A NUMERICAL INVARIANT FOR LINEAR REPRESENTATIONS OF FINITE GROUPS

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WITH AN APPENDIX BY JULIA PEVTSOVA AND ZINOVY REICHSTEIN

Abstract. We study the notion of essential dimension for a linear representation of a finite group. In characteristic zero we relate it to the canonical dimension of certain products of Weil transfers of generalized Severi-Brauer varieties. We then proceed to compute the canonical dimension of a broad class of varieties of this type, extending earlier results of the first author. As a consequence, we prove analogues of classical theorems of R. Brauer and O. Schilling about the Schur index, where the Schur index of a representation is replaced by its essential dimension. In the last section we show that in the modular setting $\mathrm{ed}(\rho)$ can be arbitrary large (under a mild assumption on G). Here G is fixed, and ρ is allowed to range over the finite-dimensional representations of G. The appendix gives a constructive version of this result.

1. Introduction

Let K/k be a field extension, G be a finite group of exponent e, and $\rho\colon G \to \mathrm{GL}_n(K)$ be a non-modular representation of G whose character takes values in k. (Here "non-modular" means that $\mathrm{char}(k)$ does not divide $|G|$.) A theorem of Brauer says that if k contains a primitive eth root of unity ζ_e then ρ is defined over k, i.e., ρ is K-equivalent to a representation $\rho'\colon G \to \mathrm{GL}_n(k)$; see, e.g. [34, §12.3]. If $\zeta_e \notin k$, we would like to know "how far" ρ is from being defined over k. In the case, where ρ is absolutely irreducible, a classical answer to this question is given by the Schur index of ρ, which is the smallest degree of a finite field extension l/k such that ρ is defined over l. Some background material on the Schur index and further references can be found in Section 2.

In this paper we introduce and study another numerical invariant, the essential dimension $\mathrm{ed}(\rho)$, which measures "how far" ρ is from being defined over k in a different way. Here ρ is not assumed to be irreducible; for the definition of $\mathrm{ed}(\rho)$, see Section 6. In Section 8 we show that the maximal value of $\mathrm{ed}(\rho)$, as ρ ranges over representations with

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a fixed character $\chi\colon G \to k$, which we denote by $\mathrm{ed}(\chi)$, can be expressed as the canonical dimension of a certain product of Weil transfers of generalized Severi-Brauer varieties. We use this to show that $\mathrm{ed}(\rho) \leqslant |G|/4$ for any n, k, and K/k in Section 9 and to prove a variant of a classical theorem of Brauer in Section 10. In Section 11 we compute the canonical dimension of a broad class of Weil transfers of generalized Severi-Brauer varieties, extending earlier results of the first author from [20] and [22]. This leads to a formula for the essential p-dimension of an irreducible character in terms of its decomposition into absolutely irreducible components; see Corollary 12.3. As an application we prove a variant of a classical theorem of Schilling in Section 13.

In Section 14 we show that in the modular setting $\mathrm{ed}(\rho)$ can be arbitrary large (under a mild assumption on G). Here G is assumed to be fixed, and ρ is allowed to range over the finite-dimensional representations of G. The appendix proves a constructive version of this result.

2. Notation and representation-theoretic preliminaries

Throughout this paper G will denote a finite group of exponent e, k a field, $\overline{k}$ an algebraic closure of k, K and F field extensions of k, ζ_d a primitive dth root of unity, ρ a finite-dimensional representation of G, and χ a character of G. In this section we will assume that $\mathrm{char}(k)$ does not divide the order of G.

2a. **Characters and character values.** A function $\chi\colon G \to k$ is said to be *a character* of G, if χ is the character of some representation $\rho\colon G \to \mathrm{GL}_n(K)$ for some field extension K/k.

If $\chi\colon G \to \overline{k}$ is a character, and F/k is a field, we set

$$F(\chi) := F(\chi(g) \,|\, g \in G) \subset F(\zeta_e).$$

Since $F(\zeta_e)$ is an abelian extension of F, so is $F(\chi)$. Moreover, $F(\chi)$ is stable under automorphisms $F(\zeta_e)/F$.

Two characters, $\chi, \chi'\colon G \to \overline{k}$ are said to be conjugate over F if there exists an F-isomorphism of fields $\sigma\colon F(\chi) \to F(\chi')$ such that $\sigma \circ \chi = \chi'$.

Lemma 2.1. *(a) Let $\chi, \chi'\colon G \to \overline{k}$ be characters and F/k be a field extension. Then*

(a) every automorphism $h \in \mathrm{Gal}(F(\chi)/F)$ leaves $k(\chi)$ invariant.

(b) If χ and χ' are conjugate over F then they are conjugate over k.

(c) Suppose k is algebraically closed in F. Then the converse to part (b) also holds. That is, if χ, χ' are conjugate over k then they are conjugate over F.

Proof. (a) It is enough to show that $h(\chi(g)) \in k(\chi)$ for every $g \in G$. Since the sequence of Galois groups

$$1 \to \mathrm{Gal}(F(\zeta_e)/F(\chi)) \to \mathrm{Gal}(F(\zeta_e)/F) \to \mathrm{Gal}(F(\chi)/F) \to 1$$

is exact, h can be lifted to an element of $\mathrm{Gal}(F(\zeta_e)/F)$. By abuse of notation, we will continue to denote this element of $\mathrm{Gal}(F(\zeta_e)/F)$ by h. The eigenvalues of $\rho(g)$ are of the form $\zeta_e^{i_1}, \dots, \zeta_e^{i_n}$ for some integers $i_1, \dots, i_n$. The automorphism h sends ζ_e to another primitive eth root of unity ζ_e^j for some integer j. Then

$$h(\chi(g)) = h(\zeta_e^{i_1} + \dots + \zeta_e^{i_n}) = \zeta_e^{ji_1} + \dots + \zeta_e^{ji_n} = \chi(g^j) \in k(\chi)\,,$$

as desired.

(b) is an immediate consequence of (a).

(c) If k is algebraically closed in F, then the homomorphism

$$\mathrm{Gal}(F(\chi)/F) \to \mathrm{Gal}(k(\chi)/k)$$

given by $\sigma \mapsto \sigma_{|k(\chi)}$ is surjective; see [28, Theorem VI.1.12]. □

2b. **The envelope of a representation.** If $\rho\colon G \to \mathrm{GL}_n(F)$ is a representation over some field F/k, we define the *k-envelope* $\mathrm{Env}_k(\rho)$ as the k-linear span of $\rho(G)$ in $\mathrm{M}_n(F)$. Note that $\mathrm{Env}_k(\rho)$ is a k-subalgebra of $\mathrm{M}_n(F)$.

Lemma 2.2. *For any integer $s \geqslant 1$, the k-algebras $\mathrm{Env}_k(s\cdot\rho)$ and $\mathrm{Env}_k(\rho)$ are isomorphic.*

Proof. The diagonal embedding $\mathrm{M}_n(F) \hookrightarrow \mathrm{M}_n(F) \times \cdots \times \mathrm{M}_n(F)$ (s times) induces an isomorphism between $\mathrm{Env}_k(\rho)$ and $\mathrm{Env}_k(s \cdot \rho)$. □

Lemma 2.3. *Assume the character χ of $\rho\colon G \to \mathrm{GL}_n(F)$ is k-valued. Then the natural homomorphism $\mathrm{Env}_k(\rho) \otimes_k F \to \mathrm{Env}_F(\rho)$ is an isomorphism of F-algebras.*

Proof. It suffices to show that if $\rho(g_1), \dots, \rho(g_r)$ are linearly dependent over F for some elements $g_1, \dots, g_r \in G$, then they are linearly dependent over k. Indeed, suppose

$$a_1\rho(g_1) + \cdots + a_r\rho(g_r) = 0$$

in $\mathrm{M}_n(F)$ for some $a_1, \dots, a_r \in F$, such that $a_i \neq 0$ for some i. Then

$$\mathrm{tr}((a_1\rho(g_1) + \cdots + a_r\rho(g_r)) \cdot \rho(g)) = 0$$

for every $g \in G$, which simplifies to

$$a_1\chi(g_1g) + \cdots + a_r\chi(g_rg) = 0.$$

The homogeneous linear system

$$x_1\chi(g_1g) + \cdots + x_r\chi(g_rg) = 0$$

in variables $x_1, \dots, x_r$ has coefficients in k and a non-trivial solution in F. Hence, it has a non-trivial solution $b_1, \dots, b_r$ in k, and we get that

$$\mathrm{tr}((b_1\rho(g_1) + \cdots + b_r\rho(g_r)) \cdot \rho(g)) = 0$$

for every $g \in G$.

Note that $\mathrm{Env}_k(\rho)$ is, by definition, a homomorphic image of the group ring $k[G]$. Hence, $\mathrm{Env}_k(\rho)$ is semisimple and consequently, the trace form in $\mathrm{Env}_k(\rho)$ is non-degenerate. It follows that the elements $\rho(g_1), \dots, \rho(g_r)$ are linearly dependent over k, as desired. □

2c. **The Schur index.** Suppose K/k is a field extension, and $\rho_1\colon G \to \mathrm{GL}_n(K)$ is an absolutely irreducible representation with character $\chi_1\colon G \to K$. By taking $F = \overline{K}$ in Lemma 2.3, one easily deduces that $\mathrm{Env}_{k(\chi_1)}(\rho_1)$ is a central simple algebra of degree n over $k(\chi_1)$. The index of this algebra is called *the Schur index* of ρ_1. We will denote it by $m_k(\rho_1)$.

In the sequel we will need the following properties of the Schur index.

Lemma 2.4. *Let K be a field, G be a finite group such that* $\mathrm{char}(K)$ *does not divide* $|G|$*, and* $\rho\colon G \to \mathrm{GL}_n(K)$ *be an irreducible representation. Denote the character of* ρ *by* χ*.*

(a) Over the algebraic closure $\overline{K}$*,* ρ *decomposes as*

$$\rho_{\overline{K}} \simeq m(\rho_1 \oplus \cdots \oplus \rho_r), \tag{2.5}$$

where $\rho_1, \dots, \rho_r$ *are pairwise non-isomorphic irreducible representations of* G *defined over* $\overline{K}$*, and* m *is their common Schur index* $m_K(\rho_1) = \cdots = m_K(\rho_r)$*.*

(b) For $i = 1, \dots, r$ *and* ρ_i *as in (a), let* $\chi_i\colon G \to \overline{K}$ *be the character of* ρ_i*. Then* $K(\chi_1) = \cdots = K(\chi_r)$ *is an abelian extension of* K *of degree* r*. Moreover,* $\mathrm{Gal}(K(\chi_1)/K)$ *transitively permutes* $\chi_1, \dots, \chi_r$*.*

(c) Conversely, every irreducible representation $\rho_1\colon G \to \mathrm{GL}_1(\overline{K})$ *occurs as an irreducible component of a unique* K*-irreducible representation* $\rho\colon G \to \mathrm{GL}_n(K)$*, as in* (2.5)*.*

(d) The center Z *of* $\mathrm{Env}_K(\rho)$ *is* K*-isomorphic to* $K(\chi_1) = K(\chi_2) = \cdots = K(\chi_r)$*.* $\mathrm{Env}_K(\rho)$ *is a central simple algebra over* Z *of index* m*.*

(e) The multiplicity of ρ_1 *in any representation of* G *defined over* K *is a multiple of* $m_K(\rho_1)$*. Consequently,* $m_K(\rho_1)$ *divides* $m_k(\rho_1)$ *for any field extension* K/k*.*

(f) m *divides* $\dim(\rho_1) = \cdots = \dim(\rho_r)$*.*

Proof. See [14, Theorem 74.5] for parts (a)-(d), and [13, Corollary 74.8] for parts (e) and (f). □

Corollary 2.6. *Let* K/k *be a field extension,* $\rho\colon G \to \mathrm{GL}_n(K)$ *be a representation, whose character takes values in* k*, and*

$$\rho = d_1\rho_1 \oplus \cdots \oplus d_r\rho_r$$

be the irreducible decomposition of ρ *over the algebraic closure* $\overline{K}$*. Then the following conditions are equivalent.*

(1) ρ *can be realized over* k*, i.e.,* ρ *is* K*-equivalent to a representation* $\rho'\colon G \to \mathrm{GL}_n(k)$*.*

(2) The Schur index $m_k(\rho_i)$ *divides* d_i *for every* $i = 1, \dots, r$*.*

Proof. Each $\rho_i\colon G \to \mathrm{GL}_{n_i}(\overline{K})$ is $\overline{K}$-equivalent to some $\rho_i'\colon G \to \mathrm{GL}_{n_i}(\overline{k})$. Let $\rho' := d_1\rho_1' \oplus \cdots \oplus d_r\rho_r'\colon G \to \mathrm{GL}_n(\overline{k})$. Since ρ and ρ' have the same character, ρ can be realized over k if and only if ρ' can be realized over k. Hence, we may replace ρ by ρ' and thus assume that $K = \overline{k}$ from now on.

Denote the character of ρ by χ and the character of ρ_i by χ_i. Since χ takes values in k, $d_i = d_j$ whenever χ_i and χ_j are conjugate over k.

(1) $\Longrightarrow$ (2). Suppose ρ can be realized over k. Decomposing ρ as a direct sum of k-irreducibles, we see that it suffices to prove (2) in the case where ρ is k-irreducible. In this case (2) holds by Lemma 2.4(a).

