SUPPORT CONES FOR INFINITESIMAL GROUP SCHEMES

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Abstract. We verify that the construction of *support cone* for infinite dimensional modules, introduced in [14], extends to modules over any infinitesimal group scheme and satisfies all good properties of support varieties for finite dimensional modules, thereby extending the results of the author for infinite dimensional modules of Frobenius kernels [14]. We show, using an alternative description of support cones in terms of Rickard idempotents, that for an algebraic group G over an algebraically closed field k of positive characteristic p and a point s in the cohomological support variety of a Frobenius kernel $G_{(r)}$, the orbit $G \cdot s$ can be realized as a support cone of a rational G-module.

0. Introduction

The theory of support varieties for finite dimensional modules for finite groups ([1],[5],[2]) or restricted Lie algebras ([7],[8],[11]) has drawn considerable attention over the last twenty years. One of its most attractive features is the elegant connection it provides between the cohomological behaviour and intrinsic representation-theoretic properties of a module. This connection proves to be crucial in establishing basic properties of support varieties such as good behavior with respect to tensor products or the property of detecting projectivity of a module on its support variety.

Motivated by the work of Benson, Carlson and Rickard ([4]) for finite groups, we seek to associate a geometric object (the "support cone") to an arbitrary module M of an infinitesimal group scheme G. It turns out that the original, cohomological, approach to support varieties does not provide a good generalization for infinite dimensional modules. For this reason, in our study of geometric properties of infinite dimensional modules for infinitesimal group schemes we take the representation-theoretic approach developed in [17],[18] and define the *support cone* of a module M in purely representation-theoretic terms. The lack of a cohomological description, though, makes it more difficult to show that one of the most fundamental properties of support varieties, detection of projectivity, is satisfied by our construction. The proof of this fact for infinitesimal group schemes, built upon an earlier result for Frobenius kernels, occupies §2.

In the spirit of work of Benson, Carlson and Rickard for infinite dimensional modules for finite groups, we provide an equivalent description of support cones in terms of Rickard idempotents, the universal modules corresponding to tensor-ideal thick subcategories of the stable module category. In the last section we give an example of the interplay of these two approaches which allows us to prove certain "realization" results.

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This short note is complementary to [14], where the author studied possible extensions of the notion of support variety to infinite dimensional modules for Frobenius kernels. We have tried to make this note self-contained by recalling all the main ingredients going into the definition of support cones and construction of Rickard idempotents. At the same time, many technical details closely mimic those provided in [14] and are simply omitted here.

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Throughout the paper k will denote an algebraically closed field of positive characteristic p. All group schemes are assumed to be defined over k.

1. Support cones for Frobenius kernels

In this section we recall various definitions and results concerning infinitesimal group schemes, leading up to the construction of support cones.

Definition 1.1. A finite group scheme G over k is a functor $G : \{\text{k - alg}\} \to \{\text{groups}\}$ from the category of finitely generated commutative k-algebras to the category of groups which is represented by a finite-dimensional commutative k-algebra (denoted $k[G]$). A finite group scheme G is infinitesimal if $k[G]$ is a local ring.

Let I be the augmentation ideal of the coordinate algebra $k[G]$ of an infinitesimal group G. The *height* of G is the minimal integer r such that for any $x \in I$, $x^{p^r} = 0$.

Example 1.2. Let G be an affine algebraic group. We denote by $G_{(1)}$ the scheme-theoretic kernel of the Frobenius map

$$G_{(1)} = \ker\{F : G \to G^{(1)}\},$$

where $G^{(1)}$ is the base change of G via the Frobenius map (i.e., the p-th power map) on k. For example, $GL_{n(1)}$ is the group scheme given by $GL_{n(1)}(A) = \{(a_{ij}) \in M_n(A) : a_{ij}^p = \delta_{ij} \text{ for all } 1 \leq i, j \leq n\}$ for any finitely generated commutative k-algebra A. The r-th Frobenius kernel of G, denoted $G_{(r)}$, is the scheme-theoretic kernel of $F^r : G \to G^{(r)}$. The height of $G_{(r)}$ is precisely r. Moreover, any infinitesimal group scheme of height r can be embedded into the r-th Frobenius kernel of GL_n for an appropriate n.

Example 1.3. For the additive group $\mathbb{G}_{(a)}$, we have $k[\mathbb{G}_{a(r)}] = k[T]/T^{p^r}$. We fix notation for the dual algebra $k[\mathbb{G}_{a(r)}]^\#$ which will be used later in the text. Let $v_0, \ldots, v_{p^r-1}$ be the basis of $k[\mathbb{G}_{a(r)}]^\# = (k[T]/T^{p^r})^\#$ dual to the basis of $k[T]/T^{p^r}$ consisting of powers of T. Denote v_{p^i} by u_i. Then

$$k[\mathbb{G}_{a(r)}]^\# = k[u_0, \ldots, u_{r-1}]/(u_0^p, \ldots, u_{r-1}^p).$$

A *1-parameter subgroup* of height r of an affine group scheme G is a homomorphism $\mathbb{G}_{a(r)} \to G$. We say that a 1-parameter subgroup is injective if this homomorphism is a closed embedding of group schemes.

