CONSTRUCTIONS FOR INFINITESIMAL GROUP SCHEMES

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Abstract. We introduce the universal \( p \)-nilpotent operator \( \Theta_G \in k[V(G)] \otimes kG \) for an infinitesimal group scheme \( G \), where \( V(G) \) is the affine scheme representing 1-parameter subgroups of \( G \). The action of this operator on \( k[V(G)] \otimes M \) encodes the "local Jordan type" of a \( kG \)-module \( M \). This action also determines an operator \( \tilde{\Theta}_G \) on the free coherent sheaf \( \mathcal{M} = \mathcal{O}_{\text{Proj} V(G)} \otimes M \). If \( M \) is a \( kG \)-module of constant Jordan type, then the kernels of powers of \( \tilde{\Theta}_G \), \( \ker \{ \tilde{\Theta}_G^j \cdot M \} \), are vector bundles on \( \text{Proj} V(G) \). As seen in simple examples, two \( kG \)-modules of the same local Jordan type can sometimes be distinguished by these bundles.

We also introduce various refinements of the (cohomological) support variety construction applicable to infinitesimal group schemes. Four typical examples illustrate our constructions.

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0. Introduction

In recent years, techniques have been developed by the authors to investigate representations of an arbitrary finite group scheme over a field \( k \) of characteristic \( p > 0 \) in a manner which extends earlier work for finite groups and \( p \)-restricted finite dimensional Lie algebras [9], [11]. One general class of such finite group schemes is the class of infinitesimal group schemes, which includes Frobenius kernels of algebraic groups over \( k \) such as those corresponding to \( p \)-restricted Lie algebras (arising as infinitesimal group schemes of height 1). Although the class of infinitesimal group schemes is not as familiar or as tractable as the class of finite groups, it has its own special features as well as much relevance to the representation theory of algebraic groups.

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Our goal in this paper is two-fold: to provide examples of and new invariants for representations of infinitesimal group schemes; and to introduce algebraic vector bundles associated to special representations (those of “constant j-rank”) which are generalizations of modules of constant Jordan type considered in earlier work of the authors and Jon Carlson [3], [4].

Recall that a finite group scheme \( G \) has a finite dimensional commutative coordinate algebra \( k[G] \) whose \( k \)-linear dual \( kG \) is a cocommutative Hopf algebra, the group algebra of \( G \). In particular, a representation of \( G \) (over \( k \)) is precisely a \( kG \)-module. A finite group scheme is said to be infinitesimal if its coordinate algebra \( k[G] \) is a local ring. Thus, within finite group schemes, infinitesimal group schemes are at the opposite end of the spectrum from finite groups (whose coordinate algebras are products of the ground field). Infinitesimal 1-parameter subgroups as considered in [17], [18] determine natural representatives of equivalence classes of \( \pi \)-points. The representability of the functor of 1-parameter subgroups associated to an infinitesimal group scheme provides a universal \( p \)-nilpotent operator which we exploit. The special features of infinitesimal group schemes enable us to provide constructions associated to their representations which are not available for finite groups and other types of finite group schemes.

In Section 1, we recall some of the highlights from [17], [18] concerning the cohomology and theory of supports of finite dimensional \( kG \)-modules for an infinitesimal group scheme \( G \). A key result summarized in Theorem 1.15 is the close relationship between the spectrum \( \text{Spec} H^\bullet(G, k) \) of the cohomology of \( G \) and the scheme \( V(G) \) representing 1-parameter subgroups of an infinitesimal group scheme \( G \). We illustrate these general results with four basic examples of infinitesimal group schemes, examples which are carried along throughout this paper.

In the second section, we define the universal \( p \)-nilpotent operator

\[
\Theta_G \in \text{Hom}_k(k[G], k[V(G)]) \simeq k[V(G)] \otimes kG,
\]

for an infinitesimal group scheme \( G \). For any finite dimensional \( kG \)-module \( M \), \( \Theta_G \) determines a \( p \)-nilpotent endomorphism of the free \( k[V(G)] \) module \( k[V(G)] \otimes M \).

We establish in Proposition 2.6 that \( \Theta_G \) is homogeneous of degree \( p^{r-1} \), where \( k[V(G)] \) is equipped with its natural grading and \( V(G) \) is identified with the scheme of 1-parameter subgroups \( V_r(G) \) of height \( r \) for any choice of \( r \) with \( r \geq \text{ht}(G) \).

In the third section, we verify that specializations \( \theta_v \) of \( \Theta_G \) at points \( v \in V(G) \) when applied to a finite dimensional \( kG \)-module \( M \) determine the local Jordan type of \( M \). Theorem 3.6 can be viewed as providing an algorithm for obtaining the local Jordan type in terms of the representation \( G \to GL_N \) defining the \( kG \)-module \( M \). We utilize \( \Theta_G \) and its specializations to establish conditions on a \( kG \)-module \( M \) which prevent \( M \) from being a module of constant rank (and thus of constant Jordan type).

In the second half of this paper, we give constructions for infinitesimal group schemes which may lead to analogues for a general finite group scheme. With this in mind, we provide in the fourth section a dictionary between 1-parameter subgroups for infinitesimal group schemes and \( \pi \)-points for general finite group schemes. In particular, we recall from [11] the existence of a bijective morphism of projective schemes \( \text{Proj} V(G) \to \Pi(G) \). Here, \( \Pi(G) \) is the scheme of equivalence classes of \( \pi \)-points introduced in [11]. Given a finite dimensional \( kG \)-module \( M \), we
consider a projectivization of the operator $\Theta_G$,

$$\tilde{\Theta}_G : \mathcal{O}_{\text{Proj} V(G)} \otimes M \to \mathcal{O}_{\text{Proj} V(G)}(p^{j-1}) \otimes M,$$

a $p$-nilpotent operator on the free, coherent sheaf $\mathcal{O}_{\text{Proj} V(G)} \otimes M$ on $\text{Proj} V(G)$. We verify in Corollary 4.6 that $\tilde{\Theta}_G$ determines via base change the local Jordan type $\text{JType}(M, \theta_r)$ of a $kG$-module $M$ at any 1-parameter subgroup $\mu_o : \mathbb{G}_{a(r), k^o} \to G$.

In the fifth section, we associate to those special $kG$-modules which have constant $j$-rank various vector bundles on $\text{Proj} V(G)$ defined as kernels of iterates of $\tilde{\Theta}_G$ on $\mathcal{O}_{\text{Proj} V(G)} \otimes M$. More generally, Theorem 4.12 shows that the condition that $M$ be of constant $j$-rank is equivalent to the condition that the coherent sheaf $\text{Im} \tilde{\Theta}_G$ be locally free. We not only consider kernels and images of powers of the $p$-nilpotent operator $\Theta_G$, but also cohomological analogues motivated by a construction of M. Duflo and V. Serganova for Lie superalgebras (in [5]). We see in simple examples that the isomorphism class of the kernel vector bundle $\text{Ker} \tilde{\Theta}_G$ can distinguish certain modules of constant Jordan type which have the same local Jordan type.

Finally, in the last section, we introduce refinements of support varieties (initially considered for finite groups, but extended to infinitesimal group schemes in [18]) and non-maximal support varieties introduced in [12]. As one should require, the generalized (affine) rank variety $V^j(G)_M \subset V(G)$ is empty if and only if the $kG$-module $M$ has constant $j$-rank.

Throughout, $k$ will denote an arbitrary field of characteristic $p > 0$. Unless explicit mention is made to the contrary, $G$ will denote an infinitesimal group scheme over $k$. If $M$ is a $kG$-module and $K/k$ is a field extension, then we denote by $M_K$ the $K$-module obtained by base extension.

1. Infinitesimal Group Schemes

The purpose of this first section is to summarize the important role played by (infinitesimal) 1-parameter subgroups of an infinitesimal group scheme as presented in [17]. The four representative examples of Example 1.4, (g, $\mathbb{G}_{a(r)}$, GL$_n(r)$, SL$_2(2)$), and their associated schemes of 1-parameter subgroups discussed in Example 1.11 will serve as explicit models to which we will frequently return.

**Definition 1.1.** A finite group scheme $G$ over $k$ is a group scheme over $k$ whose coordinate algebra $k[G]$ is finite dimensional over $k$.

Equivalently, $G$ is a functor from commutative $k$-algebras to groups, $R \mapsto G(R)$, represented by a finite dimensional commutative $k$-algebra, the coordinate algebra $k[G]$ of $G$.

Associated to $G$, we have its group algebra $kG = \text{Hom}_k(k[G], k)$; more generally, for any commutative $k$-algebra $R$, we have the $R$-group algebra $RG = \text{Hom}_k(k[G], R)$.

Observe that the $R$-group algebra of $G$ consists of all $k$-linear homomorphisms, whereas $G(R) = \text{Hom}_{k-\text{alg}}(k[G], R)$ is the subgroup of $RG$ consisting of $k$-algebra homomorphisms.

**Definition 1.2.** Let $G$ be a finite group scheme over $k$ and $M$ a $k$-vector space. Then a $kG$-module structure on $M$ is given by one of the following equivalent sets of data (see, for example, [15]):

- The structure $M \to k[G] \otimes M$ of a $k[G]$-comodule on $M$. 

The structure \( kG \otimes M \to M \) of a \( kG \)-module on \( M \).

- A functorial (with respect to \( R \)) group action \( G(R) \times (R \otimes M) \to (R \otimes M) \).

For most of this paper we shall restrict our consideration to infinitesimal group schemes, a special class of finite group schemes which we now define.

**Definition 1.3.** An infinitesimal group scheme \( G \) (over \( k \)) of height \( \leq r \) is a finite group scheme whose coordinate algebra \( k[G] \) is a local algebra with maximal ideal \( \mathfrak{m} \) such that \( x^p = 0 \) for all \( x \in \mathfrak{m} \).

**Example 1.4.** We shall frequently consider the following four examples.

1. A finite dimensional \( p \)-restricted Lie algebra \( \mathfrak{g} \) corresponds naturally with a height 1 infinitesimal group scheme which we denote \( \mathfrak{g} ([15,1.8.5]) \). The group algebra of \( \mathfrak{g} \) is the restricted enveloping algebra \( u(\mathfrak{g}) \) of \( \mathfrak{g} \). If \( \mathfrak{g} \) is the Lie algebra of a group scheme \( \mathfrak{G} \), then the coordinate algebra of \( \mathfrak{g} \) is given by \( k[\mathfrak{G}]/(x^p, x \in \mathfrak{m}) \), where \( \mathfrak{m} \) is the maximal ideal of \( k[\mathfrak{G}] \) at the identity of \( \mathfrak{G} \).

2. Let \( \mathcal{G}_a \) denote the additive group, so that \( k[\mathcal{G}_a] = k[t] \) with coproduct defined by \( \nabla(t) = t \otimes 1 + 1 \otimes t \). As a functor, \( \mathcal{G}_a : \text{comm } k-\text{alg} \to \text{grps} \) sends an algebra \( R \) to its underlying abelian group. For any \( r \geq 1 \), we consider the \( r \)-th Frobenius kernel of \( \mathcal{G}_a \),

\[
\mathcal{G}_{a(r)} \equiv \text{Ker}\{F^r : \mathcal{G}_a \to \mathcal{G}_a\}.
\]

Here \( F : \mathcal{G}_a \to \mathcal{G}_a \) is the (geometric) Frobenius specified by its map on coordinate algebras \( k[t] \to k[t] \) given as the \( k \)-linear map sending \( t \) to \( t^p \). The coordinate algebra of \( \mathcal{G}_{a(r)} \) is given by \( k[\mathcal{G}_{a(r)}] = k[t]/t^p \), whereas the group algebra of \( \mathcal{G}_{a(r)} \) is given by

\[
(1.4.1) \quad k[\mathcal{G}_{a(r)}] \to ^a k[u_0, \ldots, u_{r-1}]/(u_0^p, \ldots, u_{r-1}^p),
\]

where \( u_i \) is a linear dual to \( t^p \), \( 0 \leq i \leq r-1 \).

3. Let \( \text{GL}_n \) denote the general linear group, the representable functor sending a commutative algebra \( R \) to the group \( \text{GL}_n(R) \). For any \( r \geq 1 \), we consider the \( r \)-th Frobenius kernel of \( \text{GL}_n \),

\[
\text{GL}_{n(r)} \equiv \text{Ker}\{F^r : \text{GL}_n \to \text{GL}_n\},
\]

where the geometric Frobenius

\[
F : \text{GL}_n(R) \to \text{GL}_n(R)
\]

is defined by raising each matrix entry to the \( p \)-th power. The coordinate algebra of \( \text{GL}_{n(r)} \) is given by

\[
k[\text{GL}_{n(r)}] = \frac{k[\text{X}_{ij}]}{(\text{X}_{ij}^p - \delta_{ij})}_{1 \leq i,j \leq n},
\]

whereas the group algebra of \( \text{GL}_{n(r)} \) is given as

\[
k[\text{GL}_{n(r)}] = \text{Hom}_k(k[\text{GL}_{n(r)}], k),
\]

the \( k \)-space of linear functionals \( k[\text{GL}_{n(r)}] \) to \( k \). The coproduct

\[
\nabla : k[\text{GL}_{n(r)}] \to k[\text{GL}_{n(r)}] \otimes k[\text{GL}_{n(r)}]
\]

is given by sending \( X_{ij} \) to \( \sum_k X_{ik} \otimes X_{kj} \).
(4) The height 2 infinitesimal group scheme $\text{SL}_{2(2)}$ is essentially a special case of $\text{GL}_{n(r)}$. This is once again defined as the kernel of an iterate of Frobenius

$$\text{SL}_{2(2)} = \text{Ker}\{F^2 : \text{SL}_2 \rightarrow \text{SL}_2\}.$$  

The coordinate algebra of $\text{SL}_{2(2)}$ is given by

$$k[\text{SL}_{2(2)}] = \frac{k[X_{11}, X_{12}, X_{21}, X_{22}]}{(X_{11}X_{22} - X_{12}X_{21} - 1, X_{12}^p - \delta_{11})}$$

whereas the group algebra of $\text{SL}_{2(2)}$ is given as

$$k \text{SL}_{2(2)} = k\{e, f, h, e^{(p)}, f^{(p)}, h^{(p)}\}/\langle\text{relations}\rangle$$

with $e$, $f$, $h$, $e^{(p)}$, $f^{(p)}$, $h^{(p)}$ the dual basis vectors to $X_{12}$, $X_{21}$, $X_{11} - 1$, $X_{12}^p$, $X_{21}^p$, $(X_{11} - 1)^p$ respectively.

**Definition 1.5.** A (infinitesimal) 1-parameter subgroup of height $r$ of an affine group scheme $G_R$ over a commutative $k$-algebra $R$ is a homomorphism of $R$-group schemes $G_{a(r), R} \rightarrow G_R$.

We recall the description of height $r$ 1-parameter subgroups of $\text{GL}_n$ given in [17].

**Proposition 1.6.** [17, 1.2] If $G = \text{GL}_n$ and if $R$ is a commutative $k$-algebra, then a 1-parameter subgroup of $\text{GL}_{n, R}$ of height $r$, $f : G_{a(r), R} \rightarrow \text{GL}_{n, R}$, is naturally (with respect to $R$) equivalent to a comodule map

$$\Delta_f : R^n \rightarrow R[t]/(t^r \otimes_R R^n), \quad \Delta_f(v) = \sum_{j=0}^{r-1} t^j \otimes \beta_j(v), \quad \beta_j \in M_n(R)$$

satisfying the constraints of being counital and coassociative. This in turn is equivalent to specifying an $r$-tuple of matrices $\alpha_0 = \beta_0, \alpha_1 = \beta_1, \ldots, \alpha_{r-1} = \beta_{r-1}$ in $M_n(R)$ such that each $\alpha_i$ has $p^i$ power 0 and such that the $\alpha_i$'s pairwise commute. The other coefficient matrices $\beta_j$ are given by the formula

$$\beta_j = \frac{\alpha_0 \cdots \alpha_{j-1}}{(j_0! \cdots j_{r-1}!)} \in M_n(R), \quad j = \sum_{i=0}^{r-1} j_i p^i \text{ with } 0 \leq j_i < p.$$  

As shown in [17], Proposition 1.6 implies the following representability of the functor of 1-parameter subgroups of height $r$.

**Theorem 1.7.** [17, 1.5] For any affine group scheme $G$, the functor from commutative $k$-algebras to sets

$$R \mapsto \text{Hom}_{\text{grp sch}}(G_{a(r), R}, G_R)$$

is representable by an affine scheme $V_r(G) = \text{Spec} k[V_r(G)]$. Namely, this functor is naturally isomorphic to the functor

$$R \mapsto \text{Hom}_{\text{alg}}(k[V_r(G)], R).$$  

By varying $r$, we can associate a family of affine schemes to an affine group scheme $G$. In the following remark we make explicit the relationship between various $V_r(G)$ for the same $G$ and varying $r$'s.
Remark 1.8. For $r > s \geq 1$, let $p_{r,s} : G_{s(r)} \to G_{s(s)}$ be the canonical projection given by the natural embedding of the coordinate algebras

$$p_{r,s} : k[G_{s(s)}] = k[t]/t^{r-s} \to k[t]/t^{r} = k[G_{s(r)}].$$

The corresponding map on group algebras

$$kG_{s(r)} \simeq k[u_0, \ldots, u_{r-1}]/(u_0^p, \ldots, u_{r-1}^p) p_{r,s} \mapsto kG_{s(s)} \simeq k[v_0, \ldots, v_{s-1}]/(v_0^p, \ldots, v_{s-1}^p)$$

sends $\{u_0, \ldots, u_{r-s-1}\}$ to $\{0, \ldots, 0\}$, and $\{u_{r-s}, \ldots, u_{r-1}\}$ to $\{v_0, \ldots, v_{s-1}\}$.

Precomposition with $p_{r,s}$ determines a canonical embedding of affine schemes

$$\xymatrix{i_{s,r} : V_s(G) \ar[r] & V_r(G),}$$

where a one-parameter subgroup $\mu : G_{s(s), R} \to G_R$ of height $s$ is sent to the one-parameter subgroup $\mu \circ p_{r,s} : G_{s(r), R} \to G_{s(s), R} \to G_R$ of height $r$. The construction is transitive, that is, we have $\nu_{s,t} = \nu_{s',r} \circ \nu_{s,s'}$ for $s \leq s' \leq r$. Hence, we have an inductive system

$$V_1(G) \subset V_2(G) \subset \cdots \subset V_r(G) \subset \cdots$$

Conversely, any one-parameter subgroup $G_{s(s'), R} \to G_R$ can be decomposed as

$$G_{s(s'), R} \xrightarrow{p_{r,s'}} G_{s(s), R} \xrightarrow{p_{r,s}} G_R$$

for some $s \leq s'$. If $G$ is an infinitesimal group scheme of height $\leq r$ then, we must have $s \leq r$. This justifies the following definition

Definition 1.9. Let $G$ be an infinitesimal group scheme. Then the embedding $\nu_{r,r'} : V_r(G) \subset V_{r'}(G)$ for $r' > r$ is an equality provided the height of $G$ is $\leq r$. We denote by $V(G)$ the stable value of $V_r(G)$,

$$V(G) \equiv \lim_{r \to \infty} V_r(G).$$

We next recall the construction of 1-parameter subgroups for $GL_n$. This construction can be applied to any affine group scheme of exponential type (see [17, §1] and also [16] for an extended list of groups of exponential type). We define the homomorphism

$$\exp_\Delta : G_{s(s), R} \to GL_{n,R}$$

of $R$-group schemes corresponding to an $r$-tuple $\Delta = (\alpha_0, \ldots, \alpha_{r-1}) \in M_n(R)^{\times r}$ of pairwise commuting $p$-nilpotent matrices to be the natural transformation of group-valued functors on commutative $R$-algebras $S$ sending any $s \in S$ with $s^{p^r} = 0$ to

$$\exp(s^{\alpha_0}) \cdot \exp(s^{\alpha_1}) \cdots \cdot \exp(s^{\alpha_{r-1}}) \in GL_n(S).$$

where for any $p$-nilpotent matrix $A \in GL_n(S)$ we set

$$\exp(A) = 1 + A + \frac{A^2}{2} + \cdots + \frac{A^{p-1}}{(p-1)!}.$$

Proposition 1.10. [17, 1.2] The scheme of one-parameter subgroups $V_r(GL_n)$ is isomorphic to the scheme of $r$-tuples of pairwise commuting $p$-nilpotent $n \times n$ matrices $N_p^{(r)}(gl_n)$; the identification is given by sending $\Delta = (\alpha_0, \ldots, \alpha_{r-1}) \in N_p^{(r)}(gl_n)(R)$ to the one-parameter subgroup $\exp_\Delta : G_{s(s), R} \to GL_{n,R}$. 
Example 1.11. We describe \( V(G) \) in each of the four examples of Example 1.4.

(1) \( V(\mathfrak{g}) \simeq N_p(\mathfrak{g}) \), the closed subvariety of the affine space underlying \( \mathfrak{g} \) consisting of \( p \)-nilpotent elements \( x \in \mathfrak{g} \) (that is, \( x^{[p]} = 0 \)). Let \( g_a \) be the Lie algebra of the additive group \( G_a \). Note that \( g_a \) is a one-dimensional restricted Lie algebra with trivial \( p \)-restriction. Each \( p \)-nilpotent element \( x \in g_R = g \otimes_k R \) determines a map of \( p \)-restricted Lie algebras over \( R \) where \( R \) is a commutative \( k \)-algebra: \( g_a,R \to g_R \).

The corresponding map of height 1 infinitesimal group schemes \( G_a(1),R \to \mathcal{G}_R \) is the associated 1-parameter subgroup of \( g_a \).

(2) \( V(G_a(v)) \simeq \mathbb{A}^r \). The \( r \)-tuple \( \underline{a} = (a_0, \ldots, a_{r-1}) \in R^{\times r} = \mathbb{A}^r(R) \) corresponds to the 1-parameter subgroup \( \mu_{\underline{a}} : G_a(v),R \to G_a(v),R \) whose map on coordinate algebras \( R[t]/t^{p^r} \to R[t]/t^{p^r} \) sends \( t \) to \( \sum a_i t^{p^i} \) ([17, 1.10]).

(3) By Proposition 1.10, \( V(GL_n(v)) = N^{|p|}(gl_n) \), the variety of \( r \)-tuples of pairwise commuting, \( p \)-nilpotent \( n \times n \) matrices. The embedding \( i_{r+1} : V_r(GL_n) \simeq N_p^{[p]}(gl_n) \subset V_{r+1}(GL_n) \simeq N_p^{[p+1]}(gl_n) \) described in Remark 1.8 is by sending an \( r \)-tuple \((\alpha_0, \ldots, \alpha_{r-1})\) to the \((r+1)\)-tuple \((0, \alpha_0, \ldots, \alpha_{r-1})\).