(2) $\Longrightarrow$ (1). If a representation ρ satisfies condition (2), then ρ is a direct sum of representations of the form $\lambda = m_k(\chi_1)(\rho_1 \oplus \cdots \oplus \rho_s)$, where $\rho_1, \dots, \rho_s$ are absolutely irreducible representations of G and the characters $\chi_1, \dots, \chi_s$ of $\rho_1, \dots, \rho_s$ are transitively permuted by $\mathrm{Gal}(\overline{k}/k)$. By Lemma 2.4(c), every representation of this form is defined over k. □

3. Preliminaries on essential and canonical dimension

3a. **Essential dimension.** Let $\mathcal{F} : \mathbf{Fields}_k \to \mathbf{Sets}$ be a covariant functor, where $\mathbf{Fields}_k$ is the category of field extensions of k and $\mathbf{Sets}$ is the category of sets. We think of the functor $\mathcal{F}$ as specifying the type of algebraic objects under consideration, $\mathcal{F}(K)$ as the set of algebraic objects of this type defined over K, and the morphism $\mathcal{F}(i)\colon \mathcal{F}(K) \to \mathcal{F}(L)$ associated to a field extension

$$k \subset K \overset{i}{\hookrightarrow} L \tag{3.1}$$

as "base change". For notational simplicity, we will write $\alpha_L \in \mathcal{F}(L)$ instead of $\mathcal{F}(i)(\alpha)$.

Given a field extension L/K, as in (3.1), an object $\alpha \in \mathcal{F}(L)$ is said to *descend* to K if it lies in the image of $\mathcal{F}(i)$. The *essential dimension* $\operatorname{ed}(\alpha)$ is defined as the minimal transcendence degree of K/k, where α descends to K. The essential dimension $\operatorname{ed}(\mathcal{F})$ of the functor $\mathcal{F}$ is the supremum of $\operatorname{ed}(\alpha)$ taken over all $\alpha \in \mathcal{F}(K)$ and all K.

Usually $\operatorname{ed}(\alpha) < \infty$ for every $\alpha \in \mathcal{F}(K)$ and every K/k; see [8, Remark 2.7]. On the other hand, $\operatorname{ed}(\mathcal{F}) = \infty$ in many cases of interest; for example, see Theorem 14.1.

The essential dimension $\operatorname{ed}_p(\alpha)$ of α at a prime integer p is defined as the minimal value of $\operatorname{ed}(\alpha_{L'})$, as L' ranges over all finite field extensions L'/L such that p does not divide the degree $[L' : L]$. The essential dimension $\operatorname{ed}_p(\mathcal{F})$ is then defined as the supremum of $\operatorname{ed}_p(\alpha)$ as K ranges over all field extensions of k and α ranges over $\mathcal{F}(K)$.

For generalities on essential dimension, see [3, 8, 30, 32].

3b. **Canonical dimension.** An interesting example of a covariant functor $\mathbf{Fields}_k \to \mathbf{Sets}$ is the "detection functor" $\mathcal{D}_X$ associated to an algebraic k-variety X. For a field extension K/k, we define

$$\mathcal{D}_X(K) := \begin{cases} \text{a one-element set, if } X \text{ has a } K\text{-point, and} \\ \emptyset, \text{ otherwise.} \end{cases}$$

If $k \subset K \overset{i}{\hookrightarrow} L$ then $0 \leqslant |\mathcal{D}_X(K)| \leqslant |\mathcal{D}_X(L)| \leqslant 1$. Thus there is a unique morphism of sets $\mathcal{D}_X(K) \to \mathcal{D}_X(L)$, which we define to be $\mathcal{D}_X(i)$.

The essential dimension (respectively, the essential p-dimension) of the functor $\mathcal{D}_X$ is called the *canonical dimension* of X (respectively, *the canonical p-dimension* of X) and is denoted by $\operatorname{cd}(X)$ (respectively, $\operatorname{cd}_p(X)$). If X is smooth and projective, then $\operatorname{cd}(X)$ (respectively, $\operatorname{cd}_p(X)$) equals the minimal dimension of the image of a rational self-map $X \dashrightarrow X$ (respectively, of a correspondence $X \rightsquigarrow X$ of degree prime to p). In particular,

$$0 \leqslant \operatorname{cd}_p(X) \leqslant \operatorname{cd}(X) \leqslant \dim(X) \tag{3.2}$$

for any prime p. If $\operatorname{cd}(X) = \dim(X)$, we say that X is incompressible. If $\operatorname{cd}_p(X) = \dim(X)$, we say that X is p-incompressible. For details on the notion of canonical dimension for algebraic varieties, we refer the reader to [30, §4].

We will say that smooth projective varieties X and Y defined over K are *equivalent* if there exist rational maps $X \dashrightarrow Y$ and $Y \dashrightarrow X$. Similarly, we will say that X and Y are p-equivalent for a prime integer p, if there exist correspondences $X \rightsquigarrow Y$ and $Y \rightsquigarrow X$ of degree prime to p.

Lemma 3.3. *(a) If X and Y are equivalent, then* $\mathrm{cd}(X) = \mathrm{cd}(Y)$.

(b) If X and Y are p-equivalent for some prime p, then $\mathrm{cd}_p(X) = \mathrm{cd}_p(Y)$.

Proof. (a) Let K/k be a field extension. By Nishimura's lemma, X has a K-point if and only if so does Y; see [33, Proposition A.6]. Thus the detection functors $\mathcal{D}_X$ and $\mathcal{D}_Y$ are isomorphic, and $\mathrm{cd}(X) = \mathrm{ed}(\mathcal{D}_X) = \mathrm{ed}(\mathcal{D}_Y) = \mathrm{cd}(Y)$.

For a proof of part (b) see [26, Lemma 3.6 and Remark 3.7]. □

4. Balanced algebras

Let Z/k be a Galois field extension, and A be a central simple algebra over Z. Given $\alpha \in \mathrm{Gal}(Z/k)$, we will denote the "conjugate" Z-algebra $A \otimes_Z Z$, where the tensor product is taken via $\alpha\colon Z \to Z$, by ${}^{\alpha}A$. We will say that A is *balanced over k* if ${}^{\alpha}A$ is Brauer-equivalent to a tensor power of A for every $\alpha \in \mathrm{Gal}(Z/k)$.

Note that A is balanced, if the Brauer class of A descends to $k\colon {}^{\alpha}A$ is then isomorphic to A for any α. In this section we will consider another family of balanced algebras.

Let K/k be a field extension, $\rho\colon G \to \mathrm{GL}_n(K)$ be an irreducible representation whose character χ is k-valued. Recall from Lemma 2.4 that $\mathrm{Env}_k(\rho)$ is a central simple algebra over $Z \simeq k(\chi_1) = \cdots = k(\chi_r)$.

Proposition 4.1. $\mathrm{Env}_k(\rho)$ *is balanced over k.*

Proof. Recall from [37, p. 14] that a cyclotomic algebra B/Z is a central simple algebra of the form

$$B = \bigoplus_{g \in \mathrm{Gal}(Z(\zeta)/Z)} Z(\zeta) u_g \,,$$

where ζ is a root of unity, $Z(\zeta)$ is a maximal subfield of B, and the basis elements u_g are subject to the relations

$$u_g x = g(x) u_g \text{ and } u_g u_h = \beta(g,h) u_{gh} \quad \text{for every } x \in Z(\zeta) \text{ and } g, h \in \mathrm{Gal}(Z(\zeta)/Z).$$

Here $\beta\colon \mathrm{Gal}(Z(\zeta)/Z) \times \mathrm{Gal}(Z(\zeta)/Z) \to Z(\zeta)^*$ is a 2-cocycle whose values are powers of ζ. Following the notational conventions in [37], we will write $B := (\beta, Z(\zeta)/Z)$.

By the Brauer-Witt Theorem [37, Corollary 3.11], $\mathrm{Env}_k(\rho)$ is Brauer-equivalent to some cyclotomic algebra B/Z, as above. Thus it suffices to show that every cyclotomic algebra is balanced over k, i.e., ${}^{\alpha}B$ is Brauer-equivalent to a power of B over Z for every $\alpha \in \mathrm{Gal}(Z/k)$.

By Lemma 2.4(d), Z is k-isomorphic to $k(\chi_1)$, which is, by definition a subfield of $k(\zeta_e)$, where e is the exponent of G. Thus there is a root of unity ϵ such that

$$Z(\zeta) \subset k(\zeta, \zeta_e) = k(\epsilon)$$

and both ζ and ζ_e are powers of ϵ. Note that $k(\epsilon)/k$ is an abelian extension, and the sequence of Galois groups

$$1 \to \mathrm{Gal}(k(\epsilon)/Z) \to \mathrm{Gal}(k(\epsilon)/k) \to \mathrm{Gal}(Z/k) \to 1$$

is exact. In particular, every $\alpha \in \mathrm{Gal}(Z/k)$ can be lifted to an element of $\mathrm{Gal}(k(\epsilon)/k)$, which we will continue to denote by α. Then $\alpha(\epsilon) = \epsilon^t$ for some integer t. Since ζ is a power of ϵ, and each $\beta(g,h)$ is a power of ζ, we have

$$\alpha(\beta(g,h)) = \beta(g,h)^t \text{ for every } g, h \in \mathrm{Gal}(Z(\zeta)/k). \tag{4.2}$$

We claim that ${}^{\alpha}B$ is Brauer-equivalent to $B^{\otimes t}$ over Z. Indeed, since

$$B = (\beta, Z(\zeta)/Z),$$

we have ${}^{\alpha}B = (\alpha(\beta), Z(\zeta)/Z)$. By (4.2), ${}^{\alpha}B = (\alpha(\beta), Z(\zeta)/Z) = (\beta^t, Z(\zeta)/Z)$, and $(\beta^t, Z(\zeta)/Z)$ is Brauer-equivalent to $B^{\otimes t}$, as desired. □

5. Generalized Severi-Brauer varieties and Weil transfers

Suppose Z/k is a finite Galois field extension and A is a central simple algebra over Z. For $1 \leqslant m \leqslant \deg(A)$, we will denote by $\mathrm{SB}(A, m)$ the generalized Severi-Brauer variety (or equivalently, the twisted Grassmannian) of $(m-1)$-dimensional subspaces in $\mathrm{SB}(A)$. The Weil transfer $R_{Z/k}(\mathrm{SB}(A, m))$ is a smooth projective absolutely irreducible k-variety of dimension $[Z : k] \cdot m \cdot (\deg(A) - m)$. For generalities on $\mathrm{SB}(A, m)$, see [5]. For generalities on the Weil transfer, see [17].

Proposition 5.1. *Let Z, k and A be as above, $X := R_{Z/k}(\mathrm{SB}(A, m))$ for some $1 \leqslant m \leqslant \deg(A)$, and K/k be a field extension.*

(a) Write $K_Z := K \otimes_k Z$ as a direct product $K_1 \times \cdots \times K_s$, where $K_1/Z, \dots, K_s/Z$ are field extensions. Then X has a K-point if and only if the index of the central simple algebra $A_{K_i} := A \otimes_Z K_i$ divides m for every $i = 1, \dots, s$.

(b) Assume that m divides $\mathrm{ind}(A)$, A is balanced and $K = k(X)$ is the function field of X. Then $K_Z = K \otimes_k Z$ is a field, and $A \otimes_k K \simeq A \otimes_Z K_Z$ is a central simple algebra over K_Z of index m.

Proof. First note that $A \otimes_k K \simeq A \otimes_Z K_Z$.

(a) By the definition of the Weil transfer, $X = R_{Z/k}(\mathrm{SB}(A, m))$ has a K-point if and only if $\mathrm{SB}(A, m)$ has a K_Z-point or equivalently, if and only if $\mathrm{SB}(A, m)$ has a K_i-point for every $i = 1, \dots, s$. On the other hand, by [5, Proposition 3], $\mathrm{SB}(A, m)$ has a K_i-point if and only if the index of A_{K_i} divides m.

(b) Since X is absolutely irreducible, K_Z is Z-isomorphic to the function field of the Z-variety

$$X_Z := X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(Z) = \prod_{\alpha \in \mathrm{Gal}(Z/k)} \mathrm{SB}({}^{\alpha}A, m)\,,$$

see [6, §2.8]. Set $F := Z(\mathrm{SB}(A, m))$. By [35, Corollary 1],

$$\mathrm{ind}(A \otimes_Z F) = m\,.$$

Since A is balanced, i.e., each algebra ${}^{\alpha}A$ is a power of A, $\mathrm{ind}({}^{\alpha}A \otimes_Z F)$ divides m for every $\alpha \in \mathrm{Gal}(Z/k)$. By [5, Proposition 3], each $\mathrm{SB}({}^{\alpha}A, m)_F$ is rational over F. Thus the natural projection of Z-varieties

$$X_Z = \prod_{\alpha \in \mathrm{Gal}(Z/k)} \mathrm{SB}({}^{\alpha}A, m) \to \mathrm{SB}(A, m)$$

induces a purely transcendental extension of function fields $F \hookrightarrow K_Z$. Consequently,

$$\mathrm{ind}(A \otimes_Z K_Z) = \mathrm{ind}(A \otimes_Z F) = m\,,$$

as claimed. □

6. The essential dimension of a representation

Let us now fix a finite group G and an arbitrary field k, and consider the covariant functor

$$\mathbf{Rep}_{G,k} : \mathbf{Fields}_k \to \mathbf{Sets}$$

defined by $\mathbf{Rep}_{G,k}(K) := \{K\text{-isomorphism classes of representations } G \to \mathrm{GL}_n(K)\}$ for every field K/k. Here $n \geqslant 1$ is allowed to vary.

The essential dimension $\mathrm{ed}(\rho)$ of a representation $\rho\colon G \to \mathrm{GL}_n(K)$ is defined by viewing ρ as an object in $\mathbf{Rep}_{G,k}(K)$, as in Section 3. That is, $\mathrm{ed}(\rho)$ is the smallest transcendence degree of an intermediate field $k \subset K_0 \subset K$ such that ρ is K-equivalent to a representation $\rho'\colon G \to \mathrm{GL}_n(K_0)$. To illustrate this notion, we include an example, where $\mathrm{ed}(\rho)$ is positive, and three elementary lemmas.

