(G,k) \to \mathrm{Ext}^*_G(M,M)) = \mathrm{Ann}_{H^{ev}(G,k)}(\mathrm{Ext}^*_G(M,M))$ is contained in the left hand side, and the right hand side equals μ_s, we conclude that $\mathrm{Ann}_{H^{ev}(G,k)}(\mathrm{Ext}^*_G(M,M)) \subset \mu_s$.

□

Question. *Is it right that the closure of $V_G(M)$ coincides with $V(Ann_{H^{ev}(G,k)}(Ext^*_G(M,M))$?*

The following result, which is immediately implied by the proposition above and Theorem 3.3.2.3, will be used in the next section to show that the "tensor product property" does not hold for the extension to infinite dimensional modules of the cohomological definition of support variety.

Corollary 3.4.2. *Let G be an infinitesimal group satisfying the hypotheses of Theorem 2.3.4 and M be a G-module. If $V(Ann_{H^{ev}(G,k)}(Ext^*_G(M,M))) \subset Spec\, H^{ev}(G,k)$ is 0, then M is projective.*

In this section we will also give a proof of the Prop. 2.2.3. We first recall the statement:

Proposition. *Let G be an infinitesimal group scheme and M be a locally projective G-module. Let $G \hookrightarrow G'$ be a closed embedding of G into some Frobenius kernel of the same height. Then $Ind_G^{G'}(M)$ is (locally) projective as a G'-module.*

Proof. It is sufficient to show that $V(G')_{\mathrm{Ind}_G^{G'}(M)} = 0$ (cf. Theorem 3.3.2.3). Let s be a point in $V(G')$. It corresponds to a 1-parameter subgroup $\nu_s : \mathbb{G}_{a(r)} \otimes K \to G' \otimes K$. Since Ind commutes with extension of scalars and since we are going to study the behavior of M only at this one particular point s, we may assume that everything is defined over the ground field k.

Let $k[\mathbb{G}_{a(r)}]^{\#} = k[u_0, \dots, u_{r-1}]/(u_0^p, \dots, u_{r-1}^p)$. To show that $s \notin V(G')_{\mathrm{Ind}_G^{G'}(M)}$, one needs to show that the restriction of $\mathrm{Ind}_G^{G'}(M)$ to $k[u_{r-1}]/(u_{r-1}^p)$ is projective. By lowering the height of $\mathbb{G}_{a(r)}$, is necessary, we can assume that the map ν_s is an embedding. (This will involve factoring through the "projection" map $p_{r,r'} : \mathbb{G}_{a(r)} \to \mathbb{G}_{a(r')}$, which takes generator u_{r-1} of $k[\mathbb{G}_{a(r)}]^{\#}$ to the generator $u_{r'-1}$ of $k[\mathbb{G}_{a(r')}]^{\#}$)

Consider the following Cartesian square of group schemes:

$$\begin{tikzcd} \mathbb{G}_{a(t)} \arrow[r, hook] \arrow[d, hook] & G \arrow[d, hook] \\ \mathbb{G}_{a(r)} \arrow[r, hook, "\nu_s"] & G' \end{tikzcd}$$

Set $\Lambda = \mathrm{End}_k(M, M)$. Λ is an associative unital G-algebra. Moreover, restricted to $\mathbb{G}_{a(t)}$ via the pull-back of the map induced by ν_s, Λ is projective. Indeed, $H^1(\mathbb{G}_{a(t)}, \Lambda) = \mathrm{Ext}^1_{\mathbb{G}_{a(t)}}(M, M) = 0$, since M is locally projective G-module. Since $\mathbb{G}_{a(t)}$ is unipotent, vanishing of the first cohomology group implies that Λ is projective as a $\mathbb{G}_{a(t)}$-module. Thus, $\mathrm{Ind}^{\mathbb{G}_{a(r)}}_{\mathbb{G}_{a(t)}}(\Lambda)$, which is again an associative unital $\mathbb{G}_{a(r)}$ - algebra, is projective.

The natural map of $\mathbb{G}_{a(r)}$-algebras $\mathrm{Ind}^{G'}_{G}(\Lambda) \to \mathrm{Ind}^{\mathbb{G}_{a(r)}}_{\mathbb{G}_{a(t)}}(\Lambda)$ (determined by the adjointness of Ind and Res) is surjective and has a nilpotent kernel (cf. [SFB2;4.3]). Denote the kernel by I. Projectivity of $\mathrm{Ind}^{\mathbb{G}_{a(r)}}_{\mathbb{G}_{a(t)}}(\Lambda)$ as a $\mathbb{G}_{a(r)}$-module implies that it is projective restricted further to $k[u_{r-1}]/(u^p_{r-1})$. Thus,

$$H^*(k[u_{r-1}]/(u^p_{r-1}), \mathrm{Ind}^{\mathbb{G}_{a(r)}}_{\mathbb{G}_{a(t)}}(\Lambda)) = 0 \text{ for } * > 0.$$

Therefore, the long exact sequence in cohomology corresponding to the short exact sequence

$$0 \to I \to \mathrm{Ind}^{G'}_{G}(\Lambda) \to \mathrm{Ind}^{\mathbb{G}_{a(r)}}_{\mathbb{G}_{a(t)}}(\Lambda) \to 0$$

of modules gives an isomorphism

$$H^*(k[u_{r-1}]/(u^p_{r-1}), \mathrm{Ind}^{G'}_{G}(\Lambda)) \cong H^*(k[u_{r-1}]/(u^p_{r-1}), I)$$

in positive degrees. The ideal I is nilpotent, so the algebra without unit $H^*(k[u_{r-1}]/(u_{r-1}^p), I)$ is also nilpotent. The isomorphism above implies that the augmentation ideal $H^{*>0}(k[u_{r-1}]/(u_{r-1}^p), \mathrm{Ind}_G^{G'}(\Lambda))$ is nilpotent. On the other hand, the map of algebras $k \to \mathrm{Ind}_G^{G'}(\Lambda)$ induces an action of $H^{ev}(k[u_{r-1}]/(u_{r-1}^p), k) \cong k[x]$ (where x is a generator in degree two) on $H^*(k[u_{r-1}]/(u_{r-1}^p), \mathrm{Ind}_G^{G'}(\Lambda))$. The action of x, in particular, induces a periodicity isomorphism

$$H^i(k[u_{r-1}]/(u_{r-1}^p), \mathrm{Ind}_G^{G'}(\Lambda)) \cong H^{i+2}(k[u_{r-1}]/(u_{r-1}^p), \mathrm{Ind}_G^{G'}(\Lambda)) \text{ for all } i > 0$$

(cf.). The image of x in $H^2(k[u_{r-1}]/(u_{r-1}^p), \mathrm{Ind}_G^{G'}(\Lambda))$ under the map of algerbas $H^{ev}(k[u_{r-1}]/(u_{r-1}^p), k) \to H^*(k[u_{r-1}]/(u_{r-1}^p), \mathrm{Ind}_G^{G'}(\Lambda))$ is nilpotent (since all elements of positive degree are nilpotent in the target), therefore, the periodicity isomorphism induced by the action of x is trivial. Hence,

$$H^{*>0}(k[u_{r-1}]/(u_{r-1}^p), \mathrm{Ind}_G^{G'}(\Lambda)) = 0.$$

There is a natural action of Λ on M compatible with their G-module structure. This induces an action of $\mathrm{Ind}_G^{G'}(\Lambda)$ on $\mathrm{Ind}_G^{G'}(M)$ compatible with their structure as G'-modules, and, therefore, $k[u_{r-1}]/(u_{r-1}^p)$-modules. Hence, the action of $H^{ev}(k[u_{r-1}]/(u_{r-1}^p), k) \cong k[x]$ on $H^*(k[u_{r-1}]/(u_{r-1}^p), \mathrm{Ind}_G^{G'}(M))$ factors through the action of $H^*(k[u_{r-1}]/(u_{r-1}^p), \mathrm{Ind}_G^{G'}(\Lambda))$. The latter is zero in positive degrees. This implies that the action of $x \in H^2(k[u_{r-1}]/(u_{r-1}^p), k)$, which, again, induces periodicity isomorphism, is trivial. Therefore, $H^1(k[u_{r-1}]/(u_{r-1}^p), \mathrm{Ind}_G^{G'}(M)) = 0$.