For a field extension K/k we shall use the subscript $_K$ to denote the extension of scalars from k to K.

1-parameter subgroups constitute a detecting family of small subgroups for an infinitesimal group, analogous to the family of "shifted cyclic subgroups" of an elementary abelian p-group. We make this detecting property precise in the following theorem, which will be generalized to all infinitesimal group schemes in the next section.

Theorem 1.4. [14, 1.7] *Let $G_{(r)}$ be the r-th Frobenius kernel of an algebraic group G and let M be a $G_{(r)}$-module such that for any field extension K/k and any subgroup scheme $H_K \hookrightarrow G_{(r),K}$ isomorphic to $\mathbb{G}_{a(s),K}$, $s \leq r$, the restriction of M_K to H_K is projective. Then M is projective as a $G_{(r)}$-module.*

Remark 1.5. Considering field extensions is essential here when the module in question is allowed to be infinite dimensional. We refer the reader to [4] or [14] for examples of modules which are projective restricted to every 1-parameter subgroup (or cyclic shifted subgroup in the case of a module for a finite group) defined over the ground field k, but not projective as $G_{(r)}$-modules.

Following [17], we define a functor

$$V_r(G) : (\text{comm } k\text{-alg}) \to (\text{sets})$$

by setting

$$V_r(G)(A) = \operatorname{Hom}_{Gr/A}(\mathbb{G}_{a(r)} \otimes_k A, G \otimes_k A).$$

This functor is representable by an affine scheme of finite type over k, which we will still denote $V_r(G)$. Indeed, the following holds:

Theorem 1.6. [17, 1.5] *The functor $V_r(G)$ is represented by an affine scheme of finite type over k. Moreover, $G \to V_r(G)$ is a covariant functor from the category of affine group schemes over k of height $\leq r$ to the category of affine schemes of finite type over k, which takes closed embeddings to closed embeddings.*

Thus, a point $s \in V_r(G)$ defines a canonical $k(s)$-rational point of $V_r(G)$ and the associated 1-parameter subgroup defined over $k(s)$:

$$\nu_s : \mathbb{G}_{a(r),k(s)} \to G_{k(s)}.$$

For G an infinitesimal group scheme of height r, we will write $V(G) = V_r(G)$. We can now define the *support cone* of a G-module.

Definition 1.7. Let G be an infinitesimal k-group scheme of height r and let M be a G-module. The *support cone* of M is the following subset of $V(G)$:

$$V(G)_M = \{s \in V(G) : M_{k(s)} \text{ is not projective as a module for the subalgebra}$$
$$k(s)[u_{r-1}]/(u_{r-1}^p) \subset k(s)[u_0, \ldots, u_{r-1}]/(u_0^p, \ldots, u_{r-1}^p) = k(s)[\mathbb{G}_{a(r)}]^{\#}\}.$$

We remark that by a "subset" of an affine scheme $X = \operatorname{Spec} A$ we mean simply a set of prime ideals in A. We shall often use the same notation for a point in X and the corresponding prime ideal in A. The algebra $k[V(G)]$ is graded connected, thus, there is a well-defined map $V(G) - \{0\} \xrightarrow{\pi} \operatorname{Proj} V(G)$, where $\operatorname{Proj} V(G)$ denotes the projective spectrum of $k[V(G)]$. We call a subset of $V(G)$ *conical* if it coincides with a full preimage of a subset in $\operatorname{Proj} V(G)$ with added $\{0\}$.

Definition 1.7 was first introduced in [18] for finite dimensional modules where it was further shown that $V(G)_M$ is a closed subvariety of $V(G)$ and, furthermore, is naturally homeomorphic to the cohomological support variety of M, i.e. the variety of the ideal $\operatorname{Ann}_{H^{ev}(G,k)}(\operatorname{Ext}^*_G(M,M))$ in $\operatorname{Specm} H^{ev}(G,k)$ (respectively $\operatorname{Specm} H^*(G,k)$ if $p = 2$), the maximal ideal spectrum of the cohomology algebra of G. In particular, $V(G)$ is naturally identified with $\operatorname{Spec} H^{ev}(G,k)$.

It is shown in [14] that support cones for Frobenius kernels satisfy most of the standard properties of support varieties, except for being closed.