Example 6.1. Let $\mathbb{H} = (-1,-1)$ be the algebra of Hamiltonian quaternions over $k = \mathbb{R}$, i.e., the 4-dimensional $\mathbb{R}$-algebra given by two generators i, j, subject to relations, $i^2 = j^2 = -1$ and $ij = -ji$. The multiplicative subgroup $G = \{\pm 1, \pm i, \pm j, \pm ij\}$ of $\mathbb{H}^*$ is the quaternion group of order 8. Let $K = \mathbb{R}(\mathrm{SB}(\mathbb{H}))$, where $\mathrm{SB}(\mathbb{H})$ denotes the Severi-Brauer variety of $\mathbb{H}$. The representation $\rho\colon G \hookrightarrow \mathbb{H} \hookrightarrow \mathbb{H} \otimes_{\mathbb{R}} K \simeq \mathrm{M}_2(K)$ is easily seen to be absolutely irreducible. We claim that $\mathrm{ed}(\rho) = 1$. Indeed, $\mathrm{trdeg}_{\mathbb{R}}(F) = 1$, for any intermediate extension $\mathbb{R} \subset F \subset K$, unless $F = \mathbb{R}$. On the other hand, ρ cannot descend to $\mathbb{R}$, because $\mathrm{Env}_{\mathbb{R}}(\rho) = \mathbb{H}$, and thus $m_{\mathbb{R}}(\rho) = \mathrm{ind}(\mathbb{H}) = 2$ by Lemma 2.4(e). □

Lemma 6.2. *Let G be a finite group, K/k be a field, $\rho_i\colon G \to \mathrm{GL}_{n_i}(K)$ be representations of G over K (for $i = 1, \dots, s$) and $\rho \simeq a_1\rho_1 \oplus \dots \oplus a_s\rho_s$, where $a_1, \dots, a_s \geqslant 1$ are integers. Then $\mathrm{ed}(\rho) \leqslant \mathrm{ed}(\rho_1) + \dots + \mathrm{ed}(\rho_s)$.*

Proof. Suppose ρ_i descends to an intermediate field $k \subset K_i \subset K$, where $\mathrm{trdeg}_k(K_i) = \mathrm{ed}(\rho_i)$. Let K_0 be the subfield of K generated by $K_1, \dots, K_s$. Then ρ descends to K_0 and $\mathrm{ed}(\rho) \leqslant \mathrm{trdeg}_k(K_0) \leqslant \mathrm{trdeg}_k(K_1) + \dots + \mathrm{trdeg}_k(K_s) = \mathrm{ed}(\rho_1) + \dots + \mathrm{ed}(\rho_s)$. □

Lemma 6.3. *Let $k \subset K$ be fields, G be a finite group, and $\rho\colon G \to \mathrm{GL}_n(K)$ be a representation. Let $k' := k(\chi) \subset K$, where χ is the character of ρ. Then the essential dimension of ρ is the same, whether we consider it as an object on $\mathbf{Rep}_{K,k}$ or $\mathbf{Rep}_{K,k'}$.*

Proof. If ρ descends to an intermediate field $k \subset F \subset K$, then F automatically contains k'. Moreover, $\mathrm{trdeg}_k(F) = \mathrm{trdeg}_{k'}(F)$. The rest is immediate from the definition. □

Lemma 6.4. *Assume that $\mathrm{char}(k)$ does not divide $|G|$ and the Schur index $m_k(\lambda)$ equals 1 for every absolutely irreducible representation λ of G. Then $\mathrm{ed}(\rho) = 0$ for any representation $\rho\colon G \to \mathrm{GL}_n(L)$ over any field L/k. In other words, $\mathrm{ed}(\mathbf{Rep}_{G,k}) = 0$.*

Proof. Let χ be the character of ρ and $k' := k(\chi)$. By Lemma 2.4(e), $m_{k'}(\lambda) = 1$ for every absolutely irreducible representation $\lambda\colon G \to \mathrm{GL}_n(K)$ of G. By Lemma 6.3 we may replace k by $k' = k(\chi)$ and thus assume that χ is k-valued. Corollary 2.6 now tells us that ρ descends to k. □

Remark 6.5. The condition of Lemma 6.4 is always satisfied if $\mathrm{char}(k) > 0$; see [14, Theorem 74.9]. This tells us that for non-modular representations the notion of essential dimension is only of interest when $\mathrm{char}(k) = 0$. The situation is drastically different in the modular setting; see Section 14.

7. Irreducible characters

In view of Remark 6.5, we will now assume that $\mathrm{char}(k) = 0$. In this setting there is a tight connection between representations and characters.

Lemma 7.1. *Suppose F_1/k, F_2/k are field extensions, and*

$$\rho_1 \colon G \to \mathrm{GL}_n(F_1), \quad \rho_2 \colon G \to \mathrm{GL}_n(F_2)$$

are representations of a finite group G, with the same character $\chi \colon G \to k$. Then the k-algebras $\mathrm{Env}_k(\rho_1)$ and $\mathrm{Env}_k(\rho_2)$ are isomorphic.

Proof. Let F/k be a field containing both F_1 and F_2. Then ρ_1 and ρ_2 are equivalent over F, because they have the same character. Thus $\mathrm{Env}_k(\rho_1)$ and $\mathrm{Env}_k(\rho_2)$ are conjugate inside $\mathrm{M}_n(F)$. □

Given a representation $\rho \colon G \to \mathrm{GL}_n(F)$, with a k-valued character $\chi \colon G \to k$, Lemma 7.1 tells us that, up to isomorphism, the k-algebra $\mathrm{Env}_k(\rho)$ depends only on χ and not on the specific choice of F and ρ. Thus we may denote this algebra by $\mathrm{Env}_k(\chi)$.

If ρ is absolutely irreducible (and the character χ is not necessarily k-valued), it is common to write $m_k(\chi)$ for the index of $\mathrm{Env}_{k(\chi)}(\chi)$ instead of $m_k(\rho)$.

Let $\chi \colon G \to k$ be a character of G. Write

$$\chi = \sum_{i=1}^{r} m_i \chi_i \,, \tag{7.2}$$

where $\chi_1, \dots, \chi_r \colon G \to \overline{k}$ are absolutely irreducible and distinct and $m_1, \dots, m_r$ are positive integers. Since χ is k-valued, $m_i = m_j$ whenever χ_i and χ_j are conjugate over k.

Lemma 7.3. *Let $\chi = \sum_{i=1}^{r} m_i \chi_i \colon G \to k$ be a character of G, as in* (7.2)*. Then the following are equivalent.*

(a) χ is the character of a K-irreducible representation $\rho \colon G \to \mathrm{GL}_n(K)$ for some field extension K/k.

(b) $\chi_1, \dots, \chi_r$ form a single $\mathrm{Gal}(k(\chi_1)/k)$-orbit and $m_1 = \dots = m_r$ divides $m_k(\chi_1) = \dots = m_k(\chi_r)$.

Proof. (a) $\Longrightarrow$ (b): By Lemma 2.4(a) and (b), $\chi = m(\chi_1 + \dots + \chi_r)$, where $\chi_1, \dots, \chi_r$ are absolutely irreducible characters transitively permuted by $\mathrm{Gal}(K(\chi_1)/K)$, and $m = m_K(\chi_1) = \dots = m_K(\chi_r)$. By Lemma 2.1(b), $\chi_1, \dots, \chi_r$ are also transitively permuted by $\mathrm{Gal}(k(\chi_1)/k)$. Moreover, by Lemma 2.4(e), m divides $m_k(\chi_1) = \dots = m_k(\chi_r)$.

(b) $\Longrightarrow$ (a): Let K be the function field of the Weil transfer variety $R_{Z/k}(\mathrm{SB}(A, m))$, where A is the underlying division algebra, Z is the center of $\mathrm{Env}_k(\chi)$, and

$$m := m_1 = \dots = m_r \,.$$

Since the variety $R_{Z/k}(\mathrm{SB}(A, m))$ is absolutely irreducible, k is algebraically closed in K. Lemma 2.1(c) now tells us that $\chi_1, \dots, \chi_r$ are conjugate over K. By Lemma 2.4(c) there exists an irreducible K-representation ρ whose character is $m_K(\chi_1)(\chi_1 + \dots + \chi_r)$. It remains to show that $m_K(\chi_1) = m$. Indeed,

$$m_K(\chi_1) = \mathrm{ind}(\mathrm{Env}_K(\chi)) = \mathrm{ind}(\mathrm{Env}_k(\chi) \otimes_k K) = m \,.$$

Here the first equality follows from Lemma 2.4(d), the second from Lemma 2.3, and the third from Proposition 5.1(b). □

We will say that a character $\chi\colon G \to k$ is *irreducible over* k if it satisfies the equivalent conditions of Lemma 7.3.

8. The essential dimension of a character

In this section we will assume that $\mathrm{char}(k) = 0$ and consider subfunctors

$$\mathbf{Rep}_\chi : \mathbf{Fields}_k \to \mathbf{Sets}$$

of $\mathbf{Rep}_{G,k}$ given by

$$K \quad \mapsto \quad \{K\text{-isomorphism classes of representations } \rho\colon G \to \mathrm{GL}_n(K) \text{ with character } \chi\}$$

for every field K/k. Here $\chi\colon G \to k$ is a fixed character and $n = \chi(1_G)$. The assumption that χ takes values in k is natural in view of Lemma 6.3, and the assumption that $\mathrm{char}(k) = 0$ in view of Remark 6.5. Since any two K-representations with the same character are equivalent, $\mathbf{Rep}_\chi(K)$ is either empty or has exactly one element. We will say that χ *can be realized over* K/k if $\mathbf{Rep}_\chi(K) \neq \emptyset$. In particular, $\mathbf{Rep}_\chi$ and $\mathbf{Rep}_{\chi'}$ are isomorphic if and only if χ and χ' can be realized over the same fields K/k.

Definition 8.1. Let $\chi\colon G \to k$ be a character of a finite group G and p be a prime integer. We will refer to the essential dimension of $\mathbf{Rep}_\chi$ as the *essential dimension of* χ and will denote this number by $\mathrm{ed}(\chi)$. Similarly for the essential p-dimension:

$$\mathrm{ed}(\chi) := \mathrm{ed}(\mathbf{Rep}_\chi) \text{ and } \mathrm{ed}_p(\chi) := \mathrm{ed}_p(\mathbf{Rep}_\chi).$$

We will say that characters χ and λ of G, are *disjoint* if they have no common absolutely irreducible components.

Lemma 8.2. *(a) If the characters $\chi, \lambda\colon G \to k$ are disjoint then $\mathbf{Rep}_{\chi+\lambda} \simeq \mathbf{Rep}_\chi \times \mathbf{Rep}_\lambda$.*

(b) Suppose a character $\chi\colon G \to k$ decomposes as $\sum_{i=1}^s m_i\chi_i$, as in (7.2). Set $\chi' := \sum_{i=1}^s m'_i\chi_i$, where m'_i is the greatest common divisor of m_i and $m_k(\chi_i)$. Then $\mathbf{Rep}_\chi \simeq \mathbf{Rep}_{\chi'}$.

Proof. Let K be a field extension of k.

(a) By Corollary 2.6, $\chi + \lambda$ can be realized over K if and only if both χ and λ can be realized over K.

(b) By Corollary 2.6

(i) χ can be realized over K if and only if

(ii) $m_K(\chi_i)$ divides m_i, for every $i = 1, \dots, s$.

By Lemma 2.4(e), $m_K(\chi_i)$ divides $m_k(\chi_i)$. Thus (ii) is equivalent to

(iii) $m_K(\chi_i)$ divides m'_i, for every $i = 1, \dots, s$.

Applying Corollary 2.6 one more time, we see that (iii) is equivalent to

(iv) χ' can be realized over K.

In summary, χ can be realized over K if and only if χ' can be realized over K, as desired. □

Remark 8.3. Note that the character χ' in Lemma 8.2(b) is a sum of pairwise disjoint k-irreducible characters (see the discussion of k-irreducible characters at the end of Section 7). In other words, we can replace any character $\chi\colon G \to k$ by a sum of pairwise disjoint k-irreducible characters without changing the functor $\mathbf{Rep}_\chi$.

As we observed above, $\mathbf{Rep}_\chi(K)$ has at most one element for every field K/k. In other words, $\mathbf{Rep}_\chi$ is a *detection functor* in the sense of [24] or [30, Section 4a]. We saw in Section 3b that to every algebraic variety X defined over k, we can associate the detection functor $\mathcal{D}_X$, where $\mathcal{D}_X(K)$ is either empty or has exactly one element, depending on whether or not X has a K-point. Given a character $\chi\colon G \to k$, it is thus natural to ask if there exists a smooth projective k-variety X_χ such that the functors $\mathbf{Rep}_\chi$ and $\mathcal{D}_{X_\chi}$ are isomorphic. The rest of this section will be devoted to showing that this is, indeed, always the case. We begin by defining X_χ.

Definition 8.4. (a) Let G be a finite group and $\chi := m(\chi_1 + \cdots + \chi_r)\colon G \to k$ be an irreducible character of G, where $\chi_1, \dots, \chi_r$ are $\mathrm{Gal}(k(\chi_1)/k)$-conjugate absolutely irreducible characters, and $m \geqslant 1$ divides $m_k(\chi_1) = \cdots = m_k(\chi_r)$. We define the k-variety X_χ as the Weil transfer $R_{Z/k}(\mathrm{SB}(A_\chi, m))$, where Z is the center and A_χ is the underlying division algebra of $\mathrm{Env}_k(\chi)$.