Thus, $\mathrm{Ind}_G^{G'}(M)$ is projective as a $k[u_{r-1}]/(u_{r-1}^p)$-module. We conclude that $s \notin V(G)_{\mathrm{Ind}_G^{G'}(M)}$. The statement follows.

$\square$

CHAPTER 4

Rickard Idempotent modules

In this chapter we give a different interpretation of support cones and connect them to the work of Benson, Carlson and Rickard on infinite dimensional modules of finite groups. G will denote an infinitesimal group scheme of height r.

4.1. Stable Module Category of an infinitesimal group scheme

In this section we will give a brief account of the structure of the Stable Module category of G, a triangulated quotient category of the category of G-modules. The key point that makes this construction work is the fact that injective G-modules are projective, which holds for any finite-dimensional cocommutative Hopf algebra. For a much better exposition in the practically identical case of an abstract finite group, one may consult [**BCR1**], [**BCR2**], [**R2**]. In the monograph on axiomatic stable homotopy theory [**HPS**], a very general and more formal treatment of the subject can be found. For a discussion of triangulated categories, see [**W**].

We say that a map $f : M \to N$ of two G-modules is stably equivalent to the trivial map if it factors through a projective G-module. Let $\mathrm{PHom}_G(M, N)$ denote the set of all such maps. Two G-module maps $f, g : M \to N$ are said to be stably equivalent if their difference is in $\mathrm{PHom}_G(M, N)$. Define the *Stable Module* *category* of G, StMod$G$, to be the category which has G-modules as its objects, and equivalence classes of maps with respect to the stable equivalence defined above as morphisms. Denote the set of morphisms in StMod$G$ by $\underline{\mathrm{Hom}}(M, N)$. Thus, by definition,

$$\underline{\mathrm{Hom}}(M, N) = \mathrm{Hom}_G(M, N)/\mathrm{PHom}_G(M, N).$$

We shall use the notation "$\cong$" for stable isomorphisms.

Proposition 4.1.1. *StModG is a triangulated category. The shift operator is given by the Heller operator $\Omega^{-1} : StModG \to StModG$ (cf. section 2.1). Exact triangles are those stably isomorphic to short exact sequences of G-modules.*

Proof. We give a sketchy proof and lazily omit checking the octahedral axiom.

Let $i : M \hookrightarrow I$ be an embedding of M into an injective module. Then the cokernel of I differs from the cokernel of the embedding of M into its injective hull by an injective summand. Thus, $\Omega^{-1}M \cong \mathrm{coker}i$. Similary, to determine ΩM, the kernel of the projection onto M of the projective cover of M, up to a stable isomorphism, we can take the kernel of any surjective map to M from a projective module. Since projective and injective modules coincide, we conclude that $M \cong \Omega\Omega^{-1}M$ and $N \cong \Omega^{-1}\Omega N$ and, hence, the Heller operator $\Omega^{-1} : StModG \to StModG$ defines a self-equivalence of the category StMod$G$.

There is a canonical isomorphism

$$\underline{\mathrm{Hom}}(N, \Omega^{-1}M) \simeq \mathrm{Ext}^1_G(N, M) \qquad (*)$$

Indeed, let $M \to I^0 \xrightarrow{d^1} I^1 \to \dots$ be an injective resolution of M. By definition, d^1 factors through $\Omega^{-1}M$: $d^1 : I^0 \twoheadrightarrow \Omega^{-1}M \hookrightarrow I^1$. Let $\alpha \in \mathrm{Hom}(N, \Omega^{-1}M)$. Extend α to a map $N \to I^1$ by composing with the embedding $\Omega^{-1}M \hookrightarrow I^1$. This is automatically a cocycle, since the composition $\Omega^{-1}M \hookrightarrow I^1 \to I^2$ is zero. Thus, we defined a map $\mathrm{Hom}_G(N, \Omega^{-1}M) \to \mathrm{Ext}^1_G(N, M)$ and it is functorial in both M and N. The only thing left is to identify the kernel. Suppose $\alpha : N \to \Omega^{-1}M \to I^1$ is a coboundary. Then α factors through I^0 which means that $\alpha \in \mathrm{PHom}_G(M, N)$. On the other hand, if α factors through some injective module, then it factors through the I^0, since $I^0 \to \Omega^{(-1)}M$ is surjective. Thus, the kernel is exactly $\mathrm{PHom}_G(M, N)$, and the isomorphism follows.

Using the isomorphism (*), we associate to any exact sequence of G-modules $0 \to M \to L \to N \to 0$ an exact triangle in StMod$G$: $M \to L \to N \to \Omega^{-1}M$. Next, we check the axioms.

- *TR1: every map $f : M \to N$ fits in an exact triangle.*

Let P be the injective hull of M, and extend f to a stably equivalent map $f' : M \to N \oplus I$. This map is injective, so we can complete f to a short exact sequence:

$$0 \to M \xrightarrow{f'} N \oplus P \xrightarrow{g} L \to 0.$$

In fact, to get an exact triangle fitting f, we take the extension $\Omega N \to L \to M$ corresponding to the map $f : M \to N \cong \Omega^{-1}\Omega N$ under the isomorphism (*) and shift it once.

- *TR2: shifts of exact triangles are exact.*

Let $0 \to M \xrightarrow{\alpha} L \to N \to 0$ be an exact sequence of G-modules. It defines a canonical map $N \to \Omega^{-1}M$ and we have to show that $L \to N \to \Omega^{-1}M \to \Omega^{-1}L$ is again an exact triangle. Under the canonical isomorphism $\mathrm{Ext}^1_G(\Omega^{-1}M, L) \cong \underline{\mathrm{Hom}}(M, L)$ the map α corresponds to an extension $L \to \tilde{N} \to \Omega^{-1}M$ which we can write explicitly as a push-out of the short exact sequence $0 \to M \to I^0 \to \Omega^{-1}M \to 0$ (I^0 is the first term of an injective resolution of M):

$$\begin{tikzcd}
M \arrow[r] \arrow[d, hook, "\alpha"] & I^0 \arrow[r] \arrow[d] & \Omega^{-1}M \arrow[d, equal] \\
L \arrow[r] & \widetilde{N} \arrow[r] & \Omega^{-1}M
\end{tikzcd}$$

Since $\widetilde{N}$ is a push-out, the cokernels of α and the map $I^0 \to \widetilde{N}$ coincide. The latter is stably isomorphic to $\widetilde{N}$, since I^0 is injective, and $\mathrm{coker}\alpha = N$. Thus, $N \cong \widetilde{N}$ and the sequence $L \to N \to \Omega^{-1}M \to \Omega^{-1}L$ is stably isomorphic to the exact triangle $L \to \widetilde{N} \to \Omega^{-1}M \to \Omega^{-1}L$. TR2 follows.

TR3 is straightforward once we view exact triangles as short exact sequences.

□

Lemma 4.1.2. *Two G-modules M and N are stably isomorphic if and only if there exist projective modules P and Q such that $M \oplus P \cong N \oplus Q$ in the category of G-modules. Moreover, we always can choose one of P and Q to be zero.*

Proof. Every G-module M has a maximal projective submodule, which splits off as a direct summand since projective = injective. The compliment is a module with no projective submodules, stably isomorphic to the original module. We will call this submodule the "projective-free" part of M, denoted M_{pf}. We can reformulate the statement of the lemma in the following way: two modules are stably isomorphic if and only if their projective-free parts are isomorphic. The "if" part is clear. Now assume that both M and N are projective-free and let $\alpha : M \to N$ be a stable isomorphism. The embedding $\ker(\alpha) \hookrightarrow M$ is stably zero, i.e. factors through a projective map. Let S be a simple submodule of $\ker(\alpha)$. Then the inclusion $S \hookrightarrow M$ factors through the injective hull of S, $I(S)$, and since $I(S)$ is indecomposable, the map $I(S) \to M$ is again an embedding. This contradicts the "projective-free" assumption on M. Thus, α is injective. Applying the same argument to the cokernel, one shows that it is surjective. Hence, α is an isomorphism. $\square$

We shall denote by stmod(G) the full triangulated subcategory of StMod(G) whose objects are represented by finite dimensional modules. This subcategory is equivalent to the usual stable module category of finite dimensional G-modules (i.e. the category of finite dimensional G-modules whose maps are equivalence classes of G-homomorphisms where two maps are equivalent if their difference factors through a finite dimensional projective G-module). The lemma above implies that the construction of support cones descends nicely to StMod$G$. Tensor products are also well-defined up to a stable isomorphism. One can formally rewrite properties of support cones from Theorem 3.3.2 for StMod$G$. One special property, which is a consequence of Theorem 3.3.2, is that Heller operator does not affect support cones: $V_G(M) = V_G(\Omega^{-1}M)$.