Theorem 1.8. [14, 2.6] *Let $G_{(r)}$ be the r-th Frobenius kernel of an algebraic group G and M and N be $G_{(r)}$-modules. Support cones satisfy the following properties:*

(1) *$V(G)_M$ is a conical subset of $V(G)$.*
(2) *"Naturality." Let $f : H \to G$ be a homomorphism of infinitesimal group schemes of height $\leq r$. Denote by $f_* : V(H) \to V(G)$ the associated morphism of schemes. Then*

$$f_*^{-1}(V(G)_M) = V(H)_M,$$

where M is considered as an H-module via f.
(3) *$V(G)_M = 0$ if and only if M is projective.*
(4) *"Tensor product property." $V(G)_{(M\otimes N)} = V(G)_M \cap V(G)_N$.*
(5) *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of G-modules. Then for any permutation (ijk) of (123) we have*

$$V(G)_{M_i} \subset V(G)_{M_j} \cup V(G)_{M_k}.$$

(6) *For any collection of G-modules $\{M_\alpha\}$, we have*

$$V(G)_{\bigoplus\limits_\alpha M_\alpha} = \bigcup_\alpha V(G)_{M_\alpha}.$$

2. Detection of projectivity.

In this section we prove that for any infinitesimal group scheme G and any G-module M, projectivity of M can be detected via restricting to 1-parameter subgroups of G, building upon Theorem 1.4.

Proposition 2.1. *Let G be an infinitesimal group scheme and let M be a G-module such that for any field extension K/k and any subgroup scheme $H_K \hookrightarrow G_K$ isomorphic to $\mathbb{G}_{a(r),K}$, the restriction of M_K to H_K is projective. Let $G \hookrightarrow G'$ be a closed embedding of G into some Frobenius kernel of the same height as G. Then $Ind_G^{G'}(M)$ is projective as a G'-module.*

Proof. By Theorem 1.8.3, it suffices to show that $V(G')_{\mathrm{Ind}_G^{G'}(M)} = 0$. Let s be a point in $V(G')$, corresponding to a 1-parameter subgroup $\nu_s : \mathbb{G}_{a(r),K} \to G'_K$. By definition of the support cone, to show that $s \notin V(G')_{\mathrm{Ind}_G^{G'}(M)}$, we have to show that the restriction of $(\mathrm{Ind}_G^{G'}(M))_K$ to $K[u_{r-1}]/(u_{r-1}^p) \subset K[u_0, \dots, u_{r-1}]/(u_0^p, \dots, u_{r-1}^p) = K[\mathbb{G}_{a(r)}]^\#$ via ν_s is projective. Since Ind commutes with extension of scalars, we may assume that everything is defined over the ground field k, i.e. $K = k$. By lowering the height of $\mathbb{G}_{a(r)}$, if necessary, we can further assume that the map ν_s is an embedding. This will involve factoring through the "projection" map $p_{r,r'} : \mathbb{G}_{a(r)} \to \mathbb{G}_{a(r')}$, which takes generator u_{r-1} of $k[\mathbb{G}_{a(r)}]^\#$ to the generator $u_{r'-1}$ of $k[\mathbb{G}_{a(r')}]^\#$, and, thus, will not affect the projectivity of the restriction of $\mathrm{Ind}_G^{G'}(M)$ to the corresponding subalgebra.

Let $\mathbb{G}_{a(t)}$ be a 1-parameter subgroup of G defined as $G \cap \mathbb{G}_{a(r)} \subset G'$. Consider the following Cartesian square of group schemes:

$$\begin{array}{ccc} \mathbb{G}_{a(t)} & \hookrightarrow & G \\ \downarrow & & \downarrow \\ \mathbb{G}_{a(r)} & \xrightarrow{\nu_s} & G' \end{array}$$

Let $\Lambda = \mathrm{End}_k(M, M)$. Then Λ is an associative unital G-algebra. Observe that $H^1(\mathbb{G}_{a(t)}, \Lambda) = \mathrm{Ext}^1_{\mathbb{G}_{a(t)}}(M, M) = 0$, since the restriction of M to any injective 1-parameter subgroup of G is projective. Since $\mathbb{G}_{a(t)}$ is unipotent, vanishing of the first cohomology group implies that Λ is projective as a $\mathbb{G}_{a(t)}$-module. Thus, $\mathrm{Ind}_{\mathbb{G}_{a(t)}}^{\mathbb{G}_{a(r)}}(\Lambda)$, which is again an associative unital $\mathbb{G}_{a(r)}$ - algebra, is projective as a $\mathbb{G}_{a(r)}$-module.

