# COHOMOLOGY OF FINITE DIMENSIONAL POINTED HOPF ALGEBRAS

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Abstract. We prove finite generation of the cohomology ring of any finite dimensional pointed Hopf algebra, having abelian group of grouplike elements, under some mild restrictions on the group order. The proof uses the recent classification by Andruskiewitsch and Schneider of such Hopf algebras. Examples include all of Lusztig's small quantum groups, whose cohomology was first computed explicitly by Ginzburg and Kumar, as well as many new pointed Hopf algebras.

We also show that in general the cohomology ring of a Hopf algebra in a braided category is braided commutative. As a consequence we obtain some further information about the structure of the cohomology ring of a finite dimensional pointed Hopf algebra and its related Nichols algebra.

## 1. Introduction

Venkov [28] and Evens [16] independently proved that the cohomology ring of a finite group with coefficients in a field of positive characteristic is finitely generated. This fundamental result opened the door to using geometric methods in the study of cohomology and modular representations of finite groups. This geometric approach was pioneered by Quillen and expanded by Carlson [13], and Avrunin and Scott [6]. Friedlander and Suslin [18] vastly generalized the result of Venkov and Evens, proving that the cohomology ring of any finite group scheme (equivalently, finite dimensional cocommutative Hopf algebra) over a field of positive characteristic is finitely generated. In a different direction, Ginzburg and Kumar [19] computed the cohomology ring of each of Lusztig's small quantum groups $u_q(\mathfrak{g})$, defined over $\mathbb{C}$, under some restrictions on the parameters; it is the coordinate ring of the nilpotent cone of the Lie algebra $\mathfrak{g}$, and thus is finitely generated. The result is parallel to an earlier result of Friedlander and Parshall [17] who computed the cohomology ring of a restricted Lie algebra in positive characteristic. Recently, Bendel, Nakano, Parshall, and Pillen [8] calculated the cohomology of a small quantum group under significantly loosened restrictions on the parameters. The broad common feature of these works is that they investigate cohomology of certain nonsemisimple finite dimensional Hopf algebras. Hence, these results lead one to ask whether the cohomology ring of any finite dimensional Hopf algebra is

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finitely generated. A positive answer would simultaneously generalize the known results for cocommutative Hopf algebras and for small quantum groups. More generally, Etingof and Ostrik [15] conjectured finite generation of cohomology in the context of finite tensor categories.

In this paper, we begin the task of proving this conjecture for more general classes of noncocommutative Hopf algebras over a field of characteristic 0. We prove finite generation of the cohomology ring of any finite dimensional pointed Hopf algebra, with abelian group of grouplike elements, under some mild restrictions on the group order. Pointed Hopf algebras are precisely those whose coradicals are group algebras, and in turn these groups determine a large part of their structure. We use the recent classification of these Hopf algebras by Andruskiewitsch and Schneider [5]. Each has a presentation by generators and relations similar to those of the small quantum groups, yet they are much more general. Due to this similarity, some of the techniques of Ginzburg and Kumar [19] yield results in this general setting. However some differences do arise, notably that cohomology may no longer vanish in odd degrees, and that useful connections to Lie algebras have not been developed. Thus we must take a somewhat different approach in this paper, which also yields new proofs of some of the results in [19]. Since we have less information available in this general setting, we prove finite generation without computing the full structure of the cohomology ring. However we do explicitly identify a subalgebra over which the cohomology is finite, and establish some results about its structure.

These structure results follow in part from our general result in Section 3, that the cohomology ring of a Hopf algebra in a braided category is always braided graded commutative. This generalizes the well-known result that the cohomology ring of a Hopf algebra is graded commutative, one proof of which follows from the existence of two definitions of its multiplication. We generalize that proof, giving a braided version of the Eckmann-Hilton argument, from which follows the braided graded commutativity result. We apply this to a Nichols algebra in Section 5, thus obtaining some details about the structure of the finitely generated cohomology ring of the corresponding pointed Hopf algebra.

Of course one hopes for results in yet greater generality. However the structure of finite dimensional noncommutative, noncocommutative, nonsemisimple Hopf algebras, other than those treated in this paper, is largely unknown. There are a very small number of known (nontrivial) finite dimensional pointed Hopf algebras having nonabelian groups of grouplike elements (see for example [1, 2]). Even fewer examples are known of nonpointed, nonsemisimple Hopf algebras [12]. To prove finite generation of cohomology in greater generality, it may be necessary to find general techniques, such as the embedding of any finite group scheme into $\mathrm{GL}_n$ used by Friedlander and Suslin [18], rather than depending on structural knowledge of the Hopf algebras as we do here.

Our proof of finite generation is a two-step reduction to type $A_1 \times \cdots \times A_1$, in which case the corresponding Nichols algebra is a quantum complete intersection. For these algebras, we compute cohomology explicitly via a resolution constructed in Section 4. This resolution is adapted from [10, 21] where similar algebras were considered (but in positive characteristic). Each of our two reduction steps involves a spectral sequence associated to an algebra filtration. In the first step (Section 5) a Radford biproduct of the form $\mathcal{B}(V)\#k\Gamma$, for a group $\Gamma$ and Nichols algebra $\mathcal{B}(V)$, has a filtration for which the associated graded algebra has type $A_1 \times \cdots \times A_1$. This filtration is generalized from De Concini and Kac [14]. We identify some permanent cycles and apply a lemma adapted from Friedlander and Suslin [18] to conclude finite generation. In the second reduction step (Section 6), any of Andruskiewitsch and Schneider's pointed Hopf algebras $u(\mathcal{D}, \lambda, \mu)$ has a filtration for which the associated graded algebra is a Radford biproduct, whose cohomology features in Section 5. Again we identify some permanent cycles and conclude finite generation. As a corollary we show that the Hochschild cohomology ring of $u(\mathcal{D}, \lambda, \mu)$ is also finitely generated.

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## 2. Definitions and Preliminary Results

Let $k$ be a field, usually assumed to be algebraically closed and of characteristic 0. All tensor products are over $k$ unless otherwise indicated. Let $\Gamma$ be a finite group.

**Hopf algebras in Yetter-Drinfeld categories.** A *Yetter-Drinfeld module* over $k\Gamma$ is a $\Gamma$-graded vector space $V = \oplus_{g\in\Gamma} V_g$ that is also a $k\Gamma$-module for which $g \cdot V_h = V_{ghg^{-1}}$ for all $g, h \in \Gamma$. The grading by the group $\Gamma$ is equivalent to a $k\Gamma$-comodule structure on $V$, that is a map $\delta : V \to k\Gamma \otimes V$, defined by $\delta(v) = g \otimes v$ for all $v \in V_g$. Let ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ denote the category of all Yetter-Drinfeld modules over $k\Gamma$. The category has a tensor product: $(V \otimes W)_g = \oplus_{xy=g} V_x \otimes W_y$ for all $g \in \Gamma$, and $\Gamma$ acts diagonally on $V \otimes W$, that is, $g(v \otimes w) = gv \otimes gw$ for all $g \in \Gamma$, $v \in V$, and $w \in W$. There is a *braiding* $c : V \otimes W \xrightarrow{\sim} W \otimes V$ for all $V, W \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ as follows: Let $g \in \Gamma$, $v \in V_g$, and $w \in W$. Then $c(v \otimes w) = gw \otimes v$. Thus ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ is a braided monoidal category. (For details on the category ${}^{\Gamma}_{\Gamma}\mathcal{YD}$, including the connection to Hopf algebras recalled below, see for example [5].)

Let $\mathcal{C}$ be any braided monoidal category. For simplicity, we will always assume the tensor product is strictly associative with strict unit object $I$. An *algebra in* $\mathcal{C}$ is an object $R$ together with morphisms $u : I \to R$ and $m : R \otimes R \to R$ in $\mathcal{C}$, such that $m$ is associative in the sense that $m(m \otimes 1_R) = m(1_R \otimes m)$, and $u$ is a unit in the sense that $m(u \otimes 1_R) = 1_R = m(1_R \otimes u)$. The definition of a coalgebra in $\mathcal{C}$ is similar, with the arrows going in the opposite direction. Thus, an algebra (resp. coalgebra) in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ is simply an ordinary algebra (resp. coalgebra) with multiplication (resp. comultiplication) a graded and equivariant map. A *braided Hopf algebra in* $\mathcal{C}$ is an algebra as well as coalgebra in $\mathcal{C}$ such that its comultiplication and counit are algebra morphisms, and such that the identity morphism $\mathrm{id}\colon R \to R$ has a convolution inverse $s$ in $\mathcal{C}$. When we say that the comultiplication $\Delta\colon R \to R \otimes R$ should be an algebra morphism, the braiding $c$ of $\mathcal{C}$ arises in the definition of the algebra structure of $R \otimes R$ (so in particular, a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ is not an ordinary Hopf algebra). More generally, if $A, B$ are two algebras in $\mathcal{C}$, their tensor product $A \otimes B$ is defined to have multiplication $m_{A\otimes B} = (m_A \otimes m_B)(1_A \otimes c \otimes 1_B)$.

An example of a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ is the *Nichols algebra* $\mathcal{B}(V)$ associated to a Yetter-Drinfeld module $V$ over $k\Gamma$; $\mathcal{B}(V)$ is the quotient of the tensor algebra $T(V)$ by the largest homogeneous braided Hopf ideal generated by homogeneous elements of degree at least 2. For details, see [4, 5]. In this paper, we need only the structure of $\mathcal{B}(V)$ in some cases as are explicitly described below.

If $R$ is a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$, then its *Radford biproduct* (or *bosonization*) $R\#k\Gamma$ is a Hopf algebra in the usual sense (that is, a Hopf algebra in ${}^{k}_{k}\mathcal{YD}$). As an algebra, $R\#k\Gamma$ is just a *skew group algebra*, that is, $R\#k\Gamma$ is a free $R$-module with basis $\Gamma$ on which multiplication is defined by $(rg)(sh) = r(g\cdot s)gh$ for all $r, s \in R$ and $g, h \in G$. Comultiplication is given by $\Delta(rg) = r^{(1)}(r^{(2)})_{(-1)}g \otimes (r^{(2)})_{(0)}g$, for all $r \in R$ and $g \in \Gamma$, where $\Delta(r) = \sum r^{(1)} \otimes r^{(2)}$ in $R$ as a Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ and $\delta(r) = \sum r_{(-1)} \otimes r_{(0)}$ denotes the $k\Gamma$-comodule structure.

**The pointed Hopf algebras of Andruskiewitsch and Schneider.** A *pointed* Hopf algebra $H$ is one for which all simple comodules are one dimensional. This is equivalent to the condition $H_0 = k\Gamma$ where $\Gamma = G(H)$ is the group of grouplike elements of $H$ and $H_0$ is the coradical of $H$ (the initial term in the coradical filtration).

The Hopf algebras of Andruskiewitsch and Schneider in [5] are pointed, and are deformations of Radford biproducts. They depend on the following data: Let $\theta$ be a positive integer. Let $(a_{ij})_{1\le i,j\le\theta}$ be a *Cartan matrix of finite type*, that is the Dynkin diagram of $(a_{ij})$ is a disjoint union of copies of some of the diagrams $A_\bullet$, $B_\bullet$, $C_\bullet$, $D_\bullet$, $E_6$, $E_7$, $E_8$, $F_4$, $G_2$. In particular, $a_{ii} = 2$ for $1 \le i \le \theta$, $a_{ij}$ is a nonpositive integer for $i \ne j$, and $a_{ij} = 0$ implies $a_{ji} = 0$. Its *Dynkin diagram* is a graph with vertices labeled $1, \dots, \theta$: The vertices $i$ and $j$ are connected by $a_{ij}a_{ji}$ edges, and if $|a_{ij}| > |a_{ji}|$, there is an arrow pointing from $j$ to $i$.

Now assume $\Gamma$ is *abelian*, and denote by $\hat{\Gamma}$ its dual group of characters. For each $i$, $1 \leq i \leq \theta$, choose $g_i \in \Gamma$ and $\chi_i \in \hat{\Gamma}$ such that $\chi_i(g_i) \neq 1$ and

$$\chi_j(g_i)\chi_i(g_j) = \chi_i(g_i)^{a_{ij}} \tag{2.0.1}$$

(the *Cartan condition*) holds for $1 \leq i, j \leq \theta$. Letting $q_{ij} = \chi_j(g_i)$, this becomes $q_{ij}q_{ji} = q_{ii}^{a_{ij}}$. Call

$$\mathcal{D} = (\Gamma, (g_i)_{1\leq i\leq\theta}, (\chi_i)_{1\leq i\leq\theta}, (a_{ij})_{1\leq i,j\leq\theta}) \tag{2.0.2}$$

a *datum of finite Cartan type* associated to $\Gamma$ and $(a_{ij})$. The Hopf algebras of interest will be generated as algebras by $\Gamma$ and symbols $x_1, \ldots, x_\theta$.

Let $V$ be the vector space with basis $x_1, \ldots, x_\theta$. Then $V$ has a structure of a Yetter-Drinfeld module over $k\Gamma$: $V_g = \mathrm{Span}_k\{x_i \mid g_i = g\}$ and $g(x_i) = \chi_i(g)x_i$ for $1 \leq i \leq \theta$ and $g \in \Gamma$. This induces the structure of an algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ on the tensor algebra $T(V)$. In particular $\Gamma$ acts by automorphisms on $T(V)$, and $T(V)$ is a $\Gamma$-graded algebra in which $x_{i_1} \cdots x_{i_s}$ has degree $g_{i_1} \cdots g_{i_s}$. The braiding $c\colon T(V) \otimes T(V) \to T(V) \otimes T(V)$ is induced by $c(x_i \otimes y) = g_i(y) \otimes x_i$. Moreover, $T(V)$ can be made a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ if we define comultiplication as the unique algebra map $\Delta\colon T(V) \to T(V) \otimes T(V)$ satisfying $\Delta(v) = v \otimes 1 + 1 \otimes v$ for all $v \in V$. We define the braided commutators

$$\mathrm{ad}_c(x_i)(y) = [x_i, y]_c := x_i y - g_i(y)x_i,$$

for all $y \in T(V)$, and similarly in quotients of $T(V)$ by homogeneous ideals.

Let $\Phi$ denote the root system corresponding to $(a_{ij})$, and let $\Pi$ denote a fixed set of simple roots. If $\alpha_i, \alpha_j \in \Pi$, write $i \sim j$ if the corresponding vertices in the Dynkin diagram of $\Phi$ are in the same connected component. Choose scalars $\lambda = (\lambda_{ij})_{1\leq i<j\leq\theta,\ i\not\sim j}$, called *linking parameters*, such that

$$\lambda_{ij} = 0 \quad \text{if} \quad g_i g_j = 1 \quad \text{or} \quad \chi_i\chi_j \neq \varepsilon,$$

where $\varepsilon$ is the identity element in the dual group $\hat{\Gamma}$ (equivalently $\varepsilon$ is the counit on $k\Gamma$, $\varepsilon(g) = 1$ for all $g \in \Gamma$). The Hopf algebra $U(\mathcal{D}, \lambda)$ defined by Andruskiewitsch and Schneider [5] is the quotient of $T(V)\#k\Gamma$ by relations corresponding to the equations

$$\begin{array}{llcll}
\textit{(group action)} & g x_i g^{-1} & = & \chi_i(g)x_i & (g \in \Gamma, 1 \leq i \leq \theta), \\
\textit{(Serre relations)} & (\mathrm{ad}_c(x_i))^{1-a_{ij}}(x_j) & = & 0 & (i \neq j,\ i \sim j), \\
\textit{(linking relations)} & (\mathrm{ad}_c(x_i))(x_j) & = & \lambda_{ij}(1 - g_i g_j) & (i < j,\ i \not\sim j).
\end{array}$$

The coalgebra structure of $U(\mathcal{D}, \lambda)$ is given by

$$\Delta(g) = g \otimes g, \quad \Delta(x_i) = x_i \otimes 1 + g_i \otimes x_i,$$

$\varepsilon(g) = 1$, $\varepsilon(x_i) = 0$, $s(g) = g^{-1}$, $s(x_i) = -g_i^{-1}x_i$, for all $g \in \Gamma$, $1 \leq i \leq \theta$.

**Example 2.1.** Let $\mathfrak{g}$ be a simple Lie algebra of rank $n$, let $(a_{ij})$ be *two* block diagonal copies of the corresponding Cartan matrix, and let $\theta = 2n$. Let $q$ be a primitive $\ell$th root of unity, $\ell$ odd. Let $\Gamma = (\mathbb{Z}/\ell\mathbb{Z})^n$ with generators $g_1, \ldots, g_n$ and define $\chi_i \in \hat{\Gamma}$ by $\chi_i(g_j) = q^{\langle \alpha_i,\, \alpha_j \rangle}$. Let $g_{i+n} = g_i$ and $\chi_{i+n} = \chi_i^{-1}$ for $1 \le i \le n$. Let $\lambda_{ij} = (q^{-1} - q)^{-1}\delta_{j,i+n}$ for $1 \le i < j \le n$. Then $U(\mathcal{D}, \lambda)$ is a quotient of the quantum group $U_q(\mathfrak{g})$. An epimorphism $U_q(\mathfrak{g}) \to U(\mathcal{D}, \lambda)$ is given by $K_i \mapsto g_i$, $E_i \mapsto x_i$ and $F_i \mapsto x_{i+n}g_i^{-1}$.