(b) More generally, suppose $\chi := \lambda_1 + \cdots + \lambda_s$, where $\lambda_1, \dots, \lambda_s\colon G \to k$ are pairwise disjoint and irreducible over k. Then we define $X_\chi := X_{\lambda_1} \times_k \cdots \times_k X_{\lambda_r}$, where each X_{λ_i} is a Weil transfer of a generalized Severi-Brauer variety, as in part (a).

Theorem 8.5. *Let G be a finite group and $\chi := \lambda_1 + \cdots + \lambda_s$ be a character, where*

$$\lambda_1, \dots, \lambda_s\colon G \to k$$

are pairwise disjoint and irreducible over k. Let X_χ be the k-variety, as in Definition 8.4. Then the functors $\mathbf{Rep}_\chi$ and $\mathcal{D}_{X_\chi}$ are isomorphic. Consequently $\mathrm{ed}(\chi) = \mathrm{cd}(X_\chi)$ and $\mathrm{ed}_p(\chi) = \mathrm{cd}_p(X_\chi)$ for any prime p.

Proof. In view of Lemma 8.2(a) we may assume that χ is irreducible over k, i.e., $s = 1$ and $\chi = \lambda_1$. Write $\chi := m(\chi_1 + \cdots + \chi_r)$, where $\chi_1, \dots, \chi_r\colon G \to \overline{k}$ are the absolutely irreducible components of χ. Let K/k be a field extension. By Corollary 2.6 the following conditions are equivalent:

(i) $\mathbf{Rep}_\chi(K) \neq \emptyset$, i.e., χ can be realized over K,

(ii) $m_K(\chi_j)$ divides m for $j = 1, \dots, r$.

Note that while the characters $\chi_1, \dots, \chi_r$ are conjugate over k, they may not be conjugate over K. Denote the orbits of the $\mathrm{Gal}(\overline{K}/K)$-action on $\chi_1, \dots, \chi_r$ by $\mathcal{O}_1, \dots, \mathcal{O}_t$, and set $\mu_i := \sum_{\chi_j \in \mathcal{O}_i} \chi_j$, so that $\chi = m(\mu_1 + \cdots + \mu_t)$.

Denote the center of the central simple algebra $\mathrm{Env}_k(\chi)$ by Z. Write $K_Z := K \otimes_k Z$ as a direct product $K_1 \times \cdots \times K_s$, where $K_1/Z, \dots, K_s/Z$ are field extensions, as in Proposition 5.1. By Lemma 2.3,

(8.6)

$$\mathrm{Env}_K(\chi) \simeq \mathrm{Env}_k(\chi) \otimes_k K \simeq \mathrm{Env}_k(\chi) \otimes_Z K_Z \simeq (\mathrm{Env}_K(\chi) \otimes_Z K_1) \times \cdots \times (\mathrm{Env}_K(\chi) \otimes_Z K_s)\,,$$

where $\simeq$ denotes isomorphism of K-algebras. On the other hand, since $\mu_1, \dots, \mu_t$ are K-valued characters,

$$\mathrm{Env}_K(\chi) \simeq \mathrm{Env}_K(m\mu_1) \times \dots \times \mathrm{Env}_K(m\mu_t)\,. \tag{8.7}$$

Suppose $\chi_j \in \mathcal{O}_i$. Then by Lemma 2.2 $\mathrm{Env}_K(m\mu_i) \simeq \mathrm{Env}_K(\mu_i) \simeq \mathrm{Env}_K(m_K(\chi_j)\mu_i)$, and by Lemma 2.4(d), $\mathrm{Env}_K(m_K(\chi_j)\mu_i)$ is a central simple algebra of index $m_K(\chi_j)$. Comparing (8.6) and (8.7), we conclude that $s = t$, and after renumbering $K_1, \dots, K_s$, we may assume that $\mathrm{Env}_K(m\mu_i) \simeq \mathrm{Env}_K(\chi) \otimes_Z K_i$. Thus (ii) is equivalent to

(iii) the index of $\mathrm{Env}_K(\chi) \otimes_Z K_i$ divides m for every $i = 1, \dots, s$.

By Proposition 5.1(a), (iii) is equivalent to

(iv) X_χ has a K-point, i.e., $\mathcal{D}_{X_\chi}(K) \neq \emptyset$.

The equivalence of (i) and (iv) shows that the functors $\mathbf{Rep}_\chi$ and $\mathcal{D}_{X_\chi}$ are isomorphic. Now

$$\mathrm{ed}(\chi) \stackrel{\mathrm{def}}{=} \mathrm{ed}(\mathbf{Rep}_\chi) = \mathrm{ed}(\mathcal{D}_{X_\chi}) \stackrel{\mathrm{def}}{=} \mathrm{cd}(X_\chi)$$

and similarly for the essential dimension at p. □

Remark 8.8. Theorem 8.5 can, in fact, be applied to an arbitrary k-valued character $\chi\colon G \to k$. Indeed, the character χ' of Lemma 8.2(b) is a sum of pairwise disjoint k-irreducible characters; see Remark 8.3. Thus $\mathbf{Rep}_\chi \simeq \mathbf{Rep}_{\chi'}$ by Lemma 8.2, and $\mathbf{Rep}_{\chi'} \simeq \mathcal{D}_{X_{\chi'}}$ by Theorem 8.5.

9. Upper bounds

If G is generated by r elements $g_1, \dots, g_r$, then any representation $\rho\colon G \to \mathrm{GL}_n(K)$ defined over a field K/k descends to the subfield K_0 generated over k by the rn^2 matrix entries of $\rho(g_1), \dots, \rho(g_r)$. Thus

$$\mathrm{ed}(\rho) \leqslant \mathrm{trdeg}_k(K_0) \leqslant rn^2\,.$$

In this section we will improve on this naive upper bound, under the assumption that $\mathrm{char}(k) = 0$.

Our starting point is the following inequality, which is an immediate corollary of Theorem 8.5 and the inequality (3.2).

Corollary 9.1. *Let G be a finite group and $\chi = m(\chi_1 + \dots + \chi_r)\colon G \to k$ be an irreducible character over k, as in Section 7. Then* $\mathrm{ed}(\chi) \leqslant \dim(X_\chi) = rm(m_k(\chi_1) - m)$. □

We are now in a position to prove the main result of this section.

Proposition 9.2. *Let G be a finite group, k be a field of characteristic 0, and K/k be a field extension. Let $\rho\colon G \to \mathrm{GL}_n(K)$ be a representation of G. Then*

(a) $\mathrm{ed}(\rho) \leqslant \dfrac{n^2}{4}$.

(b) $\mathrm{ed}(\rho) \leqslant \sum_\lambda \lfloor \dfrac{m_k(\lambda)^2}{4} \rfloor \leqslant \dfrac{|G|}{4}$. *Here the sum is taken over the distinct absolutely irreducible $\overline{K}$-subrepresentations λ of ρ, and $\lfloor x \rfloor$ denotes the integer part of x.*

(c) $\operatorname{ed}(\mathbf{Rep}_\chi) \leqslant \dfrac{\chi(1)^2}{4}$ *and* $\operatorname{ed}(\mathbf{Rep}_{G,k}) \leqslant \sum_\lambda \lfloor \dfrac{m_k(\lambda)^2}{4} \rfloor \leqslant \dfrac{|G|}{4}$ *for any base field* k *and any* k*-valued character* $\chi\colon G \to k$*. Here* $\mathbf{Rep}_{G,k}$ *is the functor defined at the beginning of Section 6, and the sum is taken over all absolutely irreducible representations* λ *of* G *defined over* $\overline{k}$*.*

Proof. (a) Suppose $\rho \simeq \rho_1 \oplus \rho_2$ over K, where $\dim(\rho_1) = n_1$, $\dim(\rho_2) = n_2$ and $n = n_1 + n_2$. If we can prove the inequality of part (a) for ρ_1 and ρ_2, then by Lemma 6.2,

$$\operatorname{ed}(\rho) \leqslant \operatorname{ed}(\rho_1) + \operatorname{ed}(\rho_2) \leqslant \frac{n_1^2}{4} + \frac{n_2^2}{4} \leqslant \frac{n^2}{4}$$

so that the desired inequality holds for ρ. Thus we may assume without loss of generality that ρ is K-irreducible.

By Lemma 6.3 we may also assume that the character χ of ρ is k-valued. By Lemma 7.3, χ is an irreducible character over k. Write $\chi = m(\chi_1 + \cdots + \chi_r)$, where $m \geqslant 1$ divides $m_k(\chi_1) = \cdots = m_k(\chi_r)$. By Corollary 9.1

$$\operatorname{ed}(\rho) \leqslant rm(m_k(\chi_1) - m) \leqslant r\frac{m_k(\chi_1)^2}{4}\,. \tag{9.3}$$

Now recall that by Lemma 2.4(d), $\operatorname{Env}_k(\rho)$ is a central simple algebra of index $m_k(\chi_1)$ over a field Z such that $[Z : k] = r$. Thus

$$rm_k(\chi_1)^2 \leqslant r\dim_Z(\operatorname{Env}_k(\rho)) = \dim_k(\operatorname{Env}_k(\rho)) = \dim_K(\operatorname{Env}_K(\rho)) \leqslant n^2\,. \tag{9.4}$$

Here the equality $\dim_k(\operatorname{Env}_k(\rho)) = \dim_K(\operatorname{Env}_K(\rho))$ follows from Lemma 2.3, and the inequality $\dim_K(\operatorname{Env}_K(\rho)) \leqslant n^2$ follows from the fact that $\operatorname{Env}_K(\rho)$ is a K-subalgebra of $\mathrm{M}_n(K)$. Combining (9.3) and (9.4), we obtain $\operatorname{ed}(\rho) \leqslant n^2/4$.

(b) Decompose ρ as a direct sum $a_1\rho_1 \oplus \cdots \oplus a_s\rho_s$, where $\rho_1, \dots, \rho_s$ are pairwise non-isomorphic K-irreducibles. Over $\overline{K}$, we can further decompose each ρ_i as

$$\rho_i \simeq m_i(\rho_{i1} \oplus \cdots \oplus \rho_{ir_i})\,, \tag{9.5}$$

where the $\rho_{i1}, \dots, \rho_{ir_i}$ are pairwise non-isomorphic $\overline{K}$-irreducibles. In fact, by Lemma 2.4(c), no two irreducible representations ρ_{ij} can be isomorphic over $\overline{K}$, as i ranges from 1 to s and j ranges from 1 to r_i.

Now let us sharpen (9.3) a bit. Since $m(m_k(\chi_{i1}) - m) \leqslant \dfrac{m_k(\chi_{i1})^2}{4}$ and $m(m_k(\chi_{i1}) - m)$ is an integer, we conclude that

$$\operatorname{ed}(\rho_i) \leqslant r_i \lfloor \frac{m_k(\chi_{i1})^2}{4} \rfloor = \sum_{i=1}^{r_i} \lfloor \frac{m_k(\chi_{ij})^2}{4} \rfloor\,.$$

Here the last equality follows from the fact that the characters $\chi_{i1}, \dots, \chi_{ir_i}$ of $\rho_{i1}, \dots, \rho_{ir_i}$ are conjugate over k, and consequently, $m_k(\rho_{i1}) = \cdots = m_k(\rho_{ir_i})$. Now by Lemma 6.2,

$$\operatorname{ed}(\rho) \;\leqslant\; \sum_{i=1}^{s} \operatorname{ed}(\rho_i) \;\leqslant\; \sum_{i=1}^{s}\sum_{j=1}^{r_i} \lfloor \frac{m_k(\chi_{ij})^2}{4} \rfloor\,.$$

This proves the first inequality in part (b).

To prove the second inequality, note that by Lemma 2.4(f), $m_k(\chi_{ij}) \leqslant \dim(\rho_{ij})$. Moreover, $\sum_\lambda \dim(\lambda)^2 = |G|$, where the sum is taken over the distinct absolutely irreducible representations λ of G; see, e.g., [34, Corollary 2(a), Section 2.4]. Thus

$$\sum_{i=1}^{s}\sum_{j=1}^{r_i}\lfloor\frac{m_k(\chi_{ij})^2}{4}\rfloor \leqslant \sum_{i=1}^{s}\sum_{j=1}^{r_i}\frac{m_k(\chi_{ij})^2}{4} \leqslant \sum_{i=1}^{s}\sum_{j=1}^{r_i}\frac{\dim(\rho_{ij})^2}{4} \leqslant \frac{|G|}{4}\,.$$

This completes the proof of part (b). Part (c) is an immediate consequence of (a) and (b). □

Remark 9.6. Note that absolutely irreducible representations λ of Schur index 1 do not contribute anything to the sum $\sum_\lambda\lfloor\frac{m_k(\lambda)^2}{4}\rfloor$ in part (b) and (c). In particular, in the case, where every absolutely irreducible representation of G has Schur index 1, we recover Lemma 6.4 from Proposition 9.2 (under the assumption that $\mathrm{char}(k) = 0$).

Another interesting example is obtained by setting $G = Q_8$, the quaternion group of order 8 and $k = \mathbb{Q}$ or $\mathbb{R}$. In this case G has five absolutely irreducible representations whose Schur indices are 1, 1, 1, 1 and 2; see [14, Example, p. 740]. Thus Proposition 9.2 yields

$$\mathrm{ed}(\mathbf{Rep}_{Q_8,k}) \leqslant \lfloor\frac{1^2}{4}\rfloor + \lfloor\frac{1^2}{4}\rfloor + \lfloor\frac{1^2}{4}\rfloor + \lfloor\frac{1^2}{4}\rfloor + \lfloor\frac{2^2}{4}\rfloor = 1\,.$$

Example 6.1 shows that this upper bound is sharp, i.e., $\mathrm{ed}(\mathbf{Rep}_{Q_8,k}) = 1$.