The natural map of $\mathbb{G}_{a(r)}$-algebras $\mathrm{Ind}_G^{G'}(\Lambda) \to \mathrm{Ind}_{\mathbb{G}_{a(t)}}^{\mathbb{G}_{a(r)}}(\Lambda)$ (determined by the adjointness of Induction and Restriction functors) is surjective and has a nilpotent kernel (cf. [18, 4.3]). Denote the kernel by I. Projectivity of $\mathrm{Ind}_{\mathbb{G}_{a(t)}}^{\mathbb{G}_{a(r)}}(\Lambda)$ as a $\mathbb{G}_{a(r)}$-module implies that it is projective restricted further to $k[u_{r-1}]/(u_{r-1}^p)$. Thus,

$$H^*(k[u_{r-1}]/(u_{r-1}^p), \mathrm{Ind}_{\mathbb{G}_{a(t)}}^{\mathbb{G}_{a(r)}}(\Lambda)) = 0 \text{ for } * > 0.$$

Therefore, the long exact sequence in cohomology corresponding to the short exact sequence

$$0 \to I \to \mathrm{Ind}_G^{G'}(\Lambda) \to \mathrm{Ind}_{\mathbb{G}_{a(t)}}^{\mathbb{G}_{a(r)}}(\Lambda) \to 0$$

of modules gives an isomorphism

$$H^*(k[u_{r-1}]/(u_{r-1}^p), \mathrm{Ind}_G^{G'}(\Lambda)) \cong H^*(k[u_{r-1}]/(u_{r-1}^p), I)$$

in positive degrees. The ideal I is nilpotent, so the algebra without unit $H^*(k[u_{r-1}]/(u_{r-1}^p), I)$ is also nilpotent. The isomorphism above implies that the augmentation ideal $H^{*>0}(k[u_{r-1}]/(u_{r-1}^p), \mathrm{Ind}_G^{G'}(\Lambda))$ is nilpotent. On the other hand, the map of algebras $k \to \mathrm{Ind}_G^{G'}(\Lambda)$ induces an action of $H^{ev}(k[u_{r-1}]/(u_{r-1}^p), k) \cong k[x]$ (where x is a generator in degree two) on $H^*(k[u_{r-1}]/(u_{r-1}^p), \mathrm{Ind}_G^{G'}(\Lambda))$. The action of x, in particular, induces a periodicity isomorphism

$$H^i(k[u_{r-1}]/(u_{r-1}^p), \mathrm{Ind}_G^{G'}(\Lambda)) \cong H^{i+2}(k[u_{r-1}]/(u_{r-1}^p), \mathrm{Ind}_G^{G'}(\Lambda)) \text{ for all } i > 0$$

Since image of x in $H^2(k[u_{r-1}]/(u_{r-1}^p), \mathrm{Ind}_G^{G'}(\Lambda))$ under the map of algebras $H^{ev}(k[u_{r-1}]/(u_{r-1}^p), k) \to H^*(k[u_{r-1}]/(u_{r-1}^p), \mathrm{Ind}_G^{G'}(\Lambda))$ is nilpotent, the periodicity isomorphism induced by the action of x is trivial. Hence,

$$H^{*>0}(k[u_{r-1}]/(u_{r-1}^p), \mathrm{Ind}_G^{G'}(\Lambda)) = 0.$$

There is a natural action of Λ on M compatible with the G-structure. This induces an action of $\mathrm{Ind}_G^{G'}(\Lambda)$ on $\mathrm{Ind}_G^{G'}(M)$ compatible with their structure as G'-modules, and, therefore, $k[u_{r-1}]/(u_{r-1}^p)$-modules. Hence, the action of $H^{ev}(k[u_{r-1}]/(u_{r-1}^p), k) \cong k[x]$ on $H^*(k[u_{r-1}]/(u_{r-1}^p), \mathrm{Ind}_G^{G'}(M))$ factors through the action of $H^*(k[u_{r-1}]/(u_{r-1}^p), \mathrm{Ind}_G^{G'}(\Lambda))$. Since the latter vanishes in positive degrees, the action of x on $H^*(k[u_{r-1}]/(u_{r-1}^p), \mathrm{Ind}_G^{G'}(M))$ is trivial. On the other hand, it induces a periodicity isomorphism. We conclude that $H^1(k[u_{r-1}]/(u_{r-1}^p), \mathrm{Ind}_G^{G'}(M)) = 0$ and, hence, $\mathrm{Ind}_G^{G'}(M)$ is projective as a $k[u_{r-1}]/(u_{r-1}^p)$-module. Hence, $s \notin V(G)_{\mathrm{Ind}_G^{G'}(M)}$. The statement follows.

□

Theorem 2.2. *Let G be an infinitesimal group scheme and let M be a G-module such that for any field extension K/k and any subgroup scheme $H_K \hookrightarrow G_K$ isomorphic to $\mathbb{G}_{a(r),K}$ the restriction of M_K to H_K is projective. Then M is projective as a G-module.*

Proof. Embed G into some Frobenius kernel G'. By Proposition 2.1, $\mathrm{Ind}_G^{G'}(M)$ is a projective G'-module. Therefore, $H^*(G, M) = H^*(G', \mathrm{Ind}_G^{G'}(M)) = 0$ for $* > 0$. Applying the same argument to all modules of the form $M \otimes S^\#$ for all simple G-modules S, we get $\mathrm{Ext}_G^*(S, M) = 0$ for $* > 0$. Applying Lemma 1.2 in [14], we conclude that M is projective.