The Hopf algebra $U(\mathcal{D}, \lambda)$ has finite dimensional quotients that we discuss next. As in [5] we make the assumptions:

(2.1.1) *The order of $\chi_i(g_i)$ is odd for all $i$, and is prime to 3 for all $i$ in a connected component of type $G_2$.*

It then follows from the Cartan condition (2.0.1) that the order of $\chi_i(g_i)$ is the same as the order of $\chi_j(g_j)$ if $i \sim j$. That is, this order is constant in each connected component $J$ of the Dynkin diagram; denote this common order by $N_J$. It will also be convenient to denote it by $N_{\beta_j}$ or $N_j$ for each positive root $\beta_j$ in $J$ (this standard notation is defined below). Let $\alpha \in \Phi^+$, $\alpha = \sum_{i=1}^{\theta} n_i\alpha_i$, and let $\mathrm{ht}(\alpha) = \sum_{i=1}^{\theta} n_i$,

$$g_\alpha = \prod_{i=1}^{\theta} g_i^{n_i}, \quad \text{and} \quad \chi_\alpha = \prod_{i=1}^{\theta} \chi_i^{n_i}. \tag{2.1.2}$$

There is a unique connected component $J_\alpha$ of the Dynkin diagram of $\Phi$ for which $n_i \neq 0$ implies $i \in J_\alpha$. We write $J = J_\alpha$ when it is clear which $\alpha$ is intended.

Let $W$ be the Weyl group of the root system $\Phi$. Let $w_0 = s_{i_1} \cdots s_{i_r}$ be a reduced decomposition of the longest element $w_0 \in W$ as a product of simple reflections. Let

$$\beta_1 = \alpha_{i_1}, \quad \beta_2 = s_{i_1}(\alpha_{i_2}), \ \ldots, \ \beta_r = s_{i_1}s_{i_2}\cdots s_{i_{r-1}}(\alpha_{i_r}).$$

Then $\beta_1, \ldots, \beta_r$ are precisely the positive roots $\Phi^+$ [24]. Corresponding root vectors $x_{\beta_j} \in U(\mathcal{D}, \lambda)$ are defined in the same way as for the traditional quantum groups: In case $\mathcal{D}$ corresponds to the data for a quantum group $U_q(\mathfrak{g})$ (see Example 2.1), let

$$x_{\beta_j} = T_{i_1}T_{i_2}\cdots T_{i_{j-1}}(x_{i_j}),$$

where the $T_{i_j}$ are Lusztig's algebra automorphisms of $U_q(\mathfrak{g})$ [24]. The $x_{\beta_j}$ may be expressed as iterated braided commutators. If $\beta_j$ is a simple root $\alpha_l$, then $x_{\beta_j} = x_l$. In our more general setting, as in [5], define the $x_{\beta_j}$ to be the analogous iterated braided commutators. More precisely, we describe how to obtain these elements in case the Dynkin diagram is connected; in the general case construct the root vectors separately for each connected component. Let $I$ be the ideal of $T(V)$ generated by elements corresponding to the Serre relations. Then [5, Lemmas 1.2 and 1.7] may be used to obtain a *linear* isomorphism between $T(V)/I$ and the upper triangular part of some $U_q(\mathfrak{g})$ with the same Dynkin diagram. This linear isomorphism preserves products and braided commutators up to nonzero scalar multiples, and thus yields $x_{\beta_j}$ in our general setting as an iterated braided commutator. (See the proof of Lemma 2.4 below for more details on this linear isomorphism.)

Choose scalars $(\mu_\alpha)_{\alpha\in\Phi^+}$, called *root vector parameters*, such that

$$\mu_\alpha = 0 \text{ if } g_\alpha^{N_\alpha} = 1 \text{ or } \chi_\alpha^{N_\alpha} \neq \varepsilon. \tag{2.1.3}$$

The finite dimensional Hopf algebra $u(\mathcal{D}, \lambda, \mu)$ is the quotient of $U(\mathcal{D}, \lambda)$ by the ideal generated by all

$$\textit{(root vector relations)} \qquad x_\alpha^{N_\alpha} - u_\alpha(\mu) \qquad (\alpha \in \Phi^+) \tag{2.1.4}$$

where $u_\alpha(\mu) \in k\Gamma$ is defined inductively on $\Phi^+$ in [5, Defn. 2.14]. In this paper, we do not need the details of the construction of the elements $u_\alpha(\mu)$ in the group algebra. We only need the fact that if $\mu_\alpha = 0$ for all $\alpha \in \Phi^+$, then $u_\alpha(\mu) = 0$ for all $\alpha \in \Phi^+$ [5] (see for example Lemma 6.1 below). It is interesting to note that if $\alpha$ is a *simple* root, then $u_\alpha(\mu) := \mu_\alpha(1 - g_\alpha^{N_\alpha})$.

**Example 2.2.** Let $\mathcal{D}$, $\lambda$ be the data from Example 2.1. Then there is an isomorphism $u_q(\mathfrak{g}) \simeq u(\mathcal{D}, \lambda, 0)$, induced by the epimorphism $U_q(\mathfrak{g}) \to U(\mathcal{D}, \lambda)$ given in that example.

The following theorem is [5, Classification Theorem 0.1], and requires $k$ to be algebraically closed of characteristic 0.

**Theorem 2.3** (Andruskiewitsch-Schneider)**.** *The Hopf algebras $u(\mathcal{D}, \lambda, \mu)$ are finite dimensional and pointed. Conversely, if $H$ is a finite dimensional pointed Hopf algebra having abelian group of grouplike elements with order not divisible by primes less than 11, then $H \simeq u(\mathcal{D}, \lambda, \mu)$ for some $\mathcal{D}$, $\lambda$, $\mu$.*

**Two filtrations.** Note that $u(\mathcal{D}, \lambda, \mu)$ is a (coradically) filtered Hopf algebra, with $\deg(x_i) = 1$ $(1 \leq i \leq \theta)$ and $\deg(g) = 0$ $(g \in \Gamma)$. The associated graded Hopf algebra is isomorphic to the graded Hopf algebra $u(\mathcal{D}, 0, 0)$. There is an isomorphism $u(\mathcal{D}, 0, 0) \simeq \mathcal{B}(V)\#k\Gamma$, the Radford biproduct (or bosonization) of the Nichols algebra $\mathcal{B}(V)$ of the Yetter-Drinfeld module $V$ over $k\Gamma$. For details, see [5]. Note that $\mathcal{B}(V)$ is isomorphic to the subalgebra of $u(\mathcal{D}, 0, 0)$ generated by $x_1, \ldots, x_\theta$.

In Section 6, we prove that the cohomology of $u(\mathcal{D}, \lambda, \mu)$ is finitely generated by using a spectral sequence relating its cohomology to that of $\mathcal{B}(V)\#k\Gamma$. In Section 5, we prove that the cohomology of this (coradically) graded bialgebra $\mathcal{B}(V)\#k\Gamma$ is finitely generated, by using a spectral sequence relating its cohomology to that of a much simpler algebra: We put a different filtration on it (see Lemma 2.4 below) for which the associated graded algebra has type $A_1 \times \cdots \times A_1$. (That is, the Dynkin diagram is a disjoint union of diagrams $A_1$.) In Section 4, we give the cohomology for type $A_1 \times \cdots \times A_1$ explicitly as a special case of the cohomology of a quantum complete intersection.

By [5, Thm. 2.6], the Nichols algebra $\mathcal{B}(V)$ has PBW basis all

$$\mathbf{x}^{\mathbf{a}} = x_{\beta_1}^{a_1} \cdots x_{\beta_r}^{a_r} \quad (0 \le a_i < N_{\beta_i}), \tag{2.3.1}$$

and further

$$[x_\alpha, x_\beta^{N_\beta}]_c = 0 \tag{2.3.2}$$

for all $\alpha, \beta \in \Phi^+$. As in [14], put a total order on the PBW basis elements as follows: The degree of such an element is

$$d(x_{\beta_1}^{a_1} \cdots x_{\beta_r}^{a_r}) = \Big(\prod a_i \operatorname{ht}(\beta_i), a_r, \dots, a_1\Big) \in \mathbb{N}^{r+1}.$$

Order the elements (2.3.1) lexicographically by degree where

$$(0, \dots, 0, 1) < (0, \dots, 0, 1, 0) < \cdots < (1, 0, \dots, 0).$$

**Lemma 2.4.** *In the Nichols algebra $\mathcal{B}(V)$, for all $i < j$,*

$$[x_{\beta_j}, x_{\beta_i}]_c = \sum_{\mathbf{a} \in \mathbb{N}^p} \rho_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$$

*where the $\rho_{\mathbf{a}}$ are scalars for which $\rho_{\mathbf{a}} = 0$ unless $d(\mathbf{x}^{\mathbf{a}}) < d(x_{\beta_i} x_{\beta_j})$.*

*Proof.* First note that if $i \not\sim j$, then $[x_{\beta_j}, x_{\beta_i}]_c = 0$ by the Serre relations, since $a_{ij} = 0$. Thus we may assume now that $i \sim j$. Then the lemma is just the translation of [14, Lemma 1.7], via twisting by a cocycle and specializing $q$, into this more general setting. The twisting method is described in [5] and is used there to prove that (2.3.1) is a basis of $\mathcal{B}(V)$. In particular, [5, Lemma 2.3] states that there exist integers $d_i \in \{1, 2, 3\}$, $1 \le i \le \theta$, and $q \in k$ such that for all $1 \le i, j \le \theta$, $q_{ii} = q^{2d_i}$ and $d_i a_{ij} = d_j a_{ji}$. This allows us to define a matrix $(q'_{ij})$ by $q'_{ij} = q^{d_i a_{ij}}$, so that $q_{ij} q_{ji} = q'_{ij} q'_{ji}$ and $q_{ii} = q'_{ii}$. Let $V' \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ with basis $x'_1 \dots, x'_\theta$ and structure given by the data $\{q'_{ij}\}$. Now $\{q'_{ij}\}$ are classical quantum group parameters for the positive part of $U_q(\mathfrak{g})$, where $\mathfrak{g}$ corresponds to the Dynkin diagram. Thus [14, Lemma 1.7] applies. Further, [5, Lemma 1.2] states that there is a cocycle $\sigma : \mathbb{Z}\Pi \times \mathbb{Z}\Pi \to k^\times$ and a $k$-linear isomorphism $\phi : T(V) \to T(V')$ with $\phi(x_i) = x'_i$ and such that if $x \in T(V)_{g_i}$ and $y \in T(V)_{g_j}$, then

$$\phi(xy) = \sigma(\alpha_i, \alpha_j)\phi(x)\phi(y) \quad \text{and} \quad \phi([x, y]_c) = \sigma(\alpha_i, \alpha_j)[\phi(x), \phi(y)]_{c'}.$$

Thus $\phi$ preserves the PBW basis up to nonzero scalar multiples, and it preserves the total order on the PBW basis. Since it also preserves braided commutators up to nonzero scalar, the lemma now holds as a consequence of the same result [14, Lemma 1.7] for the parameters $\{q'_{ij}\}$. $\square$

By Lemma 2.4, the above ordering induces a filtration $F$ on $\mathcal{B}(V)$ for which the associated graded algebra $\operatorname{Gr}\mathcal{B}(V)$ has relations $[x_{\beta_j}, x_{\beta_i}]_c = 0$ for all $i < j$, and $x_{\beta_i}^{N_i} = 0$. In particular, $\operatorname{Gr}\mathcal{B}(V)$ is of type $A_1 \times \cdots \times A_1$. We may put a corresponding Hopf algebra structure on $(\operatorname{Gr}\mathcal{B}(V))\#k\Gamma$ as follows. If $\beta_i = \sum_{j=1}^{\theta} n_j\alpha_j$, let $g_{\beta_i} = g_1^{n_1}\cdots g_\theta^{n_\theta}$ as in (2.1.2). Now identify $\beta_1, \dots, \beta_r$ with the simple roots of type $A_1 \times \cdots \times A_1$. For $i < j$, define $q_{\beta_i\beta_j} = \chi_{\beta_j}(g_{\beta_i})$, $q_{\beta_j\beta_i} = q_{\beta_i\beta_j}^{-1}$, and $q_{\beta_i\beta_i} = 1$. Then the Cartan condition (2.0.1) holds for the scalars $q_{\beta_i\beta_j}$ in this type $A_1 \times \cdots \times A_1$, and $(\operatorname{Gr}\mathcal{B}(V))\#k\Gamma$ is a Hopf algebra for which

$$\Delta(x_{\beta_i}) = x_{\beta_i} \otimes 1 + g_{\beta_i} \otimes x_{\beta_i}.$$

For our spectral sequence constructions, we rewrite the total order on the PBW basis elements (2.3.1) explicitly as an indexing by positive integers. We may set

$$\deg(x_{\beta_1}^{a_1}\cdots x_{\beta_r}^{a_r}) = N_{\beta_1}\cdots N_{\beta_r}\prod a_i \operatorname{ht}(\beta_i) + N_{\beta_1}\cdots N_{\beta_{r-1}}a_r + \cdots + N_{\beta_1}a_2 + a_1.$$

A case-by-case argument shows that $d(x_{\beta_1}^{a_1}\cdots x_{\beta_r}^{a_r}) < d(x_{\beta_1}^{b_1}\cdots x_{\beta_r}^{b_r})$ if, and only if, $\deg(x_{\beta_1}^{a_1}\cdots x_{\beta_r}^{a_r}) < \deg(x_{\beta_1}^{b_1}\cdots x_{\beta_r}^{b_r})$.

**Hochschild cohomology.** In this paper, we are interested in the cohomology ring $\mathrm{H}^*(u(\mathcal{D},\lambda,\mu),k) := \operatorname{Ext}^*_{u(\mathcal{D},\lambda,\mu)}(k,k)$, where $k$ is the trivial module given by the counit $\varepsilon$. If $A$ is any $k$-algebra with an augmentation $\varepsilon : A \to k$, note that $\operatorname{Ext}^*_A(k,k)$ is isomorphic to Hochschild cohomology with trivial coefficients, $\operatorname{Ext}^*_{A\otimes A^{op}}(A,k)$: This is due to an equivalence of the bar complexes for computing these Ext algebras. That is, these bar complexes are each equivalent to the reduced complex

$$(2.4.1)\quad C^\bullet : \qquad 0 \to \operatorname{Hom}_k(k,k) \to \operatorname{Hom}_k(A^+,k) \to \operatorname{Hom}_k((A^+)^2,k) \to \cdots,$$

where $A^+ = \ker\varepsilon$ is the augmentation ideal of $A$. The differential is given by $\delta_{n+1}(f) = \sum_{i=0}^{n-1}(-1)^{i+1} f\circ(1^i\otimes m\otimes 1^{n-i-1})$ for all $f : (A^+)^{\otimes n} \to k$. We will exploit this equivalence. This complex arises, for example, by applying $\operatorname{Hom}_A(-,k)$ to the free $A$-resolution of $k$:

$$\cdots \to A\otimes (A^+)^{\otimes 2} \xrightarrow{\partial_2} A\otimes A^+ \xrightarrow{\partial_1} A \xrightarrow{\varepsilon} k \to 0,$$

where

$$(2.4.2)\qquad \partial_i(a_0\otimes\cdots\otimes a_i) = \sum_{j=0}^{i-1}(-1)^j a_0\otimes\cdots\otimes a_j a_{j+1}\otimes\cdots\otimes a_i.$$

Then $\partial^*_{n+1} = \delta_{n+1}$. Equivalently, we may apply $\operatorname{Hom}_{A\otimes A^{op}}(-,k)$ to the free $A\otimes A^{op}$-resolution of $A$:

$$\cdots \to A\otimes (A^+)^{\otimes 2}\otimes A \xrightarrow{d_2} A\otimes A^+\otimes A \xrightarrow{d_1} A\otimes A \xrightarrow{\varepsilon} A \to 0,$$

$d_i(a_0\otimes\cdots\otimes a_{i+1}) = \sum_{j=0}^{i}(-1)^j a_0\otimes\cdots\otimes a_j a_{j+1}\otimes\cdots\otimes a_{i+1}$.

**A finite generation lemma.** In Sections 5 and 6, we will need the following general lemma adapted from [18, Lemma 1.6]. Recall that an element $a \in E_r^{p,q}$ is called a *permanent cycle* if $d_i(a) = 0$ for all $i \geq r$.