10. A variant of a theorem of Brauer

A theorem of R. Brauer [7] asserts for every integer $l \geqslant 1$ there exists a number field k, a finite group G and a k-valued absolutely irreducible character χ such that the Schur index $m_k(\chi) = l$. For an alternative proofs of Brauer's theorem, see [4] or [36].

In this section we will prove an analogous statement with the Schur index replaced by the essential dimension. Note however, that the analogy is not perfect. Our character χ will be reducible and $\mathbb{Q}$-valued for every $l \geqslant 2$, while Brauer's theorem will fail if we insist that k should be the same for all l, or that χ should be real-valued. (These assertions follow from the Benard-Schacher theorem [37, Theorem 6.8]; see also [14, Section 74C].)

Proposition 10.1. *For every integer $l \geqslant 0$ there exists a finite group G, and a character $\chi\colon G \to \mathbb{Q}$ such that $\mathrm{ed}_{\mathbb{Q}}(\chi) = l$.*

Proof. The proposition is obvious for $l = 0$; just take χ to be the trivial character, for any group G. We may thus assume that $l \geqslant 1$. Choose l distinct prime integers $p_1, \dots, p_l \equiv 3 \pmod 4$, and let A_i be the quaternion algebra $(-1, p_i)$ over $\mathbb{Q}$.

Lemma 10.2. *The classes of $A_1, \dots, A_l$ in $\mathrm{Br}(\mathbb{Q})$ are linearly independent over $\mathbb{Z}/2\mathbb{Z}$.*

Proof. Assume the contrary. Then after renumbering $A_1, \dots, A_l$, we may assume that $A_1 \otimes_k \dots \otimes_k A_s$ is split over $\mathbb{Q}$ for some $s \geqslant 1$. Since $[(a, c)] \otimes [(b, c)] = [(ab, c)]$ in $\mathrm{Br}(\mathbb{Q})$, we see that the quaternion algebra $(-1, p_1 \dots p_s)$ is split over $\mathbb{Q}$. Equivalently, $p_1 \dots p_s$ is a norm in $\mathbb{Q}(\sqrt{-1})/k$ (see, e.g., [27, Theorem 2.7]), i.e., $p_1 \dots p_s$ can be written as a sum of two rational squares. Now recall that by a classical theorem of Fermat, a positive integer n can be written as a sum of two rational squares if and only if it can be written

as a sum of two integer squares if and only if every prime p which is $\equiv 3 \pmod 4$ occurs to an even power in the prime decomposition of n. In our case $n = p_1 \dots p_s$ does not satisfy this condition. Hence, $p_1 \dots p_s$ cannot be written as a sum of two rational squares, a contradiction. □

We now return to the proof of Proposition 10.1. By a theorem of M. Benard [1] there exist finite groups $G_1, \dots, G_l$, number fields $F_1, \dots, F_l$, and 2-dimensional absolutely irreducible representations $\rho_i \colon G_i \to \mathrm{GL}_2(F_i)$ such that $A_i := \mathrm{Env}_k(\rho_i)$. (In fact, since $\mathbb{Q}(\sqrt{-1})$ splits every A_i, we may take $F_1 = \dots = F_l = \mathbb{Q}(\sqrt{-1})$.) We will view each ρ_i as a representation of $G = G_1 \times \dots \times G_l$ via the natural projection $G \to G_i$. Let χ_i be the character of ρ_i and $\chi := \chi_1 + \dots + \chi_r \colon G \to \mathbb{Q}$. By Theorem 8.5

$$\mathrm{ed}(\chi) = \mathrm{cd}(X_\chi)\,,$$

where $X_\chi := X_{\chi_1} \times_k \dots \times_k X_{\chi_l}$, and X_{χ_i} is the 1-dimensional Severi-Brauer variety $\mathrm{SB}(A_i)$ over $\mathbb{Q}$. Since the Brauer classes of $A_1, \dots, A_l$ in $\mathrm{Br}(\mathbb{Q})$ are linearly independent over $\mathbb{Z}/2\mathbb{Z}$, [25, Theorem 2.1] tells us that $\mathrm{cd}(X_\chi) = l$, as desired. (For an alternative proof of [25, Theorem 2.1], see [23, Corollary 4.1 and Remark 4.2].) □

Remark 10.3. Proposition 10.1 implies that there exists a field $K/\mathbb{Q}$ and a linear representation $\rho \colon G \to \mathrm{GL}_{2l}(K)$ such that $\mathrm{ed}_{\mathbb{Q}}(\rho) = l$. Note however, that ρ is not the same as $\rho_1 \times \dots \times \rho_l \colon G \to \mathrm{GL}_{2l}(\mathbb{Q}(\sqrt{-1}))$, even though ρ and $\rho_1 \times \dots \times \rho_l$ have the same character. Indeed, since each ρ_i is defined over $\mathbb{Q}(\sqrt{-1})$, $\mathrm{ed}_{\mathbb{Q}}(\rho_1 \times \dots \times \rho_l) = 0$. Under the isomorphism of functors $\mathbf{Rep}_\chi \simeq \mathcal{D}_{X_\chi}$ of Theorem 8.5, $\rho_1 \times \dots \times \rho_l$ corresponds to a $\mathbb{Q}(\sqrt{-1})$-point of X_χ, while ρ corresponds to the generic point.

11. Computation of canonical p-dimension

This section aims to determine canonical p-dimension of a broad class of Weil transfers of generalized Severi-Brauer varieties. Here p is a fixed prime integer. The base field k is allowed to be of arbitrary characteristic.

Let Z/k be a finite Galois field extension (not necessarily abelian). We will work with Chow motives with coefficients in a finite field of p elements; see [15, §64]. For a motive M over Z, $R_{Z/k}M$ is the motive over k given by the Weil transfer of M introduced in [17]. Although the coefficient ring is assumed to be $\mathbb{Z}$ in [17], and the results obtained there over $\mathbb{Z}$ do not formally imply similar results for other coefficients, the proofs go through for an arbitrary coefficient ring.

For any finite separable field extension K/k and a motive M over K, the *corestriction* of M is a well-defined motive over k; see [19].

Lemma 11.1. *Let Z/k be an arbitrary finite Galois field extension and let $M_1, \dots, M_m$ be $m \geqslant 1$ motives over Z. Then the motive $R_{Z/k}(M_1 \oplus \dots \oplus M_m)$ decomposes in a direct sum*

$$R_{Z/k}(M_1 \oplus \dots \oplus M_m) \simeq R_{Z/k}M_1 \oplus \dots \oplus R_{Z/k}M_m \oplus N,$$

where N is a direct sum of corestrictions to k of motives over fields K with $k \subsetneq K \subset Z$.

Proof. For $m = 1$ the statement is void. For $m = 2$ use the same argument as in [20, Proof of Lemma 2.1] or see below. For $m \geqslant 3$ argue by induction.

For the reader's convenience, we supply a proof for $m = 2$. First we recall that the Weil transfer $R_{Z/k}X$ of a Z-variety X is characterized by the property that there exists an isomorphism of Z-varieties $(R_{Z/k}X)_Z \simeq \prod_{\sigma\in\mathrm{Gal}(Z/k)} {}^\sigma X$ commuting with the action of the Galois group. Here ${}^\sigma X$ is the conjugate variety and $\mathrm{Gal}(Z/k)$ acts on the product $\prod {}^\sigma X$ by permutation of the factors.

We start with the case where M_1 and M_2 are the motives of some smooth projective Z-varieties X and Y. The Weil transfer $R_{Z/k}(M_1 \oplus M_2)$ is then the motive of the k-variety $R_{Z/k}(X \coprod Y)$. We have

$$\prod {}^\sigma(X \coprod Y) = \prod({}^\sigma X \coprod {}^\sigma Y) = (\prod {}^\sigma X) \coprod (\prod {}^\sigma Y) \coprod \dots,$$

where the dots stand for a disjoint union of products none of which is stable under the action of $\mathrm{Gal}(Z/k)$. It follows that $R_{Z/k}(X \coprod Y)$ is a disjoint union of $R_{Z/k}X$, $R_{Z/k}Y$, and corestrictions of some K-varieties with some $k \subsetneq K \subset Z$. This gives the required motivic formula in the particular case under consideration.

In the general case, we have $M_1 = (X, [\pi])$ and $M_2 = (Y, [\tau])$ for some algebraic cycles π and τ ($[\pi]$ and $[\tau]$ are their classes modulo rational equivalence). We recall that the Weil transfer of the motive $(X, [\pi])$ is defined as $(R_{Z/k}X, [R_{Z/k}\pi])$, where $R_{Z/k}\pi$ is the algebraic cycle determined by $(R_{Z/k}\pi)_Z = \prod {}^\sigma\pi$. Computing $R_{Z/k}(M_1 \oplus M_2)$ this way, we get the desired formula. □

Now recall from Section 3b that a k-variety X is called incompressible if $\mathrm{cd}(X) = \dim(X)$ and p-incompressible if $\mathrm{cd}_p(X) = \dim(X)$.

Theorem 11.2. *Let p be a prime number, Z/k a finite Galois field extension of degree p^r for some $r \geqslant 0$, D a balanced central division Z-algebra of degree p^n for some $n \geqslant 0$, and X the generalized Severi-Brauer variety $\mathrm{SB}(D, p^i)$ of D for some $i = 0, 1, \dots, n$. Then the k-variety $R_{Z/k}X$, given by the Weil transfer of X, is p-incompressible.*

Note that in the case, where Z/k is a quadratic Galois extension, D is balanced if the k-algebra given by the norm of D is Brauer-trivial; ${}^\alpha D$ for $\alpha \neq 1$ is then opposite to D. In this special case Theorem 11.2 was proved in [20, Theorem 1.1].

Proof of Theorem 11.2. In the proof we will use Chow motives with coefficients in a finite field of p elements. Therefore the Krull-Schmidt principle holds for direct summands of motives of projective homogeneous varieties by [12] (see also [22]).

We will prove Theorem 11.2 by induction on $r + n$. The base case, where $r + n = 0$, is trivial. Moreover, in the case where $r = 0$ (and n is arbitrary), we have $Z = k$ and thus $R_{Z/k}X = X$ is p-incompressible by [22, Theorem 4.3]. Thus we may assume that $r \geqslant 1$ from now on.

If $i = n$, then $X = \mathrm{Spec}\, Z$, $R_{Z/k}X = \mathrm{Spec}\, k$, and the statement of Theorem 11.2 is trivial. We will thus assume that $i \leqslant n - 1$ and, in particular, that $n \geqslant 1$.

Let k' be the function field of the variety $R_{Z/k}\,\mathrm{SB}(D, p^{n-1})$. Set $Z' := k' \otimes_k Z$. By Proposition 5.1(b), the index of the central simple Z'-algebra $D_{Z'} = D \otimes_Z Z' = D \otimes_k k'$ is p^{n-1}. Thus there exists a central division Z'-algebra D' such that the algebra of $(p \times p)$-matrices over D' is isomorphic to $D_{Z'}$. Let $X' = \mathrm{SB}(D', p^i)$. By [16, Theorem 10.9 and Corollary 10.19] (see also [11]), the motive of the variety $X_{Z'}$ decomposes in a direct sum

$$M(X_{Z'}) \simeq M(X') \oplus M(X')(p^{i+n-1}) \oplus M(X')(2p^{i+n-1}) \oplus \dots \oplus M(X')((p-1)p^{i+n-1}) \oplus N,$$

where N is a direct sum of shifts of motives of certain projective homogeneous Z'-varieties Y under the direct product of p copies of $\mathrm{PGL}_1(D')$ such that the index of $D'_{Z'(Y)}$ divides p^{i-1}. (If $i=0$, then $N=0$.) It follows by [22, Theorems 3.8 and 4.3] that

$$M(X_{Z'}) \simeq U(X') \oplus U(X')(p^{i+n-1}) \oplus U(X')(2p^{i+n-1}) \oplus \cdots \oplus U(X')((p-1)p^{i+n-1}) \oplus N,$$

where $U(X')$ is the upper motive of X' and N is now a direct sum of shifts of upper motives of the varieties $\mathrm{SB}(D', p^j)$ with $j < i$. Therefore, by Lemma 11.1 and [17, Theorem 5.4], the motive of the variety $(R_{Z/k}X)_{k'} \simeq R_{Z'/k'}(X_{Z'})$ decomposes in a direct sum

$$\begin{aligned} M(R_{Z/k}X)_{k'} \simeq{}& R_{Z'/k'}U(X') \oplus R_{Z'/k'}U(X')(p^{r+i+n-1}) \oplus \\ & R_{Z'/k'}U(X')(2p^{r+i+n-1}) \oplus \cdots \oplus R_{Z'/k'}U(X')((p-1)p^{r+i+n-1}) \oplus N \oplus N', \end{aligned} \tag{11.3}$$

where now N is a direct sum of shifts of $R_{Z'/k'}U(\mathrm{SB}(D', p^j))$ with $j < i$, and N' is a direct sum of corestrictions of motives over fields K with $k' \subsetneq K \subset Z'$. By the induction hypothesis, the variety $R_{Z'/k'}X'$ is p-incompressible. By [18, Theorem 5.1], this means that no positive shift of the motive $U(R_{Z'/k'}X')$ is a direct summand of the motive of $R_{Z'/k'}X'$. It follows by [19] that $R_{Z'/k'}U(X')$ is a direct sum of $U(R_{Z'/k'}X')$, of shifts of $U(R_{Z'/k'}\mathrm{SB}(D', p^j))$ with $j < i$, and of corestrictions of motives over fields K with $k' \subsetneq K \subset Z'$. Therefore we may exchange $R_{Z'/k'}$ with U in (11.3) and get a decomposition of the form

$$\begin{aligned} M(R_{Z/k}X)_{k'} \simeq{}& U(R_{Z'/k'}X') \oplus U(R_{Z'/k'}X')(p^{r+i+n-1}) \oplus \\ & U(R_{Z'/k'}X')(2p^{r+i+n-1}) \oplus \cdots \oplus U(R_{Z'/k'}X')((p-1)p^{r+i+n-1}) \oplus N \oplus N', \end{aligned} \tag{11.4}$$

where N is now a direct sum of shifts of some $U(R_{Z'/k'}\mathrm{SB}(D', p^j))$ with $j < i$, and N' is a direct sum of corestrictions of motives over fields K with $k' \subsetneq K \subset Z'$. Note that the first p summands of decomposition (11.4) (that is, all but the last two) are shifts of an indecomposable motive; moreover, no shift of this motive is isomorphic to a summand of N or of N'. Since the variety $R_{Z'/k'}X'$ is p-incompressible, we have

$$\dim U(R_{Z'/k'}X') = \dim R_{Z'/k'}X' = [Z' : k'] \cdot \dim X' = p^r \cdot p^i(p^{n-1} - p^i)\,.$$

(We refer the reader to [18, Theorem 5.1] for the definition of the dimension of the upper motive, as well as its relationship to the dimension and p-incompressibility of the corresponding variety). Note that the shifting number of the p-th summand in (11.4) plus $\dim R_{Z'/k}X'$ equals $\dim R_{Z/k}X$:

$$(p-1)p^{r+i+n-1} + p^r p^i(p^{n-1} - p^i) = p^r p^i(p^n - p^i).$$

We want to show that the variety $R_{Z/k}X$ is p-incompressible. In other words, we want to show that $\dim U(R_{Z/k}X) = \dim R_{Z/k}X$. Let l be the number of shifts of $U(R_{Z'/k'}X')$ contained in the complete decomposition of the motive $U(R_{Z/k}X)_{k'}$. Clearly, $1 \leqslant l \leqslant p$ and it suffices to show that $l = p$ because in this case the p-th summand of (11.4) is contained in the complete decomposition of $U(R_{Z/k}X)_{k'}$.