□

Theorem 2.3. *Let G be an infinitesimal group scheme, and let M, N be G-modules. Support cones $V(G)_M, V(G)_N$ satisfy properties (1)-(6) of Theorem 1.8.*

We omit the proof of this theorem since (1),(2) and (4)-(6) were proved in [14] and (3) is a straightforward application of Theorem 2.2.

3. Rickard idempotents.

Theorem 2.2 allows us to extend the description of support cones in terms of Rickard idempotents given in [14] for Frobenius kernels to any infinitesimal group scheme. We begin by briefly recalling the notion of Rickard idempotent modules ([15]) and then state Theorem 3.3 which provides an alternative description of the support cones. This approach to supports of infinite dimensional modules for finite groups was introduced by Benson, Carlson and Rickard in [4] and it works equally well in our context of infinitesimal group schemes. This section does not have any proofs since the existence of Rickard idempotent modules is a general statement about Bousfield localization (cf. [15], [16] or [10]) and the proof of Theorem 3.3 goes exactly as in the case of Frobenius kernels which is presented in [14].

We shall denote by StMod(G) the stable category of all G-modules. Recall that objects of $\mathrm{StMod}(G)$ are G-modules and maps are equivalence classes of G-module homomorphisms where two maps are equivalent if their difference factors through a projective G-module.

The fact that in the category of G-modules projectives are injectives and vice versa (cf. [9]) implies the existence of a triangulated structure on $\mathrm{StMod}(G)$. The shift operator in $\mathrm{StMod}(G)$ is given by the Heller operator $\Omega^{-1} : \mathrm{StMod}(G) \to \mathrm{StMod}(G)$ (cf., for example, [3] for the definition of Ω) and distinguished triangles come from short exact sequences in $\mathrm{Mod}(G)$.

We shall denote by $\mathrm{stmod}(G)$ the full triangulated subcategory of $\mathrm{StMod}(G)$ whose objects are represented by finite dimensional modules. A full triangulated subcategory $\mathcal{C}$ of $\mathrm{stmod}(G)$ (respectively $\mathrm{StMod}(G)$) is called *thick* if it is closed under taking direct summands (respectively taking direct summands and arbitrary direct sums). It is called *tensor-ideal* if it is closed under taking tensor products with any G-module. We shall use the notation $\underline{\mathrm{Hom}}$ for $\mathrm{Hom}_{\mathrm{StMod}}$ and "$\cong$" for stable isomorphisms.

Two modules are stably isomorphic (i.e. isomorphic in $\mathrm{StMod}(G)$) if and only if they become isomorphic after adding projective summands to them. This implies that support cones are well-defined in $\mathrm{StMod}(G)$.

Let $\mathcal{C}$ be a thick subcategory of stmod(G). Denote by $\vec{\mathcal{C}}$ the full triangulated subcategory of $\mathrm{StMod}(G)$ whose objects are filtered colimits of objects in $\mathcal{C}$. ($\vec{\mathcal{C}}$ coincides with the smallest full triangulated subcategory of $\mathrm{StMod}(G)$ which contains $\mathcal{C}$ and is closed under taking direct summands and arbitrary direct sums (cf. [15]).)

The following theorem introduces the universal modules $E(W)$ and $F(W)$ and establishes some of their properties.

Theorem 3.1. *Let W be a subset of $V(G)$ and let $\mathcal{C}_W$ be the subcategory of stmod(G) consisting of all finite dimensional modules M such that $V(G)_M \subset W$. Then*

(1) *$\mathcal{C}_W$ is a tensor-ideal thick subcategory of stmod(G).*

(2) *There exists a distinguished triangle*

$$T(W) : E(W) \xrightarrow{\epsilon} k \xrightarrow{\eta} F(W) \to \Omega^{-1} E(W)$$

in StMod(G), satisfying the following universal properties for any G-module M:

(i) $E(W) \otimes M \in \vec{\mathcal{C}_W}$;

(ii) the map $\epsilon \otimes id_M$ is the universal map in StMod(G) from an object in $\vec{\mathcal{C}_W}$ to M, i.e. for any $C \in \vec{\mathcal{C}_W}$, $\epsilon \otimes id_M$ induces an isomorphism

$$\underline{Hom}(C, E(W) \otimes M) \simeq \underline{Hom}(C, M);$$

(iii) the map $\eta \otimes id_M : M \to F(W) \otimes M$ is the universal map in StMod(G) from M to a $\mathcal{C}_W$-local object (where N is called a $\mathcal{C}_W$-local object iff $\underline{Hom}(M, N) = 0$ for any $M \in \mathcal{C}_W$).