Let \( X_{ij} \) be the coordinate functions of \( R[GL_n(v)] \simeq R[X_{ij}]/(X_{ij}^{p^r} - \delta_{ij}) \). Then \( \exp^*_\underline{a} : R[GL_n(v)] \to R[G_a(v)] \) is given by sending \( X_{ij} \) for some \( 1 \leq i,j \leq n \) to the \((i,j)\)-entry of the polynomial \( p_\underline{a}(t) \) with matrix coefficients whose coefficient of \( t^a \) is computed as the multiple of \( s^\alpha \) in the \((i,j)\)-entry of the matrix (1.9.1).

Upon performing the indicated multiplication in (1.9.1), the coefficient of \( p_\underline{a}(t) \) multiplying \( s_\alpha \) is \( \alpha_\ell \) for \( 0 \leq \ell \leq r \), whereas coefficients of \( p_\underline{a}(t) \) multiplying \( s_\alpha \) for \( n \) not a power of \( p \) are determined as in formula (1.6.1). Consequently, we conclude that \( \exp^*_\underline{a}(X_{ij}) \) is a polynomial in \( t \) whose coefficient multiplying \( t^{\alpha_\ell} \) is \( (\alpha_\ell)_{i,j} \) for \( 0 \leq \ell \leq r \).

(4) Since \( \text{SL}_2(2) \) is a group scheme with an embedding of exponential type (see [17, 1.8]), its variety admits a description similar to the one of \( GL_n(v) \). Namely, \( V(\text{SL}_2(2)) \) is the variety of pairs of \( p \)-nilpotent proportional \( 2 \times 2 \) matrices \( \underline{a} = (\alpha_0, \alpha_1) \). This variety is given explicitly as the affine scheme with coordinate algebra \( k[V(\text{SL}_2(2))] = k[x_0, y_0, z_0, x_1, y_1, z_1]/(x_1 y_1 - z_1^2, x_1 y_2 - x_2 y_1, y_1 z_2 - z_2 y_1, x_1 z_2 - x_2 z_1). \)

We give an explicit description of the map on coordinate algebras \( \exp^*_\underline{a} : R[\text{SL}_2(2)] \to R[G_a(2)] \simeq R[t]/t^{p^2} \) induced by the one-parameter subgroup \( \exp\underline{a} : G_a(2),R \to \text{SL}_2(2),R \). This description follows immediately from the general discussion in the previous example. Let \( \underline{a} = \begin{bmatrix} a_0 & a_1 \\ b_0 & -c_0 \end{bmatrix} \), \( \begin{bmatrix} c_1 & a_1 \\ b_1 & -c_1 \end{bmatrix} \) \( \in N^{[2]}(sl_2) \). Then \( \exp^*_\underline{a} \) is determined by the formulae:

\[
\begin{align*}
X_{11} &\mapsto 1 + c_0 t + c_1 t^p, \\
X_{12} &\mapsto a_0 t + a_1 t^p, \\
X_{21} &\mapsto b_0 T + b_1 t^p, \\
X_{22} &\mapsto 1 - c_0 t - c_1 t^p,
\end{align*}
\]

where \( X_{ij} \) are the standard polynomial generators of \( k[\text{SL}_2(2)] \simeq k[X_{11}, X_{12}, X_{21}, X_{22}]/(\det -1, X_{ij}^{p^2} - \delta_{ij}) \).

Remark 1.12. If \( k(v) \) denotes the field of definition of the point \( v \in V(G) \) for an infinitesimal group scheme \( G \), then we have a naturally associated map \( \text{Spec } k(v) \to \)
$V(G)$ and, hence, an associated group scheme homomorphism over $k(v)$ (for $r$ sufficiently large):

$$
\mu_v : G_{a(r), k(v)} \longrightarrow G_{k(v)}.
$$

Note that if $K/k$ is a field extension and $\mu : G_{a(r), K} \rightarrow G_K$ is a group scheme homomorphism, then this data defines a point $v \in V(G)$ and a field embedding $k(v) \hookrightarrow K$ such that $\mu$ is obtained from $\mu_v$ via scalar extension from $k(v)$ to $K$.

We next recall the rank variety and cohomological support variety of a $kG$-module of an infinitesimal group scheme. We denote by

$$
H^\bullet(G, k) = \begin{cases} 
H^\bullet(G, k), & \text{if } p = 2, \\
H^{ev}(G, k) & \text{if } p > 2.
\end{cases}
$$

The map of $R$-algebras (but not of Hopf algebras for $r > 1$),

$$
(1.12.1) \quad \epsilon : R[u]/u^p \longrightarrow R[u_0, \ldots, u_{r-1}]/(u_p^r) \simeq RG_{a(r)},
$$

makes its first appearance in the following definition and will recur throughout this paper.

**Definition 1.13.** Let $G$ be a finite group scheme and $M$ a finite dimensional $kG$-module. We define the cohomological support variety for $M$ to be

$$
|G|_M \equiv V(\text{ann}_{H^\bullet(G, k)} \text{Ext}^*_{kG}(M, M)),
$$

the reduced closed subscheme of $|G| = \text{Spec } H^\bullet(G, k)_{\text{red}}$ given as the variety of the annihilator ideal of $\text{Ext}^*_{kG}(M, M)$.

**Definition 1.14.** Let $G$ be an infinitesimal group scheme and $M$ a finite dimensional $kG$-module. We define the rank variety for $M$ to be the reduced closed subscheme $V(G)_M$ whose points are given as follows:

$$
V(G)_M = \{v \in V(G) : (\mu_{v, *} \circ \epsilon^*)(M_{k(v)}) \text{ is not free as a } k[u]/u^p \text{- module} \}.
$$

Proposition [18, 6.2] asserts that $V(G)_M$ is a closed subvariety of $V(G)$. A key result of [18] is the following theorem relating the scheme of 1-parameter subgroups $V(G)$ to the cohomology of $G$.

**Theorem 1.15.** ([18, 5.2, 6.8, 7.5]) Let $G$ be an infinitesimal group scheme of height $\leq r$. There is a natural homomorphism of $k$-algebras

$$
\psi : H^\bullet(G, k) \rightarrow k[V(G)]
$$

with nilpotent kernel and image containing the $p^r$-th power of each element of $k[V(G)]$. Hence, the associated morphism of schemes

$$
\Psi : V(G) \rightarrow \text{Spec } H^\bullet(G, k)
$$

is a $p$-isogeny.

If $M$ is a finite dimensional $kG$-module, then $\Psi$ restricts to a homeomorphism

$$
\Psi_M : V(G)_M \sim \rightarrow |G|_M.
$$

Furthermore, every closed conical subspace of $V(G)$ is of the form $V(G)_M$ for some finite dimensional $kG$-module $M$.

In the special case of of $G = GL_{n(r)}$ the isogeny $\Psi$ has an explicitly constructed inverse.
Theorem 1.16. ([17, 5.2]) There exists a homomorphism of $k$-algebras
\[ \tilde{\phi} : k[V(GL_n(k))] \to H^*(GL_n(k), k) \]
such that $\psi \circ \tilde{\phi}$ is the $r^{th}$ iterate of the $k$-linear Frobenius map. Hence, the associated morphisms of schemes
\[ \Psi : V(GL_n(k)) \to \text{Spec} H^*(GL_n(k), k), \quad \Phi : \text{Spec} H^*(GL_n(k), k) \to V(GL_n(k)) \]
are mutually inverse homeomorphisms.

Example 1.17. We investigate $V(G)_M$ for the four examples of Example 1.4.

1. Let $M$ be a $p$-restricted $\mathfrak{g}$-module of dimension $m$, given by the map of $p$-restricted Lie algebras $\rho : \mathfrak{g} \to \text{End}_k(M) \simeq \mathfrak{gl}_m$. Then $V(\mathfrak{g})_M \subset V(\mathfrak{gl}_m)$ consists of those $p$-nilpotent elements of $\mathfrak{g}$ whose Jordan type (as an $m \times m$-matrix in $\mathfrak{gl}_m$) has at least one block of size $< p$ (see [7]).

2. For $G = \mathbb{G}_a(k)$, $kG \simeq kE$ where $E$ is an elementary abelian $p$-group of rank $r$. The rank variety of a $kE$-module was first investigated in [2]. We consider directly the rank variety $V(\mathbb{G}_a(r))_M$ of a finite dimensional $k\mathbb{G}_a(r)$-module $M$. The data of such a module is the choice of $r$ $p$-nilpotent endomorphisms $\tilde{u}_0, \ldots, \tilde{u}_{r-1} \in \text{End}_k(M)$, given as the image of the distinguished generators of $k\mathbb{G}_a(r)$ as in (1.4.1). A 1-parameter subgroup of $\mathbb{G}_a(r)$ has the form $\mu_a : \mathbb{G}_a(r) \to \mathbb{G}_a(r)_K$ for some $r$-tuple $a = (a_0, \ldots, a_{r-1})$ of $K$-rational points as in Example 1.11(2). The condition that $\mu_a$ be a point of $V(\mathbb{G}_a(r))_M$ is the condition that $(\mu_a \circ \epsilon)^*(M_K)$ is not free as a $K[u]/u^p$-module, where $u = a_{r-1}u_0 + a_{r-2}u_1 + \cdots + a_0 \tilde{u}_{r-1} \in \text{End}_K(M_K)$ (see [17, 6.5]).

3. Let $M$ be a finite dimensional $kG$-module with $G = GL_n(k)$. By Theorem 1.15, $V(GL_n(k)_M \subset V(GL_n(k))$ is the closed subvariety whose set of points in a field $K/k$ are 1-parameter subgroups $\exp \gamma : \mathbb{G}_a(r)_K \to GL_n(k)_K$ indexed by $r$-tuples $a = (a_0, \ldots, a_{r-1}) \in M_n(K)$ of $p$-nilpotent, pairwise commuting matrices such that $(\exp \gamma \circ \epsilon)^*(M_K)$ is not a free as a $K[u]/u^p$-module. The action of $u$ on $M_K$ is determined utilizing Example 1.11(3). Namely, the action of $u$ is given by composing the coproduct $M_K \to K[GL_n(r)] \otimes M_K$ defining the $GL_n(r)$-module structure on $M_K$ with the linear functional $\epsilon \circ \exp \gamma : K[GL_n(r)] \to K$. In §3, we shall investigate this case in more detail by considering some concrete examples.

4. A complete description of support varieties for simple modules for $SL_2(k)$ can be found in [18, §7]. We describe the situation for $G = SL_2(k)$. Let $S_\lambda$ be irreducible modules of highest weight $\lambda$, where $0 \leq \lambda \leq p^2 - 1$. For $\lambda < p - 1$, the module $S_\lambda$ has dimension less than $p$ and thus $V(G)S_\lambda = V(G)S^{(1)}_\lambda \subset V(G)$. For $\lambda = p - 1$, the restriction of $S_{p-1}$ to $SL_2(1) \subset SL_2(2)$ is projective (the Steinberg module for $SL_2(1)$ but $S_{p-1}$ is not itself projective. Hence, $V(G)S_{p-1}$ is a proper non-trivial subvariety of $V(G)$. Using the notation introduced in Example 1.11(4), we have
\[ V(G)S_{p-1} = \{(a_0, 0) \mid a_0 \in N(sl_2)\} \subset V(G), \]
and
\[ V(G)S^{(1)}_{p-1} = \{(0, a_1) \mid a_1 \in N(sl_2)\} \subset V(G) \]
(see [18, 6.10]). $V(G)S_{p-1}$ can be described as a subscheme of $V(G)$ defined by the equations $x_2 = y_2 = 0$. For $\lambda = \lambda_0 + \lambda_1p$ where $\lambda_0, \lambda_1 \leq p - 1$ we have
For \( \lambda \) the support variety of \( S \)

\[ \Phi \]

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2. Universal \( p \)-nilpotent operators

Let \( G \) be an affine group scheme over \( k \).

The natural isomorphism of covariant functors on commutative \( k \)-algebras \( R \)

\[ \text{Hom}_{\text{grp sch}}(\mathbb{G}_{a(r), R}, G_R) \sim \text{Hom}_{k - \text{alg}}(k[V_r(G)], R) \]

given in Theorem 1.7 implies the existence of a universal 1-parameter subgroup of height \( r \)

\[ U_{G,r} : \mathbb{G}_{a(r), k[V_r(G)]} \to G_{k[V_r(G)]}. \]

Definition 2.1. Let \( G \) be an infinitesimal group scheme of height \( \leq r \). We define the universal \( p \)-nilpotent operator for \( G \) to be

\[ \Theta_{G,r} \equiv U_{G,r}(u_{r-1}) = (U_{G,r, \epsilon} \circ \epsilon)(u) \in k[V_r(G)] \otimes kG, \]

where \( \epsilon : k[u]/u^p = \mathbb{G}_{a(1)} \to k\mathbb{G}_{a(r)} = k[u_0, \ldots, u_{r-1}]/(u_0^p) \) is the map of \( k \)-algebras introduced in (1.12.1).

If \( G \) is infinitesimal of height \( \leq r \), then by Remark 1.8 \( V_r(G) \) is essentially independent of \( r \). The following proposition justifies our using the the simpler notation

\[ U_G : \mathbb{G}_{a(r), k[V(G)]} \to G_{k[V(G)]}, \quad \Theta_G \in k[V(G)] \otimes kG. \]

Proposition 2.2. Let \( G \) be an affine group scheme and let \( r' \geq r \). For notational convenience, set \( A_r = k[V_r(G)] \), \( A_{r'} = k[V_{r'}(G)] \). Let \( i_r : A_r \to A_{r'} \) be the projection corresponding to the embedding \( i_r : V_r(G) \hookrightarrow V_{r'}(G) \) which is induced by the canonical projection \( p_{r', r} : \mathbb{G}_{a(r')} \to \mathbb{G}_{a(r)} \) (see Remark 1.8). Consider \( A_r \) as an \( A_{r'} \)-module via \( i_r^* \).

If \( G \) is an infinitesimal group scheme, then

\[ \Theta_{G,r} = \Theta_{G,r'} \otimes_{A_{r'}} 1 \in A_{r'} G \otimes_{A_{r'}} A_r \simeq A_r G. \]

Moreover, if \( G \) is an infinitesimal group scheme of height \( \leq r \), then \( \Theta_{G,r} \) is thereby naturally identified with \( \Theta_{G,r'} \).

Proof. Consider the composition

\[ \mathbb{G}_{a(r'), A_r} \xrightarrow{U_{G,r \circ p_{r', r}}} G_{A_{r'}} \]

where \( \mathbb{G}_{a(r'), A_r} \) is the Steinberg module for \( SL_2 \).

Overall, we get

\[ V(G)_{S_\lambda} = \begin{cases} N[2](s_2), & \text{if } \lambda_0, \lambda_1 \neq p - 1, \\ \{(\alpha_0, 0) | \alpha_0 \in N(s_2)\} & \text{if } \lambda_0 = p - 1, \lambda_1 \neq p - 1, \\ \{(0, \alpha_1) | \alpha_1 \in N(s_2)\} & \text{if } \lambda_0 \neq p - 1, \lambda_1 = p - 1, \\ 0 & \text{if } \lambda = p^2 - 1. \end{cases} \]
Since \( U_{G,r} \in V_r(G)(A_r) \simeq \text{Hom}(A_r, A_r) \) corresponds to the identity map on \( A_r \), and \( p_{r',r} \) is the map that induces \( i^*: A_{r'} \to A_r \), we conclude that the composition \( U_{G,r} \circ p_{r',r} \in V_r(G) \simeq \text{Hom}(A_r, A_r) \) corresponds to \( i^* \). Hence, the universality of \( U_{G,r'} \) implies that \( U_{G,r'} \circ p_{r',r} \) is obtained by pulling back the universal one-parameter subgroup \( U_{G,r'} \) via \( i^* : A_{r'} \to A_r \). Therefore, we conclude

\[
U_{G,r} \circ p_{r',r} = U_{G,r'} \otimes_{A_{r'}} A_r
\]

which implies the equality of maps of group algebras

\[
U_{G,r'} \otimes_{A_{r'}} A_r = U_{G,r'} \otimes_{\theta_{G,r'}} A_r G.
\]

Since \( p_{r',r}(u_r) = u_r \in k\mathfrak{g}(\mathfrak{a}(r)) \), we conclude \( (U_{G,r'} \circ p_{r',r})(u_r) = U_{G,r'}(u_r) = \theta_{G,r'} \), whereas \( (U_{G,r'} \otimes_{A_{r'}} A_r)(u_r') = U_{G,r'}(u_r') \otimes_{A_{r'}} 1 \). The second statement follows immediately from the fact that for \( G \) of height \( \leq r \), the map \( i^* : A_{r'} \to A_r \) is an isomorphism as shown in Remark 1.8.

\[\square\]

**Example 2.3.** We describe the universal \( p \)-nilpotent operator \( \Theta_G \) in each of the four examples of Example 1.4.

1. Let \( G = \mathfrak{g} \) for some finite dimensional \( p \)-restricted Lie algebra \( \mathfrak{g} \) embedded as a \( p \)-restricted subalgebra of some \( \mathfrak{gl}_m \). Then \( k[V(\mathfrak{g})] = k[N_p(\mathfrak{g})] \), which we view as a quotient of \( k[X_{ij}] \) (where \( 1 \leq i, j \leq m \)) by the ideal of relations that a general \( m \times m \) matrix must satisfy to be a \( p \)-nilpotent element of \( \mathfrak{g} \subset \mathfrak{gl}_m \).

Then \( \Theta_{\mathfrak{g}} \in k[N_p(\mathfrak{g})] \otimes \mathfrak{u}(\mathfrak{g}) \) has image in \( k[N_p(\mathfrak{g})] \otimes \mathfrak{u}(\mathfrak{gl}_m) \) equal to the image of \( \Theta_{\mathfrak{gl}_m} \in k[N_p(\mathfrak{gl}_m)] \otimes \mathfrak{u}(\mathfrak{gl}_m) \). Moreover, \( \Theta_{\mathfrak{gl}_m} \) is given explicitly as the image of the generic matrix in \( k[X_{ij}] \otimes \mathfrak{u}(\mathfrak{gl}_m) \).

In other words,

\[
(2.3.1) \quad \Theta_{\mathfrak{g}} = \sum_{x_i} \tilde{x}_i \otimes x_i \in k[N_p(\mathfrak{g})] \otimes \mathfrak{u}(\mathfrak{g})
\]

where \( \{x_i\} \) is a basis of \( \mathfrak{g} \) and \( \tilde{x}_i \) denotes the image of the dual basis element to \( x_i \) under the quotient map \( S^*(\mathfrak{g}^\#) \to k[N_p(\mathfrak{g})] \).

We record an explicit formula for the universal \( p \)-nilpotent operator in the case of \( \mathfrak{g} = sl_2 \) for future reference. We have \( k[N_p(sl_2)] \simeq k[x, y, z]/(xy + z^2) \). Let \( e, f, h \) be the standard basis of the \( p \)-restricted Lie algebra \( sl_2 \). Then

\[
\Theta_{sl_2} = xe + yf + zh.
\]

Observe that this formula agrees with the presentation of a “generic” \( \pi \)-point for \( u(sl_2) \) as given in [11, 2.5].

2. Take \( G = \mathcal{G}_{a(r)} \). Then \( k[V(\mathcal{G}_{a(r)})] \simeq k[x_0, \ldots, x_{r-1}] \) is graded in such a way that \( x_i \) has degree \( p^i \) (see Proposition 2.6 below). As shown in [18, 6.5.1], the map \( U_{\mathcal{G}_{a(r)}^*} : k[x_0, \ldots, x_{r-1}][u]/u^p \to k[x_0, \ldots, x_{r-1}][u_0, \ldots, u_{r-1}]/(u_0^p, \ldots, u_{r-1}^p) \)

sends \( u \) to \( u_{r-1} \) to

\[
(2.3.2) \quad (i_0, \ldots, i_{r-1})x_0^{i_0} \cdots x_{r-1}^{i_{r-1}} \cdot v_i
\]
where \((i_0, \ldots, i_{r-1}) = \frac{(i_1 + \cdots + i_{r-1})!}{i_1! \cdots i_{r-1}!}\) and \(v_0, \ldots, v_{p^r-1}\) is the standard basis of \(k[\mathcal{G}_{a(t)}]\). Here,
\[v_i = \frac{u_0^{(i)}}{i!} \cdots \frac{u_{p^r-1}^{(i)}}{(p^r-1)!},\]
where \(i = i^{(0)} + i^{(1)}p + \cdots + i^{(r-1)}p^{r-1}\) \((0 \leq i^{(j)} \leq p - 1)\) is the \(p\)-adic expansion of \(i\).

The linear part of \(\Theta_{\mathcal{G}_{a(t)}} = \mathcal{U}_{\mathcal{G}_{a(t)}}(u)\) (the part of the image of \(u\) linear in the \(u_i\)'s) equals
\[x_{r-1}u_0 + x_{r-2}^pu_1 + \cdots + x_0^{p^r-1}u_{r-1}.\]

(3) Let \(G = GL_n(r)\). Recall that \(V(G)\) is the scheme of \(r\)-tuples of \(p\)-nilpotent, pair-wise commuting matrices. For notational convenience, let \(A\) denote \(k[\mathcal{V}(G)]\). Then \(\mathcal{U}_{GL_n(r)} : \mathcal{G}_{a(t)}, A \rightarrow GL_n(r), A\) is specified by the \(A\)-linear map on coordinate algebras
\[\mathcal{U}_{GL_n(r)} : A[GL_n(r)] \rightarrow A[T]/T^{p^r}, \quad X_{a,b} \mapsto \sum_{j=0}^{p^r-1} (\beta_j)_{a,b} T^j\]
where \(\{X_{a,b} : 1 \leq a, b \leq n\}\) are the coordinate functions of \(GL_n\), where \(\beta_j\) is given as in formula (1.6.1) in terms of the matrices \(a_0, \ldots, a_{r-1} \in M_n(A)\), and \(a_i = \beta_p\) have coordinate functions which generate \(A\). (In other words, the \(n^2r\) entries of \(a_0, \ldots, a_{r-1}\) viewed as variables generate \(A\), with relations given by the conditions that these matrices must be \(p\)-nilpotent and pairwise commuting.)

The \(p\)-nilpotent operator
\[\Theta_{GL_n(r)} = (\mathcal{U}_{GL_n(r)} \circ \epsilon)(u) \in \text{Hom}_k(k[GL_n(r)], A) = k[GL_n(r)] \otimes k GL_n(r)\]
is given by the \(k\)-linear functional sending a polynomial in the matrix coefficients \(P(X_{a,b}) \in k[GL_n(r)]\) to the coefficient of \(T^{p^r-1}\) of the sum of products corresponding to the polynomial \(P\) given by replacing each \(X_{a,b}\) by \(\sum_{j=0}^{p^r-1} (\beta_j)_{a,b} T^j\) (when taking products of matrix coefficients, one uses the usual rule for matrix multiplication).