**Lemma 2.5.** (a) *Let* $E_1^{p,q} \Longrightarrow E_\infty^{p+q}$ *be a multiplicative spectral sequence of* $k$*-algebras concentrated in the half plane* $p + q \geq 0$*, and let* $A^{*,*}$ *be a bigraded commutative* $k$*-algebra concentrated in even (total) degrees. Assume that there exists a bigraded map of algebras* $\phi : A^{*,*} \longrightarrow E_1^{*,*}$ *such that*

(i) $\phi$ *makes* $E_1^{*,*}$ *into a Noetherian* $A^{*,*}$*-module, and*
(ii) *the image of* $A^{*,*}$ *in* $E_1^{*,*}$ *consists of permanent cycles.*

*Then* $E_\infty^*$ *is a Noetherian module over* $\mathrm{Tot}(A^{*,*})$.

(b) *Let* $\widetilde{E}_1^{p,q} \Longrightarrow \widetilde{E}_\infty^{p+q}$ *be a spectral sequence that is a bigraded module over the spectral sequence* $E^{*,*}$*. Assume that* $\widetilde{E}_1^{*,*}$ *is a Noetherian module over* $A^{*,*}$ *where* $A^{*,*}$ *acts on* $\widetilde{E}_1^{*,*}$ *via the map* $\phi$*. Then* $\widetilde{E}_\infty^*$ *is a finitely generated* $E_\infty^*$*-module.*

*Proof.* Let $\Lambda_r^{*,*} \subset E_r^{*,*}$ be the bigraded subalgebra of permanent cycles in $E_r^{*,*}$. Observe that $d_r(E_r^{*,*})$ is an $A^{*,*}$-invariant left ideal of $\Lambda_r^{*,*}$. Indeed, let $a \in A^{p,q}$ and $x \in E_r^{s,t}$. We have $d_r(\overline{\phi(a)}x) = d_r(\overline{\phi(a)})x + (-1)^{p+q}\overline{\phi(a)}d_r(x) = \overline{\phi(a)}d_r(x)$ since $\overline{\phi(a)} \in A^{*,*}$ is assumed to be a permanent cycle of even total degree. A similar computation shows that $\Lambda_1^{*,*}$ is an $A^{*,*}$-submodule of $E_1^{*,*}$. By induction, $\Lambda_{r+1}^{*,*} = \Lambda_r^{*,*}/d_r(E_r^{*,*})$ is an $A^{*,*}$-module for any $r \geq 1$. We get a sequence of surjective maps of $A^{*,*}$-modules:

$$\Lambda_1^{*,*} \twoheadrightarrow \cdots \twoheadrightarrow \Lambda_r^{*,*} \twoheadrightarrow \Lambda_{r+1}^{*,*} \twoheadrightarrow \cdots \tag{2.5.1}$$

Since $\Lambda_1^{*,*}$ is an $A^{*,*}$-submodule of $E_1^{*,*}$, it is Noetherian as an $A^{*,*}$-module. Therefore, the kernels of the maps $\Lambda_1^{*,*} \twoheadrightarrow \Lambda_r^{*,*}$ are Noetherian for all $r \geq 1$. These kernels form an increasing chain of submodules of $\Lambda_1^{*,*}$, hence, by the Noetherian property, they stabilize after finitely many steps. Consequently, the sequence (2.5.1) stabilizes after finitely many steps. We conclude that $\Lambda_r^{*,*} = E_\infty^{*,*}$ for some $r$. Therefore $E_\infty^{*,*}$ is a Noetherian $A^{*,*}$-module. By [16, Proposition 2.1], $E_\infty^*$ is a Noetherian module over $\mathrm{Tot}(A^{*,*})$, finishing the proof of (a).

Let $\widetilde{\Lambda}_r^{*,*} \subset \widetilde{E}_r^{*,*}$ be the subspace of permanent cycles. Arguing as above we can show that $\widetilde{\Lambda}_r^{*,*}$ is an $A^{*,*}$-submodule of $\widetilde{E}_r^{*,*}$, and, moreover, that there exists $r$ such that $\widetilde{\Lambda}_r^{*,*} = \widetilde{E}_\infty^{*,*}$. Hence, $\widetilde{E}_\infty^{*,*}$ is Noetherian over $A^{*,*}$. Applying [16, Proposition 2.1] once again, we conclude that $\widetilde{E}_\infty^*$ is Noetherian and hence finitely generated over $\mathrm{Tot}(A^{*,*})$. Therefore, it is finitely generated over $E_\infty^*$. $\square$

## 3. Braided graded commutativity of cohomology

We will show that Hochschild cohomology of a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$ is a braided graded commutative algebra (more precisely, its opposite algebra is). This will be done in a more general categorical setting, so it will make sense to relax the overall requirement that $k$ is a field.

To prepare for the main result of this section, we need some information on the Alexander-Whitney map. We follow Weibel [29, Section 8.5]. The case of a tensor product of simplicial modules over a commutative ring (rather than simplicial objects in a tensor category, or bisimplicial objects) can also be found in Mac Lane [25, Ch. VII, Section 8].

Consider a bisimplicial object $A=(A_{k\ell},\partial^h,\partial^v)$ in an abelian category. Associated to it we have the diagonal simplicial object $\operatorname{diag} A$ with components $A_{nn}$; to this in turn we associate the usual chain complex, whose differentials we denote by $d_n^{\mathrm{diag}}$. Also associated to $A$ is the double complex $CA$, whose horizontal and vertical differentials we denote by $d^h_{k,\ell}\colon A_{k\ell}\to A_{k-1,\ell}$ and $d^v_{k,\ell}\colon A_{k\ell}\to A_{k,\ell-1}$. There is a natural chain morphism, unique up to natural chain homotopy,

$$f\colon \operatorname{diag} A\to \operatorname{Tot} CA$$

for which $f_0$ is the identity. We note that $f$ being a chain morphism means that the components $f_{k\ell}\colon A_{nn}\to A_{k\ell}$ for $n=k+\ell$ satisfy

$$f_{k\ell}\, d^{\mathrm{diag}}_{n+1}=d^h_{k+1,\ell}f_{k+1,\ell}+d^v_{k,\ell+1}f_{k,\ell+1}.$$

One possible choice for $f$ is the Alexander-Whitney map, whose components are

$$f_{k\ell}=\partial^h_{k+1}\dots\partial^h_n\underbrace{\partial^v_0\dots\partial^v_0}_{k}\colon A_{nn}\to A_{k\ell}$$

for $n=k+\ell$. We claim that

$$f'_{k\ell}=(-1)^{k\ell}\partial^h_0\dots\partial^h_0\partial^v_{\ell+1}\dots\partial^v_n$$

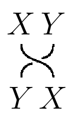
is another choice, and therefore chain homotopic to $f$. For this it suffices to check that $f'$ (which is clearly natural) is a chain morphism. Instead of doing this directly (by more or less the same calculations usually done for the Alexander-Whitney map), we consider the front-to-back dual version of $A$. We denote this by $\tilde{A}$. The horizontal and vertical face operators are $\tilde{\partial}^h_k=\tilde{\partial}^h_{n-k}$ and $\tilde{\partial}^v_k=\tilde{\partial}^v_{n-k}$. The horizontal and vertical differentials in $C\tilde{A}$ are $\tilde{d}^h_{k,\ell}=(-1)^k d^h_{k,\ell}$ and $\tilde{d}^v_{k,\ell}=(-1)^\ell d^v_{k,\ell}$, the differentials in $\operatorname{diag}\tilde{A}$ are $\tilde{d}^{\mathrm{diag}}_n=(-1)^n d^{\mathrm{diag}}_n$. The Alexander-Whitney map $\tilde{f}$

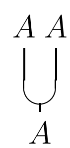
of $\tilde{A}$ is related to $f'$ by $f'_{k\ell} = (-1)^{k\ell}\tilde{f}_{k\ell}$. Therefore

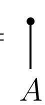
$$\begin{aligned}
f'_{k\ell}\, d^{\mathrm{diag}}_{n+1} &= (-1)^{k\ell}\tilde{f}_{k\ell}(-1)^{n+1}\tilde{d}^{\mathrm{diag}}_{n+1} \\
&= (-1)^{k\ell+k+\ell+1}(\tilde{d}^h_{k+1,\ell}\tilde{f}_{k+1,\ell} + \tilde{d}^v_{k,\ell+1}\tilde{f}_{k,\ell+1}) \\
&= (-1)^{k+1}\tilde{d}^h_{k+1,\ell}(-1)^{(k+1)\ell}\tilde{f}_{k+1,\ell} + (-1)^{\ell+1}\tilde{d}^v_{k,\ell+1}(-1)^{k(\ell+1)}\tilde{f}_{k,\ell+1} \\
&= d^h_{k+1,\ell}f'_{k+1,\ell} + d^v_{k,\ell+1}f'_{k,\ell+1}.
\end{aligned}$$

We will use the above to treat the cohomology of braided Hopf algebras with trivial coefficients.

Let $\mathcal{C}$ be a braided monoidal category with braiding $c$; we denote the unit object by $I$. We will assume that the tensor product in $\mathcal{C}$ is strictly associative and unital, so we can perform some calculations in the graphical calculus (see for example [30] for more extensive examples of graphical calculations or [22] for more rigorous exposition). Our standard notations for braiding, multiplication, and unit will be

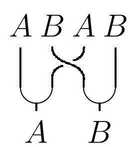
$c = c_{XY} =$ 

, $\quad m = m_A =$ 

, $\quad \eta = \eta_A =$ 

.

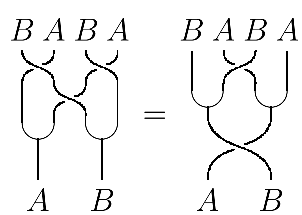
Thus, if $A, B$ are algebras in $\mathcal{C}$, their tensor product $A \otimes B$ is defined to have multiplication

$m_{A\otimes B} =$ 

.

We will need the following universal property of the tensor product algebra: Given an algebra $R$ in $\mathcal{C}$ and algebra morphisms $f_X \colon X \to R$ for $X \in \{A, B\}$ satisfying $m_R(f_B \otimes f_A) = m_R(f_A \otimes f_B)c$, there exists a unique algebra morphism $f \colon A \otimes B \to R$ with $f_A = f(1_A \otimes \eta_B)$ and $f_B = f(\eta_A \otimes 1_B)$, namely $f = m_R(f_A \otimes f_B)$.

We will denote by $A \overline{\otimes} B$ the tensor product algebra of $A, B \in \mathcal{C}$ taken with respect to the inverse of the braiding (i.e. in the braided category $(\mathcal{B}, c^{-1})$.). It possesses an obvious analogous universal property. The equation




shows that $c \colon B \overline{\otimes} A \to A \otimes B$ is an isomorphism of algebras in $\mathcal{C}$.

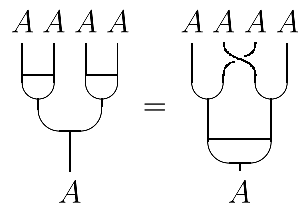
The following lemma is a version of the Eckmann-Hilton argument for algebras in braided monoidal categories.

**Lemma 3.1.** *Let $(A, m)$ be an algebra in the braided monoidal category $\mathcal{C}$, equipped with a second multiplication $\nabla\colon A \otimes A \to A$ that shares the same unit $\eta\colon I \to A$. Suppose that $\nabla\colon A \otimes A \to A$ is multiplicative with respect to $m$. Then the two multiplications coincide, and they are commutative.*

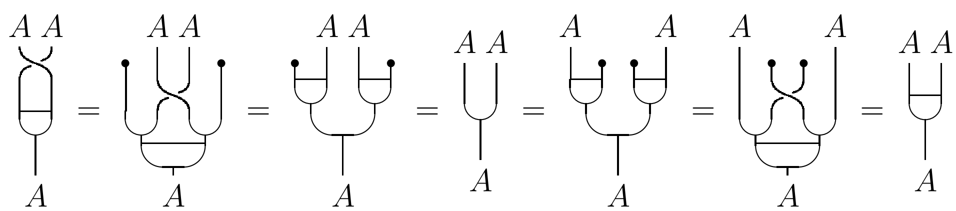
*Proof.* We use the graphical symbols


to distinguish the two multiplications.

The condition that $\nabla$ is multiplicative then reads




which we use twice in the calculation




$\square$

**Remark 3.2.** It is not used in the proof that $(A, m)$ is an (associative) algebra. It would suffice to require that we are given two multiplications that share the same unit, and that one is multiplicative with respect to the other according to the same formula that results from the definition of a tensor product of algebras.

The above generalities on tensor products of algebras in a braided category will be applied below to an $\mathbb{N}$-graded algebra that occurs as the cohomology of an algebra. To fix the terminology, we will denote the monoidal category of $\mathbb{N}$-graded objects in $\mathcal{C}$ by $\mathcal{C}^{\mathbb{N}}$, with braiding $c_{\mathrm{gr}}$, and we will call a commutative algebra in $\mathcal{C}^{\mathbb{N}}$ a braided graded commutative algebra.

**Definition 3.3.** Let $\mathcal{C}$ be a monoidal category, and $A$ an augmented algebra in $\mathcal{C}$. The simplicial object $S_\bullet A$ is defined by $S_n A = A^{\otimes n}$ with the faces $\partial_k^n\colon S_n A \to S_{n-1} A$ defined by $\partial_0^n = \varepsilon \otimes (1_A)^{\otimes(n-1)}$, $\partial_i^n = (1_A)^{\otimes(i-1)} \otimes m \otimes (1_A)^{\otimes(n-i-1)}$ for $0 < i < n$ and $\partial_n^n = (1_A)^{\otimes(n-1)} \otimes \varepsilon$ and the degeneracies $\sigma_k^n\colon S_n A \to S_{n+1} A$ defined by inserting units in appropriate places.

If $\mathcal{C}$ is abelian, then $S_\bullet A$ has an associated chain complex, which we will also denote by $S_\bullet A$.

Recall that a monoidal category $\mathcal{C}$ is (right) closed if for each object $V$, the endofunctor $-\otimes V$ is left adjoint. Its right adjoint is denoted $\underline{\mathrm{hom}}(V,—)$ and called an inner hom-functor. If $\mathcal{C}$ is right closed, then $\underline{\mathrm{hom}}(V,Y)$ is a bifunctor, covariant in $Y$ and contravariant in $V$.

More generally, one can denote by $\underline{\mathrm{hom}}(V,Y)$ an object, if it exists, such that $\mathcal{C}(X\otimes V,Y)\simeq\mathcal{C}(X,\underline{\mathrm{hom}}(V,Y))$ as functors of $X\in\mathcal{C}$. By Yoneda's Lemma, such hom-objects are unique, and bifunctorial in those objects $V$ and $Y$ for which they exist.

Now consider a monoidal functor $\mathcal{F}\colon\mathcal{C}\to\mathcal{C}'$. If $V,Y\in\mathcal{C}$ are such that the hom objects $\underline{\mathrm{hom}}(V,Y)$ in $\mathcal{C}$ and $\underline{\mathrm{hom}}(\mathcal{F}V,\mathcal{F}Y)$ in $\mathcal{C}'$ exist, then there is a canonical morphism $\mathcal{F}\underline{\mathrm{hom}}(V,Y)\to\underline{\mathrm{hom}}(\mathcal{F}V,\mathcal{F}Y)$. If this is an isomorphism, we say that $\mathcal{F}$ preserves the hom object $\underline{\mathrm{hom}}(V,Y)$; if both categories are right closed, and all hom objects are preserved, we say that $\mathcal{F}$ preserves inner hom-functors.

**Definition 3.4.** Let $\mathcal{C}$ be an abelian monoidal category, and $A$ an augmented algebra in $\mathcal{C}$ such that all hom objects $\underline{\mathrm{hom}}(A^{\otimes n},I)$ exist (e.g. if $\mathcal{C}$ is closed).

The Hochschild cohomology of $A$ in $\mathcal{C}$ (with trivial coefficients) is the cohomology of the cochain complex $\underline{\mathrm{hom}}(S_\bullet A,I)$. Thus the Hochschild cohomology consists of objects in the category $\mathcal{C}$. We denote it by $\mathrm{H}^*(A)$.

**Remark 3.5.** We will only need the results in this section for the case $\mathcal{C}={}^{\Gamma}_{\Gamma}\mathcal{YD}$, but we will indicate more generally how they apply to ordinary algebras and (braided) Hopf algebras. For this, it is not necessary always to stick to the paper's general assumption that we work over a base *field*, therefore we assume now that $k$ is an arbitrary commutative base ring.