(3) *There are stable isomorphisms:*

$$E(W) \otimes E(W) \cong E(W) \text{ and } F(W) \otimes F(W) \cong F(W)$$

and $E(W) \otimes F(W)$ is projective;

(4) *For a G-module M, the following are equivalent:*

- *$M \in \vec{\mathcal{C}_W}$*
- *$M \otimes E(W)$ is stably isomorphic to M*
- *$M \otimes F(W)$ is projective.*

The modules $E(W)$ and $F(W)$ were introduced by J. Rickard ([15]) for finite groups and are thereby called *Rickard idempotent modules*. It is not hard to see that the universal properties (2) determine $E(W), F(W)$ uniquely up to a stable isomorphism.

Let V be a closed conical subset of $V(G)$. Denote by V' the subset of V consisting of all points of V except for generic points of irreducible components of V. Define

$$\kappa(V) \stackrel{def}{=} E(V) \otimes F(V').$$

As a tensor product of idempotent modules, $\kappa(V)$ is again idempotent, i.e. $\kappa(V) \otimes \kappa(V) \cong \kappa(V)$.

Note that the generic point of an irreducible closed conical subvariety is a homogeneous prime ideal, so that there is a natural 1-1 correspondence between homogeneous prime ideals of $k[V(G)]$ and closed irreducible conical subvarieties of $V(G)$.

For an irreducible closed conical set V with the generic point s we shall use $\kappa(s)$ to denote $\kappa(V)$.

We conclude the review of the properties of Rickard idempotents with the following lemma establishing their good behaviour with respect to restriction to a subgroup scheme (cf. [4], [14]).

Lemma 3.2. *Let G be an infinitesimal group scheme, H be a closed subgroup scheme of G and W be a subset of $V(G)$. Let $i_* : V(H) \hookrightarrow V(G)$ be the embedding of schemes induced by the inclusion $i : H \hookrightarrow G$. Then the following two distinguished triangles in StMod(H) are stably isomorphic:*

$$T(i_*^{-1}(W)) : E(i_*^{-1}(W)) \to k \to F(i_*^{-1}(W)) \to \Omega^{-1}E(i_*^{-1}(W))$$

and

$$T(W)\downarrow_H : E(W)\downarrow_H \to k \to F(W)\downarrow_H \to \Omega^{-1}E(W)\downarrow_H .$$

For a conical subset W in $V(G)$ we denote by **Proj W**, the "projectivization" of W, the set of points in W which correspond to homogeneous prime ideals of $k[V(G)]$ excluding the augmentation ideal. $\operatorname{Proj} W$ can be viewed as a subset of the scheme Proj $k[V(G)]$. There is 1-1 correspondence between conical subsets of $V(G)$ and their "projectivizations", i.e. a conical subset is completely determined by its homogeneous ideals. Therefore, the standard properties of support cones, described in Theorem 1.8, apply to their "projectivizations".

The proof of the following theorem can be found in [14].

Theorem 3.3. *Let G be an infinitesimal group scheme.*

(1) *Let s be a point in $V(G)$ corresponding to a homogeneous prime ideal. Then $\operatorname{Proj} V(G)_{\kappa(s)} = \{s\}$.*
(2) *Let M be a G-module. Then*

$$\operatorname{Proj}\ V(G)_M = \{s \in \operatorname{Proj}\ V(G) : M \otimes \kappa(s) \text{ is not projective as a } G\text{-module}\}.$$

The following "realization" statement is an immediate application of the first part of Theorem 3.3 and Theorem 2.3.6.

Corollary 3.4. *Let W be any subset of $\operatorname{Proj} V(G)$. Then there exists a G-module M such that $\operatorname{Proj} V(G)_M = W$.*

4. Support Cones and Induction.

In this last section we establish some properties of support cones with respect to induction functor. As an application, we show that for an algebraic group G, any G-invariant conical subset of $V(G_{(r)})$ can be realized as a support cone of a $G_{(r)}$-module admitting a compatible G-structure. We point out that even for closed G-invariant subsets our construction yields infinite dimensional modules. For realization by finite dimensional modules using different methods see [12] and [13].

A group scheme is called unipotent if it can be embedded in $U_N \subset GL_N$, the subgroup scheme of upper triangular matrices in the general linear group. Recall that a module M of a unipotent group scheme U is injective if and only if $H^1(U, M) = 0$.

Proposition 4.1. *Let $i : H \hookrightarrow G$ be a closed embedding of infinitesimal group schemes and assume further that H is unipotent. Then for any H-module M,*

$$V(G)_{Ind_H^G(M)} = V(H)_M .$$

Proof. Since H is unipotent, the isomorphism $H^*(G, \operatorname{Ind}_H^G(N)) \simeq H^*(H,N)$ implies that projectivity of $\operatorname{Ind}_H^G(N)$ as a G-module implies projectivity of N as an H-module. Conversely, since Induction takes injectives to injectives, projectivity of N implies projectivity of $\operatorname{Ind}_H^G(N)$. Thus, for any H-module N, N is projective if and only if $\operatorname{Ind}_H^G(N)$ is projective.