The coaction \(k^n \rightarrow k^n \otimes k[GL_n]\) corresponding to the natural representation of \(GL_n\) on \(k^n\) determines an action of \(\text{Hom}_k(k[GL_n(r)], A) \subset \text{Hom}_k(k[GL_n], A)\) on \(A^n\), so that we may associate to \(\Theta_G\) an \(A\)-linear endomorphism of \(A^n\) given in matrix form by \((\Theta_G(X_{a,b}))\).

(4) We compute \(\Theta_{SL_2(2)}\) explicitly. Recall that \(V(SL_2(2))\) is the variety of pairs of commuting, trace 0, nilpotent matrices with coordinate algebra \(A = k[V(SL_2(2))]\) as determined in Example 1.11(4). A 1-parameter subgroup \(G_{a(2)}, R \rightarrow SL_2(2), R\) is specified by a map on coordinate algebras \(R[SL_2(2)] \rightarrow R[t]/t^2\) as described in Example 1.11(4).

As in Example 1.4(4), write \(e, f, h, e^{(p)}, f^{(p)}, h^{(p)}\) for the generators of \(k SL_2(2)\) and set
\[e(i) = e^i, \quad f(i) = f^i, \quad \frac{h}{i} = \frac{h(h-1)(h-2)\ldots(h-i+1)}{i!}\]
for \(i < p\). Fix the linear basis of \(k[SL_2(2)]\) given by powers of \(X_{12}, X_{21}, X_{11} - 1\) (in this fixed order). Then the element of \(k SL_2(2)\) dual to \(X_{12}X_{21}(X_{11} - 1)\) for
i + j + ℓ ≤ p is given by

\[(X_{12}X_{21}(X_{11} - 1)^\ell)\]  

(where \(i/j\) is identified with \(h/p\) by definition).

With these conventions \(\Theta_{\text{SL}_2(\ell)} \in k[V(\text{SL}_2(\ell))] \otimes k\text{SL}_2(\ell)\) equals

\[\begin{equation}
(2.3.3) \quad x_1 e + y_1 f + z_1 h + x_0^p e^{(p)} + y_0^p f^{(p)} + z_0^p h^{(p)} + \sum_{i+j+\ell=p, i,j,\ell < p} x_i^p y_i^p z_0^p e^{(i)} f^{(j)} h^\ell.
\end{equation}\]

Let \(G\) be an infinitesimal group scheme, and \(M\) be a \(k\)-module. Applying base change to the given \(G\)-action \(kG \otimes M \to M\), we obtain a \(k[V(G)]\)-linear action of \(k[V(G)] \otimes kG\) on \(k[V(G)] \otimes M\):

\[(k[V(G)] \otimes kG) \otimes_{k[V(G)]} (k[V(G)] \otimes M) \to k[V(G)] \otimes M.\]

Thus, \(\Theta_G \in k[V(G)] \otimes kG\) determines the \(k[V(G)]\)-linear map

\[(2.4.4) \quad \Theta_G : k[V(G)] \otimes M \to k[V(G)] \otimes M, \quad a \otimes m \mapsto \Theta_G(a \otimes m) = a \cdot \Theta_G(1 \otimes m).\]

To complement Example 2.3, we make explicit the action of \(\Theta_G\) on some \(kG\)-representation for each of the four types of finite group schemes we have been considering in examples.

**Example 2.4.**

1. Let \(G = \mathfrak{g}\) and let \(M = \mathfrak{g}^{ad}\) denote the adjoint representation of the \(p\)-restricted Lie algebra \(\mathfrak{g}\); let \(\{x_i\}\) be a basis for \(\mathfrak{g}\). We identify \(\Theta_\mathfrak{g}\) as the \(k\)-linear endomorphism

\[\Theta_\mathfrak{g} : k[N_p(\mathfrak{g})] \otimes \mathfrak{g}^{ad} \to k[N_p(\mathfrak{g})] \otimes \mathfrak{g}^{ad}, \quad 1 \otimes x \mapsto \sum_i \bar{x}_i \otimes [x_i, x],\]

where \(\bar{x}_i\) is the image under the projection \(S^*(\mathfrak{g}^{\#}) \to k[N_p(\mathfrak{g})]\) of the dual basis element to \(x_i\).

2. Let \(M\) denote the cyclic \(k\text{GL}_a(r)\)-module

\[M = k[u_0, \ldots, u_{r-1}]/(u_0, u_1, \ldots, u_{p-1}) = k[u_1, \ldots, u_{r-1}]/(u_0^p, \ldots, u_{p-1}).\]

As recalled in Example 2.3(2), \(k[V(\text{GL}_a(r))] = k[\mathfrak{a}^r] = k[a_0, \ldots, a_{r-1}], k\text{GL}_a(r) = k[u_0, \ldots, u_{r-1}]/(u_0^p),\) and

\[\Theta_{\text{GL}_a(r)} \in A \otimes k[u_0, \ldots, u_{r-1}]/(u_0^p)\]

is given by the complicated, but explicit formula (2.3.2). We conclude that

\[\Theta_{\text{GL}_a(r)} : A \otimes M \to A \otimes M\]

is the \(A\) linear endomorphism sending \(u_i\) to \(\bar{\Theta}_G \cdot u_i\), where \(\bar{\Theta}_G\) is the image of \(\Theta_G\) under the projection \(A \otimes k[u_0, \ldots, u_{r-1}]/(u_0^p) \to A \otimes M\).

3. Let \(M\) be the restriction to \(\text{GL}_n(r)\) of the canonical \(n\)-dimensional rational \(\text{GL}_n\)-module \(V_n\). By Example 1.11(3), \(A = k[V(\text{GL}_n(r))]\) is the quotient of \(k[gl_n]^{\otimes r}\) by the ideal generated by the equations satisfied by an \(r\)-tuple of \(n \times n\)-matrices with the property that each matrix is \(p\)-nilpotent and that the matrices pair-wise commute. The complexity of the map

\[\Theta_{\text{GL}_n(r)} : A \otimes V_n \to A \otimes V_n\]
is revealed even in the case \( n = 2 \) which is worked out explicitly below.

(4) Let \( M \) be the restriction to \( \text{SL}_2(2) \) of the rational \( \text{GL}_2 \) representation \( V_2 \). Then Example 1.11(4) gives an explicit description of \( A = k[V(\text{SL}_2(2))] \) as a quotient of \( k[x_0, y_0, z_0, x_1, y_1, z_1] \) and (2.3.3) gives \( \Theta_{\text{SL}_2(2)} \) explicitly. The divided powers \( e^{(p)} \), \( f^{(p)} \) and \( h^{(p)} \) as well as all products of the form \( e^{(i)} f^{(j)} h^{(k)} \) act trivially on \( M \). Hence, the map

\[
\Theta_{\text{SL}_2(2)} : A \otimes M \to A \otimes M
\]

is given by the matrix

\[
A^2 \begin{bmatrix}
z_1^p & x_1^p \\
-1 & y_1^p
\end{bmatrix}
\]

A^{2,2}.

For any affine group scheme \( G \), the \( k \)-algebra \( k[V_r(G)] \) is provided with a natural grading determined by the action of \( \mathbb{A}^1 \cong V_r(\mathbb{G}_a(1)) \subseteq V_r(\mathbb{G}_a(r)) \) on \( V_r(G) \) (see [17, 1.12]). From the point of view of functors on commutative \( k \)-algebra \( R \), this grading is determined by pre-composition

\[
V_r(G)(R) \times V_r(\mathbb{G}_a(1))(R) \to V_r(G)(R) \times V_r(\mathbb{G}_a(r))(R) \to V_r(G)(R).
\]

Observe that this grading is functorial with respect to homomorphisms \( G \to G' \) of group schemes.

When viewing group schemes as functors, it is often convenient to think of \( G_k[V(G)] \) as \( G \times V(G) \) (i.e., \( G \times V(G) = \text{Spec} k[V(G)] \otimes k[G] \)). From this point of view, \( \mathcal{U}_G \) has the form

\[
\mathcal{U}_G : \mathbb{G}_a(r) \times V(G) \to G \times V(G).
\]

**Lemma 2.5.** Let \( f : G \to G' \) be an embedding of infinitesimal group schemes inducing the map \( \phi : V(G) \to V(G') \) whose map on coordinate algebras we denote by \( \phi^* : A' \to A \). Then the following square commutes

\[
\begin{array}{ccc}
\mathbb{G}_a(r) \times V(G) & \xrightarrow{\mathcal{U}_A} & G \times V(G) \\
\text{id} \times \phi & \downarrow & \phi \times \text{id} \\
\mathbb{G}_a(r) \times V(G') & \xrightarrow{\mathcal{U}_{A'}} & G' \times V(G')
\end{array}
\]

**Proof.** By universality of \( \mathcal{U}_A' \), the composition \( (f \times \text{id}) \circ \mathcal{U}_A : \mathbb{G}_a(r) \times V(G) \to G' \times V(G) \) is obtained by pull-back of \( \mathcal{U}_A' \) via some morphism \( V(G) \to V(G') \). By checking at field valued points, we verify that this morphism must be \( \phi \). This implies the commutativity of (2.5.1).

If \( G = \text{GL}_N \), then an \( R \)-valued point of \( V_r(\text{GL}_N) \) is given by an \( r \)-tuple of \( N \times N \) pair-wise commuting, \( p \)-nilpotent matrices with entries in \( R \), \((\alpha_0, \ldots, \alpha_{r-1})\). The coordinate functions of the matrix \( \alpha_i \) have grading \( p^i \); in other words, the action of \( c \in V(\mathbb{G}_a(1))(R) \) on \((\alpha_0, \alpha_1, \ldots, \alpha_{r-1}) \in V(\text{GL}_N(r))(R) \) is given by the formula

\[
c \cdot (\alpha_0, \alpha_1, \ldots, \alpha_{r-1}) = (\alpha_0 c^p, \alpha_1 c^{p^2}, \ldots, \alpha_{r-1} c^{p^{r-1}}).
\]

More generally, if \( G \) is a closed subgroup scheme of \( \text{GL}_N \) of height \( \leq r \), then the embedding \( G \subset \text{GL}_N \) induces \( V(G) \to V(\text{GL}_N(r)) \) whose associated map on coordinate algebras \( k[V(G)] \to k[V(\text{GL}_N(r))] \) is a map of graded algebras.
Proposition 2.6. For any infinitesimal group scheme $G$, $\Theta_G \in k[V(G)] \otimes kG$ is homogeneous. More precisely, if $G$ has height $\leq r$ and if we let $A = k[V_r(G)]$, then $U_{A,+} : A_{\mathfrak{a}(r)} \rightarrow A$ sends $u_{r-1} \in \operatorname{Hom}_{k}(k[A_{\mathfrak{a}(r)}], A)$ to a linear combination in $A \otimes kG$ with coefficients from $A$ all having degree $p^{r-1}$.

Proof. Let $(\lambda_i)$ be a set of linear generators of $k[G]$, and $(\hat{\lambda}_i)$ be the dual set of linear generators of $kG$. Then $U_{A,+}(u_{r-1}) = \sum \hat{\lambda}_i \otimes f_i$ if and only if $u_{r-1}(U_{A}^\ast(\lambda_i)) = f_i$ if and only if $U_{A}^\ast(\lambda_i) = \ldots + f_i T^{p^{r-1}} + \ldots$. Hence, the assertion that $\theta_A$ is homogeneous of degree $p^{r-1}$ is equivalent to showing that the map $k[G] \rightarrow A$ defined by reading off the coefficient of

$U_{A}^\ast : k[G] \rightarrow A \otimes k[G] \rightarrow A \otimes k[A_{\mathfrak{a}(r)}] \rightarrow A[T]/T^{p^{r}}$

of the monomial $T^{p^{r-1}}$ is homogeneous of degree $p^{r-1}$.

The coordinate algebra $k[A_{\mathfrak{a}(r)}] \simeq k[T]/T^{p}$ has a natural grading with $T$ assigned degree 1. This grading corresponds to the monoidal action of $\mathbb{A}^1$ on $A_{\mathfrak{a}(r)}$ by multiplication:

$$
\mathbb{A}^1 \times A_{\mathfrak{a}(r)} \xrightarrow{s \times a \mapsto sa} A_{\mathfrak{a}(r)}
$$

We proceed to prove that this action is compatible with the action of $\mathbb{A}^1$ on $V_r(G)$ which defines the grading on $A$ in the sense that the following diagram commutes:

\begin{equation}
\begin{array}{ccc}
\mathbb{A}^1 \times V_r(G) & \xrightarrow{\text{id} \times \text{action}} & \mathbb{A}^1 \times V_r(G) \\
\downarrow \text{action} \times \text{id} & & \downarrow \text{action} \times \text{id} \\
\mathbb{A}^1 \times V_r(G) & \xrightarrow{U_{A}} & G \times V_r(G) \\
\downarrow \text{pr}_G & & \downarrow \text{pr}_G \\
G & & G
\end{array}
\end{equation}

(2.6.1)

Commutativity of (2.6.1) is equivalent to the commutativity of the corresponding diagram of $\mathbb{S}$-valued points for any choice of finitely generated commutative $k$-algebras $\mathbb{S}$ and element $a \in \mathbb{S}$:

\begin{equation}
\begin{array}{ccc}
\mathbb{A}^1 \times V_r(G)(\mathbb{S}) & \xrightarrow{1 \times a} & \mathbb{A}^1 \times V_r(G)(\mathbb{S}) \\
\downarrow \text{ax1} & & \downarrow \text{ax1} \\
\mathbb{A}^1 \times V_r(G)(\mathbb{S}) & \xrightarrow{U_{A}(\mathbb{S})} & G(\mathbb{S}) \times V_r(G)(\mathbb{S}) \\
\downarrow \text{pr}_G & & \downarrow \text{pr}_G \\
G(\mathbb{S}) & & G(\mathbb{S})
\end{array}
\end{equation}

(2.6.2)

Choose an embedding of $G$ into some $\text{GL}_{N(r)}$. Using Lemma 2.5 and the naturality with respect to change of $G$ of the action of $\mathbb{A}^1$ on $V_r(G)$, we can compare the diagram (2.6.2) for $G$ and for $\text{GL}_{N(r)}$. The injectivity of $G(\mathbb{S}) \rightarrow \text{GL}_{N(r)}(\mathbb{S})$ implies that it suffices to assume that $G = \text{GL}_{N(r)}$. Let $s \in \mathbb{G}_{\mathfrak{a}(r)}(\mathbb{S})$, $\alpha = (\alpha_0, \ldots, \alpha_{r-1}) \in V_r(\text{GL}_{N}))(\mathbb{S})$. Then $a \circ \alpha = (a_0 \alpha_0, a^p \alpha_1, \ldots, a^{p^{r-1}} \alpha_{r-1})$, and $\exp_{\alpha}(s) = \exp(s_0 \alpha_0) \exp(s^p \alpha_1) \ldots \exp(s^{p^{r-1}} a_{r-1}) \in \text{GL}_{N(r)}(\mathbb{S})$. Thus, restricted to the point $(s, \alpha) \in (\mathbb{G}_{\mathfrak{a}(r)} \times V_r(\text{GL}_{N}))(\mathbb{S})$, (2.6.2) becomes

\begin{equation}
\begin{array}{ccc}
(s, \alpha) & \xrightarrow{1 \times a} & (s, a \circ \alpha) \\
\downarrow \text{ax1} & & \downarrow \text{ax1} \\
(as, \alpha) & \xrightarrow{U_A} & (\exp_{\alpha}(as), \alpha) \\
\downarrow \text{ax1} & & \downarrow \text{ax1} \\
(\exp_{\alpha}(as), \alpha) & \xrightarrow{U_A} & \exp_{\alpha}(as)
\end{array}
\end{equation}

(2.6.3)
Commutativity of (2.6.3) is implied by the equality \( \exp_{cog}(s) = \exp_{cog}(as) \) which follows immediately by direct inspection of the formulas in [17, p.9].

Consequently, we have a commutative diagram on coordinate algebras corresponding to (2.6.1):

\[
\begin{array}{c}
k[G_{a(r)}] \otimes k[t] \otimes A \\
\downarrow{\scriptstyle (-\otimes t)\otimes\text{id}} \\
k[G_{a(r)}] \otimes A \\
\downarrow{\scriptstyle \mathcal{U}_g} \\
k[G] \otimes A \\
\downarrow{\scriptstyle \mathcal{U}_g} \\
k[G]
\end{array}
\]

The map \( \text{act}^*: A \longrightarrow k[t] \otimes A = k[\mathbb{A}^1] \otimes A \) is the map on coordinate algebras which corresponds to the grading on \( A \). The left vertical map corresponds to the grading on \( k[G_{a(r)}] \simeq k[T]/T^p \) and is given explicitly by \( T \mapsto T \otimes 1 \).

For \( \lambda \in k[G] \), write \( \mathcal{U}_g(\lambda \otimes 1) = \sum c_i T^i \otimes f_i \in k[G_{a(r)}] \otimes A \). The composition of the lower horizontal and left vertical maps of (2.6.4) sends \( \lambda \) to \( \sum c_i T^i \otimes (\text{act}^*(f_i)) \). We conclude that

\[
t^i \otimes f_i = \text{act}^*(f_i),
\]

so that \( f_i \) is homogeneous of degree \( i \).

As a corollary (of the proof of) Proposition 2.6, we see why for \( G \) infinitesimal of height \( \leq r \) the homogeneous degree of \( \Theta_{G,r} \in k[V_r(G)] \otimes kG \) is \( p^r-1 \) whereas the homogeneous degree of \( \Theta_{G,r+1} \in k[V_{r+1}(G)] \otimes kG \) is \( p^r \).

**Corollary 2.7.** Let \( G \) be an infinitesimal group of height \( \leq r \). Then the map \( i^*: k[V_{r+1}(G)] \rightarrow k[V_r(G)] \) of Proposition 2.2 is a graded isomorphism which divides degrees by \( p \).

**Proof.** Let \( \pi^*: k[V_r(G)] \rightarrow k[V_{r+1}(G)] \) be the inverse of \( i^* \). The commutativity of (2.6.1) implies that we may compute the effect on degree of \( \pi^* \) by identifying the effect on degree of the map \( p^*: k[G_{a(r)}] = k[[t]]/t^{p^r} \rightarrow k[[t]]/t^{p^{r+1}} = k[G_{a(r+1)}] \). Yet this map clearly multiplies degree by \( p \).

\[\square\]

### 3. \( \theta_v \) and local Jordan type

The purpose of this section is to exploit our universal \( p \)-nilpotent operator \( \Theta_G \) to investigate the local Jordan type of a finite dimensional \( kG \)-module \( M \). The local Jordan type of \( M \) gives much more detailed information about a \( kG \)-module \( M \) than the information which can be obtained from the support variety (or, rank variety) of \( M \). In this section, we work through various examples, give an algorithm for computing local Jordan types, and understand the effect of Frobenius twists. Moreover, we establish restrictions on the rank and dimension of \( kG \)-modules of constant Jordan type.

**Definition 3.1.** Let \( G \) be an infinitesimal group scheme and \( v \in V(G) \). Let \( k(v) \) denote the residue field of \( V(G) \) at \( v \), and let

\[
\mu_v = k(v) \otimes_{k[V(G)]} \mathcal{U}_G : G_{a(r),k(v)} \rightarrow G_{k(v)}
\]
be the associated 1-parameter subgroup (for \( r \geq \text{ht}(G) \)). We define the local \( p \)-nilpotent operator at \( v, \theta_v \), to be

\[ \theta_v = k(v) \otimes_k [V(G)] \Theta_G = (\mu_{v^*} \circ \epsilon)(u) \in k(v)G. \]

In the special case that \( G = \text{GL}_{n(r)} \) for some \( n > 0 \), we use the alternate notation \( \theta_{\alpha} \) for the local \( p \)-nilpotent operator at \( \alpha = (\alpha_0, \ldots, \alpha_{r-1}) \in V(\text{GL}_{n(r)}) \simeq N_p^r(\mathfrak{gl}_n) \):

\[ \theta_{\alpha} = (\exp_{\alpha^*} \circ \epsilon)(u) \in k(\alpha) \text{GL}_{n(r)}, \]

where \( k(\alpha) \) is the residue field of \( \alpha \in V(\text{GL}_{n(r)}) \).

Let \( K \) be a field. Then a finite dimensional \( K[u]/u^p \)-module \( M \) is a direct sum of cyclic modules of dimension ranging from 1 to \( p \). We may thus write \( M \simeq a_1[p] + \cdots + a_1[1] \), where \( [i] \) is the cyclic \( K[u]/u^p \)-module \( K[u]/u^i \) of dimension \( i \). We refer to the \( p \)-tuple

\[ \text{JType}(M, u) = (a_1, \ldots, a_1) \]

as the Jordan type of the \( K[u]/u^p \)-module \( M \). We also refer to \( \text{JType}(M, u) \) as the Jordan type of the \( p \)-nilpotent operator \( u \) on \( M \).

For simplicity, we introduce the following notation.

**Definition 3.2.** With notation as in Definition 3.1, we set

\[ \text{JType}(M, \theta_v) \equiv \text{JType}((\mu_{v^*} \circ \epsilon)^*(M_{k(v)}), u). \]

We refer to this Jordan type as the local Jordan type of \( M \) at \( v \in V(G) \).

The following proposition will enable us to make more concrete and explicit the local Jordan type of a \( kG \)-module \( M \) at a given 1-parameter subgroup of \( G \).

**Proposition 3.3.** Let \( \alpha = (\alpha_0, \ldots, \alpha_{r-1}) \in V(\text{GL}_{n(r)}) \) be an \( r \)-tuple of \( p \)-nilpotent pair-wise commuting matrices. Let \( M \) be a \( k \text{GL}_{n(r)} \)-module of dimension \( N \), and let \( \rho : \text{GL}_{n(r)} \rightarrow \text{GL}_N \) be the associated structure map. The \((i,j)\)-matrix entry of the action of the local \( p \)-nilpotent operator \( \theta_{\alpha} \in k(\alpha) \text{GL}_{n(r)} \) of (3.1.1) on \( M \) equals the coefficient of \( t^{p^{i-1}} \) of

\[ (\exp_{\alpha^*} \circ \epsilon)(\rho^* X_{ij}) \in k(\alpha)[G_{\alpha(r)}], \]

where \( \{X_{ij}, 1 \leq i, j \leq N \} \) are the coefficient functions of \( \text{GL}_N \).