(1) In the case of an ordinary augmented $k$-algebra $A$, Definition 3.4 recovers the ordinary definition of Hochschild cohomology with trivial coefficients, that is $\mathrm{H}^*(A)=\mathrm{H}^*(A,k)$ (see §2), which motivates our notation; general coefficients are not considered here.
(2) Assume that $\mathcal{F}\colon\mathcal{C}\to\mathcal{C}'$ is an exact monoidal functor that preserves the hom objects $\underline{\mathrm{hom}}(S_\bullet A,I)$. Then $\mathcal{F}$ preserves Hochschild cohomology of $A$ in the sense that $\mathrm{H}^*(\mathcal{F}(A))\simeq\mathcal{F}(\mathrm{H}^*(A))$.
(3) The category of (say, left) $G$-modules for a $k$-Hopf algebra $G$ has inner hom-objects preserved by the underlying functor to the category of $k$-modules. Also, if $G$ is a finitely generated projective Hopf algebra over $k$, the categories of (say, left) $G$-comodules, and of Yetter-Drinfeld modules over $G$ have inner hom-objects that are preserved by the underlying functors to the category of $k$-modules. More concretely, the hom-object $\underline{\mathrm{hom}}(V,Y)$ is $\mathrm{Hom}_k(V,Y)$ with $G$-module and comodule structures induced by the antipode and its inverse.

(4) Recall that a dual object $(V^*, e, d)$ of an object $V$ is an object $V^*$ with morphisms $e\colon V^*\otimes V \to I$ and $d\colon I \to V\otimes V^*$ that satisfy $(1_V\otimes e)(d\otimes 1_V) = 1_V$ and $(e\otimes 1_{V^*})(1_{V^*}\otimes d) = 1_{V^*}$. If an object $V$ in $\mathcal{C}$ has a dual, then all the inner hom-objects $\underline{\hom}(V, Y)$ exist and are given by $\underline{\hom}(V, Y) = Y \otimes V^*$. Also, monoidal functors always preserve dual objects (by and large because these are given by morphisms and relations) and thus, if $V$ has a dual, then the inner hom-objects $\underline{\hom}(V, Y)$ are preserved by every monoidal functor.

(5) A module $V$ over a commutative ring $k$ has a dual in the category of $k$-modules if and only if it is finitely generated projective; then $V^*$ is the dual module, $e$ is the evaluation, and $d$ maps $1_k$ to the canonical element, in the field case obtained by tensoring basis elements with the elements of the dual basis. If $G$ is a $k$-Hopf algebra with bijective antipode, then in the categories of $G$-modules, $G$-comodules, and Yetter-Drinfeld modules over $G$ an object $V$ has a dual if and only if $V$ is finitely generated projective over $k$.

(6) Let $G$ be a flat Hopf algebra over $k$ with bijective antipode, so that the category $\mathcal{C}$ of Yetter-Drinfeld modules over $G$ is abelian braided monoidal with an exact underlying functor to the category $\mathcal{C}'$ of $k$-modules. Then by the above remarks the underlying functor will preserve the Hochschild cohomology of $A$ in $\mathcal{C}$ whenever $G$ is finitely generated projective over $k$, or $A$ is finitely generated projective. Thus, in either of these cases the Hochschild cohomology of $A$ defined as above within the Yetter-Drinfeld category is the same as the ordinary Hochschild cohomology with trivial coefficients, endowed with a Yetter-Drinfeld module structure induced by that of $A$.

**Remark 3.6.** In a closed monoidal category $\mathcal{C}$, there is a natural morphism $\xi\colon \underline{\hom}(B, I)\otimes\underline{\hom}(A, I) \to \underline{\hom}(A\otimes B, I)$ which is suitably coherent with respect to higher tensor products. Under the adjunction

$$\mathcal{C}(\underline{\hom}(B, I) \otimes \underline{\hom}(A, I), \underline{\hom}(A \otimes B, I)) \simeq \mathcal{C}(\underline{\hom}(B, I) \otimes \underline{\hom}(A, I) \otimes A \otimes B, I)$$

$\xi$ corresponds to

$$\underline{\hom}(B, I) \otimes \underline{\hom}(A, I) \otimes A \otimes B \xrightarrow{1\otimes e\otimes 1} \underline{\hom}(B, I) \otimes B \xrightarrow{e} I,$$

where $e\colon \underline{\hom}(X, I) \otimes X \to I$ is the natural "evaluation" morphism. Of course, to define the morphism in question, it is enough to assume that all the inner hom objects that occur in it exist.

If all inner hom functors $\underline{\hom}(V, I)$ exist, the isomorphisms $\xi$ make $\underline{\hom}(—, I)$ into a contravariant weak monoidal functor.

If $A$ and $B$ have dual objects in $\mathcal{C}$, then one can show that $\xi$ above is an isomorphism; this can be attributed to the fact that $(B^* \otimes A^*, \tilde{e}, \tilde{d})$ is a dual object for $A \otimes B$ with

$$\tilde{e} = \left(B^* \otimes A^* \otimes A \otimes B \xrightarrow{1\otimes e\otimes 1} B^* \otimes B \xrightarrow{e} I\right)$$

$$\tilde{d} = \left(I \xrightarrow{d} A \otimes A^* \xrightarrow{1\otimes d\otimes 1} A \otimes B \otimes B^* \otimes A^*\right)$$

**Definition 3.7.** The cup product on $\mathrm{H}^*(A)$ is the collection of morphisms $\mathrm{H}^n(A)\otimes \mathrm{H}^m(A) \to \mathrm{H}^{m+n}(A)$ induced by the morphisms

$$\underline{\hom}(S_nA, I) \otimes \underline{\hom}(S_mA, I) \xrightarrow{\xi} \underline{\hom}(S_mA \otimes S_nA, I) = \underline{\hom}(S_{m+n}(A), I).$$

**Remark 3.8.** Thus, in the category of modules over a commutative base ring $k$, our definition of cup product recovers *the opposite* of the usual cup product in Hochschild cohomology.

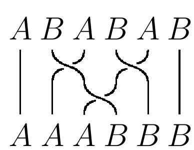
From now on we assume that $\mathcal{C}$ is braided.

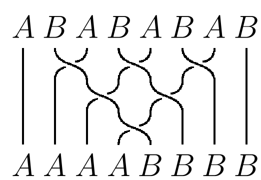
**Lemma 3.9.** *Let $A, B$ be two augmented algebras in $\mathcal{C}$. Denote by $S_\bullet A \times S_\bullet B$ the bisimplicial object obtained by tensoring the two simplicial objects $S_\bullet A$ and $S_\bullet B$. An isomorphism $S_\bullet(A \otimes B) \to \mathrm{diag}(S_\bullet A \times S_\bullet B)$ of simplicial objects is given by the morphisms*

$$g_n \colon S_n(A \otimes B) = (A \otimes B)^{\otimes n} \to A^{\otimes n} \otimes B^{\otimes n} = \mathrm{diag}_n(S_\bullet A \otimes S_\bullet B)$$

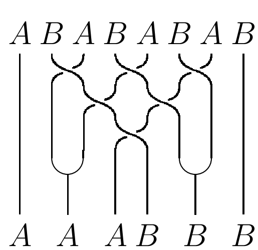
*that are composed from instances of the braiding $c_{B,A} \colon B \otimes A \to A \otimes B$.*

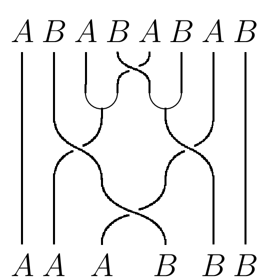
*Proof.* For example we have

$g_3 =$ 

and $g_4 =$ 


and therefore

$(\partial_2 \otimes \partial_2)g_4 =$ 

$=$ 

$= g_3\partial_2$

The general claim $(\partial_k \otimes \partial_k)g_n = g_{n-1}\partial_k$ is a merely larger version of the above example for $0 < k < n$, and simpler (with augmentations instead of multiplications) for $k \in \{0, n\}$. $\square$

According to the discussion at the beginning of the section we have two chain homotopic morphisms

$$\operatorname{diag}(S_\bullet A \times S_\bullet B) \simeq S_\bullet A \otimes S_\bullet B.$$

Here, the right hand side denotes the tensor product of chain complexes. Explicitly, the two morphisms are given by

$$f_{k\ell} = (1_A)^{\otimes k} \otimes \varepsilon^{\otimes \ell} \otimes \varepsilon^{\otimes k} \otimes (1_B)^{\otimes \ell} \colon S_n A \otimes S_n B \to S_k A \otimes S_\ell B$$
$$f'_{k\ell} = (-1)^{k\ell} \varepsilon^{\otimes \ell} \otimes (1_A)^{\otimes k} \otimes (1_B)^{\otimes \ell} \otimes \varepsilon^{\otimes k} \colon S_n A \otimes S_n B \to S_k A \otimes S_\ell B$$

for $k + \ell = n$.

Upon application of $\underline{\hom}(—, I)$, each of these morphisms, composed with the map $g^{-1}$ from the preceding lemma, yields a morphism

$$\underline{\hom}(S_\bullet B, I) \otimes \underline{\hom}(S_\bullet A, I) \to \underline{\hom}(S_\bullet A \otimes S_\bullet B, I) \to \underline{\hom}(S_\bullet(A \otimes B), I). \tag{3.9.1}$$

Both these maps, being homotopic, yield the same morphism

$$T' \colon \mathrm{H}^\bullet(B) \otimes \mathrm{H}^\bullet(A) \to \mathrm{H}^\bullet(A \otimes B),$$

which we compose with the inverse of the braiding in $\mathcal{C}$ to obtain a morphism

$$T = T' c_{\mathrm{gr}}^{-1} \colon \mathrm{H}^\bullet(A) \otimes \mathrm{H}^\bullet(B) \to \mathrm{H}^\bullet(A \otimes B).$$

**Lemma 3.10.** *The morphism*

$$T \colon \mathrm{H}^\bullet(A) \otimes \mathrm{H}^\bullet(B) \to \mathrm{H}^\bullet(A \otimes B)$$

*is a morphism of graded algebras in* $\mathcal{C}$.

*Proof.* We need to show that

$$T' \colon \mathrm{H}^\bullet(B) \overline{\otimes} \mathrm{H}^\bullet(A) \to \mathrm{H}^\bullet(A \otimes B)$$

is a morphism of algebras. To do this, we consider the algebra morphisms

$$\mathrm{H}^\bullet(A) \to \mathrm{H}^\bullet(A \otimes B) \leftarrow \mathrm{H}^\bullet(B).$$

We need to show that the composite

$$\mathrm{H}^\bullet(A) \otimes \mathrm{H}^\bullet(B) \to \mathrm{H}^\bullet(A \otimes B) \otimes \mathrm{H}^\bullet(A \otimes B) \to \mathrm{H}^\bullet(A \otimes B),$$

where the last morphism is multiplication, is the map in the statement of the lemma, while the composite

$$\mathrm{H}^\bullet(B) \otimes \mathrm{H}^\bullet(A) \to \mathrm{H}^\bullet(A \otimes B) \otimes \mathrm{H}^\bullet(A \otimes B) \to \mathrm{H}^\bullet(A \otimes B)$$

is the same, composed with the inverse braiding. But for this we only need to observe that

$$S_n(A \otimes B) = S_k(A \otimes B) \otimes S_\ell(A \otimes B) \xrightarrow{S_k(1_A \otimes \varepsilon) \otimes S_\ell(\varepsilon \otimes 1_B)} S_k(A) \otimes S_\ell(B)$$

equals

$$S_n(A \otimes B) \xrightarrow{g} S_n(A) \otimes S_n(B) \xrightarrow{f_{k\ell}} S_k(A) \otimes S_\ell(B),$$

whereas

$$S_n(A\otimes B)=S_k(A\otimes B)\otimes S_\ell(A\otimes B)\xrightarrow{S_k(\varepsilon\otimes 1_B)\otimes S_\ell(1_A\otimes\varepsilon)} S_k(B)\otimes S_\ell(A)$$

equals

$$S_n(A\otimes B)\xrightarrow{g} S_n(A)\otimes S_n(B)\xrightarrow{f'_{\ell k}} S_\ell(A)\otimes S_k(B)\xrightarrow{c_{\mathrm{gr}}^{-1}} S_k(B)\otimes S_\ell(A).$$

□

**Remark 3.11.** In the case which is of main interest to us, $k$ is a field, $\mathcal{C}$ is the category of Yetter-Drinfeld modules over a Hopf algebra $G$, and $A, B$ are finite dimensional. We briefly sketch why these conditions imply that $T$ is an isomorphism (see also [20, Prop.4.5]).

First, both in the construction of $T'$ (when homology is applied to the sequence 3.9.1) and in that of the cup product in Definition 3.7 we have tacitly used a version of a Künneth-type morphism

$$\mathrm{H}(\mathrm{C}^\bullet)\otimes\mathrm{H}(\mathrm{D}^\bullet)\to\mathrm{H}(\mathrm{C}^\bullet\otimes\mathrm{D}^\bullet)$$

for chain complexes $C^\bullet, D^\bullet$ in a closed monoidal category $\mathcal{C}$. Under suitable flatness hypotheses on the objects and homologies that occur, this map will be an isomorphism. For example, this is the case if the tensor product in $\mathcal{C}$ is assumed to be exact in each argument which holds in our situation.

Second, the morphism $\xi$ that occurs in the construction of $T'$ will be an isomorphism by Remark 3.6 since the algebras $A$ and $B$ clearly have duals in $\mathcal{C}$.

Therefore, in our case $T$ is an isomorphism.

**Theorem 3.12.** *Let $R$ be a bialgebra in the abelian braided monoidal category $\mathcal{C}$. Then Hochschild cohomology $\mathrm{H}^*(R)$ is a braided graded commutative algebra in $\mathcal{C}$, assuming it is defined (i.e. the necessary hom-objects exist).*

*Proof.* Comultiplication $\Delta$ on the bialgebra $R$ yields a second multiplication

$$\nabla := \left(\mathrm{H}^*(R)\otimes\mathrm{H}^*(R)\to\mathrm{H}^*(R\otimes R)\xrightarrow{\mathrm{H}^*(\Delta)}\mathrm{H}^*(R)\right)$$

on $\mathrm{H}^*(R)$. By construction, $\nabla$ is an algebra map with respect to the algebra structure of $\mathrm{H}^*(R)$ given by the cup product. Also, the two multiplications share the same unit, induced by the counit of $R$. Therefore, the two multiplications coincide and are commutative by Lemma 3.1. □

**Corollary 3.13.** *Let $G$ be a Hopf algebra with bijective antipode, and $R$ a bialgebra in the category $\mathcal{C}$ of Yetter-Drinfeld modules over $G$. Assume that either $G$ or $R$ is finite dimensional. Then the Hochschild cohomology of $R$ in $\mathcal{C}$ is a braided graded commutative algebra. Consequently, the opposite of ordinary Hochschild cohomology is a braided commutative Yetter-Drinfeld module algebra.*

*Proof.* In fact the assumptions on $G$ or $R$ are designed to ensure that the hypotheses on inner hom objects in the results above are satisfied: If $G$ is finitely generated projective, the category of Yetter-Drinfeld modules as a whole is closed, with inner hom-functors preserved by the underlying functors. If $G$ is not finitely generated projective, but $R$ is, then still $R$ and all its tensor powers have dual objects in the category of Yetter-Drinfeld modules over $G$, that are preserved by the underlying functors. $\square$

## 4. Cohomology of quantum complete intersections

Let $\theta$ be a positive integer, and for each $i$, $1 \le i \le \theta$, let $N_i$ be an integer greater than 1. Let $q_{ij} \in k^\times$ for $1 \le i < j \le \theta$. Let $S$ be the $k$-algebra generated by $x_1, \dots, x_\theta$ subject to the relations

$$x_i x_j = q_{ij} x_j x_i \text{ for all } i < j \quad \text{and} \quad x_i^{N_i} = 0 \text{ for all } i. \tag{4.0.1}$$

It is convenient to set $q_{ji} = q_{ij}^{-1}$ for $i < j$. For $\mathcal{B}(V)$, the Nichols algebra constructed in Section 2, Lemma 2.4 implies that $\operatorname{Gr}\mathcal{B}(V)$ has this form.

We will compute $\mathrm{H}^*(S, k) = \operatorname{Ext}^*_S(k, k)$ for use in later sections. The structure of this ring may be determined by using the braided Künneth formula of Grunenfelder and Mastnak [20, Prop. 4.5 and Cor. 4.7] or the twisted tensor product formula of Bergh and Oppermann [11, Thms. 3.7 and 5.3]. We give details, using an explicit free $S$-resolution of $k$, in order to record needed information at the chain level. Our resolution was originally adapted from [10] (see also [21]): It is a braided tensor product of the periodic resolutions

(4.0.2)
$$\cdots \xrightarrow{x_i^{N_i-1}\cdot} k[x_i]/(x_i^{N_i}) \xrightarrow{x_i\cdot} k[x_i]/(x_i^{N_i}) \xrightarrow{x_i^{N_i-1}\cdot} k[x_i]/(x_i^{N_i}) \xrightarrow{x_i\cdot} k[x_i]/(x_i^{N_i}) \xrightarrow{\varepsilon} k \to 0,$$

one for each $i$, $1 \le i \le \theta$.