Definition 4.1.3. *A full triangulated subcategory $\mathcal{C}$ of stmodG (respectively StMod G) is called thick if it is closed under taking direct summands (respectively taking direct summands and arbitrary direct sums). It is called tensor-ideal if it is closed under taking tensor products with any G-module.*

4.2. Bousfield localization

We define and prove the existence of Bousfield localization functors for certain thick subcategories of StMod$G$. To avoid introducing more notation, which will not be used later in the text, we adapt a general definition of a localization functor for a "stable homotopy category" to our special situation. A definite advantage is that we can use $\otimes$ for the smash products and Hom for morphisms.

Definition 4.2.1. *(Localization functor). Let $\mathcal{S}$ be the triangulated category StModG, $\mathcal{C}$ be a thick subcategory, and $\mathcal{F} : \mathcal{S} \to \mathcal{S}$ be an exact functor. $\mathcal{F}$ is a localization functor with respect to the category $\mathcal{C}$ (which is called localizing subcategory) if there is a natural transformation $\eta : Id_{\mathcal{S}} \to \mathcal{F}$ such that*

(i) *The natural transformation $\mathcal{F}\eta : \mathcal{F} \to \mathcal{F}^2$ is an equivalence.*

(ii) *The map $Hom_{\mathcal{S}}(\mathcal{F}(M), \mathcal{F}(N)) \xrightarrow{\eta_M^*} Hom_{\mathcal{S}}(M, \mathcal{F}(N))$ is an isomorphism.*

(iii) *$\mathcal{C}$ coincides with the full subcategory of objects (called $\mathcal{F}$ - acyclic) for which $\mathcal{F}(M) = 0$.*

Remark 4.2.2. The definition immediately implies that for any M, $\mathcal{F}(M)$ is $\mathcal{C}$ - *local*, i.e. there are no non-trivial maps from objects of $\mathcal{C}$ to $\mathcal{F}(M)$. We say that $\mathcal{F}$ is a localization "away from $\mathcal{C}$".

We give a dual definition of a *colocalization* functor:

Definition 4.2.3. *(Colocalization functor) Let $\mathcal{S}$ be the triangulated category StModG, $\mathcal{C}$ be a thick subcategory, and $\mathcal{E} : \mathcal{S} \to \mathcal{S}$ be an exact functor. $\mathcal{E}$ is a colocalization functor (with respect to the localizing category $\mathcal{C}$) if there is a natural transformation $\epsilon : \mathcal{E} \to Id_{\mathcal{S}}$ such that*

(i) *The natural transformation $\mathcal{E}\epsilon : \mathcal{E}^2 \to \mathcal{E}$ is an equivalence.*

(ii) *The map $Hom_{\mathcal{S}}(\mathcal{E}(M), \mathcal{E}(N)) \overset{\epsilon_{N*}}{\to} Hom_{\mathcal{S}}(\mathcal{E}(M), N)$ is an isomorphism.*

(iii) *$\mathcal{C}$ coincides with the full subcategory of objects (called $\mathcal{E}$-local) for which $\epsilon_M : \mathcal{E}(M) \to M$ is an isomorphism.*

Remark 4.2.4. (i) and (iii) imply that $\mathcal{E}(M) \in \mathcal{C}$ and for any $M \in \mathcal{C}$ one has $\mathcal{E}(M) \cong M$. Thus, $\mathcal{E}$ is a "projection" onto the subcategory $\mathcal{C}$. Together with (ii) this gives that $\mathcal{E}$ is the right adjoint to the inclusion functor $\mathcal{C} \hookrightarrow \mathcal{S}$

Remark 4.2.5. There is a natural equivalence between localization and colocalization functors, in which $\mathcal{F}$ and $\mathcal{E}$ correspond if and only if

$$\mathcal{E}(M) \overset{\epsilon_M}{\to} M \overset{\eta_M}{\to} \mathcal{F}(M)$$

is an exact triangle for every M (cf. [**HPS**, 3.1.6]). Of course, under this correspondence the subcategory of $\mathcal{E}$-local objects coincides with the subcategory of $\mathcal{F}$-acyclic objects. If we apply $\mathrm{Hom}(-, L)$ for any $\mathcal{C}$-local object L to the exact triangle above, we get a canonical isomorphism

$$\underline{\mathrm{Hom}}(M, L) \cong \underline{\mathrm{Hom}}(\mathcal{F}(M), L).$$

Thus, $\mathcal{F}(M)$ is not only $\mathcal{C}$-local, but the map $\eta_M : M \to \mathcal{F}(M)$ is the universal map from M to a $\mathcal{C}$-local object.

If $\mathcal{C}$ is tensor-ideal, then the value of the localization functor $\mathcal{F}$ corresponding to the localizing subcategory $\mathcal{C}$ is completely determined by its value on k, i.e. $\mathcal{F}(M) = \mathcal{F}(k) \otimes M$.

Let $\mathcal{C}$ be a thick subcategory of stmod(G). Denote by $\vec{\mathcal{C}}$ the full triangulated subcategory of $\mathrm{StMod}(G)$ whose objects are filtered colimits of objects in $\mathcal{C}$. ($\vec{\mathcal{C}}$ coincides with the smallest full triangulated subcategory of $\mathrm{StMod}(G)$ which contains $\mathcal{C}$ and is closed under taking direct summands and arbitrary direct sums (cf. [**R1**]).)

Next we sketch a construction of the pair of localization and colocalization functors corresponding to $\vec{\mathcal{C}} \subset \mathrm{StMod}G$. Since $\vec{\mathcal{C}}$ is generated by the objects of the category $\mathrm{stmod}G$, which are "small" (where an object C is small if the funtor $\mathrm{Hom}(C, ?)$ preserves arbitrary direct sums), we will be talking about the simplest case of Bousfield localization here - "finite localization". Our construction is taking from [**R1**].

A detailed discussion of Bousfield localization for any finite dimensional cocommutative Hopf algebra can be found in [**R2**] or [**HPS**].

Given a module M, we would like to construct $\mathcal{F}(M)$ with a canonical map from M and the property of being $\mathcal{C}$ - local. The idea of the construction below is to kill all the maps from $\mathcal{C}$ by consecutive approximations.

Define a homotopy colimit of the sequence $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \dots$, *hocolim*$(M_i)$, by completing the map $\bigoplus_{i=1}^{i=\infty} M_i \xrightarrow{1-f} \bigoplus_{i=1}^{i=\infty} M_i$ to an exact triangle:

$$\bigoplus_{i=1}^{i=\infty} M_i \xrightarrow{1-f} \bigoplus_{i=1}^{i=\infty} M_i \to \mathrm{hocolim}(M_i)$$

Clearly, $\mathrm{hocolim}(M_i)$ is stably isomorphic to the $\varinjlim M_i$, where $\varinjlim$ is taken in the category of G-modules.