The complete motivic decomposition of $R_{Z/k}X$ contains several shifts of $U(R_{Z/k}X)$. Let N be any of the remaining (indecomposable) summands. Then, by [19], N is either a shift of the upper motive $U(R_{Z/k}\mathrm{SB}(D, p^j))$ with some $j < i$ or a corestriction to k of a motive over a field K with $k \subsetneq K \subset Z$. It follows that the complete decomposition of $N_{k'}$

does not contain any shift of $U(R_{Z'/k'}X')$. Therefore l divides p, that is, $l = 1$ or $l = p$, and we only need to show that $l \neq 1$.

We claim that $l > 1$ provided that $\dim U(R_{Z/k}X) > \dim U(R_{Z'/k'}X')$. Indeed, by [21, Proposition 2.4], the complete decomposition of $U(R_{Z/k}X)_{k'}$ contains as a summand the motive $U(R_{Z'/k'}X')$ shifted by the difference $\dim U(R_{Z/k}X) - \dim U(R_{Z'/k'}X')$. Therefore, in order to show that $l \neq 1$ it is enough to show that

$$\dim U(R_{Z/k}X) > \dim U(R_{Z'/k'}X').$$

We already know the precise value of the dimension on the right, so we only need to find a good enough lower bound on the dimension on the left. This will be given by $\dim U((R_{Z/k}X)_{\tilde{k}})$, where $\tilde{k}/k$ is a degree p Galois field subextension of Z/k. We can determine the latter dimension using the induction hypothesis.

Indeed, since $R_{Z/k}X \simeq R_{\tilde{k}/k}R_{Z/\tilde{k}}X$, the variety $(R_{Z/k}X)_{\tilde{k}}$ is isomorphic to

$$(R_{Z/k}X)_{\tilde{k}} \simeq \prod_{\tilde{\alpha}\in\tilde{\Gamma}} {}^{\tilde{\alpha}}R_{Z/\tilde{k}}X \simeq R_{Z/\tilde{k}} \prod_{\tilde{\alpha}\in\tilde{\Gamma}} {}^{\alpha}X,$$

where Γ is the Galois group of Z/k, $\tilde{\Gamma}$ is the Galois group of $\tilde{k}/k$, and $\alpha \in \Gamma$ is a representative of $\tilde{\alpha} \in \tilde{\Gamma}$ (see [6, §2.8]). Since D is balanced, the product $\prod_{\tilde{\alpha}\in\tilde{\Gamma}} {}^{\alpha}X$ is equivalent to X. It follows that the varieties $R_{Z/\tilde{k}} \prod_{\tilde{\alpha}\in\tilde{\Gamma}} {}^{\alpha}X$ and $R_{Z/\tilde{k}}X$ are equivalent and hence, by Lemma 3.3, have the same canonical p-dimension (i.e., the dimensions of their upper motives coincide). The latter variety is p-incompressible by the induction hypothesis. Consequently,

$$\dim U(R_{Z/k}X) \geqslant \dim U((R_{Z/k}X)_{\tilde{k}}) = \dim R_{Z/\tilde{k}}X = p^{r-1} \cdot p^i(p^n - p^i).$$

The lower bound $p^{r-1} \cdot p^i(p^n - p^i)$ on $\dim U(R_{Z/k}X)$ thus obtained is good enough for our purposes, because

$$p^{r-1} \cdot p^i(p^n - p^i) > p^r \cdot p^i(p^{n-1} - p^i) = \dim U(R_{Z'/k'}X').$$

This completes the proof of Theorem 11.2. □

The following example, due to A. Merkurjev, shows that Theorem 11.2 fails if D is not assumed to be balanced.

Example 11.5. Let L be a field containing a primitive 4-th root of unity. Let Z be the field $Z := L(x, y, x', y')$ of rational functions over L in four variables x, y, x', y'. Consider the degree 4 cyclic central division Z-algebras $C := (x, y)_4$ and $C' := (x', y')_4$. Let $k \subset Z$ be the subfield Z^{α} of the elements in Z fixed under the L-automorphism α of Z exchanging x with x' and y with y'. The field extension Z/k is then Galois of degree 2, and the algebra C' is conjugate to C.

The index of the tensor product of Z-algebras $C \otimes C'^{\otimes 2}$ is 8. Let D/Z be the underlying (unbalanced!) division algebra of degree 8. Since the conjugate algebra ${}^{\alpha}D$ is Brauer-equivalent to $C' \otimes C^{\otimes 2}$, the subgroup of the Brauer group $\mathrm{Br}(Z)$ generated by the classes of D and ${}^{\alpha}D$ coincides with the subgroup generated by the classes of C and ${}^{\alpha}C = C'$. Therefore the varieties $X_1 := R_{Z/k}\,\mathrm{SB}(D)$ and $X_2 := R_{Z/k}\,\mathrm{SB}(C)$ are equivalent. Thus, by Lemma 3.3,

$$\mathrm{cd}(X_1) = \mathrm{cd}(X_2) \leqslant \dim(X_2) < \dim(X_1)$$

and consequently, X_1 is compressible (and in particular, 2-compressible).

Remark 11.6. Some generalizations of Theorem 11.2 can be found in [23].

12. Some consequences of Theorem 11.2

Theorem 11.2 makes it possible to determine the canonical p-dimension of the Weil transfer in the situation, where the degrees of Z/k and of D are not necessarily p-powers.

Corollary 12.1. *Let Z/k be a finite Galois field extension and D a balanced central division Z-algebra. For any positive integer m dividing* $\deg(D)$*, one has*

$$\operatorname{cd}_p R_{Z/k}\operatorname{SB}(D,m) = \dim R_{Z/k'}\operatorname{SB}(D',m') = [Z:k']\cdot m'(\deg D' - m'),$$

where m' is the p-primary part of m (i.e., the highest power of p dividing m), D' is the p-primary component of D, and $k' = Z^{\Gamma_p}$, where Γ_p is a Sylow p-subgroup of $\Gamma := \operatorname{Gal}(Z/k)$ (so that $[Z:k']$ is the p-primary part of $[Z:k]$).

Proof. Since the degree $[k':k]$ is prime to p, we have

$$\operatorname{cd}_p R_{Z/k}\operatorname{SB}(D,m) = \operatorname{cd}_p(R_{Z/k}\operatorname{SB}(D,m))_{k'};$$

see [29, Proposition 1.5(2)]. The k'-variety $R_{Z/k}\operatorname{SB}(D,m)_{k'}$ is isomorphic to a product of $R_{Z/k'}\operatorname{SB}(D,m)$ with several varieties of the form $R_{Z/k'}\operatorname{SB}(\tilde{D},m)$ where $\tilde{D}$ ranges over a set of conjugates of D. Since D is balanced, these algebras $\tilde{D}$ are Brauer-equivalent to powers of D. Thus the product is equivalent to the k'-variety $R_{Z/k'}\operatorname{SB}(D,m)$. We conclude by Lemma 3.3 that $\operatorname{cd}_p R_{Z/k}\operatorname{SB}(D,m) = \operatorname{cd}_p R_{Z/k'}\operatorname{SB}(D,m)$. In the sequel we will replace k by k', so that the degree $[Z:k]$ becomes a power of p.

We may now replace k by its p-special closure; see [15, Proposition 101.16]. This will not change the value of $\operatorname{cd}_p(X)$. In other words, we may assume that k is p-special. Under this assumption the algebras D and D' become Brauer-equivalent and consequently, the k-varieties $R_{Z/k}\operatorname{SB}(D,m)$ and $R_{Z/k}\operatorname{SB}(D',m')$ become equivalent. By Lemma 3.3,

$$\operatorname{cd}_p R_{Z/k}\operatorname{SB}(D,m) = \operatorname{cd}_p R_{Z/k}\operatorname{SB}(D',m').$$

Since the Z-algebra D' is balanced over k, Theorem 11.2 tells us that $R_{Z/k}\operatorname{SB}(D',m')$ is p-incompressible. That is,

$$\operatorname{cd}_p R_{Z/k}\operatorname{SB}(D',m') = \dim(R_{Z/k}\operatorname{SB}(D',m')) = [Z:k]\cdot m'(\deg D' - m'),$$

and the corollary follows. □

Remark 12.2. Corollary 12.1 can be used to compute the p-canonical dimension of $R_{Z/k}\operatorname{SB}(D,j)$ for any $j = 1,\dots,\deg(D)$, even if j does not divide $\deg(D)$. Indeed, let m be the greatest common divisor of j and $\deg(D)$. Proposition 5.1(a) tells us that for any field extension K/k, $R_{Z/k}\operatorname{SB}(D,j)$ has a K-point if and only if $R_{Z/k}\operatorname{SB}(D,m)$ has a K-point. In other words, the detection functors for these two varieties are isomorphic. Consequently,

$$\operatorname{cd}(R_{Z/k}\operatorname{SB}(D,j)) = \operatorname{cd}(R_{Z/k}\operatorname{SB}(D,m)) \text{ and } \operatorname{cd}_p(R_{Z/k}\operatorname{SB}(D,j)) = \operatorname{cd}_p(R_{Z/k}\operatorname{SB}(D,m)),$$

and the value of $\operatorname{cd}_p(R_{Z/k}\operatorname{SB}(D,m))$ is given by Corollary 12.1.

We now return to the setting of Sections 7–9. In particular, G is a finite group, and the base field k is of characteristic 0.

Corollary 12.3. *Let $\chi = m(\chi_1 + \cdots + \chi_r)\colon G \to k$ be an irreducible k-valued character, where $\chi_1, \dots, \chi_r$ are absolutely irreducible and conjugate over k, and m divides $m_k(\chi_1) = \cdots = m_k(\chi_r)$, as in Section 7.*

(a) $\operatorname{ed}_p(\chi) = r'm'(m_k(\chi_1)' - m')$. Here x' denotes the p-primary part of x (i.e., the highest power of p dividing x) for any integer $x \geq 1$.

(b) If r and $m_k(\chi_1)$ are powers of p, then $\operatorname{ed}_p(\chi) = \operatorname{ed}(\chi) = \dim(X_\chi) = rm(m_k(\chi_1) - m)$. Here X_χ is as in Definition 8.4.

Proof. (a) Let D be the underlying division algebra and Z/k be the center of $\operatorname{Env}_k(\chi)$. By Theorem 8.5, $\operatorname{ed}_p(\chi) = \operatorname{cd}_p(X_\chi)$. By Proposition 4.1, D is balanced. The desired conclusion now follows from Corollary 12.1.

(b) Here $r' = r$, $m_k(\chi_1)' = m_k(\chi)$ and thus $m' = m$. By part (a),

$$\dim(X_\chi) = rm(m_k(\chi_1) - m) = \operatorname{ed}_p(\chi) \leqslant \operatorname{ed}(\chi)\,.$$

On the other hand, by Corollary 9.1, $\operatorname{ed}(\chi) \leqslant rm(m_k(\chi_1) - m)$, and part (b) follows. □

Remark 12.4. While a priori $\operatorname{ed}_p(\chi)$ depends on k, G, and χ, Corollary 12.3(a) shows that, in fact, $\operatorname{ed}_p(\chi)$ depends only on the integers r, m, and $m_k(\chi_1)$. (Here we are assuming that χ is irreducible.) We do not know if the same is true of $\operatorname{ed}(\chi)$.

13. A variant of a theorem of Schilling

Let G be a p-group and χ_1 be an absolutely irreducible character of G. It is well known that for any field k of characteristic 0, $m_k(\chi_1) = 1$ if p is odd, and $m_k(\chi_1) = 1$ or 2 if $p = 2$. Following C. Curtis and I. Reiner, we will attribute this theorem to O. Schilling; see [14, Theorem 74.15]. For further bibliographical references, see [37, Corollary 9.8].

In this section we will use Corollary 12.3 to prove the following analogous statement, with the Schur index replaced by the essential dimension.