Specifically, let $K_\bullet$ be the following complex of free $S$-modules. For each $\theta$-tuple $(a_1, \dots, a_\theta)$ of nonnegative integers, let $\Phi(a_1, \dots, a_\theta)$ be a free generator in degree $a_1 + \cdots + a_\theta$. Thus $K_n = \oplus_{a_1+\cdots+a_\theta=n} S\Phi(a_1, \dots, a_\theta)$. For each $i$, $1 \le i \le \theta$, let $\sigma_i, \tau_i : \mathbb{N} \to \mathbb{N}$ be the functions defined by

$$\sigma_i(a) = \begin{cases} 1, & \text{if } a \text{ is odd} \\ N_i - 1, & \text{if } a \text{ is even,} \end{cases}$$

and $\tau_i(a) = \displaystyle\sum_{j=1}^{a} \sigma_i(j)$ for $a \ge 1$, $\tau(0) = 0$. Let

$$d_i(\Phi(a_1, \dots, a_\theta)) = \left( \prod_{\ell < i} (-1)^{a_\ell} q_{\ell i}^{\sigma_i(a_i)\tau_\ell(a_\ell)} \right) x_i^{\sigma_i(a_i)} \Phi(a_1, \dots, a_i - 1, \dots, a_\theta)$$

if $a_i > 0$, and $d_i(\Phi(a_1,\ldots,a_\theta)) = 0$ if $a_i = 0$. Extend each $d_i$ to an $S$-module homomorphism. Note that $d_i^2 = 0$ for each $i$ since $x_i^{N_i} = 0$ and $\sigma_i(a_i)+\sigma_i(a_i-1) = N_i$. If $i < j$, we have

$$\begin{aligned}
&d_i d_j(\Phi(a_1,\ldots,a_\theta))\\
&\quad = d_i\left(\left(\prod_{\ell<j}(-1)^{a_\ell} q_{\ell j}^{\sigma_j(a_j)\tau_\ell(a_\ell)}\right) x_j^{\sigma_j(a_j)}\Phi(a_1,\ldots,a_j-1,\ldots,a_\theta)\right)\\
&\quad = \left(\prod_{\ell<j}(-1)^{a_\ell} q_{\ell j}^{\sigma_j(a_j)\tau_\ell(a_\ell)}\right)\left(\prod_{m<i}(-1)^{a_m} q_{mi}^{\sigma_i(a_i)\tau_m(a_m)}\right)\cdot\\
&\qquad\qquad x_j^{\sigma_j(a_j)} x_i^{\sigma_i(a_i)}\Phi(a_1,\ldots,a_i-1,\ldots,a_j-1,\ldots,a_\theta).
\end{aligned}$$

Now $d_j d_i(\Phi(a_1,\ldots,a_\theta))$ is also an $S$-multiple of $\Phi(a_1,\ldots,a_i-1,\ldots,a_j-1,\ldots,a_\theta)$; comparing the two, we see that the term in which $\ell = i$ has a scalar factor that changes from $(-1)^{a_i}q_{ij}^{\sigma_j(a_j)\tau_i(a_i)}$ to $(-1)^{a_i-1}q_{ij}^{\sigma_j(a_j)\tau_i(a_i-1)}$, and $x_j^{\sigma_j(a_j)}x_i^{\sigma_i(a_i)}$ is replaced by $x_i^{\sigma_i(a_i)}x_j^{\sigma_j(a_j)} = q_{ij}^{\sigma_j(a_j)\sigma_i(a_i)}x_j^{\sigma_j(a_j)}x_i^{\sigma_i(a_i)}$. Since $\tau_i(a_i) = \tau_i(a_i-1)+\sigma_i(a_i)$, this shows that

$$d_i d_j + d_j d_i = 0.$$

Letting

$$d = d_1 + \cdots + d_\theta, \tag{4.0.3}$$

we now have $d^2 = 0$, so $K_\bullet$ is indeed a complex.

Next we show that $K_\bullet$ is a resolution of $k$ by giving a contracting homotopy: Let $\alpha \in S$, and fix $\ell$, $1 \le \ell \le \theta$. Write $\alpha = \sum_{j=0}^{N_i-1}\alpha_j x_\ell^j$ where $\alpha_j$ is in the subalgebra of $S$ generated by the $x_i$ with $i \ne \ell$. Define $s_\ell(\alpha\Phi(a_1,\ldots,a_\theta))$ to be the sum $\sum_{j=0}^{N_i-1} s_\ell(\alpha_j x_\ell^j\Phi(a_1,\ldots,a_\theta))$, where

$$\begin{aligned}
&s_\ell(\alpha_j x_\ell^j\Phi(a_1,\ldots,a_\theta))\\
&= \begin{cases}
\delta_{j>0}\left(\prod_{m<\ell}(-1)^{a_m} q_{m\ell}^{-\sigma_\ell(a_\ell+1)\tau_m(a_m)}\right)\alpha_j x_\ell^{j-1}\Phi(a_1,\ldots,a_\ell+1,\ldots,a_\theta), & \text{if } a_\ell \text{ is even}\\
\delta_{j,N_\ell-1}\left(\prod_{m<\ell}(-1)^{a_m} q_{m\ell}^{-\sigma_\ell(a_\ell+1)\tau_m(a_m)}\right)\alpha_j \Phi(a_1,\ldots,a_\ell+1,\ldots,a_\theta), & \text{if } a_\ell \text{ is odd}
\end{cases}
\end{aligned}$$

where $\delta_{j>0} = 1$ if $j > 0$ and $0$ if $j = 0$. Calculations show that for all $i$, $1 \le i \le \theta$,

$$(s_i d_i + d_i s_i)(\alpha_j x_i^j\Phi(a_1,\ldots,a_\theta)) = \begin{cases}\alpha_j x_i^j\Phi(a_1,\ldots,a_\theta), & \text{if } j>0 \text{ or } a_i>0\\ 0, & \text{if } j=0 \text{ and } a_i = 0\end{cases}$$

and $s_\ell d_i + d_i s_\ell = 0$ for all $i, \ell$ with $i \neq \ell$. For each $x_1^{j_1} \cdots x_\theta^{j_\theta} \Phi(a_1, \dots, a_\theta)$, let $c = c_{j_1,\dots,j_\theta,a_1,\dots,a_\theta}$ be the cardinality of the set of all $i$ $(1 \leq i \leq \theta)$ such that $j_i a_i = 0$. Define

$$s(x_1^{j_1} \cdots x_\theta^{j_\theta} \Phi(a_1, \dots, a_\theta)) = \frac{1}{\theta - c_{j_1,\dots,j_\theta,a_1,\dots,a_\theta}} (s_1 + \cdots + s_\theta)(x_1^{j_1} \cdots x_\theta^{j_\theta} \Phi(a_1, \dots, a_\theta)).$$

Then $sd + ds = \mathrm{id}$ on each $K_n$, $n > 0$. That is, $K_\bullet$ is exact in positive degrees. To show that $K_\bullet$ is a resolution of $k$, put $k$ in degree $-1$, and let the corresponding differential be the counit map $\varepsilon : S \to k$. It can be shown directly that $K_\bullet$ then becomes exact at $K_0 = S$: The kernel of $\varepsilon$ is spanned over the field $k$ by the elements $x_1^{j_1} \cdots x_\theta^{j_\theta} \Phi(0, \dots, 0)$, $0 \leq j_i \leq N_i$, with at least one $j_i \neq 0$. Let $x_1^{j_1} \cdots x_\theta^{j_\theta} \Phi(0, \dots, 0)$ be such an element, and let $i$ be the smallest positive integer such that $j_i \neq 0$. Then $d(x_i^{j_i - 1} \cdots x_\theta^{j_\theta} \Phi(0, \dots, 1, \dots, 0))$ is a nonzero scalar multiple of $x_i^{j_i} \cdots x_\theta^{j_\theta} \Phi(0, \dots, 0)$. Thus $\ker(\varepsilon) = \mathrm{im}(d)$, and $K_\bullet$ is a free resolution of $k$ as an $S$-module.

Next we will use $K_\bullet$ to compute $\mathrm{Ext}_S^*(k, k)$. Applying $\mathrm{Hom}_S(-, k)$ to $K_\bullet$, the induced differential $d^*$ is the zero map since $x_i^{\sigma_i(a_i)}$ is always in the augmentation ideal. Thus the cohomology *is* the complex $\mathrm{Hom}_S(K_\bullet, k)$, and in degree $n$ this is a vector space of dimension $\binom{n+\theta-1}{\theta-1}$. Now let $\xi_i \in \mathrm{Hom}_S(K_2, k)$ be the function dual to $\Phi(0, \dots, 0, 2, 0, \dots, 0)$ (the 2 in the $i$th place) and $\eta_i \in \mathrm{Hom}_S(K_1, k)$ be the function dual to $\Phi(0, \dots, 0, 1, 0, \dots, 0)$ (the 1 in the $i$th place). By abuse of notation, identify these functions with the corresponding elements in $\mathrm{H}^2(S, k)$ and $\mathrm{H}^1(S, k)$, respectively. We will show that the $\xi_i$, $\eta_i$ generate $\mathrm{H}^*(S, k)$, and determine the relations among them. In order to do this we will abuse notation further and denote by $\xi_i$ and $\eta_i$ the corresponding chain maps $\xi_i : K_n \to K_{n-2}$ and $\eta_i : K_n \to K_{n-1}$ defined by

$$\xi_i(\Phi(a_1, \dots, a_\theta)) = \prod_{\ell > i} q_{i\ell}^{N_i \tau_\ell(a_\ell)} \Phi(a_1, \dots, a_i - 2, \dots, a_\theta)$$

$$\eta_i(\Phi(a_1, \dots, a_\theta)) = \prod_{\ell < i} q_{\ell i}^{(\sigma_i(a_i) - 1)\tau_\ell(a_\ell)} \prod_{\ell > i} (-1)^{a_\ell} q_{i\ell}^{\tau_\ell(a_\ell)} x_i^{\sigma_i(a_i) - 1} \Phi(a_1, \dots, a_i - 1, \dots, a_\theta).$$

Calculations show that these are indeed chain maps. The ring structure of the subalgebra of $\mathrm{H}^*(S, k)$ generated by $\xi_i, \eta_i$ is given by composition of these chain maps. Direct calculation shows that the relations given in Theorem 4.1 below hold. (Note that if $N_i \neq 2$ the last relation implies $\eta_i^2 = 0$.) Alternatively, in case $S = \mathcal{B}(V)$, a Nichols algebra defined in Section 2, we may apply Corollary 3.13 to obtain the relations in Theorem 4.1 below, performing calculations that are a special case of the ones in the proof of Theorem 5.4. Thus any element in the algebra generated by the $\xi_i$ and $\eta_i$ may be written as a linear combination of elements of the form $\xi_1^{b_1} \cdots \xi_\theta^{b_\theta} \eta_1^{c_1} \cdots \eta_\theta^{c_\theta}$ with $b_i \geq 0$ and $c_i \in \{0, 1\}$. Such an element takes $\Phi(2b_1 + c_1, \dots, 2b_\theta + c_\theta)$ to a nonzero scalar multiple of $\Phi(0, \dots, 0)$

and all other $S$-basis elements of $K_{\sum(2b_i+c_i)}$ to 0. Since the dimension of $\mathrm{H}^n(S,k)$ is $\binom{n+\theta-1}{\theta-1}$, this shows that the $\xi_1^{b_1}\cdots\xi_\theta^{b_\theta}\eta_1^{c_1}\cdots\eta_\theta^{c_\theta}$ form a $k$-basis for $\mathrm{H}^*(S,k)$. We have proven:

**Theorem 4.1.** *Let $S$ be the $k$-algebra generated by $x_1,\dots,x_\theta$, subject to relations (4.0.1). Then $\mathrm{H}^*(S,k)$ is generated by $\xi_i$, $\eta_i$ ($i=1,\dots,\theta$) where $\deg\xi_i = 2$ and $\deg\eta_i = 1$, subject to the relations*

$$\xi_i\xi_j = q_{ji}^{N_iN_j}\xi_j\xi_i, \quad \eta_i\xi_j = q_{ji}^{N_j}\xi_j\eta_i, \quad \textit{and} \quad \eta_i\eta_j = -q_{ji}\eta_j\eta_i. \tag{4.1.1}$$

Note that although the relations (4.1.1) can be obtained as a consequence of Corollary 3.13, in order to obtain the full statement of the theorem, we needed more information.

**Remark 4.2.** We obtain [19, Prop. 2.3.1] as a corollary: In this case we replace the generators $x_i$ of $S$ by the generators $E_\alpha$ ($\alpha\in\Delta^+$) of $\mathrm{Gr}\,u_q^+$, whose relations are

$$E_\alpha E_\beta = q^{\langle\alpha,\beta\rangle}E_\beta E_\alpha \ (\alpha\succ\beta), \qquad E_\alpha^\ell = 0\ (\alpha\in\Delta^+).$$

Theorem 4.1 then implies that $\mathrm{H}^*(\mathrm{Gr}\,u_q^+)$ is generated by $\xi_\alpha,\eta_\alpha$ ($\alpha\in\Delta^+$) with relations

$$\xi_\alpha\xi_\beta=\xi_\beta\xi_\alpha,\quad \eta_\alpha\xi_\beta=\xi_\beta\eta_\alpha,\quad \eta_\alpha\eta_\beta=-q^{-\langle\alpha,\beta\rangle}\eta_\beta\eta_\alpha\ (\alpha\succ\beta),\quad \eta_\alpha^2=0,$$

which is precisely [19, Prop. 2.3.1].

Now assume a finite group $\Gamma$ acts on $S$ by automorphisms for which $x_1,\dots,x_\theta$ are eigenvectors. For each $i$, $1\le i\le\theta$, let $\chi_i$ be the character on $\Gamma$ for which $gx_i=\chi_i(g)x_i$. As the characteristic of $k$ is 0, we have

$$\mathrm{Ext}^*_{S\#k\Gamma}(k,k)\simeq \mathrm{Ext}^*_S(k,k)^\Gamma,$$

where the action of $\Gamma$ at the chain level on (4.0.2) is as usual in degree 0, but shifted in higher degrees so as to make the differentials commute with the action of $\Gamma$. Specifically, note that the following action of $\Gamma$ on $K_\bullet$ commutes with the differentials:

$$g\cdot\Phi(a_1,\dots,a_\theta)=\prod_{\ell=1}^{\theta}\chi_\ell(g)^{\tau_\ell(a_\ell)}\Phi(a_1,\dots,a_\theta)$$

for all $g\in\Gamma$, and $a_1,\dots,a_\theta\ge 0$. Then the induced action of $\Gamma$ on generators $\xi_i,\eta_i$ of the cohomology ring $\mathrm{H}^*(S,k)$ is given explicitly by

$$g\cdot\xi_i=\chi_i(g)^{-N_i}\xi_i \text{ and } g\cdot\eta_i=\chi_i(g)^{-1}\eta_i. \tag{4.2.1}$$

## 5. Coradically graded finite dimensional pointed Hopf algebras

Let $\mathcal{D}$ be arbitrary data as in (2.0.2), $V$ the corresponding Yetter-Drinfeld module, and $R = \mathcal{B}(V)$ its Nichols algebra, and described in Section 2. By Lemma 2.4, there is a filtration on $R$ for which $S = \operatorname{Gr} R$ is of type $A_1 \times \cdots \times A_1$, given by generators and relations of type (4.0.1). Thus $\mathrm{H}^*(S,k)$ is given by Theorem 4.1. As the filtration is finite, there is a convergent spectral sequence associated to the filtration (see [29, 5.4.1]):

$$E_1^{p,q} = \mathrm{H}^{p+q}(\operatorname{Gr}_{(p)} R, k) \Longrightarrow \mathrm{H}^{p+q}(R,k). \tag{5.0.2}$$

It follows that the $E_1$-page of the spectral sequence is given by Theorem 4.1 with grading corresponding to the filtration on $R$. We will see that by (5.0.6) and Lemma 5.1 below, the generators $\xi_i$ are in degrees $(p_i, 2-p_i)$, where

$$p_i = N_{\beta_1} \cdots N_{\beta_i}(N_{\beta_i} \cdots N_{\beta_r} \operatorname{ht}(\beta_i) + 1). \tag{5.0.3}$$

Since the PBW basis elements (2.3.1) are eigenvectors for $\Gamma$, the action of $\Gamma$ on $R$ preserves the filtration, and we further get a spectral sequence converging to the cohomology of $u(\mathcal{D},0,0) \simeq R\# k\Gamma$:

$$\mathrm{H}^{p+q}(\operatorname{Gr}_{(p)} R, k)^{\Gamma} \Longrightarrow \mathrm{H}^{p+q}(R,k)^{\Gamma} \simeq \mathrm{H}^{p+q}(R\# k\Gamma, k). \tag{5.0.4}$$

Moreover, if $M$ is a finitely generated $R\# k\Gamma$-module, there is a spectral sequence converging to the cohomology of $R$ with coefficients in $M$:

$$\mathrm{H}^{p+q}(\operatorname{Gr}_{(p)} R, M) \Longrightarrow \mathrm{H}^{p+q}(R,M), \tag{5.0.5}$$

also compatible with the action of $\Gamma$.

We wish to apply Lemma 2.5 to the spectral sequence (5.0.4) for the filtered algebra $R$. In order to do so, we must find some permanent cycles.