Let C_i run over a complete set of representatives of isomorphism classes of modules in $\mathcal{C}$. Let M be a G-module. Let further $C(M) = \bigoplus_i (C_i \otimes \mathrm{Hom}_G(C_i, M))$ (each summand is itself a direct sum of copies of C_i as a G-module and, hence, $C(M) \in \vec{\mathcal{C}}$). There is a natural homomorphism $C(M) \to M$, defined by sending each $c_i \otimes f$ to $f(c_i)$, which has the following property: any map $C \to M$ for $C \in \mathcal{C}$ factors through $C(M)$. Let $M = M_0$. We construct inductively a sequence $M_1, M_2, \dots$ as follows: for each M_i complete the map $C(M_i) \to M_i$ to an exact triangle

$$C(M_i) \to M_i \to M_{i+1}.$$

Set $\mathcal{F}(M) = \text{hocolim} M_i$. $\mathcal{F}(M)$ comes equipped with a map from M_0, which we denote η_M. Now, complete η_M to an exact trianlge and call the third vertex $\mathcal{E}(M)$:

$$\mathcal{E}(M) \xrightarrow{\epsilon_M} M \xrightarrow{\eta_M} \mathcal{F}(M).$$

Lemma 4.2.6. *For any $C \in \mathcal{C}$, we have $\underline{Hom}(C, \mathcal{F}(M)) = 0$ (hence, $\mathcal{F}(M)$ is $\mathcal{C}$-local).*

PROOF. Any map $f : C \to \mathcal{F}(M)$ factors through some M_i. Therefore, it factors through $C(M_i) \to M_i$ by the construction of $C(M)$. But then the composition $C \to C(M_i) \to M_i \to M_{i+1}$ is stably trivial and, hence, $f : C \to \mathcal{F}(M)$ is stably trivial as well. $\square$

Corollary 4.2.7. *$\mathcal{F}(M)$ is $\vec{\mathcal{C}}$-local.*

This is clear since a map from an object in $\vec{\mathcal{C}}$ is the same as a map from a filtered system of objects in $\mathcal{C}$.

Lemma 4.2.8. *$\mathcal{E}(M) \in \vec{\mathcal{C}}$*

PROOF. In the construction of $\mathcal{F}(M)$ each M_i comes equipped with a map from M. Define E_i by completing this map to an exact trianlge: $E_i \to M \to M_i$. Thus, we have a sequence of exact triangles

$$\begin{array}{ccccc}
0 & \longrightarrow & M & = & M_0 \\
\downarrow & & \| & & \downarrow \\
E_1 & \longrightarrow & M & \longrightarrow & M_1 \\
\downarrow & & \| & & \downarrow \\
E_2 & \longrightarrow & M & \longrightarrow & M_2 \\
\downarrow & & \| & & \downarrow \\
\vdots & & \vdots & & \vdots
\end{array}$$

By taking homotopy colimit of this sequence, we get an exact triangle which is stably isomorphic to the trianlge $\mathcal{E}(M) \to M \to \mathcal{F}(M)$. Hence, to prove the lemma it suffices to show that $E_i \in \vec{\mathcal{C}}$ for all i. We prove this by induction, the case $i = 0$ being obvious. Assume $E_i \in \vec{\mathcal{C}}$. By shifting once, we obtain two exact triangles

$$\begin{array}{ccccc}
M & \longrightarrow & M_i & \longrightarrow & \Omega^{-1}E_i \\
\| & & \downarrow & & \downarrow \\
M & \longrightarrow & M_{i+1} & \longrightarrow & \Omega^{-1}E_{i+1}
\end{array}$$

Complete the middle vertical arrow to an exact triangle $M_i \to M_{i+1} \to \Omega^{-1}C(M_i)$. By the octahedral axiom (cf. [**W**, 10.2.1]), there exists a map $\Omega^{-1}E_{i+1} \to \Omega^{-1}C(M_i)$, which completes the most right vertical arrow of the diagram above to an exact trianlge and makes the diagram commutative. Adding $\Omega^{-1}C(M_i)$ to the diagram above, we get the following commutative diagram in which all rows and columns are exact triangles:

$$\begin{array}{ccccc} M & \longrightarrow & M_i & \longrightarrow & \Omega^{-1}E_i \\ \| & & \downarrow & & \downarrow \\ M & \longrightarrow & M_{i+1} & \longrightarrow & \Omega^{-1}E_{i+1} \\ & & \downarrow & & \downarrow \\ & & \Omega^{-1}C(M_i) & = & \Omega^{-1}C(M_i) \end{array}$$

Note that $C(M_i)$ is in $\vec{\mathcal{C}}$ by construction of $C(M_i)$, and E_i is in $vec\mathcal{C}$ by induction hypothesis. Since $\vec{\mathcal{C}}$ is a thick subcategory, we conclude that $\Omega^{-1}E_{i+1}$ and, hence, E_{i+1}, belongs to $\vec{\mathcal{C}}$.

If we replace exact triangles by isomorphic exact sequences, via adding appropriate projective summands, then the statement becomes clear without explicitly appealing to the octahedral axiom. This way would not be as fancy, but might be more honest, since we did not check the octahedral axiom.

□

Applying the functor $\underline{\mathrm{Hom}}(C, ?)$ to the exact triangle $\mathcal{E}(M) \overset{\epsilon_M}{\to} M \overset{\eta_M}{\to} \mathcal{F}(M)$ and using Corollary 4.2.7, we get that $\mathcal{E}(M)$ satisfies the following universal property: for any $C \in \vec{\mathcal{C}}$, $\underline{\mathrm{Hom}}(C, \mathcal{E}(M)) \cong \underline{\mathrm{Hom}}(C, M)$, where the isomorphism is induced by ϵ_M. Proceeding as in Remark 4.2.5, we conclude that $\mathcal{F}(M)$ satisfies an analogous universal property with respect to the maps from M to $\mathcal{C}$-local objects. Thus, both $\mathcal{F}(M)$ and $\mathcal{E}(F)$ are well-defined and, furthermore, functorial on M.

Next, we check that $\mathcal{F}$ and $\mathcal{E}$ are exact. Let $M' \to M \to M''$ be an exact triangle. By functoriality, we have a commutative diagram of exact triangles

$$\begin{tikzcd}
 & \mathcal{E}(M) \arrow[r] \arrow[d] & E(M'') \arrow[d] \\
M' \arrow[r] & M \arrow[r] \arrow[d] & M'' \arrow[d] \\
 & \mathcal{F}(M) \arrow[r] & F(M'').
\end{tikzcd}$$

By completing the upper and lower rows to exact triangles we get a new commutative diagram

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & \mathcal{E}(M) \arrow[r] \arrow[d] & \mathcal{E}(M'') \arrow[d] \\
M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M'' \arrow[d] \\
F \arrow[r] & \mathcal{F}(M) \arrow[r] & \mathcal{F}(M'').
\end{tikzcd}$$

Since $\vec{\mathcal{C}}$ is a thick subcategory, and contains both $\mathcal{E}(M)$ and $\mathcal{E}(M'')$, it contains E. Applying $\mathrm{Hom}(C, ?)$, $C \in \vec{\mathcal{C}}$, to the exact trianlge in the last row, we conclude that F is $\vec{\mathcal{C}}$-local. Thus, the exact trianlge $E \to M \to F$ satisfies Corollary 4.2.7 and Lemma 4.2.8. Proceeding as above, we can show that E and F satisfy the universal properties of $\mathcal{E}(M)$ and $\mathcal{F}(M)$. Therefore, the constructed trianlge is canonically isomorphic to $\mathcal{E}(M) \to M \to \mathcal{F}(M)$. The exactness of $\mathcal{F}$ and $\mathcal{E}$ now follows.

If $M \in \vec{\mathcal{C}}$, then clearly $\mathcal{E}(M) \cong M$, since M satisfies the universal property of $\mathcal{E}(M)$. Together with Lemma 4.2.8, this implies that $\mathcal{E}(\mathcal{E}(M)) \overset{\mathcal{E}\epsilon_M}{\cong} \mathcal{E}(M)$. Similarly, $\mathcal{F}(M)$ is $\mathcal{C}$-local and, thus, shares the universal property of $\mathcal{F}(\mathcal{F}(M))$. Hence, $\mathcal{F}(M) \overset{\mathcal{F}\eta_M}{\cong} \mathcal{F}(\mathcal{F}(M))$.

Finally, we show that $\mathcal{F}$ and $\mathcal{E}$ have a special form when the thick subcategory $\mathcal{C}$ is tensor-ideal.