Proposition 13.1. *Let k be a field of characteristic 0, G be a p-group, and $\chi\colon G \to k$ be an irreducible character over k.*

(a) If p is odd then $\operatorname{ed}(\chi) = 0$.

(b) If $p = 2$ then $\operatorname{ed}_2(\chi) = \operatorname{ed}(\chi) = 0$ or 2^l for some integer $l \geqslant 0$.

(c) Moreover, every $l \geqslant 0$ in part (b) can occur with $k = \mathbb{Q}$, for suitable choices of G and χ.

Proof. Write $\chi = m(\chi_1 + \cdots + \chi_r)$, where $\chi_i\colon G \to \overline{k}$ are absolutely irreducible characters and m divides $m_k(\chi_1)$. If $m = m_k(\chi_1)$ then $\operatorname{ed}(\chi) = 0$ by Corollary 9.1.

(a) In particular, this will always be the case if p is odd. Indeed, by Schilling's theorem, $m_k(\chi_1) = 1$ and thus $m = 1$. (Also cf. Lemma 6.4.)

(b) By Schilling's theorem, $m_k(\chi_1) = 1$ or 2, and by the above argument, we may assume that $m < m_k(\chi_1)$. Thus the only case we need to consider is $m_k(\chi_1) = 2$ and $m = 1$. By Lemma 2.4(b), $r = [k(\chi_1) : k]$. Since $k(\chi_1) \subset k(\zeta_e)$, where the exponent e

of G is a power of 2, we see that r divides $[k(\zeta_e):k]$, which is, once again, a power of 2. Thus we conclude that r is a power of 2. Corollary 12.3(b) now tells us that

$$\mathrm{ed}_2(\chi) = \mathrm{ed}(\chi) = rm(m_k(\chi) - m) = r \cdot 1 \cdot (2-1) = r \tag{13.2}$$

is a power of 2, as claimed.

(c) Let $s = 2^{l+2}$, and $\sigma \in \mathrm{Gal}(\mathbb{Q}(\zeta_s)/\mathbb{Q})$ be complex conjugation, and

$$F := \mathbb{Q}(\zeta_s)^\sigma = \mathbb{Q}(\zeta_s) \cap \mathbb{R} = \mathbb{Q}(\zeta_s + \zeta_s^{-1})\,.$$

Consider the quaternion algebra $A = ((\zeta_s - \zeta_s^{-1})^2, -1)$ over F, i.e., the F-algebra generated by elements x and y, subject to the relations

$$x^2 = (\zeta_s - \zeta_s^{-1})^2,\ y^2 = -1 \text{ and } xy = -yx.$$

One readily checks that $F(\zeta_s - \zeta_s^{-1}) = \mathbb{Q}(\zeta_s)$ is a maximal subfield of A, ζ_s and y generate a multiplicative subgroup G of A of order $2s$, which spans A as an F-vector space, and the inclusion $G \hookrightarrow A^\times$ gives rise to an absolutely irreducible 2-dimensional representation

$$\rho_1 \colon G \hookrightarrow A^\times \hookrightarrow \mathrm{GL}_2(\mathbb{Q}(\zeta_s))\,.$$

Denote the character of ρ_1 by $\chi_1 \colon G \to F$. We claim that $\mathbb{Q}(\chi_1) = F$. Indeed, since A is an F-algebra, the trace of every element of A lies in F, and in particular, $\mathbb{Q}(\chi_1) \subset F$. On the other hand, $\chi_1(\zeta_s) = \zeta_s + \zeta_s^{-1}$ generates F over $\mathbb{Q}$. This proves the claim. Thus χ_1 has exactly

$$r = [F:\mathbb{Q}] = \frac{1}{2}[\mathbb{Q}(\zeta_s):\mathbb{Q}] = 2^l$$

conjugates $\chi_1, \dots, \chi_r$ over $\mathbb{Q}$, and $\chi = \chi_1 + \dots + \chi_r$ is an irreducible character over $\mathbb{Q}$.

Note that since $s = 2^{l+2} \geqslant 4$, $(\zeta_s - \zeta_s^{-1})^2 < 0$, $A \otimes_F \mathbb{R}$ is $\mathbb{R}$-isomorphic to the Hamiltonian quaternion algebra $\mathbb{H} = (-1,-1)$ and hence, is non-split. Thus $\mathrm{ind}(A) = 2$. Since $A = \mathrm{Env}_{\mathbb{Q}}(\rho)$, Lemma 2.4(d) tells us that $m_{\mathbb{Q}}(\chi_1) = 2$. Applying Corollary 12.3(b), as in (13.2), we conclude that $\mathrm{ed}_2(\chi) = \mathrm{ed}(\chi) = r = 2^l$, as desired. □

14. Essential dimension of modular representations

Let G be a finite group and $\mathbf{Rep}_{G,k}$ be the functor of representations defined at the beginning of Section 6. In the non-modular setting (where $\mathrm{char}(k)$ does not divide $|G|$), we know that

$$\mathrm{ed}(\mathbf{Rep}_{G,k}) \text{ is } \begin{cases} 0, \text{ if } \mathrm{char}(k) > 0, \text{ by Remark 6.5, and} \\ \leqslant |G|/4, \text{ if } \mathrm{char}(k) = 0, \text{ by Proposition 9.2.} \end{cases}$$

We shall now see that essential dimension of representations behaves very differently in the modular case.

Theorem 14.1. *Let k be a field of characteristic p. Suppose a finite group G contains an elementary abelian subgroup $E \simeq (\mathbb{Z}/p\mathbb{Z})^2$ of rank 2. Then* $\mathrm{ed}(\mathbf{Rep}_{G,k}) = \infty$.

It is clear from the definition of essential dimension that if $k \subset k'$ is a field extension then $\mathrm{ed}(\mathbf{Rep}_{G,k}) \geqslant \mathrm{ed}(\mathbf{Rep}_{G,k'})$. Thus for the purpose of proving Theorem 14.1 we may replace k by k'. In particular, we may assume without loss of generality that k is algebraically closed.

Following D. Quillen, we will assiciate to a finite group G the projective variety $S := \mathrm{Proj}(H^\bullet(G,k))$, where the graded ring $H^\bullet(G,k)$ is defined as the full cohomology ring $H^*(G,k)$, if $p=2$, or as the direct sum of even-dimensional cohomology groups $H^{\mathrm{even}}(G,k)$ if $p \geqslant 3$. To every representation $\rho\colon G \to \mathrm{GL}_n(K)$ defined over a field K/k (or equivalently, a finitely generated $K[G]$-module), we will denote the *support variety* of ρ by $\mathrm{Supp}(\rho)$. Note that $\mathrm{Supp}(\rho)$ is a closed subvariety of S. For a detailed discussion of this construction we refer the reader to [2, Chapter 5].

Let Z be a k-variety, and $\mathbf{Sub}_Z\colon \mathbf{Fields}_k \to \mathbf{Sets}$ be a covariant functor, given by

$$\mathbf{Sub}_Z(K) := \{\text{closed subvarieties of } Z_K\}.$$

Here subvarieties of Z_K are required to be reduced but not necessarily irreducible. Closed subvarieties $X, Y \subset Z_K$ represent the same element in $\mathbf{Sub}_Z(K)$ if $X(\overline{K}) = Y(\overline{K})$ in $Z(\overline{K})$. We will now consider the morphism of functors

$$\mathrm{Supp}\colon \mathbf{Rep}_{G,k} \to \mathbf{Sub}_S$$

which associates to a representation $\rho\colon G \to \mathrm{GL}_n(K)$ its support variety $\mathrm{Supp}(\rho)$. A theorem of J. Carlson (Carlson's realization theorem) asserts that this morphism of functors is surjective; see [2, Corollary 5.9.2]. (Note that the usual statement of Carlson's realization theorem only says that $\mathrm{Supp}(k)\colon \mathbf{Rep}_{G,k}(k) \to \mathbf{Sub}_S(k)$ is surjective; however, the proof shows that, in fact, $\mathrm{Supp}\colon \mathbf{Rep}_{G,k}(K) \to \mathbf{Sub}_S(K)$ is surjective for every field K/k.) Thus $\mathrm{ed}(\mathbf{Rep}_{G,k}) \geqslant \mathrm{ed}(\mathbf{Sub}_S)$; see [3, Lemma 1.9].

By a theorem of Quillen, the condition that G contains an elementary abelian subgroup of rank $\geqslant 2$ is equivalent to $\dim(S) \geqslant 1$; see [2, Theorem 5.3.8]. It now suffices to prove the following proposition.

Proposition 14.2. *Let Z be a projective variety of dimension $d \geqslant 1$ defined over an infinite field k. Then* $\mathrm{ed}(\mathbf{Sub}_Z) = \infty$.

Proof. We claim that there exists a surjective morphism $Z \to \mathbb{P}^d$ defined over k. Indeed, embed Z into a projective space $\mathbb{P}^N$. If $d = N$, there is nothing to prove. If $d < N$, then there exists a linear subspace of dimension $N-d-1$ defined over k which does not intersect Z. Projecting Z from this subspace to a complementary linear subspace of dimension d, we obtain a desired surjective morphism $Z \to \mathbb{P}^d$. This proves the claim.

The morphism $Z \to \mathbb{P}^d$ induces a surjective morphism of functors $\mathbf{Sub}_Z \to \mathbf{Sub}_{\mathbb{P}^d}$. Using [3, Lemma 1.9] once again, we see that it suffices to show $\mathrm{ed}(\mathbf{Sub}_{\mathbb{P}^d}) = \infty$. In other words, we may assume without loss of generality that $Z = \mathbb{P}^d$.

Let L/k be a field, $a_1, \dots, a_n \in L$, and $X[n]$ be the union of the points

$$X_1 = (1 : a_1 : 0 : \dots : 0), \dots, X_n = (1 : a_n : 0 \dots : 0) \tag{14.3}$$

in $\mathbb{P}^d$. We view $X[n]$ as an element of $\mathbf{Sub}_{\mathbb{P}^d}(L)$.

Lemma 14.4. *Suppose $X[n]$ descends to a subvariety Y defined over a subfield $K \subset L$. Then a_i is algebraic over K for every $i = 1, \dots, n$.*

Proof. Note that $X[n]$ is a subvariety of the projective line $\mathbb{P}^1 \subset \mathbb{P}^d$ given by $x_3 = \dots = x_{d+1} = 0$, where $x_1, \dots, x_{d+1}$ are the projective coordinates in $\mathbb{P}^d$. Since $X[n]$ descends to Y, we have $Y(\overline{L}) = X[n](\overline{L})$. Consequently, Y is a closed subvariety of $\mathbb{P}^1$. (Note that

here we are viewing Y as a subvariety of $\mathbb{P}^d$, not as a subscheme.) Thus for the purpose of proving Lemma 14.4 we may replace $\mathbb{P}^d$ by $\mathbb{P}^1$, i.e., assume that $d = 1$.

By the definition of the functor $\mathbf{Sub}_{\mathbb{P}^1}$, $X[n]$ descends to K if $X[n]$ can be cut out (set-theoretically) by homogeneous polynomials $f_1, \dots, f_s \in K[x_1, x_2]$. In other words, the points $X_1 = (1 : a_1), \dots, X_n = (1 : a_n)$ are the only non-trivial solutions, in the algebraic closure $\overline{L}$, of a system of homogeneous equations

$$f_1(x_1, x_2) = \dots = f_s(x_1, x_2) = 0$$

with coefficients in K. Since every solution of such a system can be found over $\overline{K}$, we have $a_1, \dots, a_n \in \overline{K}$. This completes the proof of Lemma 14.4. □

We now continue with the proof of Proposition 14.2. Taking $a_1, \dots, a_n$ to be independent variables and $L := k(a_1, \dots, a_n)$, we see that $\operatorname{trdeg}_k(K) = \operatorname{trdeg}_k(L) = n$ and thus in this case $\operatorname{ed}(X[n]) = n$. Therefore,

$$\operatorname{ed}(\mathbf{Sub}_{\mathbb{P}^d,k}) \geqslant \sup_{n \geqslant 1} \operatorname{ed}(X[n]) = \infty\,.$$

This completes the proof of Proposition 14.2 and thus of Theorem 14.1. □

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Appendix: Modular representations of high essential dimension

by Julia Pevtsova[1] and Zinovy Reichstein

Let k be a field of characteristic p, G be a finite group containing a rank 2 elementary abelian subgroup $E \simeq (\mathbb{Z}/p\mathbb{Z})^2$. Theorem 14.1 asserts that for every integer n there exists a field extension K_n/k and a representation $\rho_n \colon G \to \mathrm{GL}_{d_n}(K_n)$ such that $\operatorname{ed}_k(\rho_n) \geqslant n$. However, the proof of Theorem 14.1 in Section 14 does not tell us how to construct ρ_n or what $d_n = \dim(\rho_n)$ may be in terms of n. The purpose of this appendix is to prove the following constructive version of Theorem 14.1.

Theorem A.5. *Let k be a field of characteristic p, and G be a finite group. Suppose G contains an elementary abelian subgroup $E \simeq (\mathbb{Z}/p\mathbb{Z})^2$ of rank 2, and let $W := W_G(E) = N_G(E)/C_G(E)$ be the Weyl group of E in G. Set $K_n := k(a_1, \dots, a_n)$, where $a_1, \dots, a_n$ are independent variables. Then for every integer $n \geqslant 1$ there exists a representation $\rho_n \colon G \to \mathrm{GL}_{d_n}(K_n)$ of dimension $d_n = \dim(\rho_n) \leqslant n|G||W|/p$ such that $\operatorname{ed}_k(\rho_n) = n$.*

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The approach taken in the previous section is to use the support variety of a G-representation ρ to bound $\operatorname{ed}(\rho)$ from below. Here we will first restrict ρ to E, then use the support variety of $\rho_{|E}$ to bound $\operatorname{ed}(\rho)$ from below. Support varieties for E-representations admit an alternative description as rank varieties, due to Carlson [9] (see also [2, Section 5.8]). This makes them more amenable to explicit computations. In particular, in the course of proving Theorem A.5 we will construct an explicit representation ρ_n with $\operatorname{ed}(\rho_n) \geqslant n$ and $\dim(\rho_n) \leqslant n|G||W|/p$.