The Hochschild cohomology of $R$ in degree 2, with trivial coefficients, was studied in [26]: There is a linearly independent set of 2-cocycles $\xi_\alpha$ on $R$, indexed by the positive roots $\alpha$ ([26, Theorem 6.1.3]). We will use the notation $\xi_\alpha$ in place of the notation $f_\alpha$ used there. As shown in [26], these 2-cocycles may be expressed as functions at the chain level in the following way. Let $\mathbf{x}^{\mathbf{a}}$, $\mathbf{x}^{\mathbf{b}}$ denote arbitrary PBW basis elements (2.3.1) of $u(\mathcal{D},0,0)$. Let $\widetilde{\mathbf{x}}^{\mathbf{a}}$, $\widetilde{\mathbf{x}}^{\mathbf{b}}$ denote corresponding elements in the infinite dimensional algebra $U(\mathcal{D},0)$ arising from the section of the quotient map $U(\mathcal{D},0) \to u(\mathcal{D},0,0)$ for which PBW basis elements are sent to PBW basis elements. Then

$$\xi_\alpha(\mathbf{x}^{\mathbf{a}} \otimes \mathbf{x}^{\mathbf{b}}) = c_\alpha \tag{5.0.6}$$

where $c_\alpha$ is the coefficient of $\widetilde{x}_\alpha^{N_\alpha}$ in the product $\widetilde{\mathbf{x}}^{\mathbf{a}} \cdot \widetilde{\mathbf{x}}^{\mathbf{b}}$, and $\mathbf{x}^{\mathbf{a}}$, $\mathbf{x}^{\mathbf{b}}$ range over all pairs of PBW basis elements. A direct proof that these are 2-cocycles is in [26, §6.1]; alternatively the proof in Lemma 6.2 below, that analogous functions $f_\alpha$ in higher degrees are cocycles for $u(\mathcal{D}, \lambda, 0)$, applies with minor modifications to the $\xi_\alpha$ in this context.

We wish to relate these functions $\xi_\alpha$ to elements on the $E_1$-page of the spectral sequence (5.0.2), found in the previous section. Recall that $\beta_1, \dots, \beta_r$ enumerate the positive roots, and the integer $p_i$ is defined in (5.0.3). By (5.0.6) and Lemma 2.4, we have $\xi_{\beta_i} \downarrow_{F_{p_i-1}(R\otimes R)} = 0$ but $\xi_{\beta_i} \downarrow_{F_{p_i}(R\otimes R)} \neq 0$. Note that the filtration on $R$ induces a filtration on $R^+$, and thus a (decreasing) filtration on the complex $C^\bullet$ defined in (2.4.1), given by $F^pC^n = \{f : (R^+)^{\otimes n} \to k \,|\, f \downarrow_{F_{p-1}((R^+)^{\otimes n})} = 0\}$. We conclude that $\xi_{\beta_i} \in F^{p_i}C^2$ but $\xi_{\beta_i} \notin F^{p_i+1}C^2$. Denoting the corresponding cocycle by the same letter, we further conclude that $\xi_{\beta_i} \in \operatorname{im}\{\mathrm{H}^2(F^{p_i}C^\bullet) \to \mathrm{H}^2(C^\bullet)\} = F^{p_i}\,\mathrm{H}^2(R,k)$, but $\xi_{\beta_i} \notin \operatorname{im}\{\mathrm{H}^2(F^{p_i+1}C^\bullet) \to \mathrm{H}^2(C^\bullet)\} = F^{p_i+1}\,\mathrm{H}^2(R,k)$. Hence, $\xi_{\beta_i}$ can be identified with the corresponding nontrivial homogeneous element in the associated graded complex:

$$\widetilde{\xi}_{\beta_i} \in F^{p_i}\,\mathrm{H}^2(R,k)/F^{p_i+1}\,\mathrm{H}^2(R,k) \simeq E_\infty^{p_i,2-p_i}.$$

Since $\xi_{\beta_i} \in F^{p_i}C^2$ but $\xi_{\beta_i} \notin F^{p_i+1}C^2$, it induces an element $\bar{\xi}_{\beta_i} \in E_0^{p_i,2-p_i} = F^{p_i}C^2/F^{p_i+1}C^2$. Since $\bar{\xi}_{\beta_i}$ is induced by an actual cocycle in $C^\bullet$, it will be in the kernels of all the differentials of the spectral sequence. Hence, the residue of $\bar{\xi}_{\beta_i}$ will be in the $E_\infty$–term where it will be identified with the non-zero element $\widetilde{\xi}_{\beta_i}$ since these classes are induced by the same cocycle in $C^\bullet$. We conclude that $\bar{\xi}_{\beta_i} \in E_0^{p_i,2-p_i}$, and, correspondingly, its image in $E_1^{p_i,2-p_i} \simeq \mathrm{H}^2(\operatorname{Gr} R, k)$ which we denote by the same symbol, is a permanent cycle.

Note that the results of [26] apply equally well to $S = \operatorname{Gr} R$ to yield similar cocycles $\hat{\xi}_\alpha$ via the formula (5.0.6), for each positive root $\alpha$ in type $A_1 \times \cdots \times A_1$. Now $S = \operatorname{Gr} R$ has generators corresponding to the root vectors of $R$, and we similarly identify elements in cohomology. Thus each $\hat{\xi}_\alpha$, $\alpha$ a positive root in type $A_1 \times \cdots \times A_1$, may be relabeled $\hat{\xi}_{\beta_i}$ for some $i$ ($1 \le i \le r$), the indexing corresponding to $\beta_1, \dots, \beta_r \in \Phi^+$. Comparing the values of $\overline{\xi}_{\beta_i}$ and $\hat{\xi}_{\beta_i}$ on basis elements $\mathbf{x}^{\mathbf{a}} \otimes \mathbf{x}^{\mathbf{b}}$ of $\operatorname{Gr} R \otimes \operatorname{Gr} R$ we conclude that they are the same function. Hence $\hat{\xi}_{\beta_i} \in E_1^{p_i,2-p_i}$ are permanent cycles.

We wish to identify these elements $\hat{\xi}_{\beta_i} \in \mathrm{H}^2(\operatorname{Gr} R, k)$ with the cohomology classes $\xi_i$ in $\mathrm{H}^*(S,k)$ constructed in Section 4, as we may thus exploit the algebra structure of $\mathrm{H}^*(S,k)$ given in Theorem 4.1. We use the same symbols $x_{\beta_i}$ to denote the corresponding root vectors in $R$ and in $S = \operatorname{Gr} R$, as this should cause no confusion.

**Lemma 5.1.** *For each $i$ ($1 \le i \le r$), the cohomology classes $\xi_i$ and $\hat{\xi}_{\beta_i}$ coincide as elements of* $\mathrm{H}^2(\operatorname{Gr} R, k)$.

*Proof.* Let $K_\bullet$ be the chain complex defined in Section 4, a projective resolution of the trivial $\operatorname{Gr} R$-module $k$. Elements $\xi_i \in \mathrm{H}^2(\operatorname{Gr} R, k)$ and $\eta_i \in \mathrm{H}^1(\operatorname{Gr} R, k)$ were defined via the complex $K_\bullet$. We wish to identify $\xi_i$ with elements of the chain complex $C^\bullet$ defined in (2.4.1), where $A = R$. To this end we define maps $F_1, F_2$

making the following diagram commute, where $S = \mathrm{Gr}\, R$:

$$\begin{array}{ccccccccccc} \cdots & \to & K_2 & \xrightarrow{d} & K_1 & \xrightarrow{d} & K_0 & \xrightarrow{\varepsilon} & k & \to & 0 \\ & & \downarrow F_2 & & \downarrow F_1 & & \| & & \| & & \\ \cdots & \to & S\otimes (S^+)^{\otimes 2} & \xrightarrow{\partial_2} & S\otimes S^+ & \xrightarrow{\partial_1} & S & \xrightarrow{\varepsilon} & k & \to & 0 \end{array}$$

In this diagram, the maps $d$ are given in (4.0.3), and $\partial_i$ in (2.4.2). Let $\Phi(\cdots 1_i \cdots)$ denote the basis element of $K_1$ having a 1 in the $i$th position, and 0 in all other positions, $\Phi(\cdots 1_i \cdots 1_j \cdots)$ (respectively $\Phi(\cdots 2_i \cdots)$) the basis element of $K_2$ having a 1 in the $i$th and $j$th positions ($i \neq j$), and 0 in all other positions (respectively a 2 in the $i$th position and 0 in all other positions). Let

$$\begin{aligned} F_1(\Phi(\cdots 1_i \cdots)) &= 1\otimes x_{\beta_i}, \\ F_2(\Phi(\cdots 2_i \cdots)) &= \sum_{a_i=0}^{N_i-2} x_{\beta_i}^{a_i}\otimes x_{\beta_i}\otimes x_{\beta_i}^{N_i-a_i-1}, \\ F_2(\Phi(\cdots 1_i\cdots 1_j \cdots)) &= 1\otimes x_{\beta_i}\otimes x_{\beta_j} - q_{ij}\otimes x_{\beta_j}\otimes x_{\beta_i}. \end{aligned}$$

Direct computations show that the two nontrivial squares in the diagram above commute. So $F_1, F_2$ extend to maps $F_i : K_i \to S\otimes (S^+)^{\otimes i}$, $i \geq 1$, providing a chain map $F_\bullet : K_\bullet \to S\otimes (S^+)^{\otimes \bullet}$, thus inducing isomorphisms on cohomology.

We now verify that the maps $F_1, F_2$ make the desired identifications. By $\xi_{\beta_i}$ we mean the function on the reduced bar complex, $\xi_{\beta_i}(1\otimes \mathbf{x^a}\otimes \mathbf{x^b}) := \xi_{\beta_i}(\mathbf{x^a}\otimes\mathbf{x^b})$, defined in (5.0.6). Then

$$\begin{aligned} F_2^*(\xi_{\beta_i})(\Phi(\cdots 2_i\cdots)) &= \xi_{\beta_i}(F_2(\Phi(\cdots 2_i \cdots))) \\ &= \xi_{\beta_i}(\sum_{a_i=0}^{N_i-2} x_{\beta_i}^{a_i}\otimes x_{\beta_i}\otimes x_{\beta_i}^{N_i-a_i-1}) \\ &= \sum_{a_i=0}^{N_i-2} \varepsilon(x_{\beta_i}^{a_i})\xi_{\beta_i}(1\otimes x_{\beta_i}\otimes x_{\beta_i}^{N_i-a_i-1}) \\ &= \xi_{\beta_i}(x_{\beta_i}\otimes x_{\beta_i}^{N_i-1}) \quad = \quad 1. \end{aligned}$$

Further, it may be checked similarly that $F_2^*(\xi_{\beta_i})(\Phi(\cdots 1_i\cdots 1_j\cdots)) = 0$ for all $i, j$ and $F_2^*(\xi_{\beta_i})(\Phi(\cdots 2_j \cdots)) = 0$ for all $j \neq i$. Therefore $F_2^*(\xi_{\beta_i})$ is the dual function to $\Phi(\cdots 2_i\cdots)$, which is precisely $\xi_i$. □

Similarly, we identify the elements $\eta_i$ defined in Section 4 with functions at the chain level in cohomology: Define

(5.1.1) $$\eta_\alpha(\mathbf{x^a}) = \begin{cases} 1, & \text{if } \mathbf{x^a} = x_\alpha \\ 0, & \text{otherwise} \end{cases}.$$

The functions $\eta_\alpha$ represent a basis of $\mathrm{H}^1(S,k) \simeq \mathrm{Hom}_k(S^+/(S^+)^2, k)$. Similarly functions corresponding to the *simple* roots $\beta_i$ only represent a basis of $\mathrm{H}^1(R,k) \simeq$

$\mathrm{Hom}_k(R^+/(R^+)^2, k)$. A computation shows that $F_1^*(\eta_{\beta_i})(\Phi(\cdots 1_j \cdots)) = \delta_{ij}$, so that $F_1^*(\eta_{\beta_i})$ is the dual function to $\Phi(\cdots 1_i \cdots)$. Therefore $\eta_i$ and $\eta_{\beta_i}$ coincide as elements of $\mathrm{H}^1(S, k)$.

For each $\alpha \in \Phi^+$, let $M_\alpha$ be any positive integer for which $\chi_\alpha^{M_\alpha} = \varepsilon$. (For example, let $M_\alpha$ be the order of $\chi_\alpha$.) Note that $\xi_\alpha^{M_\alpha}$ is $\Gamma$-invariant: By (4.2.1), $g \cdot \xi_\alpha^{M_\alpha} = \chi_\alpha^{-M_\alpha N_\alpha}(g)\xi_\alpha^{M_\alpha} = \xi_\alpha^{M_\alpha}$.

Recall the notation $u(\mathcal{D}, 0, 0) \simeq R\# k\Gamma$, where $R = \mathcal{B}(V)$ is the Nichols algebra.

**Lemma 5.2.** *The cohomology algebra* $\mathrm{H}^*(u(\mathcal{D}, 0, 0), k)$ *is finitely generated over the subalgebra generated by all* $\xi_\alpha^{M_\alpha}$ *(*$\alpha \in \Phi^+$*).*

*Proof.* Let $E_1^{*,*} \Longrightarrow \mathrm{H}^*(R, k)$ be the spectral sequence (5.0.2), and let $B^{*,*}$ be the bigraded subalgebra of $E_1^{*,*}$ generated by the elements $\xi_i$. By Lemma 5.1 and the discussion prior to it, $B^{*,*}$ consists of permanent cycles. Since $\xi_i$ is $\bar{\xi}_{\beta_i}$ by Lemma 5.1, it is in bidegree $(p_i, 2 - p_i)$. Let $A^{*,*}$ be the subalgebra of $B^{*,*}$ generated by $\bar{\xi}_\alpha^{M_\alpha}$ $(\alpha \in \Phi^+)$. Then $A^{*,*}$ also consists of permanent cycles. Observe that $A^{*,*}$ is a subalgebra of $\mathrm{H}^*((\mathrm{Gr}\, R)\# k\Gamma, k)$, which is graded commutative since $(\mathrm{Gr}\, R)\# k\Gamma$ is a Hopf algebra. Hence, $A^{*,*}$ is commutative as it is concentrated in even (total) degrees. Finally, $A^{*,*}$ is Noetherian since it is a polynomial algebra in the $\xi_\alpha^{M_\alpha}$. We conclude that the bigraded commutative algebra $A^{*,*}$ satisfies the hypotheses of Lemma 2.5. By Theorem 4.1, the algebra $E_1^{*,*} \simeq \mathrm{H}^*(\mathrm{Gr}\, R, k)$ is generated by $\xi_i$ and $\eta_i$ where the generators $\eta_i$ are nilpotent. Hence, $E_1^{*,*}$ is a finitely generated module over $B^{*,*}$. The latter is clearly a finitely generated module over $A^{*,*}$. Hence $E_1^{*,*}$ is a finitely generated module over $A^{*,*}$. Lemma 2.5 implies that $\mathrm{H}^*(R, k)$ is a Noetherian $\mathrm{Tot}(A^{*,*})$-module; moreover, the action of $\Gamma$ on $\mathrm{H}^*(R, k)$ is compatible with the action on $A^{*,*}$ since the spectral sequence is compatible with the action of $\Gamma$. Therefore, $\mathrm{H}^*(R\# k\Gamma, k) \simeq \mathrm{H}^*(R, k)^\Gamma$ is a Noetherian $\mathrm{Tot}(A^{*,*})$-module. Since $\mathrm{Tot}(A^{*,*})$ is finitely generated, we conclude that $\mathrm{H}^*(R\# k\Gamma, k)$ is finitely generated. □

We immediately have the following theorem. The second statement of the theorem follows by a simple application of the second statement of Lemma 2.5.

**Theorem 5.3.** *The algebra* $\mathrm{H}^*(u(\mathcal{D}, 0, 0), k)$ *is finitely generated. If* $M$ *is a finitely generated* $u(\mathcal{D}, 0, 0)$*-module, then* $\mathrm{H}^*(u(\mathcal{D}, 0, 0), M)$ *is a finitely generated module over* $\mathrm{H}^*(u(\mathcal{D}, 0, 0), k)$.

Thanks to Corollary 3.13, we have some information about the algebra structure of the cohomology ring of the Nichols algebra $R = \mathcal{B}(V)$: Recall that $q_{\beta\alpha} = \chi_\alpha(g_\beta)$ (see (2.1.2)). Compare the following result with the graded case, Theorem 4.1.