Now we can prove the equality of support cones in four easy steps: $s \in V_G(Ind_H^G M) \overset{\text{Th. 3.3}}{\iff} Ind_H^G M \otimes \kappa(s)$ is not projective $\iff Ind_H^G(M \otimes \kappa(s))$ is not projective (tensor identity and Lemma 3.2) $\iff M \otimes \kappa(s)$ is not projective over H $\overset{\text{Th. 3.3}}{\iff} s \in V_H(M)$.

□

To proceed, we need the following algebraic lemma.

Lemma 4.2. *Let A be a regular ring of finite Krull dimension d, $k(\mu) = A/\mu A$ be the residue field of A at a prime ideal μ, and $J^\bullet$ be a cochain complex of flat A-modules acyclic in positive degrees. Then $J^\bullet \otimes_A k(\mu)$ is also acyclic in positive degrees.*

Proof. We proceed by induction on $d = \dim A$.

Since $J^\bullet \otimes_A k(\mu) \simeq J^\bullet_\mu \otimes_{A_\mu} k(\mu)$ and localization is exact, it suffices to assume that A is a regular local ring with maximal ideal μ.

Let $d = 1$. Then A is a discrete valuation ring. Denote by π a generator of the maximal ideal of A. Since $J^\bullet$ is flat, tensoring $J^\bullet$ with the short exact sequence $0 \to A \to A \to A/\pi A \to 0$ gives a long exact sequence of complexes

$$0 \to J^\bullet \to J^\bullet \to J^\bullet \otimes_A k((\pi)) \to 0$$

and, thus, a long exact sequence in cohomology

$$\dots H^{n-1}(J^\bullet \otimes_A k((\pi))) \to H^n(J^\bullet) \to H^n(J^\bullet) \to H^n(J^\bullet \otimes_A k((\pi))) \to \dots.$$

Since $H^n(J^\bullet) = 0$ for $n > 0$, we conclude that $H^n(J^\bullet \otimes_A k((\pi))) = 0$ for $n > 0$.

Let A be a regular local ring of dimension d. Since A is regular, we can find an element t in the maximal ideal of A such that A/tA is a regular ring of dimension strictly less than dimension of A. Applying the same argument as above with $\pi = t$, we conclude that $J^\bullet/tJ^\bullet$ is acyclic in positive degrees. Since tensoring preserves flatness (cf. [6, 6.6a]), $J^\bullet/tJ^\bullet = J^\bullet \otimes_A A/tA$ is a cochain complex of flat A/tA-modules. Applying induction hypothesis, we conclude that

$$J^\bullet \otimes_A k(\mu) = J^\bullet/tJ^\bullet \otimes_{A/tA} k(\mu/t\mu)$$

is acyclic in positive degrees. □

Let G be an algebraic group over k. The action of G on $G_{(r)}$ by conjugation induces a natural action on the scheme $V(G_{(r)})$. For a subset W in $V(G_{(r)})$, we denote by $G \cdot W$ the G-orbit of W in $V(G_{(r)})$. The following result is a refinement of Proposition 1.4 in [14].

Proposition 4.3. *Let G be a connected reductive algebraic group, and M be a $G_{(r)}$-module. Then $V(G_{(r)})_{Ind_{G_{(r)}}^G(M)} = G \cdot V(G_{(r)})_M$.*

Proof. Let K/k be a field extension and $H_K \to G_{(r),K}$ be a 1-parameter subgroup. Since M is a G-module, the support cone of M is stable under the action of G. Hence, to prove the theorem it suffices to check the following:

(I). If M_K restricted to H_K is not projective, then so is $(\mathrm{Ind}^G_{G_{(r)}}(M))_K \simeq \mathrm{Ind}^{G_K}_{G_{(r),K}}(M_K)$.
(II). If M_K is projective restricted to all conjugates of H_K under the action of $G(K)$, then $(\mathrm{Ind}^G_{G_{(r)}}(M))_K$ is projective as an H_K-module.

Since Induction commutes with extension of scalars, we can assume that $K = k$ in both cases listed above. By taking the image of H in $G_{(r)}$, we can also assume that $H \to G_{(r)}$ is an embedding.