Proposition 4.2.9. *Let $\mathcal{C}$ be a tensor-ideal thick subcategory of stmodG. Then the exact trianlge $M \otimes \mathcal{E}(k) \stackrel{id_M \otimes \epsilon_k}{\to} M \stackrel{id_M \otimes \eta_k}{\to} M \otimes \mathcal{F}(k)$ is stably isomorphic to $\mathcal{E}(M) \stackrel{\epsilon_M}{\to} M \stackrel{\eta_M}{\to} \mathcal{F}(M)$.*

Proof. First note that since any G-module can be written as a colimit of finite-dimensional ones, the category $\vec{\mathcal{C}}$ is also tensor-ideal. Therefore, $M \otimes \mathcal{E}(k) \in \vec{\mathcal{C}}$.

Let L be a $\mathcal{C}$-local object and N be any finite dimensional G-module. For any $C \in \mathcal{C}$ we have $\underline{\mathrm{Hom}}(C, N \otimes L) \simeq \underline{\mathrm{Hom}}(C \otimes M^{\#}, L) = 0$. The last equality holds because L is $\mathcal{C}$-local. Therefore, $M \otimes L$ is also $\mathcal{C}$-local. Since any G-module is a direct limit of finite dimensional modules, we can make the same conclusion for an arbitrary module N. Finally, if $M \otimes L$ is $\mathcal{C}$-local, then it is $\vec{\mathcal{C}}$-local. Taking L to be $\mathcal{F}(k)$, we see that $M \otimes \mathcal{F}(k)$ is $\mathcal{C}$-local.

Using the same argument as before, we obtain that $M \otimes \mathcal{E}(k)$ and $M \otimes \mathcal{F}(k)$ share the universal properties of $\mathcal{E}(M)$ and $\mathcal{F}(M)$ and, hence, are canonically isomorphic to them. □

We summarize the preceeding discussion in the following theorem.

Theorem 4.2.10. *(Bousfield localization)*

I. Let $\mathcal{C}$ be a thick subcategory of stmod(G). There exist exact functors $\mathcal{E}_{\mathcal{C}}, \mathcal{F}_{\mathcal{C}}$: StMod(G) $\to$ StMod(G) (colocalization and localization functors with respect to $\mathcal{C}$ respectively) characterized by the following properties:

(i) For any $M \in$ StMod(G), the modules $\mathcal{E}_{\mathcal{C}}(M)$ and $\mathcal{F}_{\mathcal{C}}(M)$ fit in an exact triangle:

$$\mathcal{T}_{\mathcal{C}}(M) : \mathcal{E}_{\mathcal{C}}(M) \to M \to \mathcal{F}_{\mathcal{C}}(M) \to \Omega^{-1}\mathcal{E}_{\mathcal{C}}(M).$$

(ii) $\mathcal{E}_{\mathcal{C}}(M)$ belongs to $\vec{\mathcal{C}}$ and satisfies the following universal property: the map $\epsilon_M : \mathcal{E}_{\mathcal{C}}(M) \to M$, which occurs in the exact triangle $\mathcal{T}_{\mathcal{C}}(M)$, is the universal map in StMod(G) from an object in $\vec{\mathcal{C}}$ to M, i.e. for any $C \in \vec{\mathcal{C}}$, ϵ_m induces an isomorphism

$$\underline{Hom}(C, \mathcal{E}_{\mathcal{C}}(M)) \simeq \underline{Hom}(C, M).$$

(iii) The map $\eta_M : M \to \mathcal{F}_{\mathcal{C}}(M)$, which occurs in the exact triangle $\mathcal{T}_{\mathcal{C}}(M)$, is the universal map in StMod(G) from M to a $\mathcal{C}$-local object

II. Suppose C is also tensor-ideal. Then for any G-module M we have stable isomorphisms: $\mathcal{E}_{\mathcal{C}}(M) \cong \mathcal{E}_{\mathcal{C}}(k) \otimes M$, $\mathcal{F}_{\mathcal{C}}(M) \cong \mathcal{F}_{\mathcal{C}}(k) \otimes M$.

Remark 4.2.11. At it is easily seen from the proofs above, the exact triangle $\mathcal{T}_{\mathcal{C}}(M)$ is uniquely determined up to a stable isomorphism by the following properties: $\mathcal{E}_{\mathcal{C}}(M) \in \vec{\mathcal{C}}$ and $\mathcal{F}_{\mathcal{C}}(M)$ is $\mathcal{C}$-local.

The modules $\mathcal{E}_{\mathcal{C}}(k)$ and $\mathcal{F}_{\mathcal{C}}(k)$ were introduced by J. Rickard ([**R1**]) for finite groups and are thereby called *Rickard idempotent modules.* We justify the name in the following proposition:

Proposition 4.2.12. *Let $\mathcal{C}$ be a tensor-ideal thick subcategory of stmod(G). Then*

(a) *There are stable isomorphisms:*

$$\mathcal{E}_{\mathcal{C}}(k) \otimes \mathcal{E}_{\mathcal{C}}(k) \cong \mathcal{E}_{\mathcal{C}}(k) \text{ and } \mathcal{F}_{\mathcal{C}}(k) \otimes \mathcal{F}_{\mathcal{C}}(k) \cong \mathcal{F}_{\mathcal{C}}(k);$$

(b) $\mathcal{E}_{\mathcal{C}}(k) \otimes \mathcal{F}_{\mathcal{C}}(k)$ *is projective;*

(c) *For a finite dimensional G-module M, the following are equivalent:*

  (i) $M \in \mathcal{C}$

  (ii) $M \otimes \mathcal{E}_{\mathcal{C}}(k)$ *is stably isomorphic to M*

  (iii) $M \otimes \mathcal{F}_{\mathcal{C}}(k)$ *is projective.*

Proof. The first statement is a reformulation of the first property in the definition of localization and colocalization functor, which we showed before. To show that $\mathcal{E}_{\mathcal{C}}(k) \otimes \mathcal{F}_{\mathcal{C}}(k)$ is projective, tensor $\mathcal{E}_{\mathcal{C}}(k)$ with the exact triangle $\mathcal{T}_{\mathcal{C}}$ and apply (a). For the third claim, the equivalence of (i) and (ii) was showed before, and the equivalence of (ii) and (iii) follows immediately from the fact that $\mathcal{E}_{\mathcal{C}}(M) \to M \to \mathcal{F}_{\mathcal{C}}(M) \to \Omega^{-1}\mathcal{E}_{\mathcal{C}}(M)$ form an exact triangle. □

Lemma 4.2.13. *Let W be a subset in $V(G)$ and let $\mathcal{C}_W$ be the full subcategory of stmod(G) consisting of finitely generated modules M whose variety $V(G)_M$ is contained in W. Then $\mathcal{C}_W$ is a tensor-ideal thick subcategory of stmod(G).*

The statement of the lemma follows immediately from the standard properties of support varieties and implies the existence of the Rickard idempotents associated to the subcategory $\mathcal{C}_{\mathcal{W}}$. In this special case we shall use the following notation:

$$E(W) = \mathcal{E}_{\mathcal{C}_W}(k),\ \ F(W) = \mathcal{F}_{\mathcal{C}_W}(k) \text{ and } T(W) = \mathcal{T}_{\mathcal{C}_W}(k)$$

4.3. Support cones via Rickard idempotents

In this section we compute support cones of the Rickard Idempotent modules and give an alternative description of support cone of any G-module, using Rickard Idempotents.

Definition 4.3.1. *Let W be a subset in an affine scheme $X = Spec\,A$. W is said to be closed under specialization if for any two primes $\mu \subset \nu \subset A$, $\mu \in W$ implies $\nu \in W$.*

Being closed under specialization is equivalent to the fact that for any $s \in W$ the Zariski closure of s, denoted $\overline{s}$, is contained in W. For any $U \subset X$ we denote by $\mathrm{Cs}\,(U)$ the *closure under specialization* of U, i.e.

$$\mathrm{Cs}\,(U) = \bigcup_{s \in U} \overline{s}$$

Note that closure under specialization of a conical subset is again conical.

Let V be a closed conical subset of $V(G)$. Denote by V' the subset of V consisting of all points of V except for generic points of irreducible components of V. Define

$$\kappa(V) \overset{def}{=} E(V) \otimes F(V').$$

As a tensor product of idempotent modules, $\kappa(V)$ is again idempotent, i.e. $\kappa(V) \otimes \kappa(V) \cong \kappa(V)$.