**Theorem 5.4.** *The following relations hold in* $\mathrm{H}^*(R, k)$ *for all* $\xi_\alpha, \xi_\beta$ *(*$\alpha, \beta \in \Phi^+$*) and* $\eta_\alpha, \eta_\beta$ *(*$\alpha, \beta \in \Pi$*):*

$$\xi_\alpha \xi_\beta = q_{\beta\alpha}^{N_\alpha N_\beta} \xi_\beta \xi_\alpha, \quad \eta_\alpha \xi_\beta = q_{\beta\alpha}^{N_\beta} \xi_\beta \eta_\alpha, \ \textit{and} \quad \eta_\alpha \eta_\beta = -q_{\beta\alpha} \eta_\beta \eta_\alpha.$$

*Proof.* As we shall see, this is a consequence of Corollary 3.13, since $R$ is a braided Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$.

Note first that as a function on the Yetter-Drinfeld module $R \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ the cocycle $\xi_\alpha$ is $\Gamma$-homogeneous of degree $g_\alpha^{-N_\alpha}$ and spans a one-dimensional $\Gamma$-module with character $\chi_\alpha^{-N_\alpha}$. To see this in full detail, let us rephrase the definition (5.0.6) of $\xi_\alpha$ as follows. One can write $U(\mathcal{D}, 0)$ as a Radford biproduct $U(\mathcal{D}, 0) = \hat{R}\#k\Gamma$ with a braided Hopf algebra $\hat{R} \in {}^{\Gamma}_{\Gamma}\mathcal{YD}$ such that $R$ is a quotient of $\hat{R}$. Let $s\colon R \to \hat{R}$ be the section of the surjection $\hat{R} \to R$ that maps PBW basis elements to PBW basis elements. Note that $s$ is a map in ${}^{\Gamma}_{\Gamma}\mathcal{YD}$. Finally, let $p_\alpha\colon \hat{R} \to k$ be the function projecting $r \in \hat{R}$ to the coefficient in $r$ of the PBW basis element $\tilde{x}_\alpha^{N_\alpha}$. Then $\xi_\alpha$ is by definition the pullback of $p_\alpha$ under $R \otimes R \xrightarrow{s \otimes s} \hat{R} \otimes \hat{R} \xrightarrow{m} \hat{R}$. As a consequence, since $p_\alpha$ is clearly $\Gamma$-homogeneous of degree $g_\alpha^{-N_\alpha}$ and satisfies $g \cdot p_\alpha = \chi_\alpha^{-N_\alpha}(g)p_\alpha$ (cf. (4.2.1)), the statements on $\xi_\alpha$ follow.

On the other hand, it is easy to see from definition (5.1.1) that $\eta_\alpha$ has degree $g_\alpha^{-1}$ and spans a one-dimensional $\Gamma$-module with character $\chi_\alpha^{-1}$.

Let us denote the opposite of multiplication in $\mathrm{H}^*(R, k)$ by $\zeta \circ \theta := \theta\zeta$. According to Corollary 3.13, the opposite multiplication is braided graded commutative. In particular

$$\begin{aligned}
\xi_\alpha\xi_\beta = \xi_\beta \circ \xi_\alpha &= \left(g_\beta^{-N_\beta} \cdot \xi_\alpha\right) \circ \xi_\beta \\
&= \chi_\alpha^{-N_\alpha}\left(g_\beta^{-N_\beta}\right) \xi_\alpha \circ \xi_\beta \\
&= \chi_\alpha\left(g_\beta\right)^{N_\beta N_\alpha} \xi_\alpha \circ \xi_\beta \\
&= q_{\beta\alpha}^{N_\beta N_\alpha} \xi_\alpha \circ \xi_\beta = q_{\beta\alpha}^{N_\beta N_\alpha} \xi_\beta\xi_\alpha
\end{aligned}$$

and

$$\eta_\alpha\xi_\beta = \xi_\beta\eta_\alpha = \left(g_\beta^{-N_\beta} \cdot \eta_\alpha\right) \circ \xi_\beta = \chi_\alpha^{-1}\left(g_\beta^{-N_\beta}\right) \eta_\alpha \circ \xi_\beta = q_{\beta\alpha}^{N_\beta} \xi_\beta\eta_\alpha$$

as well as finally

$$\eta_\alpha\eta_\beta = \eta_\beta \circ \eta_\alpha = -\left(g_\beta^{-1} \cdot \eta_\alpha\right) \circ \eta_\beta = -\chi_\alpha^{-1}\left(g_\beta^{-1}\right) \eta_\alpha \circ \eta_\beta = -q_{\beta\alpha}\eta_\beta\eta_\alpha.$$

□

We give a corollary in a special case. It generalizes [19, Theorem 2.5(i)]. Recall that $\Pi$ denotes a set of simple roots in the root system $\Phi$.

**Corollary 5.5.** *Assume there are no $\Gamma$-invariants in $\mathrm{H}^*(\operatorname{Gr} R, k)$ of the form $\xi_{\beta_1}^{b_1} \cdots \xi_{\beta_r}^{b_r} \eta_{\beta_1}^{c_1} \cdots \eta_{\beta_r}^{c_r}$ for which $c_1 + \cdots + c_r$ is odd. Then*

$$\mathrm{H}^*(u(\mathcal{D}, 0, 0), k) \simeq k\langle \xi_\alpha, \eta_\beta \mid \alpha \in \Phi^+,\ \beta \in \Pi\rangle^\Gamma,$$

*with the relations of Theorem 5.4,* $\deg(\eta_\beta) = 1$, $\deg(\xi_\alpha) = 2$. *If there are no $\Gamma$-invariants with $c_1 + \cdots + c_r \neq 0$, then $\mathrm{H}^*(R\#k\Gamma, k) \simeq k[\xi_\alpha \mid \alpha \in \Phi^+]^\Gamma$.*

*Proof.* The hypothesis of the first statement implies that $\mathrm{H}^i(\mathrm{Gr}\, R, k)^\Gamma = 0$ for all odd integers $i$. Thus on the $E_1$-page, every other diagonal is 0. This implies $E_1 = E_\infty$. It follows that, as a vector space, $\mathrm{H}^*(u(\mathcal{D}, 0, 0), k)$ is exactly as stated. The algebra structure is a consequence of Theorem 5.4 and the fact that the cohomology of a Hopf algebra is graded commutative. The hypothesis of the last statement implies further that the $\Gamma$-invariant subalgebra of $\mathrm{H}^*(\mathrm{Gr}\, R, k)$ is spanned by elements of the form $\xi_{\beta_1}^{b_1} \cdots \xi_{\beta_r}^{b_r}$. By graded commutativity of the cohomology ring of a Hopf algebra and the relations of Theorem 5.4, $\mathrm{H}^*(R\#k\Gamma, k)$ may be identified with the $\Gamma$-invariant subalgebra of a polynomial ring in variables $\xi_\alpha$, with corresponding $\Gamma$-action. □

**Remark 5.6.** Assume the hypotheses of Corollary 5.5, and that $q_{\alpha\alpha} \neq -1$ for all $\alpha \in \Pi$. Then $\eta_\alpha^2 = 0$ for all $\alpha \in \Pi$, and it follows that the maximal ideal spectrum of $\mathrm{H}^*(u(\mathcal{D}, 0, 0), k)$ is $\mathrm{Spec}\, k[\xi_\alpha \mid \alpha \in \Phi^+]^\Gamma \simeq \mathrm{Spec}\, k[\xi_\alpha \mid \alpha \in \Phi^+]/\Gamma$.

We give an example to show that cohomology may in fact be nonzero in odd degrees, in contrast to that of the small quantum groups of [19]. This complicates any determination of the explicit structure of cohomology in general. The simplest example occurs in type $A_1 \times A_1 \times A_1$, where there can exist a nonzero cycle in degree 3.

**Example 5.7.** Let $\Gamma = \mathbb{Z}/\ell\mathbb{Z}$ with generator $g$. Let $q$ be a primitive $\ell$th root of unity. Let $g_1 = g_2 = g_3 = g$, and choose $\chi_1, \chi_2, \chi_3$ so that the matrix $(q_{ij}) = (\chi_j(g_i))$ is

$$\begin{pmatrix} q & q^{-1} & 1 \\ q & q^{-2} & q \\ 1 & q^{-1} & q \end{pmatrix}.$$

Let $u(\mathcal{D}, 0, 0) = R\#k\Gamma$ be the pointed Hopf algebra of type $A_1 \times A_1 \times A_1$ defined by this data. Let $\eta_1, \eta_2, \eta_3$ represent elements of $\mathrm{H}^1(R, k)$ as defined by (5.1.1). The action of $\Gamma$ is described in (4.2.1): $g_i \cdot \eta_j = q_{ij}^{-1}\eta_j$. Since the product of all entries in any given row of the matrix $(q_{ij})$ is 1, we conclude that $\eta_1\eta_2\eta_3$ is invariant under $\Gamma$. Hence, it is a nontrivial cocycle in $\mathrm{H}^3(u(\mathcal{D}, 0, 0), k)$.

We also give an example in type $A_2 \times A_1$ to illustrate, in particular, that the methods employed in [19] do not transfer to our more general setting. That is, for an arbitrary (coradically graded) pointed Hopf algebra, the first page of the spectral sequence (5.0.4) can have nontrivial elements in odd degrees. In the special case of a small quantum group (with some restrictions on the order of the root of unity), it is shown in [19] that this does not happen.

**Example 5.8.** Let $\Gamma = \mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/\ell\mathbb{Z}$, with generators $g_1, g_2$. Let $q$ be a primitive $\ell$th root of unity ($\ell$ odd), and let $\mathcal{D}$ be of type $A_2 \times A_1$ so that the Cartan matrix is

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Let $\chi_1, \chi_2$ be as for $u_q(sl_3)^+$, that is

$$\begin{array}{rclcrcl} \chi_1(g_1) & = & q^2, & \quad & \chi_1(g_2) & = & q^{-1}, \\ \chi_2(g_1) & = & q^{-1}, & \quad & \chi_2(g_2) & = & q^2. \end{array}$$

Now let $g_3 := g_1g_2$ and $\chi_3 := \chi_1^{-1}\chi_2^{-1}$. Then $\chi_3(g_3) = q^{-2} \neq 1$ and the Cartan condition holds, for example $\chi_3(g_1)\chi_1(g_3) = q^{-1}q = 1 = \chi_1(g_1)^{a_{13}}$ since $a_{13} = 0$. Let $R = \mathcal{B}(V)$, the Nichols algebra defined from this data.

The root vector corresponding to the nonsimple positive root is $x_{12} = [x_1, x_2]_c = x_1x_2 - q^{-1}x_2x_1$. The relations among the root vectors other than $x_3$ are now

$$x_2x_1 = qx_1x_2 - qx_{12}, \quad x_{12}x_1 = q^{-1}x_1x_{12}, \quad x_2x_{12} = q^{-1}x_{12}x_2.$$

The associated graded algebra, which is of type $A_1 \times A_1 \times A_1 \times A_1$, thus has relations (excluding those involving $x_3$ that do not change):

$$x_2x_1 = qx_1x_2, \quad x_{12}x_1 = q^{-1}x_1x_{12}, \quad x_2x_{12} = q^{-1}x_{12}x_2.$$

Thus the cohomology of the associated graded algebra $\mathrm{Gr}\, R\#k\Gamma$ is the subalgebra of $\Gamma$-invariants of an algebra with generators

$$\xi_1,\ \xi_{12},\ \xi_2,\ \xi_3,\ \eta_1,\ \eta_{12},\ \eta_2,\ \eta_3$$

(see Theorem 4.1). The element $\eta_1\eta_2\eta_3$ is $\Gamma$-invariant since $g \cdot \eta_i = \chi_i(g)^{-1}\eta_i$ for all $i$, and $\chi_3 = \chi_1^{-1}\chi_2^{-1}$. Therefore in the spectral sequence relating the cohomology of $\mathrm{Gr}\, R\#k\Gamma$ to that of $R\#k\Gamma$, there are some odd degree elements on the $E_1$ page.

## 6. Finite dimensional pointed Hopf algebras

Let $u(\mathcal{D}, \lambda, \mu)$ be one of the finite dimensional pointed Hopf algebras from the Andruskiewitsch-Schneider classification [5], as described in Section 2. With respect to the coradical filtration, its associated graded algebra is $\mathrm{Gr}\, u(\mathcal{D}, \lambda, \mu) \simeq u(\mathcal{D}, 0, 0) \simeq R\#k\Gamma$ where $R = \mathcal{B}(V)$ is a Nichols algebra, also described in Section 2. By Theorem 5.3, $\mathrm{H}^*(u(\mathcal{D}, 0, 0), k)$ is finitely generated. Below we apply the spectral sequence for a filtered algebra, employing this new choice of filtration:

$$E_1^{p,q} = \mathrm{H}^{p+q}(u(\mathcal{D}, 0, 0), k) \Longrightarrow \mathrm{H}^{p+q}(u(\mathcal{D}, \lambda, \mu), k). \tag{6.0.1}$$

As a consequence of the following lemma, we may assume without loss of generality that all root vector relations (2.1.4) are trivial.

**Lemma 6.1.** *(i) For all $\mathcal{D}$, $\lambda$, $\mu$, there is an isomorphism of graded algebras,*

$$\mathrm{H}^*(u(\mathcal{D},\lambda,\mu),k) \simeq \mathrm{H}^*(u(\mathcal{D},\lambda,0),k).$$

*(ii) Let $M$ be a finitely generated $u(\mathcal{D},\lambda,\mu)$-module. There exists a finitely generated $u(\mathcal{D},\lambda,0)$-module $\widetilde{M}$ and an isomorphism of $\mathrm{H}^*(u(\mathcal{D},\lambda,\mu),k)$-modules*

$$\mathrm{H}^*(u(\mathcal{D},\lambda,\mu),M) \simeq \mathrm{H}^*(u(\mathcal{D},\lambda,0),\widetilde{M}),$$

*where the action of $\mathrm{H}^*(u(\mathcal{D},\lambda,\mu),k)$ on $\mathrm{H}^*(u(\mathcal{D},\lambda,0),\widetilde{M})$ is via the isomorphism of graded algebras in (i).*

*Proof.* Define the subset $I \subset \Gamma$ as follows: $g \in I$ if and only if there exists a root vector relation $x_\alpha^{N_\alpha} - u_\alpha(\mu)$ (see (2.1.4)), such that the element $u_\alpha(\mu)$ of $k\Gamma$ has a nonzero coefficient of $g$ when written as a linear combination of group elements. If $x_\alpha^{N_\alpha} - u_\alpha(\mu)$ is a nontrivial root vector relation, then necessarily $x_\alpha^{N_\alpha}$ commutes with elements of $\Gamma$, implying that $\chi_\alpha^{N_\alpha} = \varepsilon$. From this and (2.3.2), we see that $x_\alpha^{N_\alpha}$ is central in $u(\mathcal{D},\lambda,\mu)$. Since $x_\alpha^{N_\alpha} = u_\alpha(\mu)$, this element of the group ring $k\Gamma$ must also be central. Since $\Gamma$ acts diagonally, each group element involved in $u_\alpha(\mu)$ is necessarily central in $u(\mathcal{D},\lambda,\mu)$ as well.

Let $Z = k\langle I\rangle$, a subalgebra of $u(\mathcal{D},\lambda,\mu)$, and let $\overline{u} = u(\mathcal{D},\lambda,\mu)/(g-1 \mid g \in I)$. We have a sequence of algebras (see [19, §5.2]):

$$Z \to u(\mathcal{D},\lambda,\mu) \to u(\mathcal{D},\lambda,\mu)//Z \simeq \overline{u}.$$

Hence, there is a multiplicative spectral sequence

$$\mathrm{H}^p(\overline{u},\mathrm{H}^q(Z,k)) \implies \mathrm{H}^{p+q}(u(\mathcal{D},\lambda,\mu),k).$$

Since the characteristic of $k$ does not divide the order of the group, we have $\mathrm{H}^{q>0}(Z,k) = 0$. Thus the spectral sequence collapses, and we get an isomorphism of graded algebras

$$\mathrm{H}^*(\overline{u},k) \simeq \mathrm{H}^*(u(\mathcal{D},\lambda,\mu),k). \tag{6.1.1}$$

Similarly, if $M$ is any $u(\mathcal{D},\lambda,\mu)$-module, then we have a spectral sequence of $\mathrm{H}^*(\overline{u},k)$-modules

$$\mathrm{H}^p(\overline{u},\mathrm{H}^q(Z,M)) \implies \mathrm{H}^{p+q}(u(\mathcal{D},\lambda,\mu),M).$$

The spectral sequence collapses, and we get an isomorphism

$$\mathrm{H}^*(\overline{u},M^Z) \simeq \mathrm{H}^*(u(\mathcal{D},\lambda,\mu),M) \tag{6.1.2}$$

which respects the action of $\mathrm{H}^*(u(\mathcal{D},\lambda,\mu)$, where $\mathrm{H}^*(u(\mathcal{D},\lambda,\mu)$ acts on the left side via the isomorphism (6.1.1).

Note that $Z = k\langle I\rangle$ is also a central subalgebra of $u(\mathcal{D},\lambda,0)$. Arguing exactly as above, we get an isomorphism of graded algebras

$$\mathrm{H}^*(\overline{u},k) \simeq \mathrm{H}^*(u(\mathcal{D},\lambda,0),k),$$

which implies (i).