**Proof.** Let \( \{m_i\}_{1 \leq i \leq N} \) be the basis of \( M \) corresponding to the structure map \( \rho \). The structure of \( M \) as a comodule for \( k[\text{GL}_{n(r)}] \) is given by

\[ M \rightarrow M \otimes k[\text{GL}_{n(r)}], \quad m_j \mapsto \sum_i m_i \otimes \rho^* X_{ij}, \]

and thus the comodule structure of \( M_{k(\alpha)} \) for \( k(\alpha)[G_{\alpha(r)}] \) is given by

\[ M \rightarrow M \otimes k(\alpha)[G_{\alpha(r)}], \quad m_j \mapsto \sum_i m_i \otimes \exp_{\alpha^*} \rho^* X_{ij}. \]

The proposition follows from the fact that \( u_{r-1} : k(\alpha)[G_{\alpha(r)}] \rightarrow k(\alpha) \) is given by reading off the coefficient of \( t^{p^{i-1}} \in k(\alpha)[G_{\alpha(r)}] \). \( \square \)
Example 3.4. We investigate the local Jordan type of the various representations considered in Example 2.4.

(1) Consider the adjoint representation $M = g^{ad}$ of a $p$-restricted Lie algebra $g$ and a 1-parameter subgroup

$$\mu_x : g_{a(1),K} \to g_K, \quad \text{inducing } K[u]/u^p \to u(g_K)$$

sending $u$ to some $p$-nilpotent $X \in g_K$. The local Jordan type of $g^{ad}$ at $\mu_x$ is simply the Jordan type of the endomorphism $\text{ad}_x : g^{ad}_K \to g^{ad}_K$, where

$$\text{JType}(g^{ad}, \theta_x) = \text{JType}(x).$$

(2) Let $M = k[G_{a(r)}]/(u_0) \simeq k[u_1, \ldots, u_{r-1}]/(u_1^p, \ldots, u_{p-1}^p)$ be a cyclic $kG_{a(r)} = k[u_0, \ldots, u_{r-1}]/(u_0^p, \ldots, u_{p-1}^p)$-module, and let $\mu_2 : G_{a(r)} \to G_{a(r)}$ be a 1-parameter subgroup for some $K$-rational point $\overline{g}$ of $V(G_{a(r)}) = A^r$. Then

$$\text{JType}(M, \theta_2) = \begin{cases} p^{r-2}[p], & \exists i > 0, a_i \neq 0 \\ p^{r-1}[1], & \text{otherwise}. \end{cases}$$

(3) Let $G = GL_n(r)$, and let $V_n$ be the canonical $n$-dimensional rational representation of $GL_n(r)$. We apply Proposition 3.3, observing that $\rho$ for $V_n$ is simply the natural inclusion $GL_n(r) \subset GL_n$. Since

$$\exp^*(X_{i,j}) = \sum_{\ell=0}^{p^r-1} [\beta_\ell] \cdot j^\ell,$$

where $\beta_\ell$ are matrices determined by $\alpha_i$ as in Proposition 1.6, we conclude

$$\text{JType}(V_n, \theta_2) = \text{JType}(\alpha_{r-1}).$$

Specializing to $r = 2$,

$$\text{JType}(V_n, \theta_{(\alpha, \alpha)}) = \alpha_1.$$ (3.4.1)

(4) “Specializing” to $G = SL_2(2)$, consider $\alpha = (\begin{bmatrix} a_0 & a_0 \\ b_0 & -c_0 \end{bmatrix}, \begin{bmatrix} c_1 & a_1 \\ b_1 & -c_1 \end{bmatrix})$.

Then $\text{JType}(V_2, \theta_2)$ equals the Jordan type of the matrix $\begin{bmatrix} c_1 & a_1 \\ b_1 & -c_1 \end{bmatrix}$.

We extend Example 3.4(3) by considering tensor powers $V_n^{\otimes d}$ of the canonical rational representation of $GL_n$ restricted to $GL_n(2)$. In this example, the role of both entries of the pair $\alpha = (a_0, a_1)$ is non-trivial.

Example 3.5. Consider the $N = n^d$-dimensional rational $GL_n$-module $M = V_n^{\otimes d}$ where $V_n$ is the canonical $n$-dimensional rational $GL_n$-module. Let $\rho : GL_n(2) \to GL_N$ be the representation of $M$ restricted to $GL_n(2)$. A basis of $M$ is $\{ e_{i_1} \otimes \cdots \otimes e_{i_d}; 1 \leq i_j \leq n \}$, where $\{ e_i; 1 \leq i \leq n \}$ is a basis for $V_n$. Let $\{ X_{i_1,j_1; \ldots, i_d,j_d}; 1 \leq i_j, j_j \leq n \}$ denote the matrix coefficients on $GL_N$, and let $\{ Y_{s,t}; 1 \leq s, t \leq n \}$ denote the matrix coefficients of $GL_n$.

Then $\rho^* : k[GL_N] \to k[GL_n]$ is given by

$$X_{i_1,j_1; \ldots, i_d,j_d} \mapsto Y_{i_1,j_1} \cdots Y_{i_d,j_d}.$$
Thus,
\[(\exp_2)^*(\rho^*(X_{i_1,j_1},\ldots,i_d,j_d)) = (\exp_2)^*(Y_{i_1,j_1}) \cdots (\exp_2)^*(Y_{i_d,j_d}).\]

Now, specialize to \(r = 2\) so that we can make this more explicit. Then the coefficient of \(\theta^r\) of \((\exp_{(\alpha_0,\alpha_1)})^*(\rho^*(X_{i_1,j_1},\ldots,i_d,j_d))\) is

\[
\sum_{k=1}^{d} (\alpha_1)_{i_k,j_k} + \sum_{\substack{\alpha \in J \leq p}} \frac{1}{f_1! \cdots \frac{1}{f_d!}} ((\alpha_0)^{\rho})_{i_1,j_1} \cdots ((\alpha_0)^{\rho})_{i_d,j_d}.
\]

This gives the action of \(\theta_{(\alpha_0,\alpha_1)}\) on \(M\).

To simplify matters even further, consider the special case \((\alpha_0)^2 = 0\). For \(1 \leq d < p\), \(\theta_{(\alpha_0,\alpha_1)}\) on \(M\) is given by the \(N \times N\)-matrix

\[
(i_1,j_1;\ldots;i_d,j_d) \mapsto (\sum_{k=1}^{d} (\alpha_1)_{i_k,j_k}).
\]

For \(d = p\), the action of \(\theta_{(\alpha_0,\alpha_1)}\) on \(M\) is given by the \(N \times N\)-matrix

\[
(i_1,j_1;\ldots;i_p,j_p) \mapsto (\sum_{k=1}^{p} (\alpha_1)_{i_k,j_k} + (\alpha_0)_{i_1,j_1} \cdots (\alpha_0)_{i_p,j_p}).
\]

An analogous calculation applies to the the \(d\)-fold symmetric product \(S^d(V_n)\) and \(d\)-fold exterior product \(\Lambda^d(V_n)\) of the canonical \(n\)-dimensional rational \(GL_n\)-module \(V_n\).

The proof of Proposition 3.3 applies equally well to prove the following straightforward generalization, which one may view as an algorithmic method of computing the “local Jordan type” of a \(kG\)-module \(M\) of dimension \(N\). The required input is an explicit description of the map on coordinate algebras \(\rho^*\) given by \(\rho : G \to GL_N\) determining the \(kG\)-module \(M\).

**Theorem 3.6.** Let \(G\) be an infinitesimal group scheme of height \(\leq r\), and let \(\rho : G \to GL_N\) be a representation of \(G\) on a vector space \(M\) of dimension \(N\). Consider some \(v \in V(G)\), and let \(\mu_v : G_{G(v)} \to G\) be the corresponding 1-parameter subgroup of height \(r\). Then the \((i,j)\)-matrix entry of the action of \(\theta_v \in G(v)\) on \(M\) equals the coefficient of \(\theta^r\) of

\[
(\mu_v)^*(\rho^*(X_{ij})) \in k(v)[G_{G(v)}],
\]

where \(\{X_{ij}, 1 \leq i,j \leq N\} \) are the coefficient functions of \(GL_N\).

As a simple corollary of Theorem 3.6, we give a criterion for the local Jordan type of the \(kG\)-module \(M\) to be trivial (i.e., equal to \((\dim M)[1]\)) at a 1-parameter subgroup \(\mu_v\), \(v \in V(G)\).

**Corollary 3.7.** With the hypotheses and notation of Theorem 3.6,

\[
\text{JType}(M,\theta_v) = \text{JType}(\mu_v \circ \rho)^*(M_{G(v)}),u) = (\dim M)[1]
\]

if \(\deg (\rho \circ \mu_v)^*(X_{ij}) < p^{-1}\) for all \(1 \leq i,j \leq N\).
One means of constructing $kG$-modules is by applying Frobenius twists to known $kG$-modules. Our next objective is to establish (in Proposition 3.9) a simple relationship between the $p$-nilpotent operator $\theta_\alpha$ on a $k\text{GL}_{n(r)}$-module $M$ and $\theta_\alpha$ on the $s$-th Frobenius twist $M^{(s)}$ of $M$ for any $0 \neq v \in V(\text{GL}_{n(r)})$.

Before formulating this relationship, we make explicit the definition of the Frobenius map for an arbitrary affine group scheme over $k$. Let $G$ be an affine group scheme over $k$ and define for any $s > 0$ the $s$-th Frobenius map $F^s : G \to G^{(s)}$ given by the $k$-linear algebra homomorphism

\begin{equation}
\tag{3.7.1}
F^s : k[G^{(s)}] = k \otimes_{p^s} k[G] \to k[G], \quad a \otimes f \mapsto a \cdot f^{p^s},
\end{equation}

where $k \otimes_{p^s} k[G]$ is the base change of $k[G]$ along the $p^s$-power map $k \to k$ (an isomorphism only for $k$ perfect). If $G$ is defined over $\mathbb{F}_{p^s}$ (for example, if $G = \text{GL}_n$), then we have a natural isomorphism

\begin{equation}
k[G] = k \otimes_{\mathbb{F}_{p^s}} \mathbb{F}_{p^s}[G] \cong k \otimes_{\mathbb{F}_{p^s}} k \otimes_{\mathbb{F}_{p^s}} \mathbb{F}_{p^s}[G] = k[G^{(s)}]
\end{equation}

so that $F^s$ can be viewed as a self-map of $G$.

**Definition 3.8.** If $M$ is a $kG$-module, then the $s$-th Frobenius twist $M^{(s)}$ of $M$ is the $k$-vector space $k \otimes_{p^s} M$. By naturality, $M^{(s)}$ inherits a $kG^{(s)}$-module structure. We view $M^{(s)}$ as a $kG$-module via the map $F^s : kG \to kG^{(s)}$ dual to (3.7.1).

To be more explicit, suppose the $N$-dimensional $kG$-module $M$ is given by $\rho : G \to \text{GL}_N$ (so that $M = \rho^*(V_N)$, where $V_N$ is the canonical $N$-dimensional $\text{GL}_N$-module) and assume that $G$ is defined over $\mathbb{F}_{p^s}$. Let $\mu_v : \mathbb{G}_a(r, k) \to G_K$ be a 1-parameter subgroup, corresponding to some $v \in V(G)$. Then the identification of $M^{(s)}$ with $(\rho \circ F^s)^*(V_N)$ implies that

\begin{equation}
\tag{3.8.1}
\text{JType}(M^{(s)}, \theta_\mu) = \text{JType}(M, \theta_{F^s(v)})
\end{equation}

where

\begin{equation}
\theta_{F^s(v)}(\theta_\mu) = ((\rho \circ F^s)(v) \circ \theta_\mu)(\alpha).
\end{equation}

Let $G = \text{GL}_n$, and let $R$ be a finitely generated commutative $k$-algebra. The Frobenius self-map is given explicitly on the $R$-valued of $\text{GL}_n(r)$ by the formula

\begin{equation}
F : \alpha \mapsto \phi(\alpha),
\end{equation}

where $\phi$ applied to $\alpha \in M_n(R)$ raises each entry of $\alpha$ to the $p$-th power. For $t$ in $\mathbb{G}_a(r)(R)$, we compute

\begin{equation}
(F \circ \exp(\alpha_0, \ldots, \alpha_{r-1}))(t) = F(\exp(t \alpha_0) \exp(t^p \alpha_1) \ldots \exp(t^{p^{r-1}} \alpha_{r-1})) = \\
\exp(t^p \phi(\alpha_0)) \exp(t^{p^2} \phi(\alpha_1)) \ldots \exp(t^{p^{r-1}} \phi(\alpha_{r-2})) = \exp(0, \phi(\alpha_0), \ldots, \phi(\alpha_{r-2}))(t).
\end{equation}

Iterating $s$ times, we obtain the following formula for $G = \text{GL}_n(r)$:

\begin{equation}
F^s \circ \exp(\alpha_0, \ldots, \alpha_{r-1}) = \exp(0, 0, \ldots, 0, \phi^s(\alpha_0), \ldots, \phi^s(\alpha_{r-s}))
\end{equation}

where the first non-zero entry on the right happens at the $(s + 1)$-st place.

In another special case of $G = \mathbb{G}_a(r)$ the Frobenius map $F : \mathbb{G}_a(r) \to \mathbb{G}_a(r)$ is given by raising an element $a \in \mathbb{G}_a(r)(R)$ to the $p$-th power. Let $a = (a_0, \ldots, a_{r-1})$ be a point in $V(\mathbb{G}_a(r)) \simeq \mathbb{A}^r$, and let $\mu_a : \mathbb{G}_a(r) \to \mathbb{G}_a(r)$ be the corresponding 1-parameter subgroup. For $t \in \mathbb{G}_a(r)(R)$, we have $\mu(t) = a_0 + a_1 t + \cdots + a_{r-1} t^{p-1}$ (see [17, §1]). The following formula is now immediate:

\begin{equation}
F^s \circ \mu(a_0, \ldots, a_{r-1}) = \mu(0, a_0, a_0^p, \ldots, a_0^{p^{r-s}}).
\end{equation}
Combining (3.8.1) and (3.8.2), we conclude the following proposition.

**Proposition 3.9.** Let $M$ be a finite dimensional $k\text{GL}_{n(r)}$-module and let $\alpha = (\alpha_0, \ldots, \alpha_{r-1})$ be a point in $V(\text{GL}_{n(r)})$. Then

$$\text{JType}(M^{(s)}, \theta_{\alpha}) = \text{JType}(M, \theta_{F^s \circ \phi_{\alpha}}),$$

where $F^s \circ \alpha = (0, \ldots, 0, \phi^s(\alpha_1), \ldots, \phi^s(\alpha_{r-1-s})).$

Similarly, if $M$ be a finite dimensional $k\mathbb{G}_{a(r)}$-module, and $\alpha = (a_0, \ldots, a_{r-1})$ be a point in $V(\mathbb{G}_{a(r)}) \simeq k^r$. Then

$$\text{JType}(M^{(s)}, \theta_{\alpha}) = \text{JType}(M, \theta_{F^s \circ \phi_{\alpha}}),$$

where $F^s \circ \alpha = (0, \ldots, 0, a_0^s, \ldots, a_{r-1-s}^s)$.

Proposition 3.9 has the following immediate corollary.

**Corollary 3.10.** Let $M$ be a finite dimensional $k\text{GL}_{n(r)}$-module. Then $\alpha = (\alpha_0, \ldots, \alpha_{r-1}) \in V(\text{GL}_{n(r)})$ lies in the rank variety $V(\text{GL}_{n(r)})_{M^{(s)}}$ (as defined in (1.14)) provided that $\alpha_0 = \cdots = \alpha_{r-1-s} = 0$.

The following definition introduces interesting classes of $kG$-modules which have special local behavior.

**Definition 3.11.** Let $G$ be an infinitesimal group scheme and $j$ a positive integer less than $p$. A finite dimensional $kG$-module $M$ is said to be of constant $j$-rank if and only if

$$\text{rk}(M, \theta_v) \equiv \text{rk}(\theta_v^j : M_{k(v)} \rightarrow M_{k(v)})$$

is independent of $v \in V(G) - \{0\}$, where $\theta_v$ is the local $p$-nilpotent operator at $v$ as introduced in Definition 3.1.

$M$ is said to be of constant Jordan type if and only if it is of constant $j$-rank for all $j$, $1 \leq j < p$. $M$ is said to be of constant rank if it is of constant $1$-rank.

As we see in the following example, one can have rational $\text{GL}_n$-modules of constant Jordan type when restricted to $\text{GL}_{n(r)}$ of arbitrarily high degree $d$. This should be contrasted with Corollary 3.16.

**Example 3.12.** Consider the rational $\text{GL}_n$-module $M = \text{det}^{\otimes d}$, the $d$th power of the determinant representation. This is a polynomial representation of degree $n^d$. The restriction of $M$ to any Frobenius kernel $\text{GL}_{n(r)}$ has trivial constant Jordan type, for the further restriction of $M$ to any abelian unipotent subgroup of $\text{GL}_n$ is trivial.

One method of constructing $kG$-modules of constant Jordan type is to start with some $kG$-module $M$ of constant Jordan type (for example, take $M$ to be the trivial $kG$-module $k$) and consider the $n$-th “syzygy” of $M$, $\Omega^n(M)$, $n \in \mathbb{Z}$. Recall that the syzygies $\Omega^n(M)$ of a $kG$-module are defined in terms of the minimal projective resolution of $M$ (see, for example, [1]).

**Example 3.13.** Let $\mathfrak{G}$ be a reduced, irreducible group scheme and $G = \mathfrak{G}_{(r)}$. Let $k \rightarrow k[G] \rightarrow \cdots \rightarrow k[G^{\otimes s}] \rightarrow \cdots$ be the cobar resolution of $k$ by free $k[G]$-modules, so that the dual $\cdots \rightarrow k[G^{\otimes s}] \rightarrow kG \rightarrow k$ is a resolution of $k$ by free $kG$-modules. Since $k[G]$ is self-injective, the cobar resolution is also a resolution by injective $kG$-modules. Since each $k[G^{\otimes s}]$ is a rational $\mathfrak{G}$-module and each map of the cobar
resolution is a map of $\mathfrak{S}$-modules, we conclude that the Heller shifts $\Omega^i(k)$, $i \in \mathbb{Z}$ are all rational $\mathfrak{S}$-modules. On the other hand, each $\Omega^i(k)$ has constant Jordan type as a $kG$-module, of Jordan type of the form $m[p] + [1]$ if $i$ is even and $m[p] + [p - 1]$ if $i$ is odd.

We shall see below that $kG$-modules of constant $j$-rank lead to interesting constructions of vector bundles (see Theorem 5.1). We conclude this section by establishing two constraints, Propositions 3.15 and 3.18, on $kG$-modules to be modules of constant rank.

We first need the following elementary lemma.

**Lemma 3.14.** Let $M$ be a $G_{a(r)}$-module such that the local Jordan type at every $v \in V(G_{a(r)})$ is trivial. Then $M$ is trivial.

**Proof.** The action of $G_{a(r)}$ on $M$ is given by the action of $r$ commuting $p$-nilpotent operators $\tilde{u}_i$, $0 \leq i < p$ on $M$. Moreover

$$JType(M, \theta_2) = JType(a_{r-1}\tilde{u}_0 + a_{r-2}\tilde{u}_1 + \cdots + a_0\tilde{u}_{r-1})$$

(see Example 2.3(2)). Thus, if the local Jordan type of $M$ is trivial at each $\theta = (0, \ldots, 1, 0, \ldots, 0)$, then each $\tilde{u}_i$ must act trivially on $M$ and $M$ is therefore a trivial $G_{a(r)}$-module.

**Proposition 3.15.** Let $M$ be a non-trivial rational $G_a$-module given by $\rho: G_a \to GL_N$. Let $D$ be an upper bound for the degrees of the polynomials $\rho^*(X_{ij}) \in k[G_a]$ where $\{X_{ij}\}$ are the standard polynomial generators of $k[GL_N]$. Then $M$ is not a $kG_{a(r)}$-module of constant rank provided that $r > \log_p D + 1$.

**Proof.** The condition $r > \log_p D$ implies that $M$ is not $r$-twisted (i.e., of the form $N^{(r)}$). Since $M$ is is not $r$-twisted, it is necessarily non-trivial as a $G_{a(r)}$-module. Lemma 3.14 implies that the local Jordan type of $M$ at some 1-parameter subgroup $\mu_v: G_{a(r)}, k(v) \to G_{k(v)}$ is non-trivial. On the other hand, Corollary 3.7 implies that the Jordan type of $M$ at the identity 1-parameter subgroup $id: G_{a(r)} \to G_{a(r)}$ is trivial provided that $r - 1 > \log_p D$.

The preceding theorem enables us to conclude that various rational modules $M$ for algebraic groups $\mathfrak{S}$ are not of constant Jordan type when restricted to $G_{a(r)}$ for $r$ sufficiently large. Namely, we apply Proposition 3.15 to the restriction of $M$ to some 1-parameter subgroup $G_a \to \mathfrak{S}$.

Because $SL_n$ is generated by its 1-parameter subgroups, we obtain the following corollary (which should be contrasted with Example 3.12).

**Corollary 3.16.** Let $M$ be a non-trivial polynomial $SL_n$-module of degree $D$. If $r > \log_p D + 1$, then $M$ is not a $kSL_{n(r)}$-module of constant rank.

The following lemma, which is a straightforward application of the Generalized Principal Ideal Theorem (see [6, 10.9]), shows that the dimension of a non-trivial module of constant rank of $G_{a(r)}$ cannot be “too small” compared to $r$.

**Lemma 3.17.** Let $M$ be a finite dimensional $G_{a(r)}$-module. If $M$ is a non-trivial $G_{a(r)}$-module of constant rank, then the following inequality holds:

$$(3.17.1) \quad \dim_k M \geq \sqrt{r}$$
Proof. By extending scalars if necessary we may assume that $k$ is algebraically closed. Let $m = \dim_k M$. Let $kG_{\alpha(r)} = k[u_0, \ldots, u_{r-1}]/(u_0^p, \ldots, u_{r-1}^p)$, let $K = k(s_0, \ldots, s_{r-1})$ where $s_i$ are independent variables, and let $\alpha_K : K[t]/t^p \to kG_{\alpha(r)}$ be a generic $\pi$-point given by $\alpha_K(t) = s_0u_0 + \cdots + s_{r-1}u_{r-1}$. Choose a $k$-linear basis of $M$, and let $A(s_0, \ldots, s_{r-1})$ be a nilpotent matrix in $M_n(k[s_0, \ldots, s_{r-1}])$ representing the action of $\alpha_K(t)$ on $M_K$. Let $I_n(A(s_0, \ldots, s_{r-1}))$ denote the ideal generated by all $n \times n$ minors of $A(s_0, \ldots, s_{r-1})$. By [6, 10.9], the codimension of any minimal prime over $I_n(A(s_0, \ldots, s_{r-1}))$ is at most $(m - n + 1)^2$.