Note that the generic point of an irreducible closed conical subvariety is a homogeneous prime ideal, so that there is a natural 1-1 correspondence between homogeneous prime ideals and closed irreducible conical subvarieties. For an irreducible closed conical set V with the generic point s we shall use $\kappa(s)$ to denote $\kappa(V)$. In particular, for any point $s \in V(G)$ corresponding to a homogeneous prime ideal, $\kappa(s)$ will substitute for $\kappa(\overline{s})$ to simplify notation.

Theorem 4.3.2. *Let W be a conical closed under specialization subset of $V(G)$. Then*

$$V(G)_{E(W)} = W.$$

Before proving the theorem we state an immediate corollary:

Corollary 4.3.3. *For any conical closed under specialization subset W of $V(G)$ there exists a G-module M whose support cone coincides with W.*

The statement of the corollary is an extension of the "realization" theorem for support varieties of finite dimensional modules (see [**Car2**] for finite groups, [**FP2**] for restricted Lie algebras, [**SFB2**] for arbitrary infinitesimal groups). There are many different conical closed under specialization subsets with the same closure: for example, any union of infinitely many lines through the origin in $\mathbb{A}^2$ is a conical closed under specialization non-closed subset with the closure $\mathbb{A}^2$. The theorem, thus, demonstrates that $V(G)_M$ is a "finer" invariant than one taking values in closed subsets (e.g. $V(\mathrm{Ann}_{H^{ev}(G,k)}(\mathrm{Ext}^*_G(M,M))))$.

The proof of the theorem given below is adapted to our case from the proof of the analogous result for elementary abelian groups in [**BCR2**].

Proof. Let s be a point in W. Since W is conical closed under specialization, the smallest closed conical subvariety of $V(G)$ containing s is contained in W. Denote this subvariety by V_s. By the "realization" theorem for finite dimensional modules (cf. [**SFB2**, 7.5]), there exists a finite dimensional G-module M such that $V(G)_M = V_s$. By the definition of $\mathcal{C}_W$, we have that $M \in \mathcal{C}_W$, which is equivalent to the fact that $M \otimes E(W) \cong M$ in $\mathrm{StMod}(G)$ (cf. Prop. 4.2.12). The "tensor product property" implies that $V_s = V(G)_M \subset V(G)_{E(W)}$. Since $s \in V_s$ by the construction of V_s, the inclusion $W \subset V(G)_{E(W)}$ follows.

To prove the other inclusion, choose $s \notin W$. By Theorem 4.2.10, $E(W) = \varinjlim_{i \in I} M_i$, where M_i are finite dimensional modules such that $V(G)_{M_i} \subset W$ for all i. Let $\mathbb{G}_{a(r)} \otimes k(s) \to G \otimes k(s)$ be the one-parameter subgroup corresponding to the point s. Since $s \notin V(G)_{M_i}$ for any $i \in I$, the restriction of $M_i \otimes k(s)$ to $k(s)[u_{r-1}]/(u^p_{r-1}) \subset k(s)[\mathbb{G}_{a(r)}]^\#$ (see §2 for notation) is always projective. Then the restriction of $E(W) \otimes k(s)$ to the same subalgebra is projective as a filtered colimit of projective modules (we also use that restriction commutes with colimits). Thus, $E(W) \downarrow_{k(s)[u_{r-1}]/(u^p_{r-1})}$ is projective, which implies that $s \notin V(G)_{E(W)}$. The statement follows. $\square$

Recall that for a non-zero point $s \in V(G)$ corresponding to a graded prime ideal $\mu_s \in k[V(G)]$, we denote by $L(s)$ the minimal conical subset containing s with 0 removed. Alternatively, $L(s) = \{\mu \in \operatorname{Spec} k[V(G)] : \mu$ is not homogeneous, $\mu_s \subset \mu$ and $\operatorname{ht}(\mu) = \operatorname{ht}(\mu_{\mathrm{s}}) + 1\} \cup \{\mathrm{s}\}$ (cf. Ex. 3.2.2). Let s be the generic point of a closed irreducible conical subset V of $V(G)$. Let further $V' = V \backslash \{s\}$ and let $\widetilde{V'}$ be the maximal conical closed under specialization subset in V'. It is easy to see that

$$\widetilde{V'} = V' \backslash L(s).$$

The thick subcategory $C_{V'} \subset \operatorname{stmod}(G)$, corresponding to V', coincides with the thick subcategory $C_{\widetilde{V'}}$ and, therefore, we have stable isomorphisms:

$$E(V') \cong E(\widetilde{V'}),\ F(V') \cong F(\widetilde{V'}).$$

Applying the theorem above to $E(\widetilde{V'})$, we get

$$V(G)_{E(V')} = \widetilde{V'}.$$

Thanks to our description of $\widetilde{V'}$, we can rewrite the last formula as

$$V(G)_{E(V')} = V' \backslash L(s).$$

Now we can describe support cones of F and κ - modules. We shall denote by W^c the complement of any subset W of $V(G)$.

Corollary 4.3.4. *For a conical closed under specialization subset W of $V(G)$ we have*

$$V(G)_{F(W)} = W^c \cup 0.$$

Furthermore, if V is an irreducible closed conical subset of $V(G)$ and s is the generic point of V, then

$$V(G)_{\kappa(V)} = L(s) \cup 0.$$

PROOF. The existence of the exact triangle $T(W) : E(W) \to k \to F(W) \to \Omega^{-1}E(W)$ implies that

$$V(G) \subset V(G)_{E(W)} \cup V(G)_{F(W)}$$

(cf. Theorem 3.3.2.6). Proposition 4.2.12.2 asserts that $E(W) \otimes F(W)$ is projective and hence

$$V(G)_{E(W)} \cap V(G)_{F(W)} = 0.$$

We conclude that $V(G)_{F(W)} = W^c \cup 0$.

The second statement follows immediately from the "tensor product property" and the definition of $\kappa(V)$ as $E(V) \otimes F(V')$.

□

For a conical subset W in $V(G)$ we denote by **Proj W**, the "projectivization" of W, the set of points in W which correspond to homogeneous prime ideals of $k[V(G)]$

excluding the augmentation ideal. Proj W can be viewed as a subset of the scheme Proj $k[V(G)]$.

There is 1-1 correspondence between conical subsets of $V(G)$ and their "projectivizations", i.e. a conical subset is completely determined by its homogeneous ideals. Therefore, the standard properties of support cones, described in Theorem 3.3.2, apply to their "projectivizations".

In view of this remark the next theorem is a straightforward application of the above corollary.

Theorem 4.3.5. *Let M be a G-module. Then*

$$\text{Proj } V(G)_M = \{s \in \text{Proj } k[V(G)] : M \otimes \kappa(s) \textit{ is not projective as a } G\textit{-module}\}.$$

Proof. Let s be a homogeneous prime ideal in $k[V(G)]$ such that $M \otimes \kappa(s)$ is not projective as a G-module. Then $\operatorname{Proj} V(G)_{M\otimes\kappa(s)} = \operatorname{Proj} V(G)_M \cap \operatorname{Proj} V(G)_{\kappa(s)}$ is non-empty. Since $\operatorname{Proj} V(G)_{\kappa(s)} = \operatorname{Proj}(L(s)\cup 0) = \{s\}$ in view of the Corollary 4.3.4 above, we conclude that $s \in \operatorname{Proj} V(G)_M$.

Conversely, if $s \in \operatorname{Proj} V(G)_M$, then $\operatorname{Proj} V(G)_{M\otimes\kappa(s)}$ is non-empty, which implies that $M \otimes \kappa(s)$ is not projective as a G-module. $\square$

Remark 1. *We can restate the previous theorem in terms of the affine support cones using the following notation: for any prime ideal $\mu \subset k[V(G)]$ denote by hom(μ) the maximal homogeneous prime ideal contained in μ. Note that ht(hom$(\mu)) =$ ht$(\mu)-1$ unless μ itself is homogeneous. Any conical subset containing μ contains hom(μ) and* *vice versa. Together with the theorem above this observation implies the following description of* $V(G)_M$*:*

$$V(G)_M = \{s \in V(G) : M \otimes \kappa(hom(s)) \textit{ is not projective as a } G\textit{-module}\}.$$

As another application of the Corollary 4.3.4, we can generalize our "realization" statement to arbitrary conical sets. We shall utilize the notation $\hom(s)$ introduced in the remark above.