Let $M$ be a $u(\mathcal{D},\lambda,\mu)$-module. Let $\widetilde{M}$ be a $u(\mathcal{D},\lambda,0)$-module which we get by inflating the $\overline{u}$-module $M^Z$ via the projection $u(\mathcal{D},\lambda,0) \twoheadrightarrow \overline{u}$. Since $(\widetilde{M})^Z \simeq M^Z$ by construction, we get an isomorphism of $u(\mathcal{D},\lambda,0)$-modules

$$\mathrm{H}^*(\overline{u}, M^Z) \simeq \mathrm{H}^*(u(\mathcal{D},\lambda,0), \widetilde{M})$$

using another spectral sequence argument. Combining with the isomorphism (6.1.2), we get (ii).

□

By Lemma 6.1, it suffices to work with the cohomology of $u(\mathcal{D},\lambda,0)$, in which all the root vectors are nilpotent. In this case we define some permanent cycles: As before, for each $\alpha \in \Phi^+$, let $M_\alpha$ be any positive integer for which $\chi_\alpha^{M_\alpha} = \varepsilon$ (for example, take $M_\alpha$ to be the order of $\chi_\alpha$). If $\alpha = \alpha_i$ is simple, then $\chi_\alpha(g_\alpha)$ has order $N_\alpha$, and so $N_\alpha$ divides $M_\alpha$.

We previously identified an element $\xi_\alpha$ of $\mathrm{H}^2(R,k)$, where $R = \mathcal{B}(V)$. Now $\mathcal{B}(V)$ is no longer a subalgebra of $u(\mathcal{D},\lambda,0)$ in general, due to the potential existence of nontrivial linking relations, but we will show that still there is an element analogous to $\xi_\alpha^{M_\alpha}$ in $\mathrm{H}^{2M_\alpha}(u(\mathcal{D},\lambda,0),k)$. For simplicity, let $U = U(\mathcal{D},\lambda)$ and $u = u(\mathcal{D},\lambda,0) \simeq U(\mathcal{D},\lambda)/(x_\alpha^{N_\alpha} \mid \alpha \in \Phi^+)$ (see Section 2). Let $U^+$, $u^+$ denote the augmentation ideals of $U$, $u$.

We will use a similar construction as in Section 4 of [26], defining functions as elements of the bar complex (2.4.1). For each $\alpha \in \Phi^+$, define a $k$-linear function $\widetilde{f}_\alpha : (U^+)^{2M_\alpha} \to k$ by first letting $r_1, \ldots, r_{2M_\alpha}$ be PBW basis elements (2.3.1) and requiring

$$\widetilde{f}_\alpha(r_1 \otimes \cdots \otimes r_{2M_\alpha}) = \gamma_{12}\gamma_{34}\cdots\gamma_{2M_\alpha-1,2M_\alpha}$$

where $\gamma_{ij}$ is the coefficient of $x_\alpha^{N_\alpha}$ in the product $r_i r_j$ as a linear combination of PBW basis elements. Now define $\widetilde{f}_\alpha$ to be 0 whenever a tensor factor is in $k\Gamma \cap \ker\varepsilon$, and

$$\widetilde{f}_\alpha(r_1 g_1 \otimes \cdots \otimes r_{2M_\alpha} g_{2M_\alpha}) = \widetilde{f}_\alpha(r_1 \otimes {}^{g_1}r_2 \otimes \cdots \otimes {}^{g_1\cdots g_{2M_\alpha-1}}r_{2M_\alpha})$$

for all $g_1, \ldots, g_{2M_\alpha} \in \Gamma$. It follows from the definition of $\widetilde{f}_\alpha$ and the fact that $\chi_\alpha^{M_\alpha} = \varepsilon$ that $\widetilde{f}_\alpha$ is $\Gamma$-invariant. We will show that $\widetilde{f}_\alpha$ factors through the quotient $u^+$ of $U^+$ to give a map $f_\alpha : (u^+)^{\otimes 2M_\alpha} \to k$. Precisely, it suffices to show that $\widetilde{f}_\alpha(r_1 \otimes \cdots \otimes r_{2M_\alpha}) = 0$ whenever one of $r_1, \ldots, r_{2M_\alpha}$ is in the kernel of the quotient map $\pi : U(\mathcal{D},\lambda) \to u(\mathcal{D},\lambda,0)$. Suppose $r_i \in \ker\pi$, that is $r_i = x_{\beta_1}^{a_1}\cdots x_{\beta_r}^{a_r}$ and for some $j$, $a_j \geq N_{\beta_j}$. Since $x_{\beta_j}^{N_{\beta_j}}$ is braided-central, $r_i$ is a scalar multiple of $x_{\beta_j}^{N_{\beta_j}} x_{\beta_1}^{b_1}\cdots x_{\beta_r}^{b_r}$ for some $b_1, \ldots, b_r$. Now $\widetilde{f}_\alpha(r_1 \otimes \cdots \otimes r_{2M_\alpha})$ is the product of the coefficients of $x_\alpha^{N_\alpha}$ in $r_1 r_2, \ldots, r_{2M_\alpha-1}r_{2M_\alpha}$. However, the coefficient of $x_\alpha^{N_\alpha}$ in each of $r_{i-1}r_i$ and $r_i r_{i+1}$ is 0: If $\alpha = \beta_i$, then since $r_{i-1}, r_{i+1} \in U^+$, this product cannot have a nonzero coefficient for $x_\alpha^{N_\alpha}$. If $\alpha \neq \beta_i$, the same is true since $x_{\beta_j}^{N_{\beta_j}}$ is a factor of $r_{i-1}r_i$ and of $r_i r_{i+1}$. Therefore $\widetilde{f_\alpha}$ factors to give a linear map $f_\alpha : (u^+)^{2M_\alpha} \to k$. In calculations, we define $f_\alpha$ via $\widetilde{f_\alpha}$ and a choice of section of the quotient map $\pi : U \to u$.

**Lemma 6.2.** *For each $\alpha \in \Phi^+$, $f_\alpha$ is a cocycle. The $f_\alpha$ ($\alpha \in \Phi^+$) represent a linearly independent subset of $\mathrm{H}^*(u(\mathcal{D}, \lambda, 0), k)$.*

*Proof.* We first verify that $\widetilde{f_\alpha}$ is a cocycle on $U$: Let $r_0, \dots, r_{2M_\alpha} \in U^+$, of positive degree. Then

$$d(\widetilde{f_\alpha})(r_0 \otimes \cdots \otimes r_{2M_\alpha}) = \sum_{i=0}^{2M_\alpha - 1} (-1)^{i+1} \widetilde{f_\alpha}(r_0 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_{2M_\alpha}).$$

By definition of $\widetilde{f_\alpha}$, note that the first two terms cancel:

$$\widetilde{f_\alpha}(r_0 r_1 \otimes r_2 \otimes \cdots \otimes r_{2M_\alpha}) = \widetilde{f_\alpha}(r_0 \otimes r_1 r_2 \otimes \cdots \otimes r_{2M_\alpha}),$$

and similarly for all other terms, so $d(\widetilde{f_\alpha})(r_0 \otimes \cdots \otimes r_{2M_\alpha}) = 0$. A similar calculation shows that $d(\widetilde{f_\alpha})(r_0 g_0 \otimes \cdots \otimes r_{2M_\alpha - 1} g_{2M_\alpha - 1}) = 0$ for all $g_0, \dots, g_{2M_\alpha - 1} \in \Gamma$. If there is an element of $k\Gamma \cap \ker \varepsilon$ in one of the factors, we obtain 0 as well by the definition of $\widetilde{f_\alpha}$, a similar calculation.

Now we verify that $f_\alpha$ is a cocycle on the quotient $u$ of $U$: Let $r_0, \dots, r_{2M_\alpha} \in u^+$. Again we have

$$d(f_\alpha)(r_0 \otimes \cdots \otimes r_{2M_\alpha}) = \sum_{i=0}^{2M_\alpha - 1} (-1)^{i+1} f_\alpha(r_0 \otimes \cdots \otimes r_i r_{i+1} \otimes \cdots \otimes r_{2M_\alpha}).$$

We will show that $f_\alpha(r_0 r_1 \otimes r_2 \otimes \cdots \otimes r_{2M_\alpha}) = f_\alpha(r_0 \otimes r_1 r_2 \otimes \cdots \otimes r_{2M_\alpha})$, and similarly for the other terms. Let $\widetilde{r}_i$ denote the element of $U$ corresponding to $r_i$ under a chosen section of the quotient map $\pi : U \to u$. Note that $\widetilde{r}_0 \cdot \widetilde{r}_1 = \widetilde{r_0 r_1} + y$ and $\widetilde{r}_1 \cdot \widetilde{r}_2 = \widetilde{r_1 r_2} + z$ for some $y, z \in \ker \pi$. So

$$\begin{aligned}
f_\alpha(r_0 r_1 \otimes r_2 \otimes \cdots \otimes r_{2M_\alpha}) &= \widetilde{f_\alpha}(\widetilde{r_0 r_1} \otimes \widetilde{r}_2 \otimes \cdots \otimes \widetilde{r}_{2M_\alpha}) \\
&= \widetilde{f_\alpha}((\widetilde{r}_0 \cdot \widetilde{r}_1 - y) \otimes \widetilde{r}_2 \otimes \cdots \otimes \widetilde{r}_{2M_\alpha}) \\
&= \widetilde{f_\alpha}(\widetilde{r}_0 \cdot \widetilde{r}_1 \otimes \widetilde{r}_2 \otimes \cdots \otimes \widetilde{r}_{2M_\alpha}) \\
&= \widetilde{f_\alpha}(\widetilde{r}_0 \otimes \widetilde{r}_1 \cdot \widetilde{r}_2 \otimes \cdots \otimes \widetilde{r}_{2M_\alpha}) \\
&= \widetilde{f_\alpha}(\widetilde{r}_0 \otimes (\widetilde{r}_1 \cdot \widetilde{r}_2 + z) \otimes \cdots \otimes \widetilde{r}_{2M_\alpha}) \\
&= \widetilde{f_\alpha}(\widetilde{r}_0 \otimes \widetilde{r_1 r_2} \otimes \cdots \otimes \widetilde{r}_{2M_\alpha}) \\
&= f_\alpha(r_0 \otimes r_1 r_2 \otimes \cdots \otimes r_{2M_\alpha}).
\end{aligned}$$

Other computations for this case are similar to those for $U$. Thus $f_\alpha$ is a cocycle on $u$.

We prove that in a given degree, the $f_\alpha$ in that degree represent a linearly independent set in cohomology: Suppose $\sum_\alpha c_\alpha f_\alpha = \partial h$ for some scalars $c_\alpha$ and linear map $h$. Then for each $\alpha$,

$$\begin{aligned} c_\alpha &= (\sum c_\alpha f_\alpha)(x_\alpha \otimes x_\alpha^{N_\alpha - 1} \otimes \cdots \otimes x_\alpha \otimes x_\alpha^{N_\alpha -1}) \\ &= \partial h(x_\alpha \otimes x_\alpha^{N_\alpha - 1} \otimes \cdots \otimes x_\alpha \otimes x_\alpha^{N_\alpha -1}) \\ &= -h(x_\alpha^{N_\alpha} \otimes x_\alpha \otimes \cdots \otimes x_\alpha^{N_\alpha -1}) + \cdots - h(x_\alpha \otimes \cdots \otimes x_\alpha^{N_\alpha-1} \otimes x_\alpha^{N_\alpha}) \quad = \quad 0 \end{aligned}$$

since $x_\alpha^{N_\alpha} = 0$ in $u = u(\mathcal{D}, \lambda, 0)$. □

These functions $f_\alpha$ correspond to their counterparts $\xi_\alpha^{M_\alpha}$ defined on $u(\mathcal{D}, 0, 0)$, in the $E_1$-page of the spectral sequence (6.0.1), by observing what they do as functions at the level of chain complexes (2.4.1).

**Theorem 6.3.** *The algebra* $\mathrm{H}^*(u(\mathcal{D}, \lambda, \mu), k)$ *is finitely generated. If $M$ is a finitely generated $u(\mathcal{D}, \lambda, \mu)$-module, then* $\mathrm{H}^*(u(\mathcal{D}, \lambda, \mu), M)$ *is a finitely generated module over* $\mathrm{H}^*(u(\mathcal{D}, \lambda, \mu), k)$.

*Proof.* By Lemma 6.1, it suffices to prove the statements in the case $\mu = 0$. We have $E_1^{*,*} \simeq \mathrm{H}^*(u(\mathcal{D}, 0, 0), k)$, where $u(\mathcal{D}, 0, 0) \simeq R\#k\Gamma$, $R = \mathcal{B}(V)$. By Lemma 5.2, $E_1^{*,*}$ is finitely generated over its subalgebra that is generated by all $\xi_\alpha^{M_\alpha}$. By Lemma 6.2 and the above remarks, each $\xi_\alpha^{M_\alpha}$ is a permanent cycle, corresponding to the cocycle $f_\alpha$ on $u(\mathcal{D}, \lambda, \mu)$. The rest of the proof is an application of Lemma 2.5, where $A^{*,*}$ is the subalgebra of $E_1^{*,*}$ generated by the $\xi_\alpha^{M_\alpha}$ ($\alpha \in \Phi^+$), similar to the proof of Lemma 5.2 and Theorem 5.3. □

**Corollary 6.4.** *The Hochschild cohomology ring* $\mathrm{H}^*(u(\mathcal{D}, \lambda, \mu), u(\mathcal{D}, \lambda, \mu))$ *is finitely generated.*

*Proof.* Apply Theorem 6.3 to the finitely generated $u(\mathcal{D}, \lambda, \mu)$-module $u(\mathcal{D}, \lambda, \mu)$, under the adjoint action. By [19, Prop. 5.6], this is isomorphic to the Hochschild cohomology ring of $u(\mathcal{D}, \lambda, \mu)$. □

In the special case of a small quantum group, we obtain the following finite generation result (cf. [8, Thm. 1.3.4]).

**Corollary 6.5.** *Let $u_q(\mathfrak{g})$ be a quantized restricted enveloping algebra such that the order $\ell$ of $q$ is odd and prime to 3 if $\mathfrak{g}$ is of type $G_2$. Then* $\mathrm{H}^*(u_q(\mathfrak{g}), k)$ *is a finitely generated algebra. Moreover, for any finitely generated $u_q(\mathfrak{g})$-module $M$,* $\mathrm{H}^*(u_q(\mathfrak{g}), M)$ *is a finitely generated module over* $\mathrm{H}^*(u_q(\mathfrak{g}), k)$.

**Remark 6.6.** The restrictions on $\ell$ correspond to the assumptions (2.1.1). However our techniques and results are more general: The restrictions are used in the classification of Andruskiewitsch and Schneider, but not in our arguments. We need only the filtration lemma of De Concini and Kac [14, Lemma 1.7] as generalized to our setting (Lemma 2.4) to guarantee existence of the needed spectral sequences. Our results should hold for all small quantum groups $u_q(\mathfrak{g})$ having such a filtration, including those at even roots of unity $q$ for which $q^{2d} \neq 1$ (see [7] for the general theory at even roots of unity).

We illustrate the connection between our results and those in [19] with a small example. Our structure results are not as strong, however our finite generation result is much more general.

**Example 6.7.** As an algebra, $u_q(sl_2)$ is generated by $E, F, K$, with relations $K^\ell = 1$, $E^\ell = 0$, $F^\ell = 0$, $KEK^{-1} = q^2 E$, $KFK^{-1} = q^{-1}F$, and

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}, \tag{6.7.1}$$

where $q$ is a primitive $\ell$th root of 1, $\ell > 2$. Consider the coradical filtration on $u_q(sl_2)$, in which $\deg(K) = 0$, $\deg(E) = \deg(F) = 1$. Note that $\mathrm{Gr}\, u_q(sl_2)$ is generated by $E, F, K$, with all relations being the same except that (6.7.1) is replaced by $EF - FE = 0$. This is an algebra of the type featured in Section 4: $\mathrm{H}^*(\mathrm{Gr}\, u_q(sl_2), k) \simeq k[\xi_1, \xi_2, \eta_1\eta_2]/((\eta_1\eta_2)^2)$, since these are the invariants, under the action of $\Gamma = \langle K \rangle$, of the cohomology of the subalgebra of $\mathrm{Gr}\, u_q(sl_2)$ generated by $E, F$. By [19],

$$\mathrm{H}^*(u_q(sl_2), k) \simeq k[\alpha, \beta, \gamma]/(\alpha\beta + \gamma^2),$$

the coordinate algebra of the nilpotent cone of $sl_2$. Identify $\alpha \sim \xi_1$, $\beta \sim \xi_2$, $\gamma \sim \eta_1\eta_2$: Then as maps, $\deg(\alpha) = \ell$, $\deg(\beta) = \ell$, $\deg(\gamma) = 2$, so $\alpha\beta + \gamma^2 = 0$ will imply $\gamma^2 = 0$ in the associated graded algebra, as expected.

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