Assume that (3.17.1) does not hold, that is, $m < \sqrt{d}$. Hence, $(m - n + 1)^2 < r$ for any $1 \leq n \leq m$. The variety of $I_n(A(s_0, \ldots, s_{r-1}))$ is a subvariety inside $\text{Spec} k[s_0, \ldots, s_{r-1}] \cong \mathbb{A}^r$ which has dimension $r$. Since the codimension of the variety of $I_n(A(s_0, \ldots, s_{r-1}))$ is at most $(m - n + 1)^2$, we conclude that the dimension is at least $r - (m - n + 1)^2 \geq 1$. Hence, the minors of dimension $n \times n$ have a common non-trivial zero. Taking $n = 1$, we conclude that $A(b_0, \ldots, b_{r-1})$ is a zero matrix for some non-zero specialization $b_0, \ldots, b_{r-1}$ of $s_0, \ldots, s_{r-1}$. Consequently, $M$ is trivial at the $\pi$-point of $G_{\alpha(r)}$ corresponding to $b_0, \ldots, b_{r-1}$. Since $M$ is non-trivial, Lemma 3.14 implies that $M$ is not a module of constant rank.

As an immediate corollary, we provide an additional necessary condition for a $kG_{\alpha(r)}$-module to have constant rank.

Proposition 3.18. Let $\mathfrak{g}$ be a (reduced) affine algebraic group and $M$ be a rational representation of $\mathfrak{g}$. Assume that $\mathfrak{g}$ admits a 1-parameter subgroup $\mu : \mathbb{G}_a(r) \to \mathfrak{g}$ such that $\mu^* M$ is a non-trivial $kG_{\alpha(r)}$-module. If $r \geq (\dim M)^2 + 1$, then $M$ is not a $kG_{\alpha(r)}$-module of constant rank.

4. $\pi$-points and $\text{Proj} V(G)$

In a series of earlier papers, we have considered $\pi$-points for a finite group scheme $G$ (as recalled in Definition 4.1) and investigated finite dimensional $kG$-modules $M$ using the “Jordan type of $M$” at various $\pi$-points. In particular, in [12], we verified that this Jordan type is independent of the equivalence class of the $\pi$-point provided that either the $\pi$-point is generic or the Jordan type of $M$ at some representative of the equivalence class is maximal.

As we recall below, whenever $G$ is an infinitesimal group scheme, then the $\pi$-point space $\Pi(G)$ of equivalence classes of $\pi$-points is essentially the projectivization of $V(G)$. The purpose of the first half of this section is to relate the discussion of the previous section concerning the local Jordan type of a finite $kG$-module to our earlier work formulated in terms of $\pi$-points for general finite group schemes.

One special aspect of an infinitesimal group scheme $G$ is that equivalence classes of $\pi$-points of $G$ have canonical (up to scalar multiple) representatives.

Unless otherwise specified (as in Definition 4.1 immediately below), $G$ will denote an infinitesimal group scheme over $k$, and $V(G)$ will denote $V_r(G)$ for some $r \geq \text{ht}(G)$. Throughout this section we assume that $\dim V(G) \geq 1$, and work with $\text{Proj} V(G)$.

Definition 4.1. (see [11]) Let $G$ be a finite group scheme.

1. A $\pi$-point of $G$ is a (left) flat map of $K$-algebras $\alpha_K : K[t]/t^p \to KG$ for some field extension $K/k$ with the property that there exists a unipotent
abelian closed subgroup scheme \( i : C_K \subset G_K \) defined over \( K \) such that \( \alpha_K \) factors through \( i_* : KCG \to KG \). (2) If \( \beta_L : L[t]/t^p \to LG \) is another \( \pi \)-point of \( G \), then \( \alpha_K \) is said to be a specialization of \( \beta_L \), written \( \beta_L \Downarrow \alpha_K \), provided that for any finite dimensional \( kG \)-module \( M, \alpha_K(M) \) being free as \( K[t]/t^p \)-module implies that \( \beta_L(M) \) is free as \( L[t]/t^p \)-module.

(3) Two \( \pi \)-points \( \alpha_K : K[t]/t^p \to KG, \beta_L : L[t]/t^p \to LG \) are said to be equivalent, written \( \alpha_K \sim \beta_L \), if \( \alpha_K \Downarrow \beta_L \) and \( \beta_L \Downarrow \alpha_K \).

(4) A \( \pi \)-point of \( G, \alpha_K : K[t]/t^p \to KG \), is said to be generic if there does not exist another \( \pi \)-point \( \beta_L : L[t]/t^p \to LG \) which specializes to \( \alpha_K \) but is not equivalent to \( \alpha_K \).

(5) If \( M \) is a finite dimensional \( kG \)-module and \( \alpha_K : K[t]/t^p \to KG \) a \( \pi \)-point of \( G \), then the Jordan type of \( M \) at \( \alpha_K \) is by definition the Jordan type of \( \alpha_K(M) \) as \( K[t]/t^p \)-module.

Because the group algebra of a finite group scheme is always faithfully flat over the group algebra of a subgroup scheme (see [19, 14.1]), the condition on a flat map \( \alpha_K : K[t]/t^p \to KG \) is equivalent to the existence of a factorization \( i_* \circ \alpha_K \) with \( i_* \circ \alpha_K : K[t]/t^p \to KCG \) flat.

**Definition 4.2.** Let \( G \) be an infinitesimal scheme, and let \( v \in V(G) \) be the point associated to the 1-parameter subgroup \( \mu_v : G_{a(v),k(v)} \to G_{k(v)} \). Then the \( \pi \)-point of \( G \) associated to \( v \) is

\[
\mu_{v,*} \circ \epsilon : k(v)[t]/t^p \to k(v)G.
\]

The following theorem is a complement to Theorem 1.15, revealing that spaces of (equivalence) classes of \( \pi \)-points are very closely related to (cohomological) support varieties.

**Theorem 4.3.** ([11, 7.5]) Let \( G \) be an finite group scheme. Then the set of equivalence classes of \( \pi \)-points can be given a scheme structure, denoted \( \Pi(G) \), which is defined in terms of the representation theory of \( G \). Moreover, there is an isomorphism of schemes

\[
\text{Proj } \Pi^\bullet(G,k) \simeq \Pi(G).
\]

If \( G \) is an infinitesimal group scheme so that \( \Pi^\bullet(G,k) \) is related to \( k[V(G)] \) as in Theorem 1.15, then the resulting homeomorphism

\[
\text{Proj } V(G)_{\text{red}} \longrightarrow \Pi(G)_{\text{red}} \tag{4.3.1}
\]

is given on points by sending \( x \in \text{Proj } V(G) \) to the equivalence class of the \( \pi \)-point \( \mu_{v,*} \circ \epsilon \) for any \( v \in V(G) \setminus \{0\} \) projecting to \( x \). In particular, equivalence classes of generic \( \pi \)-points of \( G \) are represented by \( (\mu_{v,*} \circ \epsilon) \) as \( v \in V(G) \) runs through the (scheme-theoretic) generic points of \( V(G) \).

Furthermore, for any finite dimensional \( kG \)-module \( M \), (4.3.1) restricts to a homeomorphism of reduced, closed subvarieties

\[
\text{Proj } V(G)_M \simeq \Pi(G)_M,
\]

where \( \Pi(G)_M \) consists of those equivalence classes of \( \pi \)-points \( \alpha_K \) of \( G \) such that \( \alpha_K(M) \) is not free (as a \( K[u]/u^2 \)-module).
Generic $\pi$-points are particularly important when developing invariants of representations. The following corollary of Theorem 4.3 gives an explicit set of representatives of equivalence classes of generic $\pi$-points of $G$.

**Proposition 4.4.** Let $G$ be an infinitesimal group scheme with universal 1-parameter subgroup $U_G : G_{\pi(r), k[V(G)]} \to G_{k[V(G)]}$. For each minimal prime ideal $\mathcal{P}_i$ of $k[V(G)]$, let $K_i$ denote the field of fractions of $k[V(G)]/\mathcal{P}_i$. Then there is a well defined endomorphism (depending upon $\Theta_k$) for the fiber of the coherent sheaf $\mathcal{M}$.

**Definition 4.5.** Let $G$ be an infinitesimal group scheme of height $\leq r$ and let $M$ be a finite dimensional $kG$-module. Then we denote by

$$\bar{\Theta}_G : \mathcal{M} \equiv \mathcal{O}_{\text{Proj} V(G)} \otimes M \to \mathcal{O}_{\text{Proj} V(G)}(p^{-1}) \otimes M \equiv \mathcal{M}(p^{-1})$$

the associated homomorphism of (locally free) coherent $\mathcal{O}_{\text{Proj} V(G)}$-modules determined by the action of $\Theta_{G,r} \in k[V(G)] \otimes kG$.

For any point $x \in \text{Proj} V(G)$, we use the notation

$$M_{k(x)} = k(x) \otimes_{\mathcal{O}_{\text{Proj} V(G)}} \mathcal{M}$$

for the fiber of the coherent sheaf $\mathcal{M}$ at $x$. Here, we have identified $k(x)$ with the residue field of the stalk $\mathcal{O}_{\text{Proj} V(G),x}$.

**Proposition 4.6.** Let $G$ be an infinitesimal group scheme of height $\leq r$, and let $M$ be a finite dimensional $kG$-module. For any $v, v' \in V(G)$ projecting to the same $x \in \text{Proj} V(G)$, we have

$$\text{Im}\{\theta_v : M_{k(v)} \to M_{k(v')}\} \cong \text{Im}\{\theta_{v'} : M_{k(v')} \to M_{k(v')}\}$$

and similarly for kernels.

**Proof.** This is essentially proved in [18, 6.1].

In the next section, we shall be particularly interested in kernels and images of $\bar{\Theta}_G$. The following proposition relates the local $p$-nilpotent operator $\theta_v$ on $M$ at the point $v \in V(G)$ to the fiber of the action of $\bar{\Theta}_G$ on the coherent sheaf $\mathcal{M}$.

**Proposition 4.7.** Let $G$ be an infinitesimal group scheme of height $\leq r$, let $M$ be a finite dimensional $kG$-module, and let $s \in \Gamma(\text{Proj} V(G), \mathcal{O}_{\text{Proj} V(G)}(p^{-1}))$ be a non-zero global section with zero locus $Z(s) \subset \text{Proj} V(G)$. Set $U = \text{Proj} V(G) \setminus Z(s)$. Then there is a well defined endomorphism (depending upon $s$)

$$\bar{\Theta}_G/s : \mathcal{M}|_U \to \mathcal{M}|_U.$$

Moreover, the image and kernel of the induced map $\theta_x/s : M_{k(x)} \to M_{k(x)}$ on fibers at $x \in U \subset \text{Proj} V(G)$ is independent of $s$ and satisfies

$$\text{Im}\{\theta_x/s : M_{k(x)} \to M_{k(x)}\} \cong \text{Im}\{\theta_v : M_{k(v)} \to M_{k(v)}\}$$

and

$$\text{Ker}\{\theta_x/s : M_{k(x)} \to M_{k(x)}\} \cong \text{Ker}\{\theta_v : M_{k(v)} \to M_{k(v)}\}$$

for any $v \in V(G) \setminus \{0\}$ that projects onto $x$. 
Then we define
\[ \tilde{\Theta}_G/s \equiv 1/s \otimes (\tilde{\Theta}_G)|_U : M|_U \to \mathcal{O}_X(-p^{r-1}))|_U \otimes M(p^{r-1})|_U \cong M|_U. \]

The second statement is essentially proved in [18, 6.1].

**Remark 4.8.** For a finite group $G$, there is no natural choice of $\pi$-point representing a typical equivalence class $x \in \Pi(G)$ such that $\text{Proj} \, H^*(G, k)$ of $\pi$-points. As seen in elementary examples, the Jordan type of a $kG$-module $M$ typically can be different for two equivalent $\pi$-points representing the same point $x \in \Pi(G)$.

**Remark 4.9.** Proposition 4.7 immediately generalizes to $\Theta_j^G$ for any $1 \leq j \leq p-1$. It implies the following isomorphisms for any $x \in X = \text{Proj} V(G)$, $v \in V(G)$ projecting onto $x$, and a global section $s$ of $\mathcal{O}_X(jp^{r-1})$ such that $s(x) \neq 0$:

\[
\text{Im}\{(\theta/s)^j : M_k(x) \to M_k(x)\} \cong \text{Im}\{\theta_x^j : M_k(x) \to M_k(x)\} \cong \text{Im}\{\tilde{\Theta}_G : k(x) \otimes \mathcal{O}_X \to k(x) \otimes \mathcal{O}_X(p^{r-1})\},
\]

and similarly for kernels. In what follows, we shall use the following abbreviations:

\[
\text{Im}\{\tilde{\Theta}_G, M\} \equiv \text{Im}\{\tilde{\Theta}_G : M \to M(p^{r-1})\},
\]

\[
\text{Im}\{\theta_x^j, M_k(x)\} \equiv \text{Im}\{(\theta_x/s)^j : M_k(x) \to M_k(x)\},
\]

\[
\text{Ker}\{\tilde{\Theta}_G, M\} \equiv \text{Ker}\{\tilde{\Theta}_G : M \to M(p^{r-1})\},
\]

\[
\text{Ker}\{\theta_x^j, M_k(x)\} \equiv \text{Ker}\{(\theta_x/s)^j : M_k(x) \to M_k(x)\}.
\]

We shall verify in Theorem 4.12 that a necessary and sufficient condition on a finite dimensional $kG$-module $M$ for $\text{Im}\{\tilde{\Theta}_G, M\}$ (and thus $\text{Ker}\{\tilde{\Theta}_G, M\}$) to be an algebraic vector bundle on $X$ is that $M$ be a module of constant $j$-type.

**Proposition 4.10.** Let $X$ be a reduced, noetherian scheme and consider two coherent $\mathcal{O}_X$-modules $\tilde{N}$, $\tilde{N}$. If $f : \tilde{M} \to \tilde{N}$ is a map of $\mathcal{O}_X$-modules whose map on fibers, $k(x) \otimes \mathcal{O}_X \to \tilde{N}_k(x)$, is injective (respectively, surjective) for every $x \in X$, then $f$ is itself injective (resp., surjective).

Consequently, for $X$ as above and $\tilde{M}$ a coherent $\mathcal{O}_X$-module, $\tilde{M}$ is locally free if and only if $\text{dim}_{k(x)}(k(x) \otimes \mathcal{O}_X \tilde{M})$ depends only upon the connected component of $x$ in $\pi_0(X)$.

**Proof.** A familiar form of Nakayama’s Lemma asserts that surjectivity of a map of finite $\mathcal{O}_X$-modules is equivalent to the surjectivity of the induced map on fibers at all closed points $x \in X$.

Now assume that $f : \tilde{M} \to \tilde{N}$ is a map of coherent $\mathcal{O}_X$ modules which induces an injective map on fibers at all points $x \in X$. To prove that $f$ itself is injective, we easily reduce to the case that $X = \text{Spec} R$ is affine, with $R$ a commutative Noetherian ring without nilpotents, and $f : M \to N$ a map of $R$-modules. Set $S \equiv \text{Ker}\{f\}$. Our hypothesis on $f$ implies that $S$ maps to 0 in $M/pM$ and thus that $S = S \cap pM$ for every prime $p \subset R$. This implies that $S = S \cap ((\cap p)M)$. On the other hand, the intersection of the prime ideals of $R$ is the nil-radical which we have assumed to be 0.
To prove the second assertion, it suffices to assume that $X$ is local, so that $X = \text{Spec} \ R$ for some local Noetherian commutative ring, and that $M$ is a finite $R$-module with the property that $\dim_{k(p)}(k(p) \otimes_R M)$ is independent of the prime $p \subset R$. To prove $M$ is free, we choose some surjective $R$-module homomorphism $g : Q \to M$ from a free $R$-module $Q$ to $M$ with the property that $\overline{g} : R/m \otimes_R Q \to R/m \otimes_R M$ is an isomorphism where $m \subset R$ is the maximal ideal. Then $g$ is surjective by Nakayama’s lemma. Right exactness of tensor product and exactness of localization implies that the induced map $k(p) \otimes_R Q \to k(p) \otimes_R M$ is surjective for all primes $p \subset R$. The condition on $\dim_{k(p)}(k(p) \otimes_R M)$ thus implies that each of the induced maps $k(p) \otimes_R Q \to k(p) \otimes_R M$ is an isomorphism. Now, by the first part of this proposition, we conclude that $g : Q \to M$ is an isomorphism.

We shall find it convenient to “localize” the notion of a $kG$-module of constant $j$-rank given in Definition 3.11 as follows.

Definition 4.11. Let $G$ be an infinitesimal group scheme, and let $M$ be a finite dimensional $kG$-module. For any open subset $U \subset \text{Proj} \ V(G)$, $M$ is said to be of constant $j$-rank when restricted to $U$ if $\text{rk}_{k(x)}((\theta/s)^j)$ is independent of $x \in U$.

Our next theorem emphasizes the local nature of the concept of constant $j$-rank.

Theorem 4.12. Let $G$ be an infinitesimal group scheme, let $M$ be a finite dimensional $kG$-module, and let $X = \text{Proj} \ V(G)$. Let $U \subset X$ be a connected open subset, and let $\tilde{\Theta}_j : \mathcal{M}_U \to \mathcal{M}(jp^{j-1})|_U$ be the restriction to $U$ of the $j$th iterate of $\tilde{\Theta}_G$ on $\mathcal{M} = \mathcal{O}_X \otimes M$ as given in (4.5.1). Then the following are equivalent for some fixed $j$:

1. $\text{Im}\{\tilde{\Theta}_j^U, \mathcal{M}_U\}$ is a locally free, coherent $\mathcal{O}_U$-module.
2. $k(x) \otimes_{\mathcal{O}_X} \text{Im}\{\tilde{\Theta}_j^G, \mathcal{M}\}$ has dimension independent of $x \in U$.
3. $\text{Im}\{\theta_j^x, M_{k(x)}\} \simeq k(x) \otimes_{\mathcal{O}_X} \text{Im}\{\tilde{\Theta}_j^G, \mathcal{M}\}$, $\forall \ x \in U$
4. $M$ has constant $j$-rank when restricted to $U$.

Moreover, each of these conditions implies that

5. $\ker\{\tilde{\Theta}_j^U, \mathcal{M}_U\}$ is a locally free, coherent $\mathcal{O}_U$-module.
6. $\ker\{\theta_j^x, M_{k(x)}\} \simeq k(x) \otimes_{\mathcal{O}_X} \ker\{\tilde{\Theta}_j^G, \mathcal{M}\}$, $\forall \ x \in U$.

Proof. Clearly, (1) implies (2), whereas Proposition 4.10 implies that (2) implies (1).

If we assume (1), we obtain a locally split short exact sequence of coherent $\mathcal{O}_U$-modules

$$0 \to \ker\{\tilde{\Theta}_j^U, \mathcal{M}_U\} \to \mathcal{M}_U \to \text{Im}\{\tilde{\Theta}_j^U, \mathcal{M}_U\} \to 0.$$ (4.12.1)

In particular, $\ker\{\tilde{\Theta}_j^U, \mathcal{M}_U\}$ is a locally free, coherent $\mathcal{O}_U$-module. Locally on $U$, $\tilde{\Theta}_j^U$ on $\mathcal{M}_U$ is isomorphic to the projection

$$pr_2 : \ker\{\tilde{\Theta}_j^U, \mathcal{M}_U\} \oplus \text{Im}\{\tilde{\Theta}_j^U, \mathcal{M}_U\} \to \text{Im}\{\tilde{\Theta}_j^U, \mathcal{M}_U\}.$$ (4.12.2)

Since $\theta_j^x$ is the base change via $\mathcal{O}_U \to k(x)$ of $\tilde{\Theta}_j^U$, $\theta_j^x$ can be identified with the base change of this projection and thus we may conclude (3).

Let us now assume (3). A simple argument using Nakayama’s Lemma as in the proof of Proposition 4.10 implies that the function $x \mapsto \text{Im}\{\theta_j^x, M_{k(x)}\}$ is lower semi-continuous on $U$ whereas the function $x \mapsto k(x) \otimes_{\mathcal{O}_X} \text{Im}\{\tilde{\Theta}_j^G, \mathcal{M}\}$ is upper...
semi-continuous on $U$. Thus, we conclude that each of these functions is constant (since $U$ is connected), thereby concluding (2).

Since $\text{rk}\{[\theta_x]\} = \dim_k(\text{Im}(\theta_x))$, (2) and (3) imply (4).

To prove that (4) implies (5), observe that if $f: V \to V$ is an endomorphism of a finite dimensional vector space then $\dim(\text{Ker} f) = \dim(\text{Im} f)$. Define $\text{Coker}\{[\theta_x]\} = \text{Coker}\{[\theta_x]\}$. The assumption that the $kG$-module $M$ has constant rank (i.e., (4)) implies that

$$\dim_k(\text{Coker}\{[\theta_x]\}) = \dim_k(\text{Ker}\{[\theta_x]\})$$

is independent of $x \in U$. On the other hand, the right exactness of $k(x) \otimes_{\mathcal{O}_X} (-)$ applied to

$$M \xrightarrow{\theta_x} M(jp^{r-1}) \to \text{Coker}\{[\theta_x]\} \to 0$$

implies that

$$\text{Coker}\{[\theta_x]\} = k(x) \otimes_{\mathcal{O}_X} \text{Coker}\{[\theta_x]\}.$$

Hence, Proposition 4.10 implies that $\text{Coker}\{[\theta_x]\}$ is a locally free coherent $\mathcal{O}_U$-module whenever $M$ is of constant rank on $U$. Thus, assuming (4), we obtain a locally split short exact sequence of coherent $\mathcal{O}_U$-modules

$$0 \to \text{Im}(\theta_x^\prime, M_{|U}) \to M(jp^{r-1})_{|U} \to \text{Coker}\{[\theta_x]\} \to 0,$$

so that $\text{Im}(\theta_x^\prime, M_{|U})$ is a locally free, coherent $\mathcal{O}_U$-module. Now, using the the short exact sequence (4.12.1) is locally split, applying $k(x) \otimes_{\mathcal{O}_X} (-)$ to (4.12.1) for any $x \in U$ yields a short exact sequence, thereby implying (6). \qed

5. Vector bundles for modules of constant $j$-rank

In this section, we initiate the study of algebraic vector bundles associated to $kG$-modules of constant $j$-rank as defined in 3.11. Our constructions have two immediate consequences. The first is that certain $kG$-modules with the same “local Jordan type” have non-isomorphic associated vector bundles, so that the isomorphism classes of these vector bundles serve as a new invariant. The second is that our construction yields vector bundles on the highly non-trivial projective schemes $\text{Proj} V(G)$.