Corollary 4.3.6. *Any conical subset of* $V(G)$ *can be realized as a support cone of a* G*-module.*

Proof. Let W be a conical subset of $V(G)$. For any $s \in W$, W contains $\hom(s)$. Furthermore, by the definition of conical subset, for any point s corresponding to a homogeneous prime ideal, W contains the entire set $L(s)$. We conclude that

$$W = \bigcup_{s \in \operatorname{Proj} W} L(s) \cup 0$$

and, therefore, W is the support cone of the module $\bigoplus\limits_{s \in \operatorname{Proj} W} \kappa(s)$. □

4.4. Applications: induction revisited, complexity.

As an application of Theorem 4.3.5 we are going to show that $V(G)_{\operatorname{Ind}_H^G(M)} \subset V(H)_M$ for an arbitrary H-module M, where H is a subgroup scheme of G. Although for finite dimensional modules this follows from the cohomological description of the support variety of M and Generalized Frobenius reciprocity, in the infinite dimensional case this approach is not available due to the lack of the cohomological description. We also give an easy argument which shows that the equality takes place when H is a unipotent group scheme. This might be as good as it gets, since for a non-unipotent H one can induce a non-projective module and get a projective one (so that the support variety definitely becomes smaller). An example of this is the Steinberg module St_r, the only projective simple module of the rth Frobenius kernel of some reductive algebraic group G. We have $St_r = \mathrm{Ind}_{B_{(r)}}^{G_{(r)}} k_{(p^r-1)\rho}$, where $k_{(p^r-1)\rho}$ is the one-dimensional $B_{(r)}$-module corresponding to the weight $(p^r-1)\rho$ (cf. [**Jan**, II.3]), and the variety of this 1-dimensional module is $V(B_{(r)}) \neq 0 = V(G_{(r)})_{St_r}$.

We shall need the following general fact about Rickard idempotents. The proof is merely a repetition of the one in [**BCR2**].

Lemma 4.4.1. *Let G be an infinitesimal group scheme, H be a closed subgroup scheme of G and W be a subset of $V(G)$. Let $i_* : V(H) \hookrightarrow V(G)$ be the embedding of schemes induced by the inclusion $i : H \hookrightarrow G$. Then the following two exact triangles in StMod(H) are stably isomorphic:*

$$T(i_*^{-1}(W)) : E(i_*^{-1}(W)) \to k \to F(i_*^{-1}(W)) \to \Omega^{-1}E(i_*^{-1}(W))$$

and

$$T(W) \downarrow_H : E(W) \downarrow_H \to k \to F(W) \downarrow_H \to \Omega^{-1}E(W) \downarrow_H .$$

Proof. We have to show that $T(W)\downarrow_H$ satisfies universal properties of the exact triangle $T(i_*^{-1}(W))$. Since $E(W) \in \vec{\mathcal{C}}_W$, Prop. 3.3.2.2 implies that $E(W)\downarrow_H \in \vec{\mathcal{C}}_{i_*^{-1}(W)}$. To check that $F(W)\downarrow_H$ is $\mathcal{C}_{i_*^{-1}(W)}$-local we note that the fact that $V(G)_{\mathrm{Ind}_H^G(M)} \subset V(H)_M$ for a finite dimensional H-module M (see, for example, [**NPV**, 2.3.1(b)]) implies that for any H-module M in $C_{i_*^{-1}(W)}$, we have $\mathrm{Ind}_H^G(M) \in \mathcal{C}_W$. Recall that for any finite dimensional G-module N, $V(G)_N = V(G)_{N^\#}$, where $N^\#$ is the k-linear dual of N. Hence, an isomorphism $\mathrm{Coind}_H^G(M) = (\mathrm{Ind}_H^G(M^\#))^\#$ (cf. [**Jan**, I.8.15]) implies that $V(G)_{\mathrm{Coind}_H^G(M)} \subset V(H)_M$ for a finite dimensional H-module M. Applying the fact that $F(W)$ is $\mathcal{C}_W$-local, we get

$$\underline{\mathrm{Hom}}_H(M, F(W)\downarrow_H) = \underline{\mathrm{Hom}}_G(\mathrm{Coind}_G^H(M), F(W)) = 0.$$

for any finite dimensional H-module M. Thus, $F(W)\downarrow_H$ is $\mathcal{C}_{i_*^{-1}(W)}$-local. In view of Remark 4.2.11, we conclude that $T(i_*^{-1}(W)) \cong T(W)\downarrow_H$.

□

Corollary 4.4.2. *Let G be an infinitesimal group scheme and H be a closed subgroup scheme of G. Let M be an H-module. Then*

$$V(G)_{Ind_H^G(M)} \subset V(H)_M.$$

Proof. The embedding of group schemes $i : H \subset G$ induces a closed embedding of affine schemes $i_* : V(H) \hookrightarrow V(G)$ (cf. [**SFB2**, 5.4]). We identify $V(H)$ with its image in $V(G)$.

Let $M = \varinjlim_{i \in I} M_i$, where M_i are finite dimensional H-modules. Then

$$\mathrm{Ind}_H^G(M) = \varinjlim_{i \in I} \mathrm{Ind}_H^G(M_i)$$

and, therefore,

$$V(G)_{\mathrm{Ind}_H^G(M)} \subset \bigcup_{i \in I} V(G)_{\mathrm{Ind}_H^G(M_i)} \subset V(H).$$

The last inclusion holds because the assertion of the corollary is known for finite dimensional modules (cf. [**NPV**, 2.3.1(b)]).

To prove the corollary it now suffices to check that for any point $s \in V(G)_{\mathrm{Ind}_G^H(M)} \subset V(H)$, corresponding to a homogeneous prime ideal in $k[V(G)]$, s is contained in $V(H)_M$. Let V be the Zariski closure of s. Since $s \in V(H)$, and the latter is closed in $V(G)$, we have $i_*^{-1}(V) = V \cap V(H) = V$. Lemma 4.4.1 implies that $\kappa(i_*^{-1}(V))$ is stably isomorphic to $\kappa(V)\downarrow_H$.

Applying Theorem 4.3.5 we get that $\mathrm{Ind}_G^H(M) \otimes \kappa(V)$ is not projective (= not injective), since $s \in V(G)_{\mathrm{Ind}_G^H(M)}$. By the tensor identity,

$$\mathrm{Ind}_G^H(M) \otimes \kappa(V) \cong \mathrm{Ind}_H^G(M \otimes \kappa(V)\downarrow_H).$$

Since induction takes injectives to injectives, we conclude that $M \otimes \kappa(V)\downarrow_H \cong M \otimes \kappa(i_*^{-1}(V))$ is not injective. Since s is a point in $V(H)$, it is still the generic point of $i_*^{-1}(V)$. Thus, $M \otimes \kappa(s)$ (where $\kappa(s)$ is now constructed in $\mathrm{StMod}(H)$) is not projective which implies, using Theorem 4.3.5 once again, that $s \in V(H)_M$.