We begin by noting that $H^\bullet(E,k)$ is a polynomial ring in two variables over k; hence, $\operatorname{Proj}(H^\bullet(E,k)) = \mathbb{P}^1$. For K/k a field extension, the support variety $\operatorname{Supp}(\rho)$ of a representation $\rho\colon E \to \mathrm{GL}_n(K)$ is thus a K-subvariety of $\mathbb{P}^1$. The Weyl group W of E in G naturally acts on E by conjugation; this induces a W-action on $H^\bullet(E,k)$ and thus on $\mathbb{P}^1$. If ρ can be lifted to a K-representation of G, then $\operatorname{Supp}(\rho)$ is easily seen to be invariant under the action of W on $\mathbb{P}^1_K$.

Let $\mathbf{Sub}_{\mathbb{P}^1,W}\colon \mathbf{Fields}_k \to \mathbf{Sets}$ be a functor, given by

$$\mathbf{Sub}_{\mathbb{P}^1,W}(K) := \{\text{closed } W\text{-invariant subvarieties of } \mathbb{P}^1_K\}.$$

Here subvarieties of $\mathbb{P}^1_K$ are required to be reduced but not necessarily irreducible, as in Section 14. Let

$$\operatorname{Supp}^E\colon \mathbf{Rep}_{G,k} \to \mathbf{Sub}_{\mathbb{P}^1,W}$$

be the morphism of functors which associates to a representation $\rho\colon G \to \mathrm{GL}_n(K)$ the support variety $\operatorname{Supp}(\rho_{|E}) \subset \mathbb{P}^1_K$. One can show that $\operatorname{Supp}^E\colon \mathbf{Rep}_{G,k} \to \mathbf{Sub}_{\mathbb{P}^1,W}$ is surjective, but we will not do that here. For the purpose of proving Theorem 14.1 the following variant of Carlson's realization theorem [10] for W-invariant subvarieties of $\mathbb{P}^1$ will suffice.

Proposition A.6. *Let K be an algebraically closed field extension of k. Let $X_1, \dots, X_m$ be distinct K-points of $\mathbb{P}^1$ such that their union $X = X_1 \cup \dots \cup X_m$ is W-invariant. Then there exists a $K[G]$-module M such that $\dim_K(M) = m|G|/p$ and $\operatorname{Supp}^E(M) = X$.*

Let g_1, g_2 be group generators of E. For any point $x = [x_1 : x_2]$ on $\mathbb{P}^1_K$, consider the element

$$\alpha_x = x_1(g_1 - 1) + x_2(g_2 - 1) + 1$$

in the group algebra $K[E]$. Since $\alpha_x^p = 1$, the element α_x generates a cyclic subgroup of $K[E]$, commonly referred to as the "cyclic shifted subgroup" corresponding to the point x (see [9, 2.11]). We denote by $K[\alpha_x]$ the subalgebra of $K[E]$ generated by α_x. By construction, $K[\alpha_x] \simeq K[\mathbb{Z}/p\mathbb{Z}] \simeq K[t]/(t^p)$.

Let $k \subset K \subset L$ be field extensions, and M be a $K[E]$-module. An L-point $x = [x_1 : x_2]$ of $\mathbb{P}^1$ belongs to the rank variety $\operatorname{Supp}^E(M)$ (defined over K) if and only if the restriction $(M \otimes_K L)\downarrow_{L[\alpha_x]}$ is not a free $L[\alpha_x]$-module (see [2, II.5.8]). If M is finite-dimensional and K is algebraically closed then it suffices to check the K-points $x = [x_1 : x_2] \in \mathbb{P}^1_K$ to determine the rank variety of M. We also note that by [9, Lemma 6.4] this description of the rank variety is independent of the choice of generators of E.

The following lemma is a very special case of [31, Prop. 4.1]. For the reader's convenience we supply a direct proof.

Lemma A.7. *Let K be an algebraically closed field, and let $x = [x_1 : x_2] \in \mathbb{P}^1$ be a K-point. Let M be a (finite dimensional) $K[\alpha_x]$-module. Then*

$$\mathrm{Supp}^E(\mathrm{Ind}_{K[\alpha_x]}^{K[E]} M) = \begin{cases} \emptyset, & \text{if } M \text{ is free} \\ x, & \text{otherwise,} \end{cases}$$

where $\mathrm{Ind}_{K[\alpha_x]}^{K[E]} M = K[E] \otimes_{K[\alpha_x]} M$ is the (tensor) induction of M from $K[\alpha_x]$ to $K[E]$.

Proof. Since rank varieties distribute over direct sums,

$$\mathrm{Supp}^E(M_1 \oplus M_2) = \mathrm{Supp}^E(M_1) \cup \mathrm{Supp}^E(M_2), \tag{A.8}$$

it suffices to prove the lemma for each of the p indecomposable $K[\alpha_x]$-modules.

If M is a free $K[\alpha_x]$-module, then the induced module $\mathrm{Ind}_{K[\alpha_x]}^{K[E]} M$ is free which implies that the rank variety is empty. Hence, it suffices to prove the lemma for the remaining $p-1$ indecomposable $K[\alpha_x]$-modules. After a linear substitution of generators $\{g_1 - 1, g_2 - 1\}$ of the augmentation ideal of the group algebra $K[E]$ we may assume that $x = [1 : 0]$. Call the new generators of the augmentation ideal s and t, so that $K[E] \cong K[s,t]/(s^p, t^p)$. The list of representatives of isomorphism classes of non-free indecomposable $K[s]/(s^p)$-modules is $\{K, K[s]/(s^2), \ldots, K[s]/(s^{p-1})\}$. Hence, the lemma is reduced to the following statement. Consider a truncated polynomial algebra $K[s,t]/(s^p, t^p)$ acting on

$$\mathrm{Ind}_{K[s]/(s^p)}^{K[E]} K[s]/(s^n) = K[s,t]/(s^p, t^p) \otimes_{K[s]/s^p} K[s]/(s^n) \cong K[t,s]/(t^p, s^n)\,,$$

$1 \leqslant n \leqslant p-1$, via the obvious projection map. Then the restriction of $K[t,s]/(t^p, s^n)$ to the subalgebra of $K[s,t]/(s^p, t^p)$ generated by $as + bt$ is free if and only if $b \neq 0$. Indeed, if $b \neq 0$, then $K[t,s]/(t^p, s^n) \cong K[as+bt, s]/((as+bt)^p, s^n) \cong \bigoplus_{i=0}^{n-1} s^i K[as+bt]/(as+bt)^p$ is a free $K[as+bt]/(as+bt)^p$-module. If $b = 0$, then $(as)^{p-1} = (as+bt)^{p-1}$ annihilates $K[t,s]/(t^p, s^n)$ since $n < p$. Therefore, $K[t,s]/(t^p, s^n)$ is not a free $K[as+bt]/(as+bt)^p$-module. □

Proof of Proposition A.6. We claim that $M := \mathrm{Ind}_E^G M_X$ has the desired properties, where $M_X := \bigoplus_{i=1}^{m} \mathrm{Ind}_{K[\alpha_{X_i}]}^{K[E]} K$. Clearly, $\dim(M_X) = mp$ and, thus,

$$\dim(M) = \frac{|G|}{p^2} \cdot \dim(M_X) = \frac{m|G|}{p}\,.$$

It remains to show that $\mathrm{Supp}^E(M) = X$. We will use the double coset formula

$$\mathrm{Res}_E^G \mathrm{Ind}_E^G M_X = \bigoplus_{g \in E\backslash G/E} \mathrm{Ind}_{E\cap E^g}^E \mathrm{Res}_{E\cap E^g}^{E^g} gM_X\,.$$

By (A.8) we only need to compute the variety for each summand in the double coset formula. Since M_X is a direct summand of $\mathrm{Res}_E^G \mathrm{Ind}_E^G M_X$, we have

$$X = \mathrm{Supp}^E(M_X) \subset \mathrm{Supp}^E(\mathrm{Ind}_E^G M_X) = \mathrm{Supp}^E(M)\,.$$

We need to prove the opposite inclusion, $\mathrm{Supp}^E(\mathrm{Ind}_{E\cap E^g}^E \mathrm{Res}_{E\cap E^g}^{E^g} gM_{X_i}) \subset X$, for each $M_{X_i} = \mathrm{Ind}_{K[\alpha_{X_i}]}^{K[E]} K$. Consider three cases:

(a) $E \cap E^g = E$, that is, $g \in N_G(E)$. Then the corresponding summand in the double coset formula becomes gM_X, the module M_X twisted by g. We have $\mathrm{Supp}^E(gM_X) = g\,\mathrm{Supp}^E(M_X) = gX = X$, since X is W-invariant.

(b) $E \cap E^g = \emptyset$. Then the corresponding summand is induced from the trivial group and, hence, is free and has empty rank variety.

(c) $E \cap E^g = \langle\sigma\rangle$, a cyclic subgroup of E. Then $\sigma \in E^g = gEg^{-1}$ and, hence, $g^{-1}\sigma g \in E$. If $g^{-1}\sigma g \notin \langle\sigma\rangle$, then $\{\sigma, g^{-1}\sigma g\}$ generate E which implies that $g \in N_G(E)$ and contradicts the assumption $E \cap E^g \neq E$. Therefore, $g^{-1}\sigma g \in \langle\sigma\rangle$. By Lemma A.7, $\mathrm{Supp}^E(\mathrm{Ind}_{K\langle\sigma\rangle}^{K[E]} gM_{X_i})$ contains at most one point: the point corresponding to the subgroup $\langle\sigma\rangle$. Moreover, this variety is non-empty only if gM_{X_i} is not free as $\langle\sigma\rangle$-module. By the definition of the action on the twisted module gM_{X_i}, this happens if and only if M_{X_i} is not free as $\langle g^{-1}\sigma g\rangle$-module. Since $\langle g^{-1}\sigma g\rangle = \langle\sigma\rangle$, this is equivalent to the restriction of M_{X_i} to $\langle\sigma\rangle$ not being free. Hence, $\mathrm{Supp}^E(\mathrm{Ind}_{K\langle\sigma\rangle}^{K[E]} gM_{X_i}) \subset \mathrm{Supp}^E(M_{X_i}) \subset X$, as desired. □

Proof of Theorem A.5. For $i = 1, \dots, n$, let $X_i = (1 : a_i)$ be a K_n-point of $\mathbb{P}^1$, and $Y[n]$ be the union of the W-orbits of $X_1, \dots, X_n$. We claim that $\mathrm{ed}(Y[n]) = n$, where we view $Y[n]$ as an object in $\mathbf{Sub}_{\mathbb{P}^1,W}(\overline{K}_n)$, where $\overline{K}_n$ be the algebraic closure of K_n.

Suppose $Y[n]$ descends to a subfield $k \subset F \subset \overline{K}_n$. Then by Lemma 14.4, $a_1, \dots, a_n$ are algebraic over F. In other words, $\overline{K}_n/F$ is an algebraic extension or, equivalently, $\mathrm{trdeg}_k(F) = n$. This shows that $\mathrm{ed}(Y[n]) = n$, as claimed.

By Proposition A.6, there exists a representation $\rho_n : G \to \mathrm{GL}_{d_n}(\overline{K}_n)$ with $\mathrm{Supp}^E(\rho_n) = Y[n]$. Thus $\mathrm{ed}_k(\rho_n) \geqslant \mathrm{ed}_k(Y[n]) \geqslant n$. Moreover, since ρ_n is defined over K_n and $\mathrm{trdeg}_k(K_n) = n$, we have $\mathrm{ed}_k(\rho_n) \leqslant n$. Thus $\mathrm{ed}_k(\rho_n) = n$, as desired.

Finally, since $Y[n]$ is a union of at most $n \cdot |W|$ K_n-points of $\mathbb{P}^1$, Proposition A.6 also tells us that $d_n = \dim(\rho_n) \leqslant n|W||G|/p$. □

Many natural questions about essential dimension of modular representations remain open. We will conclude this appendix by stating some of these questions below. In what follows we will assume that k is a field of characteristic $p > 0$, G is a finite group, and $E \simeq (\mathbb{Z}/p\mathbb{Z})^2$ is a subgroup of G. We will allow K to vary over field extensions of k and ρ to vary over finite-dimensional representation of G defined over K.

(1) Fix an integer $d \geqslant 1$. What is the maximal value of $\mathrm{ed}_k(\rho)$, where the maximum is taken over all representations ρ of G of dimension $\leqslant d$?

(2) Let $S := \mathrm{Proj}(H^\bullet(G, k))$, as in Section 14, and fix a closed subvariety $X \subset S$ defined over k. What is the maximal value of $\mathrm{ed}_k(\rho)$, where ρ is subject to the condition $\mathrm{Supp}(\rho) = X_K$?

(3) Let $W := W_G(E) = N_G(E)/C_G(E)$ be the Weyl group of E in G and X be a W-equivariant subvariety of $\mathbb{P}^1 := \mathrm{Proj}(H^\bullet(E, k))$ defined over k. What is the maximal value of $\mathrm{ed}_k(\rho)$, where ρ is subject to the condition $\mathrm{Supp}^E(\rho) = X_K$?

(4) What are the maximal values of $\mathrm{ed}_k(\rho) - \mathrm{ed}_k(\mathrm{Supp}(\rho))$ and $\mathrm{ed}_k(\rho) - \mathrm{ed}_k(\mathrm{Supp}^E(\rho))$?

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