As in §4, we assume that $\dim V(G) \geq 1$ throughout this section.

The special case in which $U = \text{Proj} V(G)$ of Theorem 4.12 is the following assertion that $kG$-modules of constant $j$-rank determine algebraic vector bundles over $\text{Proj} V(G)$.

**Theorem 5.1.** Let $G$ be an infinitesimal group scheme, let $M$ be a finite dimensional $kG$-module, and let $\mathcal{M} = \mathcal{O}_{\text{Proj} V(G)} \otimes M$ be a free coherent sheaf on $\text{Proj} V(G)$. Then $M$ has constant $j$-rank if and only if $\text{Im}(\theta_x^\prime, M)$ is an algebraic vector bundle on $\text{Proj} V(G)$.

Consequently, if $M$ has constant $j$-rank, then $\text{Ker}\{[\theta_x]\}$ also is an algebraic vector bundle on $\text{Proj} V(G)$. 
Remark 5.2. Unless \( M \) is trivial as a \( kG \)-module, \( \ker(\Theta_G : k[V(G)] \otimes M \to k[V(G)] \otimes M) \) is not projective as a \( k[V(G)] \)-module, since the local \( p \)-nilpotent operator \( \theta_0 \) at \( 0 \in V(G) \) is the 0-map.

Example 5.3. For each of our four examples of infinitesimal group schemes (initially investigated in Example 1.4), we give examples of \( kG \)-modules of constant Jordan type taken from \([4]\).

1. Let \( \mathfrak{g} \) be a finite dimensional \( p \)-restricted Lie algebra of dimension at least 2. For any Tate cohomology class of negative dimension, \( \zeta \in \hat{H}^n(u(\mathfrak{g}), k) \simeq \text{Ext}^1_{u(\mathfrak{g})}(\Omega^{n-1}(k), k) \), we consider the extension of \( u(\mathfrak{g}) \)-modules

\[
0 \longrightarrow k \longrightarrow M \longrightarrow \Omega^{n-1}(k) \longrightarrow 0
\]
determined by \( \zeta \). By \([4, 6.3]\), \( M \) is a \( u(\mathfrak{g}) \)-module of constant Jordan type. We verify by inspection that the Jordan type of \( M \) is \((a,0,\ldots,0,2)\) for some \( a > 0 \) if \( n \) is odd, and \((b,1,0,\ldots,0,1)\) for some \( b > 0 \) if \( n \) is even (see (3.1.2) for notation).

2. Let \( G = G_u(r)\), and set \( I \) equal to the augmentation ideal of \( kG \simeq k[u_0, \ldots, u_{p-1}]/(u_0^p, \ldots, u_{p-1}^p)\). As observed in \([4]\), \( I/I^t \) is a module of constant Jordan type for any \( t > i \). According to an unpublished, non-trivial computation of A. Suslin, the only ideals of \( kG_u(2) \) which are of constant Jordan type are of the form \( I^t \).

3. As observed in \([4]\), the \( n \)-th syzygy module \( \Omega^n(k) \) is a module of constant Jordan type for any infinitesimal group scheme \( G \). For \( n \) even, \( \Omega^n(k) \) has constant Jordan type \((a,0,\ldots,0,1)\) for some \( a > 0 \); whereas for \( n \) odd, \( \Omega^n(k) \) has constant Jordan type \((b, p-1,0,\ldots,0)\) for some \( b > 0 \).

4. For \( G = \text{SL}_{2}(k)\), we recall that the cohomology algebra \( H^*(G,k) \) is generated modulo nilpotents by classes \( \zeta_1, \zeta_2, \zeta_3 \in H^2(G,k) \) and classes \( \xi_1, \xi_2, \xi_3 \in H^2p(G,k) \) \(([13])\). As in \([4, 6.8]\), the \( kG \)-module

\[
M \equiv \ker\left\{ \sum \zeta_i + \sum \xi_j : (\Omega^2(k))^{\otimes 3} \oplus (\Omega^2p(k))^{\otimes 3} \to k \right\}
\]
is a \( kG \)-module of constant Jordan type \((a,0,\ldots,0,1)\) for some \( a > 0 \).

We elaborate on the Example 5.3(2), constructing \( G_u(r) \)-modules of constant \( j \)-rank for but not of constant Jordan type.

Example 5.4. We start with the following simple observation. Let \( M_1 \subset M_2 \subset M \) be a chain of \( k \)-vector spaces, and let \( \phi \) be an endomorphism of \( M_3 \) such that \( \phi(M_1) \subset M_1 \) and \( \phi(M_2) \subset M_2 \). If \( \dim(\ker \phi_{|M_1}) = \dim(\ker \phi) \) then \( \dim(\ker \phi_{|M_2}) = \dim(\ker \phi) \).

Let \( G = G_u(r) \), and set \( I \) equal to the augmentation ideal of \( kG \simeq k[u_0, \ldots, u_{p-1}]/(u_0^p, \ldots, u_{p-1}^p)\). Consider any ideal \( J \) of \( kG \) with the property that \( I^i \subset J \) for some \( i \), \( i \leq p-1 \). Note that for any \( \alpha \in k^* \), and any \( j \leq p-i \),

\[
\dim(\ker \left\{ \theta_{\alpha}^j : I^i \to I^i \right\}) = pj = \dim(\ker \left\{ \theta_{\alpha}^j : kG \to kG \right\})
\]
Indeed, since \( I^i \) is a module of constant Jordan type, it suffices to check the statement for \( \theta_{\alpha}^j = u_0 \) for which it is straightforward. The observation in the previous
paragraph together with (5.4.1) and the inclusions \( I^i \subset J \subset kG \) imply
\[
\dim(\text{Ker}\{\theta^i_j : J \to J\}) = pj
\]
for any \( j \leq p - i \) and any \( a \in K^r \). Hence, \( J \) has constant \( j \)-rank for \( 1 \leq j \leq p - i \).

In the following example, we offer a method applicable to almost all infinitesimal group schemes \( G \) of constructing \( kG \)-modules which are of constant rank but not constant Jordan type.

**Example 5.5.** Let \( G \) be an infinitesimal group scheme with the property that \( V(G) \) has dimension at least 2. Assume that \( p \) is odd, and let \( n > 0 \) be an odd positive integer. Let \( \zeta \in H^n(G, k) \) be a non-zero cohomology class and let \( M \) denote the kernel of \( \zeta : \Omega^n(k) \to k \). Then \( M \) has constant rank but not constant Jordan type. Namely, the local Jordan type of \( \zeta \) at \( 0 \) \( \neq v \in V(G) \) is \( (a, 0, 1, 0, \ldots, 0) \) if \( \zeta(v) \neq 0 \), and is \( (a - 1, 2, 0, \ldots, 0) \) if \( \zeta(v) = 0 \). These Jordan types have the same rank.

For \( G = \text{SL}_2(1) \) any module is a module of constant Jordan type (see [4]). We calculate explicitly which bundles correspond the irreducible \( \text{SL}_2(1) \)-modules.

**Example 5.6.** Let \( G = \text{SL}_2(1) \), \( A = k[V(G)] \), and let \( M \) be the canonical rational 2-dimensional \( \text{SL}_2 \)-module. We have \( A \simeq k[x, y, z]/(xy + z^2) \). Hence, \( \text{Proj} V(G) \) is a smooth projective conic and therefore is isomorphic to \( \mathbb{P}^1 \). The universal \( p \)-nilpotent operator \( \Theta_G \) is given by the formula \( \Theta_G = xe + yf + zh \), where \( (e, f, h) \) is the standard basis of \( \text{sl}_2 \) (see Example 2.3(1)). Hence, by Example 2.4, the action of \( \Theta_G \) on \( A \otimes M \simeq A^2 \) is given by the matrix
\[
\Theta_G : A^2 \xrightarrow{\left(\begin{array}{cc} z & x \\
y & -z \end{array}\right)} A^2.
\]
Let \( U_x \subset V(G) \) be the open affine defined by \( x \neq 0 \). We have
\[
\left(\begin{array}{cc} z & x \\
y & -z \end{array}\right) \sim \left(\begin{array}{cc} z \cdot \frac{z}{x} & x \cdot \frac{z}{x} \\
y & -z \end{array}\right) \sim \left(\begin{array}{cc} -y & z \\
y & -z \end{array}\right) \sim \left(\begin{array}{cc} -y & z \\
0 & 0 \end{array}\right)
\]
in the localization \( A(U_x) \). Hence, the rank of \( \Theta_G \) equals 1 in \( A(U_x) \) and the kernel is a free module of rank 1. A similar calculation shows that the same is true on the affine open \( U_y \) defined by \( y \neq 0 \). Since \( \text{Proj} V(G) = U_x \cup U_y \), we conclude that \( \text{Ker}(\Theta_G, M) \), where \( M = \mathcal{O}_{\text{Proj} V(G)} \otimes M \), is a locally free sheaf on \( \text{Proj} V(G) \) of rank 1. In fact, \( \text{Ker}(\Theta_G, M) \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \).

More generally, let \( S_\lambda \) be the irreducible \( \text{sl}_2 \)-module of highest weight \( \lambda \), \( 0 \leq \lambda \leq p - 1 \). The case of the canonical representation considered above corresponds to \( \lambda = 1 \). Let \( v_0, v_1, \ldots, v_\lambda \) be a basis for \( S_\lambda \) such that the generators \( e, f \) and \( h \) of \( \text{sl}_2 \) act as follows: \( hv_i = (\lambda - 2i)v_i \), \( ev_i = (\lambda - i + 1)v_{i-1} \) for \( i > 0 \), \( ev_0 = 0 \), and \( f v_i = (i + 1)v_{i+1} \) (see [14, 7.2]). We conclude that the operator
\[
\Theta_G : A \otimes S_\lambda \simeq A^{\lambda + 1} \longrightarrow A \otimes S_\lambda \simeq A^{\lambda + 1}
\]
Example 5.7. Let \( k \) have the same local Jordan type. We remind the reader that the local Jordan bundles constructed in Theorem 5.1 can be used to distinguish certain \( k \)-modules of constant Jordan type. Let \( S_\lambda \) be the irreducible \( SL_2 \)-module of constant Jordan type \((0,\ldots,0,1,1)\) and its linear dual \( M^\# \), where \( S_\lambda \) is represented by the matrix

\[
\begin{pmatrix}
\lambda z & \lambda x & 0 & \ldots \\
 y & (\lambda - 2)z & (\lambda - 1)x & \ldots \\
 0 & 2y & (\lambda - 4)z & (\lambda - 2)x & \ldots \\
 \ldots & \ldots & \ldots & \ldots & \lambda y & -\lambda z
\end{pmatrix}
\]

(5.6.1)

A calculation similar to the special case of \( M = S_1 \) yields that the rank of this matrix on \( U_x \) and \( U_y \) is \( \lambda \). Hence, \( \text{Ker}(\tilde{\Theta}_G, S_\lambda) \), where \( S_\lambda = \mathcal{O}_{\text{Proj} V(G)} \otimes S_\lambda \), is a locally free sheaf of rank 1. Moreover, we once again have \( \text{Ker}(\tilde{\Theta}_G, S_\lambda) \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \).

As we see in the following simple example, the isomorphism type of the vector bundles constructed in Theorem 5.1 can be used to distinguish certain \( kG \)-modules which have the same local Jordan type. We next give a somewhat more interesting example of pairs of modules of the same constant Jordan type with different associated bundles.

Example 5.8. As seen in [4], any rational \( SL_2 \)-module \( M \) restricts to a \( u(sl_2) \)-module of constant Jordan type. Let \( S_\lambda \) be the irreducible \( SL_2 \)-module of highest weight \( \lambda, 0 < \lambda < p \) (of dimension \( \lambda + 1 \)) and let \( M_\lambda = S^p(S_\lambda) \), the \( p \)-th symmetric power of \( S_\lambda \). Since \( S_\lambda \) is self-dual, the dual of \( M_\lambda \) is \( N_\lambda = \Gamma^p(S_\lambda) \), the \( p \)-th divided
power of $S_\lambda$. The modules $M_\lambda, N_\lambda$ fit in a short exact sequence of rational $SL_2$-modules

$$0 \rightarrow S^{(1)}_\lambda \rightarrow M_\lambda \rightarrow N_\lambda \rightarrow S^{(1)}_\lambda \rightarrow 0.$$  

Here, $S^{(1)}_\lambda$ is the first Frobenius twist of $S_\lambda$, thus trivial as a $u(sl_2)$-module.

The projectivized null cone $\text{Proj} N(sl_2)$ is a rational conic, isomorphic to $\mathbb{P}^1$, whose elements can be viewed as homothety classes of non-zero nilpotent elements. Let $\mathcal{M}$, $\mathcal{N}$ denote the free $O_{\text{Proj} N(sl_2)}$-modules $O_{\text{Proj} N(sl_2)} \otimes M$, $O_{\text{Proj} N(sl_2)} \otimes N$. If $0 \neq v \in N(sl_2)$ is viewed as a non-zero nilpotent element in $sl_2$, then $\theta_v$ on $M_{k(v)}$ as given by action of the corresponding nilpotent element. Thus, for an element in $\Gamma(\mathbb{P}^1, \mathcal{M}) = S^p(S_\lambda)$ to lie in $\Gamma(\mathbb{P}^1, \text{Ker}\{\tilde{\Theta}_{sl_2}, \mathcal{M}\}) \subset \Gamma(\mathbb{P}^1, \mathcal{M})$, it is necessary and sufficient for that element to be $sl_2$-invariant. We conclude that $\Gamma(\mathbb{P}^1, \text{Ker}\{\tilde{\Theta}_{sl_2}, \mathcal{M}\}) = S^{(1)}_\lambda$.

Similarly, for an element $\Gamma(\mathbb{P}^1, \mathcal{N}) = \Gamma^p(S_\lambda)$ to lie in $\Gamma(\mathbb{P}^1, \text{Ker}\{\tilde{\Theta}_{sl_2}, \mathcal{N}\})$, it is necessary and sufficient for that element to be $sl_2$-invariant. Yet the $sl_2$-invariants of $\Gamma^p(S_\lambda)$ are 0.

**Proposition 5.9.** Let $G$ be an infinitesimal group scheme, let $M$ a $kG$-module of constant Jordan type $\sum_{i=1}^p a_i[i]$, and let $\mathcal{M}$ denote the free $O_{\text{Proj} V(G)}$-module $O_{\text{Proj} V(G)} \otimes M$. Then for any $j, 1 \leq j < p$,

$$\text{rk}(\text{Im}\{\tilde{\Theta}_G^j, \mathcal{M}\}) = \sum_{i=j+1}^p a_i(i-j).$$

In particular,

$$\text{Ker}\{\tilde{\Theta}_G, \mathcal{M}\} \subset \text{Ker}\{\tilde{\Theta}_G^2, \mathcal{M}\} \subset \cdots \subset \text{Ker}\{\tilde{\Theta}_G^{i-1}, \mathcal{M}\} \subset \mathcal{M}$$

is a chain of $O_{\text{Proj} V(G)}$-submodules with $\text{rk}(\text{Ker}\{\tilde{\Theta}_G^{i-1}, \mathcal{M}\}) < \text{rk}(\text{Ker}\{\tilde{\Theta}_G^i, \mathcal{M}\})$ if and only if $a_i \neq 0$ for some $1 \leq j \leq i \leq p$.

**Proof.** The formula (5.9.1) is the formula for the rank of $u^j$ on the $k[u]/u^p$-module $\oplus_i (k[u]/u^i)^{\oplus a_i}$ of Jordan type $\sum_{i=1}^p a_i[i]$. This is therefore the dimension of the image of $\theta_v$, $0 \neq v \in V(G)$ on $M_{k(v)}$, and thus the rank of the vector bundle $\text{Im}\{\tilde{\Theta}_G, \mathcal{M}\}$.

One very simple invariant of the algebraic vector bundle $\text{Ker}\{\tilde{\Theta}_G, \mathcal{M}\}$ is the dimension of its vector space of global sections. The following proposition gives a method of determining global sections.

**Proposition 5.10.** Let $G$ be an infinitesimal group scheme, and assume that $V(G)$ is reduced. Let $M$ be a finite dimensional $kG$-module and let $\mathcal{M} = O_{\text{Proj} V(G)} \otimes M$. Then

$$\Gamma(\text{Proj} V(G), \text{Ker}\{\tilde{\Theta}_G, \mathcal{M}\}) \subset M$$

consists of those $m \in M$ such that $\theta_v^j(m) = 0$ for all $x \in \text{Proj} V(G)$.

In particular, if $KG$ is generated by $\theta_v \in k(v)G, v \in V(G)$, where $K$ the field of fractions of $k[V(G)]$, then

$$\Gamma(\text{Proj} V(G), \text{Ker}\{\tilde{\Theta}_G, \mathcal{M}\}) = H^0(G, M).$$
Proof. Recall that $\operatorname{Proj} V(G)$ is connected by [4, 3.4] and thus $\Gamma(\operatorname{Proj} V(G), \mathcal{M}) = M$. Under this identification, the global sections of $\ker \{\tilde{\Theta}^j_G, \mathcal{M} \}$ coincide with the subset
\[
\{ m \in M \mid \Theta^j_G(1 \otimes m) = 0 \}.
\]
where $\Theta_G : k[V(G)] \otimes M \longrightarrow k[V(G)] \otimes M$ is the universal $p$-nilpotent operator acting on $k[V(G)] \otimes M$ as defined in (2.3.4). Since $V(G)$ is reduced, we have $\Theta^j_G(1 \otimes m) = 0$ if and only if $\theta^j_i(1 \otimes m) = \Theta^j_G(1 \otimes m) \otimes k[V(G)] k(v) = 0$ for any $v \in V(G)$. Hence, $m \in \Gamma(\operatorname{Proj} V(G), \ker \{\tilde{\Theta}^j_G, \mathcal{M} \})$ if and only if $m \in \ker \{\theta^j_i, M_k(v) \}$ for any $v \in V(G)$.

The second assertion concerning the global sections of $\ker \{\tilde{\Theta}^j_G, \mathcal{M} \}$ follows immediately upon taking $j = 1$. \(\square\)

Combining Propositions 5.9 and 5.10 in the special case $j = 1$, yields the following criterion for the non-triviality of $\ker \{\tilde{\Theta}^j_G, \mathcal{M} \}$.

**Corollary 5.11.** Let $G$ be an infinitesimal group scheme such that $V(G)$ is reduced and positive dimensional, and assume that $KG$ is generated by $\theta_v \in k(v)G$, $v \in V(G)$ for $K$ the field of fractions of $k[V(G)]$. Let $M$ a finite dimensional $kG$-module of constant Jordan type $\sum a_i [i]$. If
\[
\dim H^0(G, M) < \sum_{i=1}^p a_i,
\]
that $\ker \{\tilde{\Theta}^j_G, \mathcal{M} \}$ is a non-trivial algebraic vector bundle over $\operatorname{Proj} V(G)$

Proof. By Proposition 5.9, the dimension of the fibers of $\ker \{\Theta_G, \mathcal{M} \}$ is $\dim M - \sum_{i=2}^p a_i(i - 1) = \sum_{i=1}^p a_i$. By Proposition 5.10, the global sections of $\ker \{\tilde{\Theta}^j_G, \mathcal{M} \}$ equal $H^0(G, M)$. Hence, the inequality $\dim H^0(G, M) < \sum_{i=1}^p a_i$ implies that the dimension of the global sections is less than the dimension of the fibers. Therefore, the sheaf is not free. \(\square\)

For any finite-dimensional $kG$-module $M$ and any $j, 1 \leq j \leq p$, we may consider
\[
\rho_j(M) \equiv \dim \Gamma(\operatorname{Proj} V(G), \ker \{\tilde{\Theta}^j_G, \mathcal{M} \})
\]
where $\mathcal{M} = \mathcal{O}_{\operatorname{Proj} V(G)} \otimes M$. We make this explicit for projective $kG$-modules, a class of $kG$-modules for which rank varieties give no information.

**Proposition 5.12.** Let $G$ be an infinitesimal group scheme. Then sending a finitely generated projective $kG$-module $P$ to $(\rho_1(P), \ldots, \rho_p(P)) \in \mathbb{N}^p$ determines a covariantly functorial homomorphism
\[
\rho = (\rho_1, \ldots, \rho_p) : K_0(kG) \rightarrow \mathbb{Z}^p.
\]

Proof. To prove that $\rho$ is well defined on $K_0(G)$, it suffices to observe that each $\rho_j$ is additive and that short exact sequences of projective $kG$-modules split. \(\square\)

**Proposition 5.13.** Let $G = sl_2$ be the infinitesimal group scheme associated to the restricted Lie algebra $sl_2$. The homomorphism $\rho$ of Proposition 5.12 is a rational isomorphism:
\[
\rho_Q : K_0(\mathfrak{g}(sl_2)) \otimes \mathbb{Q} \longrightarrow \mathbb{Q}^p
\]
Embeds as a closed subgroup scheme of an infinitesimal group. Hence, we have the following equalities:

\[ \text{sl}_2 \text{reader that if } \square \theta \overset{(5.13.1)}{\Rightarrow} \text{where the actions of } \text{SL}_2 \text{ of } \Theta \text{ and only if it contains the socle} \]

It suffices to prove the statement of the Proposition once we extend scalars over \( k \). Hence, we may assume that \( k \) is algebraically closed.

Let \( M \) be a finite-dimensional rational \( SL_2 \)-module, and let \( \Theta_G : k[V(G)] \otimes M \rightarrow k[V(G)] \otimes M \) be the universal \( p \)-nilpotent operator. We first show that the kernel of \( \Theta_G \) restricted to \( 1 \otimes M \) is an \( SL_2 \)-stable subspace of \( M \). Let \( 1 \otimes m \in 1 \otimes M \) be in the kernel of \( \Theta_G \), and let \( g \in SL_2(k) \). Let \( v \in N(sl_2) \setminus \{0\} \). We have

\[ \theta_v(1 \otimes gm) = g(\theta_v\overset{-1}{g}(1 \otimes m)) = g\theta_v\overset{-1}{g}(1 \otimes m) \]

where the actions of \( SL_2(k) \) on \( kG \simeq u(sl_2) \) and on \( N(sl_2) \) are induced by the adjoint action on \( sl_2 \). Iterating the formula, we obtain

\[ \theta_v^i(1 \otimes gm) = g\theta_v\overset{-i}{g}(1 \otimes m) \]

Hence, we have the following equalities:

\[ \{ m \in M \mid \Theta_G^i(1 \otimes m) = 0 \} = \bigcap_{0 \neq v \in V(G)} \{ m \in M \mid \theta_v^i(1 \otimes m) = 0 \} = \]

\[ \{ m \in M \mid \Theta_G^i(1 \otimes gm) = 0 \} = \bigcap_{0 \neq v \in V(G)} \{ m \in M \mid \theta_v^i(1 \otimes gm) = 0 \} = \]

where the first and the last equality follows from Proposition 5.10, the second equality follows from the transitivity of the adjoint action of \( SL_2(k) \) on \( N(sl_2) \), and the third equality is an application of (5.13.1). We conclude that \( \{ m \in M \mid \Theta_G^i(1 \otimes m) = 0 \} \) is an \( SL_2(k) \)-stable subspace of \( M \). Therefore, \( \Gamma(X, \text{Ker}\{ \Theta_G^i, \mathcal{O}_X \otimes M \}) \) is a \( G \)-stable subspace of \( M \). Here, \( X = \text{Proj} V(sl_2) \).