$\square$

Proposition 4.4.3. *Let $i : H \hookrightarrow G$ be a closed embedding of infinitesimal group schemes and let further H be unipotent. Then for any H-module M,*

$$V(G)_{Ind_H^G(M)} = V(H)_M.$$

Proof. Note that an H-module N is projective if and only if $\mathrm{Ind}_H^G(N)$ is projective. The "only if" part follows from the general fact that Induction takes injectives to injectives (hence, projectives to projectives). To prove the the other direction, we use Shapiro's lemma: $H^*(G, \mathrm{Ind}_H^G(N)) = H^*(H, N)$. If $\mathrm{Ind}_H^G(N)$ is projective, then $H^*(H, N) = 0$ for $* > 0$. Since H is unipotent, N is projective. Now we can prove the equality of varieties in 4 easy steps: $s \in V_G(Ind_H^G M) \Longleftrightarrow Ind_H^G M \otimes \kappa(s)$ is not projective $\Longleftrightarrow$
$Ind_H^G(M \otimes \kappa(s))$ is not projective (tensor identity) $\Longleftrightarrow$
$M \otimes \kappa(s)$ is not projective over $H \Longleftrightarrow s \in V_H(M)$. □

Proposition 4.4.4. *Let W be a conical closed under specialization subset in $V(G)$. Then $\vec{\mathcal{C}}_W = \{M \in StMod(G) : V(G)_M \subset W\}$.*

Proof. Suppose $M \in \vec{\mathcal{C}}_W$. We need to show that $V(G)_M \subset W$. It suffices to check this inclusion for the points corresponding to homogeneous prime ideals. By the definition of $\vec{\mathcal{C}}_W$, M is stably isomorphic to $\varinjlim_{i \in I} M_i$ for some finite dimensional modules M_i whose varieties are contained in W. Let s be a point in $\mathrm{Proj}\, V(G)$ which does not belong to W. Then the restriction of $M_i \otimes k(s)$ to $k(s)[u_{r-1}]/u_{r-1}^p \subset k(s)[\mathbb{G}_{a(r)}]^\#$, where $\mathbb{G}_{a(r)} \otimes k(s) \to G \otimes k(s)$ is the one-parameter subgroup defined by the point s, is projective for all i. Since restriction commutes with filtered colimits, and a filtered colimit of projective modules is projective, we conclude that M restricted to the same subalgebra $k(s)[u_{r-1}]/u_{r-1}^p$ is projective. Thus, $s \notin V(G)_M$. The inclusion $V(G)_M \subset W$ follows.

Next assume that $V(G)_M \subset W$. By the "tensor product property" and Corollary 4.3.4, $M \otimes F(W)$ is projective. This implies, by tensoring the exact triangle $T(W)$ with M, that $M \otimes E(W) \cong M$ in StMod(G). Let $M \cong \varinjlim_{i \in I} M_i$ for some finite dimensional modules M_i and $E(W) \cong \varinjlim_{j \in J} N_j$ for some finite dimensional modules N_j, whose support varieties $V(G)_{N_j}$ are contained in W (the latter being possible by Theorem 4.2.10.I.(ii)). Then $M \cong M \otimes E(W) \cong \varinjlim_{(i,j) \in I \times J} M_i \otimes N_j$ and the variety of $M_i \otimes N_j$, $V(G)_{M_i \otimes N_j}$, is contained in $V(G)_{N_j}$ which, in turn, is contained in W for all pairs $(i, j) \in I \times J$. Thus, $M \in \vec{\mathcal{C}}_W$.

□

The following corollary is an immediate application of the proposition above to the closure under specialization of $V(G)_M$, $\mathrm{Cs}\,(V(G)_M)$.

Corollary 4.4.5. *For any G-module M there exists a filtered system of finite dimensional G-modules $\{M_i\}_{i \in I}$ such that*

(i) $M \cong \varinjlim_{i \in I} M_i$

(ii) $V(G)_{M_i} \subset Cs(V(G)_M)$.

Recall that complexity of a finite dimensional module M is defined to be the growth of the minimal projective resolution of M. It is proved to be equal to the dimension of the support variety of M ([**AE**]). In [**BCR1**] the following extension of the definition of complexity for infinite dimensional modules is given:

Definition 4.4.6. *An arbitrary G-module M is said to have complexity c, denoted $c(M)$, if it can be realized as a filtered colimit of finite dimensional modules of complexity c but not lower.*

For a subset W of $V(G)$ we define the *subset dimension* of W as follows:

$$\mathrm{s.dim}(W) \stackrel{def}{=} \max_{s \in W} \dim(\overline{s}).$$

Note that $\mathrm{s.dim}(W) = \mathrm{s.dim}(\mathrm{Cs}\,(W))$. In particular, for a closed subvariety V, its "subset dimension" coincides with the usual Krull dimension.

Using the notion of "subset dimension" we can formulate an alternative description of the complexity of an infinite dimensional module similar to the one mentioned above for the finite dimensional case:

Corollary 4.4.7. $c(M) = s.\dim(V(G)_M)$

Proof. Let $d = \mathrm{s.dim}(V(G)_M)$. The inequality $c(M) \leq d$ follows immediately from Corollary 4.4.5.

Suppose $c(M) < d$. By our definition of subset dimension there exists a point $s \in V(G)_M$ such that $\dim(\overline{s}) = d$. Let $\mathbb{G}_{a(r)} \otimes k(s) \to G \otimes k(s)$ be the one-parameter subgroup corresponding to s. According to our definition of complexity, we can realize M as $\varinjlim_{i \in I} M_i$ for some finite dimensional modules M_i whose varieties have dimension no greater than $c(M)$. Then, clearly, $s \notin V(G)_{M_i}$, which implies that $M_i \otimes k(s)$ restricted to $k(s)[u_{r-1}]/u_{r-1}^p \subset k(s)[\mathbb{G}_{a(r)}]^\#$ is projective for any $i \in I$. Hence, the restriction of $M \otimes k(s)$ to the same subalgebra is also projective as a filtered colimit of projective modules. By the definition of a support cone, $s \notin V(G)_M$. The inequality in question follows. □

To conclude, we give as promised an example of the failure of the "tensor product property" for the extension of the cohomological definition of the "support", for which we employ the construction of Rickard idempotents in a special case of a hypersurface defined by a single homogeneous element.

Example 4.4.8. *Let $\xi \in H^n(G,k)$, where n is a positive even integer. Assume further that ξ is not nilpotent. Denote by $<\xi>$ the ideal generated by ξ and by $V(<\xi>)$ the variety of this ideal, i.e. $V(<\xi>) = \{\mu \in \mathit{Spec}\, H^{ev}(G,k) : \xi \in \mu\}$. Let F_ξ be the filtered colimit of the sequence*

$$k \to \Omega^{-n}k \to \Omega^{-2n}k \to \dots$$

where each map corresponds to ξ via the natural isomorphism

$$H^n(G,k) \cong \underline{Hom}(\Omega^{-rn}k, \Omega^{-(r+1)n}k).$$

F_ξ is well-defined up to a stable isomorphism and comes equipped with a natural map from k, $k \to F_\xi$. Complete this map to an exact triangle in StMod(G):

$$E_\xi \to k \to F_\xi \to \Omega^{-1}E_\xi.$$

It can be shown that this exact triangle satisfies the universal properties of the triangle corresponding to the thick subcategory $C_{V(<\xi>)}$, $T(V(<\xi>))$ *and, thus, is stably isomorphic to it. Hence,* $V(G)_{E_\xi} = V(<\xi>)$ *and* $V(G)_{F_\xi} = V(<\xi>)^c \cup 0$. *In particular,* E_ξ *is not projective.*

The cohomology of F_ξ *can be computed as the filtered colimit of the sequence*

$$H^*(G,k) \to H^*(G,\Omega^{-n}k) \to H^*(G,\Omega^{-2n}k) \to \dots$$

which is equivalent to

$$H^*(G,k) \xrightarrow{\times\xi} H^{*+n}(G,k) \xrightarrow{\times\xi} H^{*+2n}(G,k) \xrightarrow{\times\xi} \dots$$

The direct limit of this sequence is isomorphic to $H^*(G,k)[1/\xi]$. *The inclusion*

$$Ann_{H^{ev}(G,k)}(Ext^*_G(F_\xi,F_\xi)) \subset Ann_{H^{ev}(G,k)}(Ext^*_G(k,F_\xi)) =$$
$$Ann_{H^{ev}(G,k)}(H^*(G,k)[1/\xi]) = 0$$

implies that $|G|_{F_\xi} = V(Ann_{H^{ev}(G,k)}(Ext^*_G(F_\xi,F_\xi))) = |G|$.

Since $F_\xi \otimes E_\xi$ *is projective, the "tensor product property" for "cohomological supports", if valid, would imply that*

$$0 = |G|_{F_\xi\otimes E_\xi} = |G|_{F_\xi} \cap |G|_{E_\xi} = |G| \cap |G|_{E_\xi} = |G|_{E_\xi}$$

which, in view of Proposition 3.4.1, contradicts the fact that E_ξ *is not projective.*

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