Let $M \to I^\bullet$ be the standard $G_{(r)}$-injective resolution of M and let $J^\bullet = (\mathrm{Ind}^G_{G_{(r)}} I^\bullet)^H$. We have $H^*(H, \mathrm{Ind}^G_{G_{(r)}} M) = H^*(J^\bullet)$. The complex $J^\bullet$ is naturally a complex of flat $k[G^{(r)}] = k[G/G_{(r)}]$-modules and, moreover, for any $g \in G$ there is an isomorphism:

$$J^\bullet \otimes_{k[G^{(r)}]} k(g) \cong (I^\bullet \otimes k(g))^{g^{-1}(H\otimes k(g))g}. \qquad (*)$$

(cf. [14], p.6)

Suppose $M\downarrow_H$ is not projective. Then $J^\bullet \otimes_{k[G^{(r)}]} k \cong (I^\bullet)^H$ has non-trivial cohomology in positive degrees. Since the scheme $G^{(r)}$ is smooth, the coordinate ring $k[G^{(r)}]$ is regular, and, thus, applying Lemma 4.2, we conclude that $J^\bullet$ is not acyclic in positive degrees. Thus, $\mathrm{Ind}^G_{G_{(r)}} M$ has non-trivial cohomology in positive degrees which implies that it is not projective. We have then proved (I).

In the case described in (II), the same isomorphism (*) shows that $J^\bullet \otimes_{k[G^{(r)}]} k(g)$ is acyclic in positive degrees for all $g \in G$. Since the projection $F^r : G \to G/G_{(r)} \simeq G^{(r)}$ is a bijection on points, and the extension of scalars from $k(F^r(g))$ to $k(g)$ gives an injective map in cohomology of $J^\bullet \otimes_{k[G^{(r)}]} k(F^r(g))$, we get that for any $x \in G^{(r)}$, $J^\bullet \otimes_{k[G^{(r)}]} k(x)$ has trivial cohomology in positive degrees. Lemma 1.3 in [14] now implies that $J^\bullet$ is acyclic and, thus, $\mathrm{Ind}^G_{G_{(r)}} M$ is injective (and, hence, projective) as an H-module. We have now verified (II), which completes the proof. □

Corollary 4.4. *Let G be a connected reductive algebraic group.*

1. *For any $s \in \mathit{Proj}V(G_{(r)})$, there exists a G-rational module M such that $\mathit{Proj}V(G_{(r)})_M = G \cdot s$.*
2. *For any conical subset W of $V(G_{(r)})$, there exists a G-rational module M such that $V(G_{(r)})_M = G \cdot W$. In particular, any conical subset stable under the G-action can be realized as a support cone of a rational G-module.*

Proof. Proposition 4.3 and Theorem 3.3.1 immediately imply that the module $M = \mathrm{Ind}^G_{G_{(r)}}(\kappa(s))$ has the desired support cone. The second statement now follows by applying Theorem 2.3.6. □

For a k-rational point $s \in V(G_{(r)})$, we denote by L_s the line through s in $V(G)$. By using Proposition 4.1, we can realize the orbit of $L_s \subset V(G_{(r)})$ in a more explicit way than the one described in Corollary 4.4.

Corollary 4.5. *Let G be a connected reductive algebraic group and s be a k-rational point of $V(G_{(r)})$. Let further $\nu_s : \mathbb{G}_{a(r)} \to G_{(r)}$ be the 1-parameter subgroup corresponding to s and $M_s = k[u_0, \dots u_{r-2}]/(u_0^p, \dots u_{r-2}^p)$, where we identify $k[\mathbb{G}_{a(r)}]^\#$*

with $k[u_0, \dots u_{r-1}]/(u_0^p, \dots u_{r-1}^p)$. *Then* M_s *has a natural structure of a* $\mathbb{G}_{a(r)}$*-module as a quotient of* $k[\mathbb{G}_{a(r)}]^{\#}$. *We have*

$$V(G_{(r)})_{Ind^G_{\mathbb{G}_{a(r)}}(M_s)} = G \cdot L_s.$$

Proof. It is immediate from the definition that the support cone of M_s as a $\mathbb{G}_{a(r)}$-module is the line in $V(\mathbb{G}_{a(r)}) \simeq \mathbb{A}^r$ through the origin and the point corresponding to the 1-parameter subgroup $id : \mathbb{G}_{a(r)} \to \mathbb{G}_{a(r)}$. This line maps to L_s under $\nu_{s,*} : V(\mathbb{G}_{a(r)}) \to V(G_{(r)})$. Thus, by Proposition 4.1, $V(G_{(r)})_{\mathrm{Ind}^{G_{(r)}}_{\mathbb{G}_{a(r)}} M_s} = L_s$. The statement now follows from Proposition 4.3 and transitivity of induction.

□

Remark 4.6. For $r = 1$, the category of $G_{(1)}$ modules is equivalent to the category of restricted $g = \mathrm{Lie}\, G$-modules and the support variety $V(G)_{(1)}$ can be identified with the restricted nullcone $\mathcal{N}_p(g)$ of g ([8],[18]). Then the corollary above implies that for any $x \in N_p(g)$ we can realize the orbit of kx, $G \cdot kx \subset N_p(g)$, as the support cone of $\mathrm{Ind}^G_{\mathbb{G}_{a(1)}} k$, where $\mathbb{G}_{a(1)} \to G_{(1)} \to G$ is the 1-parameter subgroup corresponding to x.

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