The decomposition series of the projective cover \( P_\lambda \) of the irreducible \( sl_2 \)-module \( S_\lambda \) of highest weight \( \lambda \), \( 0 \leq \lambda < p - 1 \), is represented by the following diagram (see [8, 2.4]):

\[ \begin{array}{ccc}
S_{\lambda} & \overset{\rho_{\lambda}}{\longrightarrow} & S_{p-2-\lambda} \\
\downarrow & & \downarrow \\
S_{\lambda} & \overset{\rho_{\lambda}}{\longrightarrow} & S_{p-2-\lambda}
\end{array} \]

On the other hand, \( P_{p-1} = S_{p-1} \) is the Steinberg module of dimension \( p \).

Since \( \Gamma(X, \text{Ker}\{ \Theta_G^i, \mathcal{O}_X \otimes P_\lambda \}) \) is a \( G \)-stable subspace of \( P_\lambda \), it is non-trivial if and only if it contains the socle \( S_\lambda \). The simple module \( S_\lambda \) belongs to the kernel of \( \Theta_G \) if and only if it is annihilated by \( j \)-th iterations of all nilpotent elements of \( sl_2 \), which happens if and only if \( j > \lambda \).

We conclude that \( \rho_j(P_\lambda) = 0 \) for \( j \leq \lambda \) and \( \rho_{\lambda+1}(P_\lambda) > 0 \). It is now immediate that \( \rho(P_\lambda) \in \mathbb{Z}p \) are linearly independent (over \( \mathbb{Q} \)) for \( 0 \leq \lambda < p - 1 \). Hence, \( \rho \) is a rational isomorphism. \( \square \)

Proposition 5.13 has the following straightforward corollary. We remind the reader that if \( sl_2 \) embeds as a closed subgroup scheme of an infinitesimal group scheme \( G \), then \( kG \) is projective as a \( sl_2 \)-module.
Corollary 5.14. Assume that an infinitesimal group scheme $G$ has a closed subgroup, $i : H \subset G$, isomorphic to $sl_2$. Then $i_* : K_0(u(sl_2)) \to K_0(kG)$ is injective; in particular, $i_*(P_\lambda) = kG \otimes_{kH} P_\lambda$ are linearly independent in $K_0(kG)$ for $0 \leq \lambda \leq p - 1$.

Proof. As mentioned above, $kG$ is a finitely generated projective $kH$-module. The composition of the restriction and induction maps $i^* \circ i_* : K_0(u(sl_2)) \to K_0(u(sl_2))$ is multiplication by the class of $kG$ as a $kH$-module. Since $K_0(kH)$ is a free abelian group (of rank $p$, indexed by $\lambda$, $0 \leq \lambda < p$), we conclude that $i_*$ is injective. Since the classes of $P_\lambda$ for $0 \leq \lambda < p$ are linearly independent in $K_0(u(sl_2))$ by Proposition 5.13, it follows that the classes of $kG \otimes_{kH} P_\lambda$ are linearly independent in $K_0(kG)$. □

Definition 5.15. Let $G$ be an infinitesimal group scheme, $M$ a finite dimensional $kG$-module, and $j < p$ a positive integer. We say that a $kG$-module $M$ has the constant $j$-image property if there exists a subspace $I(j) \subset M$ such that for every $v \neq 0$ in $V(G)$, the image of $\theta_v^j : M_{k(v)} \to M_{k(v)}$ equals $I(j)_{k(v)}$. Similarly, we say that $M$ has constant $j$-kernel property if there exists some submodule $K(j) \subset M$ such that for every $v \neq 0$ in $V(G)$, the kernel of $\theta_v^j : M_{k(v)} \to M_{k(v)}$ equals $K(j)_{k(v)}$.

Proposition 5.16. Let $G$ be an infinitesimal group scheme, let $M$ be a finite dimensional $kG$-module which is of constant $j$-rank. Then the algebraic vector bundle $\text{im}(\tilde{\Theta}^j_G, M)$ is trivial (i.e., a free coherent sheaf) on $\text{Proj} V(G)$ if and only if $M$ has the constant $j$-image property. Similarly, $\text{Ker}(\tilde{\Theta}^j_G, M)$ is trivial if and only if $M$ has the constant $j$-kernel property.

If $M$ has a constant $j$-image property then $\text{im}(\tilde{\Theta}^j_G, M)$ is a free $O_X$-module generated by $I(j)$. Conversely, assume that $\text{im}(\tilde{\Theta}^j_G, M)$ is a free $O_X$-module. Then there exists a subspace $I(j) \subset M = \Gamma(X, M)$ which maps to and spans each fiber $\text{im}(\theta_v^j, M_{k(v)})$, for $0 \neq v \in V(G)$. The argument for kernels is similar. □

Remark 5.17. We point out the properties of constant $j$-image and constant $j$-kernel are independent of each other. Consider the module $M^\#$ of Example 5.7. As shown in that example, $\text{Ker}(\tilde{\Theta}^j_G, M^\#)$ is locally free of rank 2 but not free, since the global sections have dimension one. On the other hand, $\text{im}(\tilde{\Theta}^j_G, M^\#)$ is a free $O_X$-module generated by the global section $n_3$. In particular, $M^\#$ has constant image property but not constant kernel property.

For the module $M$ of Example 5.7, the sheaf $\text{Ker}(\tilde{\Theta}^j_G, M)$ is free of rank 2 whereas $\text{im}(\tilde{\Theta}^j_G, M)$ is locally free of rank 1 but not free since it does not have any global sections. Hence, $M$ has a constant kernel property but not constant image property.

We consider an analogue of the sheaf construction of Duflo-Serganova for Lie superalgebras [5]. This construction enables us to produce additional algebraic vector bundles on $\text{Proj} V(G)$. We implicitly use the observation $\tilde{\Theta}^j_G = 0$.

Definition 5.18. Let $G$ be an infinitesimal group scheme, let $X$ denote $\text{Proj} V(G)$, and consider a finite dimensional $kG$-module $M$. Denote $O_X \otimes M$ by $M$. For any $i$, $1 \leq i \leq p - 1$, we consider the following coherent $O_X$-modules, subquotients of $M$:

\[ M[i] \equiv \text{Ker}(\tilde{\Theta}^j_G / \text{im}(\tilde{\Theta}^{p-i}) \equiv \text{Ker}(\tilde{\Theta}^j_G, M) / \text{im}(\tilde{\Theta}^{p-i}, M(-(p-i)p^{r-1})). \]
The following simple lemma helps to motivate these subquotients.

**Lemma 5.19.** Let $V$ be a finite dimensional $k[t]/t^p$-module, and let $J T ype(V, t) = (a_0, \ldots, a_1)$ (using the notation introduced in (3.1.2)). Let

$$V^{[i]} = \ker \{ t^i : V \to V \} / \im \{ t^{p-j} : V \to V \}$$

for $j \leq p - 1$. Then

$$\dim (V^{[i]}) = \sum_{1 \leq i \leq j} ia_i + \sum_{i > j} ja_i - \sum_{i+j \geq p} (i + j - p)a_i.$$

In particular, $V$ is projective as a $k[t]/t^p$-module if and only if $V^{[1]} = 0$.

Furthermore, for $j \leq p - 1$, $V^{[j]} \cong V^{[p-j]}$ as $k[t]/t^p$-modules.

As seen in the next proposition, these subquotients can provide additional examples of algebraic vector bundles over $\Proj V(G)$.

**Proposition 5.20.** Let $G$ be an infinitesimal group scheme and let $M$ be a finite dimensional $kG$-module which is of constant $j$-rank and constant $(p-j)$-rank for some $j$, $1 \leq j < p$. Then $\mathcal{M}^{[j]}$ is a locally free $\mathcal{O}_X$-module and $k(x) \otimes_{\mathcal{O}_X} \mathcal{M}^{[j]} \to M^{[j]}_{k(x)}$ is an isomorphism for all $x \in X \equiv \Proj V(G)$.

**Proof.** For any $x \in X$, consider the map of exact sequences

$$k(x) \otimes_{\mathcal{O}_X} \text{Im } \theta^{-j} \overset{\theta}{\longrightarrow} k(x) \otimes_{\mathcal{O}_X} \ker \theta \overset{\theta}{\longrightarrow} k(x) \otimes_{\mathcal{O}_X} \mathcal{M}^{[j]} \longrightarrow 0$$

where $\text{Im } \theta$ (respectively, $\ker \theta$) is an abbreviation for $\text{Im } \{ \theta, M_{k(x)} \}$ (respectively, $\ker \{ \theta, M_{k(x)} \}$). The left and middle vertical maps are isomorphisms by Theorem 4.12. Thus, the 5-Lemma implies that the right vertical arrow is also an isomorphism. □

We next give a global version of the observation in Lemma 5.19 that $V^{[j]} \cong V^{[p-j]}$ for $j \leq p - 1$.

**Proposition 5.21.** Let $G$ be an infinitesimal group scheme. Let $M$ be a finite dimensional $kG$-module which is of constant $j$-rank and of constant $(p-j)$-rank for some $j$, $1 \leq j < p$, and let $N = \Hom_k(M, k)$ denote the $k$-linear dual of $M$. Then the locally free, coherent $\mathcal{O}_X$-module $\mathcal{N}^{[p-j]}$ is naturally isomorphic to the $\mathcal{O}_X$-linear dual $\Hom_{\mathcal{O}_X}(\mathcal{M}^{[j]}, \mathcal{O}_X)$ of $\mathcal{M}^{[j]}$, where $X \equiv \Proj V(G)$. Here, $\mathcal{M} = \mathcal{O}_X \otimes M$, $\mathcal{N} = \mathcal{O}_X \otimes N$.

**Proof.** The $\mathcal{O}_X$-linear dual of the complex of $\mathcal{O}_X$-modules

$$\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X((p-j)p^{-1}) \overset{\Theta^p_{\mathcal{O}_X}}{\longrightarrow} \mathcal{M} \overset{\Theta^p_{G}}{\longrightarrow} \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_X(jp^{-1})$$

is the complex

$$\mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{O}_X((p-j)p^{-1}) \overset{\Theta^p_{\mathcal{O}_X}}{\longleftarrow} \mathcal{N} \overset{\Theta^p_{G}}{\longleftarrow} \mathcal{N} \otimes_{\mathcal{O}_X} \mathcal{O}_X(-jp^{-1})$$

A similar statement applies with $\theta$ in place of $\Theta_G$.  

For any scheme $Y$ and any complex of $\mathcal{O}_Y$-modules

$$S_1 \xrightarrow{f} S_2 \xrightarrow{g} S_3$$

with $\mathcal{O}_Y$-linear dual

$$\text{Hom}_{\mathcal{O}_Y}(S_1, \mathcal{O}_Y) \xleftarrow{f^\#} \text{Hom}_{\mathcal{O}_Y}(S_2, \mathcal{O}_Y) \xleftarrow{g^\#} \text{Hom}_{\mathcal{O}_Y}(S_3, \mathcal{O}_Y),$$

there is a natural (evident) pairing

$$(5.21.1) \quad (\text{Ker}\{g\}/\text{Im}\{f\}) \otimes (\text{Ker}\{f^\#\}/\text{Im}\{g^\#\}) \rightarrow \mathcal{O}_Y.$$  

One readily verifies that (5.21.1) is a perfect pairing if $\mathcal{O}_Y$ is a field, so that (5.21.1) induces an isomorphism for any $x \in X$ between

$$M^{[j]}_{k(x)} = \text{Ker}\theta_x^j/\text{Im}\theta_x^{p-j}$$

and the $k(x)$-linear dual of

$$N^{[p-j]}_{k(x)} = \text{Ker}\theta_x^{p-j}/\text{Im}\theta_x^j.$$  

On the other hand, Theorem 4.12 and the right exactness of $k(x) \otimes_{\mathcal{O}_X} (-)$ imply that

$$M^{[j]}_{k(x)} \cong k(x) \otimes_{\mathcal{O}_X} M^{[j]}$$

for every $x \in X$ and similarly for $N^{[p-j]}_{k(x)}$. Thus, the map induced by (5.21.1) (with $X = Y$) is an isomorphism by Proposition 4.10. □

Consideration of $\mathcal{M}^{[1]}$ leads to another characterization of projective $kG$-modules.

**Proposition 5.22.** Let $G$ be an infinitesimal group scheme and let $M$ be a finite dimensional $kG$-module. Then $M$ is projective if and only if $M$ has constant rank, has constant $(p-1)$-rank, and satisfies $\mathcal{M}^{[1]} = 0$ (where $\mathcal{M}^{[1]}$ is defined in 5.18).

**Proof.** Assume that $M$ is a projective $kG$-module. Then $M$ has constant Jordan type (which is some multiple of $[p]$), and hence has constant rank and constant $(p-1)$-rank. For any $x \in \text{Proj} V(G) = X$, $\theta_x^j(M_{k(x)})$ is a free $k(x)[t]/t^p$-module of rank equal to $\frac{\dim(M)}{p}$. If we lift a basis of this free module to $\mathcal{M}(x) = \mathcal{O}_{X,x} \otimes_{\mathcal{O}_X} M$, then an application of Nakayama’s Lemma tells us that $\mathcal{M}(x)$ is free as an $\mathcal{O}_{X,x}[t]/t^p$-module. This readily implies that $(\mathcal{M}(x))^{[p-1]} = \text{Ker}\{\Theta_G^{-1}(x), \mathcal{M}(x)/\text{Im}\{\Theta_G(x), \mathcal{M}(x)\}\}$ vanishes. Using the exactness of localization, we conclude that $(\mathcal{M}^{[p-1]}(x)) = (\mathcal{M}(x))^{[p-1]}$. Consequently, $\mathcal{M}^{[p-1]} = 0$. By Proposition 5.21, we conclude that $\mathcal{M}^{[1]} = 0$.

Conversely, if $M$ has constant rank and constant $(p-1)$-rank and if $\mathcal{M}^{[1]} = 0$, then Proposition 5.20 tells us that $M_{k(x)} \equiv \text{Ker}\theta_x^1/\text{Im}\theta_x^{p-1}$ equals 0 for all $x \in X$. Lemma 5.19 thus implies that each $M_{k(x)}$ is projective, so that the local criterion for projectivity [17] implies that $M$ is projective. □

The following two lemmas will be applied to prove Proposition 5.25

**Lemma 5.23.** Let $R$ be a local noetherian ring with residue field $k$ and let $M$ be a finite $R[t]/t^p$-module which is free as an $R$-module. If $k \otimes_R M$ is a free $k[t]/t^p$-module, then $M$ is free as an $R[t]/t^p$-module.
Proof. Let \( m_1, \ldots, m_s \in M \) be such that \( \overline{m}_1, \ldots, \overline{m}_s \) forms a basis for \( k \otimes_R M \) as a \( k[t]/t^p \)-module. Let \( Q \) be a free \( R[t]/t^p \)-module of rank \( s \) with basis \( q_1, \ldots, q_s \) and consider the \( R[t]/t^p \)-module homomorphism \( f : Q \to M \) sending \( q_i \) to \( m_i \).

By Nakayama’s Lemma, \( f : Q \to M \) is surjective. Because \( M \) is free as an \( R \)-module, applying \( k \otimes_R - \) to the short exact sequence \( 0 \to \ker f \to Q \to M \to 0 \) determines the short exact sequence
\[
0 \to k \otimes_R \ker \{ f \} \to k \otimes_R Q \to k \otimes_R M \to 0.
\]
Consequently, \( k \otimes_R \ker \{ f \} = 0 \), so that another application of Nakayama’s Lemma implies that \( \ker f = 0 \). Hence, \( f \) is an isomorphism, and thus \( M \) is free as an \( R[t]/t^p \)-module.

Lemma 5.24. Let \( G \) be an infinitesimal group scheme and \( M \) be a finite dimensional \( kG \)-module. Set \( A = k[V(G)] \); for any \( f \in A \), set \( A_f = A[1/f] \). Assume that \( \text{Spec } A_f \subset V(G) \) has empty intersection with \( V(G)_M \). Then \( (U_G \circ \varepsilon)^*(A_f \otimes M) \) is a projective \( A_f[t]/t^p \)-module.

Proof. By definition, \( V(G)_M \) consists of those points \( v \in V(G) \) such that \( \theta_v^*(M_{k(v)}) \) is not free as a \( k(v)[t]/t^p \)-module. By the universal property of \( U_G \circ \varepsilon \), the assumption that \( \text{Spec } A_f \cap V(G)_M = \emptyset \) implies for every point \( v \in \text{Spec } A_f \) that \( \theta_v^*(M_{k(v)}) = k(v) \otimes_{A_f} (U_G \circ \varepsilon)^*(A_f \otimes M) \) is free as a \( k(v)[t]/t^p \)-module. Let \( A_v \) denote the localization of \( A \) at \( v \). Then Lemma 5.23 implies for every point \( v \in \text{Spec } A_f \) that the localization \( A_v \otimes_{A_f} (U_G \circ \varepsilon)^*(A_f \otimes M) \) is free as a \( A_v[t]/t^p \)-module. This implies that \( A_f \otimes M \) is projective (since projectivity of a module over a commutative ring is determined locally).

We conclude with a property of the (projectivized) rank variety \( \text{Proj } V(G)_M \) of a \( kG \)-module \( M \).

Proposition 5.25. Let \( G \) be an infinitesimal group scheme, \( M \) be a finite dimensional \( kG \)-module, and set \( \mathcal{M} = \mathcal{O}_{\text{Proj } V(G)} \otimes M \). Then
\[
\text{Supp} \mathcal{O}_{\text{Proj } V(G)}(\mathcal{M}^{[1]} \cap \text{Proj } V(G)_M).
\]
where \( \text{Supp} \mathcal{O}_{\text{Proj } V(G)}(\mathcal{M}^{[1]} \cap \text{Proj } V(G)_M) \) is the support of the coherent sheaf \( \mathcal{M}^{[1]} \) (the closed subset of points \( x \in \text{Proj } V(G) \) at which \( \mathcal{M}^{[1]}(x) \neq 0 \).

Proof. Let \( A \) denote \( k[V(G)] \) and let \( X \) denote \( \text{Proj } V(G) \). Consider some \( x \notin X_M \) and choose some homogeneous polynomial \( F \in A \) vanishing on \( X_M \) such that \( F(x) \neq 0 \). Thus, \( x \in \text{Spec } (A_F)_0 \subset X \) and \( \text{Spec } (A_F)_0 \cap X_M = \emptyset \), where \( (A_F)_0 \) denote the elements of degree 0 in the localization \( A_F = A[1/F] \). It suffices to prove that \( x \notin \text{Supp } (\mathcal{O}_{\text{Proj } V(G)}(\mathcal{M}^{[1]} \cap \text{Spec } (A_F)_0) \cap X_M = \emptyset \). Equivalently, it suffices to prove that \( v \notin \text{Supp } (A_F \otimes M) \) for some \( v \in \text{Spec } A_F \) mapping to \( x \).

By Lemma 5.24, \( (U_G \circ \varepsilon)^*(A_f \otimes M) \) is a projective \( A_f[t]/t^p \)-module. This implies that \( (U_G \circ \varepsilon)^*(A_f \otimes M) \) is free as a \( A_f[t]/t^p \)-module, and thus that \( v \notin \text{Supp } (A_F \otimes M) \).

Remark 5.26. The reverse inclusion \( \text{Proj } V(G)_M \subset \text{Supp } \mathcal{O}_{\text{Proj } V(G)}(\mathcal{M}^{[1]} \cap \text{Proj } V(G)_M) \) seems closely related to the condition that \( k(x) \otimes_{\mathcal{O}_{\text{Proj } V(G)}} \ker \{ \Theta_G, \mathcal{M} \} \to \ker \{ \theta_x, M_{k(x)} \} \) be surjective.
6. Generalized rank varieties

For any infinitesimal group scheme $G$ and any finite dimensional $kG$-module $M$, we have a well defined function

$$(6.0.1) \quad \text{JType} : V(G) \to \mathbb{N} \times \mathbb{P}$$

which sends a 1-parameter subgroup $\mu_\epsilon : \mathbb{G}_a(\epsilon) \to kG$ to the Jordan type of $(\mu_\epsilon \circ \epsilon)^* (M_{k(\epsilon)})$, a module for $k(\epsilon)[t]/tp$. On the other hand, it is not the case that the Jordan type at an equivalence class of $\pi$-points is independent of the representative of that $\pi$-point except in special cases: if the Jordan type of $M$ is maximal at that $\pi$-point or if the $\pi$-point is generic, then this is indeed the case according to the main theorem of [12]. What is special in the case of infinitesimal group schemes is that in each equivalence class of $\pi$-points there is a distinguished representative (uniquely defined up to scalar multiple) which is of the form $\mu_\epsilon \circ \epsilon$.

The purpose of this section is to introduce new geometric invariants for $kG$-modules with $G$ an infinitesimal group scheme, invariants which refine the “non-maximal support variety” defined in [12]. These invariants are weaker than the Jordan type function of (6.0.1), but their projectivizations should extend to finite dimensional modules over an arbitrary finite group scheme.

If $G$ is an arbitrary finite group scheme and $M$ a finite dimensional $kG$-module, then the “original” cohomological support variety for $M$ is the reduced subvariety of $\text{Spec} \, H^\bullet(G, k)$ associated to the annihilator ideal inside $H^\bullet(G, k)$.

As recalled in Theorem 1.15, this cohomological support variety is homeomorphic to the “rank variety” for $V(G)_M$ provided that $G$ is an infinitesimal group scheme. Furthermore, the projectivized rank variety $\text{Proj}(V(G)_M)$ is homeomorphic to $\Pi(G)_M$ as recalled in Theorem 4.3, which is defined in [11] for any finite group scheme $G$ and any $kG$-module $M$. One disadvantage of these support varieties is that they give no information if the dimension of the $kG$-module $M$ is relatively prime to $p$, for in this case they are equal to the full cohomological variety (or its projectivization).

In [12], a refinement $\Gamma(G)_M \subset \Pi(G)_M$ was introduced for any finite dimensional $kG$-module $M$. This “non-maximal support variety” $\Gamma(G)_M \subset \Pi(G)_M$ has the desirable property that it is always strictly contained in $\Pi(G)$ and is homeomorphic to $\Pi(G)_M$ if and only if the latter is strictly contained in $\Pi(G)$. As we see in Proposition 6.6 below, we recover $\Gamma(G)_M$ as a union of projectivizations of $V^j(G)_M, 1 \leq j < p$.

We restrict our attention to infinitesimal group schemes $G$ and introduce generalized rank varieties for a given finite dimensional $kG$-module $M$ and a given $j, 1 \leq j < p$

$$V^j(G)_M \subset V(G)_M \subset V(G).$$

By applying $\text{Proj}(-)$ to these conical affine varieties, we obtain closed subvarieties of $\text{Proj}(V(G)_M)$.

For a $K[t]/tp$-module $M_K$, we shall denote by $\text{rk}(M_K, t)$ the rank of $t$ as a nilpotent operator on $M_K$. We observe that the definition of the support variety $V(G)_M$, for a $G$-module $M$ (Definition 1.14, see also [18, 6.1]), can be reinterpreted as follows:

$$V(G)_M = \{ v \in V(G) \mid \text{rk}(M_{k(\epsilon)}, \theta_\epsilon) < \frac{\dim M}{p} \},$$
where \( \theta_v \) is as in Definition 3.1. This leads us to the notion of a non-maximal rank variety.

**Definition 6.1.** Let \( G \) be an infinitesimal group scheme and \( M \) a finite-dimensional \( G \)-module. We say that \( M \) does not have a maximal rank at the point \( v \in V(G) \) if there exists a point \( w \in V(G) \) such that \( \text{rk}(M_{k(w)}, \theta_w) < \text{rk}(M_{k(v)}, \theta_v) \). Otherwise, \( M \) is said to have maximal rank at the point \( x \).

Furthermore, we say that \( M \) does not have maximal \( j \)-rank at the point \( v \in V(G) \) if there exists a point \( w \in V(G) \) such that \( \text{rk}(M_{k(w)}, \theta_w^j) < \text{rk}(M_{k(v)}, \theta_v^j) \). Otherwise, \( M \) is said to have maximal \( j \)-rank at the point \( v \).

**Proposition 6.2.** Let \( G \) be an infinitesimal group scheme and \( M \) a finite dimensional \( kG \)-module. Set

\[
V_j^j(G)_M \equiv \{ v \in V_j(G) \mid M \text{ does not have a maximal } j - \text{rank at } v \} \cup \{ 0 \}
\]

Then \( V_j^j(G)_M \) is a proper closed conical subset of \( V(G) \) for \( 1 \leq j \leq p - 1 \).

**Proof.** We prove the statement for \( V_1^1(G)_M \); the proof for \( j > 1 \) is obtained from the proof that follows by replacing \( t \) by \( t^j \).

Since there exists at least some point where the rank is maximal, \( V_1^1(G)_M \subset V(G) \) is a proper subset. Corollary 4.6 implies that \( V_j^j(G)_M \) is conical.

To prove that \( V_1^1(G)_M \) is closed, we write \( A = k[V(G)] \) and consider \( 0 \neq v \in V(G) \). Let \( \text{Spec } R \subset V(G) \setminus \{ 0 \} \) be an affine open of \( v \). Consider the restriction of \( \Theta_G : A \otimes M \to A \otimes M \) via \( A \to R \), which we denote by \( \Theta_R : R \otimes M \to R \otimes M \). Let

\[
\theta_w = k(w) \otimes_R \Theta_R : k(w) \otimes M \to k(w) \otimes M
\]

Then the lower semi-continuity of \( w \mapsto \text{rk}(\theta_w) \) implies that the the subset of those \( w \in \text{Spec } R \) for which the rank of \( \theta_w \) is maximal is an open subset. \( \square \)

**Definition 6.3.** Let \( M \) be a finite dimensional \( kG \)-module. The variety \( V_j^j(G)_M \subset V(G) \), \( 1 \leq j \leq p - 1 \) is called the (affine) \( j \)-th non-maximal rank variety of \( M \). In particular, \( V_1^1(G)_M \) is called the (affine) non-maximal rank variety of \( M \).

**Remark 6.4.** The condition that a \( kG \)-module \( M \) be of constant \( j \)-rank (see Definition 3.11) is equivalent to the condition that \( V_j^j(G)_M = \{ 0 \} \).

Moreover, the condition \( v \in V_j^j(G)_M \) neither implies nor is implied by the condition that \( v \in V_{j+1}^j(G)_M \). For example the Jordan type \( p[1] + [p] \) has 2-rank equal to \( p - 2 \) and 1 rank \( = p - 2 \). On the other, the Jordan type \( p[2] \) has 2-rank 0 and 1-rank equal to \( p \).

We make explicit the condition on \( v \in V(G) \) that it is a point of \( V_j^j(G)_M \). This description readily yields another proof that \( V_j^j(G)_M \) is closed.

**Proposition 6.5.** Let \( M \) be a finite dimensional \( G \)-module. Then for any \( j, 1 \leq j \leq p - 1 \), \( v \in V(G) \) with \( J\text{Type}(\theta_v^j(M_{k(v)})) = \sum_{i=1}^{p} a_i[i] \) satisfies \( v \in V_j^j(G)_M \) if and only if there exists another point \( w \in V(G) \) with \( J\text{Type}(\theta_w^j(M_{k(w)})) = \sum_{i=1}^{p} b_i[i] \) satisfying the condition that

\[
\sum_{i=j+1}^{p} (a_i - j) < \sum_{i=j+1}^{p} (b_i - j).
\]

**Proof.** We easily verify that if \( (a_{s,t}) \) is a \( p \)-nilpotent \( m \times m \) matrix in Jordan canonical form with Jordan type \( \sum_{i=1}^{p} a_i[i] \), then the \( j^{\text{th}} \) power \( (a_{s,t})^j \) has rank \( \sum_{i=j+1}^{p} (a_i - j) \). \( \square \)
The following proposition follows easily from the definition of $\Gamma(M) \subset V(G)_M$, the non-maximal support variety.

**Proposition 6.6.** Let $G$ be an infinitesimal group scheme and $M$ a finite dimensional $kG$-module. Then

$$\Gamma(G)_M = \bigcup_{1 \leq j < p} \text{Proj}(V^j(G)_M)$$

as closed subsets of $\Pi(G) \cong \text{Proj} V(G)$.

**Proof.** The ordering of Jordan types is such that $\sum_{i=1}^p a_i[i] \leq \sum_{i=1}^p b_i[i]$ if and only if (6.5.1) holds for all $j$, $1 \leq j < p$. Thus, the proposition follows from Proposition 6.5, granted the definition of $\Gamma(G)_M$. □

Our first example is particularly elementary, yet still instructive.

**Example 6.7.** Let $G = \mathfrak{gl}_N = GL_N(\mathbb{C})$ and let $M$ denote the canonical rational $N$-dimensional $GL_N$-module. In this case, the maximal Jordan type of a $p$-nilpotent matrix is $r[p] + 1[N - rp]$, where $r$ is the greatest non-negative multiple of $p$ which is $\leq N$ (see [12, 4.15]). The rank of the $j$th power of this matrix equals $r[p - j] + 1[N - p - j]$ if $N - p > j$ and $r[p - j]$ otherwise.

For simplicity, assume $k$ is algebraically closed so that we need only consider $k$-rational points of $V_i(GL_N) = N_p(\mathfrak{gl}_N)$. For any $X \in N_p(\mathfrak{gl}_N)$, $\theta_X : V \rightarrow V$ is simply the endomorphism $X$ itself. Consequently, if $N > rp \leq j$, $V(\mathfrak{gl}_N)_M \subset N_p(\mathfrak{gl}_N)$ consists of 0 together with those non-zero $p$-nilpotent $N \times N$ matrices with the property that their Jordan types have strictly fewer than $r$ blocks of size $p$; if $N > rp > j$, $V(\mathfrak{gl}_N)_M$ consists of 0 together with $0 \neq X \in N_p(\mathfrak{gl}_N)$ whose Jordan type is strictly less than $r[p] + 1[N - rp]$.

We can extend the specific description of the previous special example to some other restricted representations of $p$-restricted Lie algebras.

**Proposition 6.8.** Let $\mathfrak{g}$ be a finite dimensional $p$-restricted Lie algebra and let $M$ be a finite dimensional $\mathfrak{u}(\mathfrak{g})$-module of dimension $m$. Let $\rho_M : \mathfrak{g} \rightarrow \mathfrak{gl}_m$ be the representation associated to $M$ and let $V$ be the canonical $m$-dimensional $\mathfrak{gl}_m$-module. Assume that $k$ is algebraically closed. If the image of $\rho_M$ intersects the regular nilpotent orbit of $\mathfrak{gl}_m$, then

$$V^j(\mathfrak{gl}_m) \cap V(\mathfrak{g}).$$

**Proof.** The Jordan type of $X \in \mathfrak{g}$ acting on $M$ is the same as the Jordan type of $\rho(X) \in \mathfrak{gl}_m$ acting on $V$. Since the regular nilpotent orbit $O_{reg} \subset \mathfrak{gl}_m$ is open and dense, $\rho^{-1}(O_{reg}) \subset \mathfrak{g}$ is also open and dense. Thus, the maximum of $\text{rk}\{\theta_X^k, M_{k(\xi)}\}$ as $X$ runs through $p$-nilpotent elements of $\mathfrak{g}$ equals the maximum of $\text{rk}\{Y^j : V \rightarrow V\}$ as $Y$ runs through the $p$-nilpotent elements of $\mathfrak{gl}_m$. Consequently, $0 \neq X \in \mathfrak{g}$ is a point of $V^j(\mathfrak{g})_M$ if and only if $\rho(X) \in \mathfrak{gl}_m$ is a point of $V^j(\mathfrak{gl}_m)_V$. □

The following proposition is an immediate consequence of the definitions, but it does give an intriguing “stratification” of $V_r(G)$ associated to a finite dimensional $kG$-module $M$.

**Proposition 6.9.** Let $G$ be an infinitesimal group scheme of height $\leq r$ and let $M$ be a finite dimensional $kG$-module. Then for any choice of labeling $j_1, \ldots, j_{p-1}$ of the integers $1, \ldots, p - 1$ we obtain a chain of closed subsets

$$V^{j_1}(G)_M \subset \cdots \subset V_{j_{p-1}}(G)_M \subset \cdots \subset V(G)_M \subset V_G.$$
We next work through our four familiar examples, specifically the examples of representations considered in Example 3.4.

**Example 6.10.** We compute $V^j(G)_M$ in each of the four examples considered in Example 3.4.

1. We consider a $p$-restricted Lie algebra $\mathfrak{g}$ and the adjoint representation $\mathfrak{g}^{ad}$ of $(\text{ad}_X)^j$. Let $X_0 \in \mathfrak{g}$ be a $p$-nilpotent element with the property that the rank of $(\text{ad}_X)^j$ is maximal. Then $V^j(\mathfrak{g})_\mathfrak{g}^{ad} \subset N_p(\mathfrak{g})$ consists of 0 together with those $p$-nilpotent $X \in \mathfrak{g}$ such that the endomorphism $(\text{ad}_X)^j : \mathfrak{g} \to \mathfrak{g}$ has rank strictly less than the rank of $(\text{ad}_X)^j$. If $\mathfrak{g}$ has the property that its $p$-nilpotent cone $N_p(\mathfrak{g})$ is irreducible so that there exists some $p$-nilpotent $Y_0 \in \mathfrak{g}$ with the property that the Jordan type of $\text{ad}Y_0$ is greater or equal to the Jordan type of $\text{ad}X$ for any $p$-nilpotent $X \in \mathfrak{g}$, then this condition is equivalent to the condition that the rank of $(\text{ad}_X)^j$ is strictly less than the rank of $\text{ad}Y_0$.

For example, if $j = 1$, the condition on a $p$-nilpotent $X \in \mathfrak{g}$ is that its centralizer in $\mathfrak{g}$ is greater than the centralizer of $X_0$.

2. We consider the $G = k\mathbb{G}_{a(r)}$-module $M = k[u_1, \ldots, u_{r-1}]$. Then

$$V^j(\mathbb{G}_{a(r)})_M = \{(0, a_1, \ldots, a_{r-1})\} \cong \mathbb{A}^{r-1} \subset \mathbb{A}^r$$

for any $j$, $1 \leq j < p$.

3. Let $M$ denote the canonical $n$-dimensional rational $\text{GL}_n$ representation, which we restrict to $G = \text{GL}_n(r)$. By Example 3.4, the Jordan type at $\exp(a_0, \ldots, a_{r-1})$ on $M$ equals the Jordan type of $\alpha_{r-1}$. Thus, $V^j(\text{GL}_n(r))_\text{GL}_n(r) \subset V(\text{GL}_n(r))$ consists of 0 together with those $r$-tuples $(a_0, \ldots, a_{r-1})$ of $p$-nilpotent, pairwise commuting $n \times n$-matrices with the property that the rank of $\alpha_{r-1}^j$ is not maximal. (As in Example 6.7 and also as considered in [12, 4.15], the maximal rank of $\alpha_{r-1}$ equals $r[p - j] + 1[N - p - j]$ if $N - p > j$ and $r[p - j]$ otherwise.)

4. Let $M$ denote the canonical $2$-dimensional rational $\text{GL}_2$ representation, which we restrict to $G = \text{SL}_2(2)$. This is a special case of Example 6.13. Since $\alpha_1$ (in the notation of (3)) is nilpotent, its rank on $M$ is either 0 or 1. In particular, $(\alpha_1)^j$ has rank 0 if $j > 1$. Thus, for $j > 1$, $V^j(\text{SL}_2(2))_M = \emptyset$, whereas $V^1(\text{SL}_2(2))_M \subset V(\text{SL}_2(2))$ consists of 0 together with pairs of the form $(\alpha_0, 0)$.

The following generalization of Corollary 3.10 follows immediately from Proposition 3.9. The special case follows even more immediately from Corollary 3.10.

**Corollary 6.11.** Let $G$ be an infinitesimal group scheme and let $M$ be a $kG$-module of dimension $n$ given by a representation $\rho : G \to \text{GL}_n$ defined over $\mathbb{F}_{p^r}$. Then

$$V^j(G)_{\rho(\alpha)} = (\rho^*)^{-1}(V^j(G)_M).$$

In particular, if $G = \text{GL}_n(r)$, then

$$V^j(\text{GL}_n(r))_{\rho(\alpha)} = \{(\alpha_0, \ldots, \alpha_{r-1}) : (0, \ldots, 0, \phi^*(\alpha_0), \ldots, \phi^*(\alpha_{r-1})) \in V^j(G)_{\rho(\alpha)}\}.$$

Computing examples of $V^j(G)_M$ is made easier by the presence of other structure. For example, if $G = \mathfrak{g}_{p^r}$, the $r^{th}$-Frobenius kernel of the algebraic group $\mathfrak{g}$ and if the $kG$-module $M$ is the restriction of a rational $\mathfrak{g}$-module, then we verify in the following proposition that $V^j(G)_M$ is $\mathfrak{g}$-stable, and thus a union of $\mathfrak{g}$-orbits inside $V(G)$.
Proposition 6.12. Let $G$ be an algebraic group and $G$ denote the $r^{th}$ Frobenius kernel of $G$ for some $r \geq 1$. If $M$ is a finite dimensional rational $G$-module, then each $V^j(G)_M$, $1 \leq j < p$, is a $G$-stable closed subvariety of $V(G)$.

Proof. Composition with the adjoint action of $G$ on $G$ determines an action

$$G \times V(G) \to V(G)$$

which determines an action of $G$ on $V(G)$. Observe that for any field extension $K/k$ and any $x \in G(K)$, the pull-back of $M_K$ via the conjugation action $\gamma_x : G_K \to G_K$ is isomorphic to $M_K$ as a $KG$-module. Thus, the Jordan type of $(\mu \circ \gamma \circ \epsilon)^*(M_K)$ equals that of $(\gamma_x \circ \mu \circ \epsilon)^*(M_K)$ for any 1-parameter subgroup $G_{a(r),K} \to G_K$. Applying Corollary 4.6, we conclude the proposition.

As a sample application of Proposition 6.12, we compute $V^j(G)_S$ for all simple $kG$-modules $S$ for $G = \text{SL}_2(k)$.

Exercise 6.13. Let $G = \text{SL}_2(k)$. The action of $\text{SL}_2$ on $V^3(G) = \{ (\alpha_0, \alpha_1) \}$, where $(\alpha_0, \alpha_1)$ is a pair of commuting $p$-nilpotent matrices with zero trace, is by conjugation.

Let $e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. An easy calculation shows that the orbits of $V^j(G)$ with respect to the conjugation action are parametrized by $\mathbb{P}^1_k$, where $[s_0 : s_1] \in \mathbb{P}^1_k$ corresponds to the orbit represented by the pair $(s_0e, s_1e)$.

Let $S_\lambda$ be a simple $\text{SL}_2$-module with $0 \leq \lambda \leq p^2 - 1$. Since $S_\lambda$ is a rational $\text{SL}_2$-module, the non-maximal varieties $V^j(G)_{S_\lambda}$ are $\text{SL}_2$-stable. Hence, to compute the non-maximal rank varieties for $S_\lambda$ it suffices to compute the Jordan type of $S_\lambda$ at the orbit representatives $(s_0e, s_1e)$. By the explicit formula (2.3.3), the Jordan type of $S_\lambda$ at $(s_0e, s_1e)$ is given by the Jordan type of the nilpotent operator $s_1e + s_0^p e^{(p)}$.

If $0 \leq \lambda \leq p - 1$ then the Jordan type of $e \in k \text{SL}_2(k)$ as an operator on $S_\lambda$ is $\lambda + 1$. On the other hand, the action of $e^{(p)}$ is trivial. Hence, if $j \geq \lambda + 1$, then the action $(s_1e + s_0^p e^{(p)})^j$ is trivial for any pair $(s_0, s_1)$. For $1 \leq j \leq \lambda$, the $j$-rank is maximal (and equals $\lambda + 1 - j$) whenever $s_1 \neq 0$. We conclude that for $j > \lambda$, we have $V^j(G)_{S_\lambda} = 0$, and for $1 \leq j \leq \lambda$, $V^j(G)_{S_\lambda}$ is one orbit in $V(G)$, corresponding to $[s_0 : 0]$.

Now consider $p \leq \lambda < p^2 - 1$. Let $\lambda = \lambda_0 + \lambda_1$. By the Steinberg tensor product theorem we have $S_\lambda = L(\lambda_0) \otimes L(\lambda_1)^{(1)}$. Observe that $e$ acts trivially on $L(\lambda_1)^{(1)}$ and $e^{(p)}$ acts trivially on $L(\lambda_0)$. Moreover, the Jordan type of $e^{(p)}$ as an operator on $L(\lambda_1)^{(1)}$ is the same as the Jordan type of $e$ as an operator on $L(\lambda_1)$ by Proposition 3.9. Hence, the Jordan type of $s_1e + s_0^p e^{(p)}$ as an operator on $L(\lambda_0) \otimes L(\lambda_1)^{(1)}$ is $[\lambda_0 + 1] \otimes [\lambda_1 + 1]$ when $s_0s_1 \neq 0$. If $s_0 = 0$ or $s_1 = 0$ we get the types $[\lambda_0 + 1] \otimes (\text{triv})$ or $(\text{triv}) \otimes [\lambda_1 + 1]$ respectively.

For $0 < \lambda_0, \lambda_1 < p - 1$, the tensor product formula for Jordan types (see [4, Appendix]) implies that the $j$-rank of $[\lambda_0 + 1] \otimes [\lambda_1 + 1]$ is strictly greater than that of $[\lambda_0 + 1] \otimes (\text{triv})$ or $(\text{triv}) \otimes [\lambda_1 + 1]$ for $j \leq \lambda_1 + \lambda_0 - 1$. Hence, the non-maximal $j$-rank variety in the case when $j \leq \lambda_1 + \lambda_0$ and $s_0s_1 \neq 0$ consists of 2 orbits: $\text{SL}_2 \cdot \{(e, 0) \cup (0, e)\} = \{(\alpha_0, 0)\} \cup \{(0, \alpha_1)\}$ for $\alpha_0, \alpha_1 \in N(\text{SL}_2)$. If $j > \lambda_1 + \lambda_0$ then the non-maximal rank variety is trivial since the $j$-rank is 0 at every point.

The case of $\lambda_1 = 0$ was considered above. If $\lambda_0 = 0$, then $S_\lambda \simeq L(\lambda_1)^{(1)}$. Hence, Corollary 6.11 and the computation for $S_\lambda$ for $\lambda < p$ imply that the non-maximal $j$-rank variety in this case is the orbit corresponding to $[0 : s_1]$ for $j \leq \lambda_1$ and is trivial otherwise.
For $\lambda_0 = p - 1$ or $\lambda_1 = p - 1$, the non-maximal j-rank variety is the same as the support variety for any $j$ (described in Example 1.11(4)) since the support variety is a proper subvariety of $V(G)$ in this case. Finally, for $\lambda = p^2 - 1$, the corresponding module is the Steinberg for $\text{SL}_2(2)$ and is projective, so the non-maximal varieties are all trivial.

We conclude that for $p > j > \lambda_1 + \lambda_0$, the non-maximal $j$-rank variety of $S_\lambda = L(\lambda_0 + p\lambda_1)$ is trivial. For $j \leq \lambda_1 + \lambda_0$ the answer is summarized in the following table.

$$V^j(G)_{S_\lambda} = \begin{cases} \{ (\alpha_0, 0) \} & \text{if } \lambda_1 = 0, \lambda_0 \neq 0 \text{ or } \lambda_0 = p - 1, \lambda_1 \neq p - 1 \\ \{ (0, \alpha_1) \} & \text{if } \lambda_0 = 0, \lambda_1 \neq 0 \text{ or } \lambda_1 = p - 1, \lambda_0 \neq p - 1 \\ \{ (\alpha_0, 0) \} \cup \{ (0, \alpha_1) \} & \text{if } 0 < \lambda_0, \lambda_1 < p - 1 \\ 0 & \text{if } \lambda = p^2 - 1. \end{cases}$$

Projectivizing, we get $\Gamma(G)_{S_\lambda} = \bigcup_{j \geq 1} \text{Proj}(V^j(G)_{S_\lambda})$. In particular, the only two simple $\text{SL}_2(2)$-modules of constant rank are the trivial module and the Steinberg module.

References

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