TENSOR IDEALS AND VARIETIES FOR MODULES OF QUANTUM ELEMENTARY ABELIAN GROUPS

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ABSTRACT. In a previous paper we constructed rank and support variety theories for "quantum elementary abelian groups," that is, tensor products of copies of Taft algebras. In this paper we use both variety theories to classify the thick tensor ideals in the stable module category, and to prove a tensor product property for the support varieties.

1. INTRODUCTION

This paper is a sequel to our study of rank and support varieties for "quantum elementary abelian groups" [22]. These are arguably the simplest examples of finite dimensional non-commutative non-cocommutative Hopf algebras. They come up in many different contexts and have several incarnations. For the purposes of this paper, let $\ell \geq 2$ be an integer and define A to be a semi-direct (or smash) product of a truncated polynomial algebra with an abelian group,

$$A := k[X_1, \ldots, X_n]/(X_1^\ell, \ldots, X_n^\ell) \rtimes (\mathbb{Z}/\ell\mathbb{Z})^{\times n},$$

where the characteristic of k is either zero or relatively prime to ℓ (details in Section 2). The algebra A also has the following alternative descriptions.

- A is a Borel subalgebra of the small quantum group $\mathfrak{u}_q(\mathfrak{sl}_2^{\oplus n})$, where q is a primitive ℓth root of unity.
- A is the bosonization of a Nichols algebra $\mathcal{B}(V)$ where V is the braided vector space of diagonal type with braiding matrix (q_{ij}), where q_{ij} is 1 if $i = j$, q if $i > j$, and q^{-1} if $i < j$.
- A is the tensor product of n copies of the Taft algebra of dimension ℓ^2.

In [22] we developed the theory of *rank* and *support* varieties for finitely generated A-modules. The support varieties were defined classically via the actions of a cohomology ring on the Ext-algebras for A-modules M. To define rank variety, we considered "quantum" cyclic shifted subgroups of A. The main theorem was an adaptation of the result of Avrunin and Scott [1] for elementary abelian p-groups to our quantum setting, stating that there is a homeomorphism between the rank and support varieties. In this paper we prove two standard results in the theory of varieties for which this Avrunin-Scott type

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theorem is essential: the classification of tensor ideals in the stable module category of A and the tensor product property for varieties of finitely generated A-modules. The proofs of the following two theorems are in Sections 4 and 5, respectively.

Theorem 1.1 (Tensor product theorem)**.** *Let M, N be finitely generated A-modules. Then $V_A(M \otimes N) = V_A(M) \cap V_A(N)$, where $V_A(M)$ denotes the support variety of M.*

Theorem 1.2 (Classification of tensor ideals)**.** *There is an inclusion-preserving bijection between thick tensor ideals of* stmod A *and subsets of* $\operatorname{Proj} \operatorname{H}^*(A, k)$ *closed under specialization.*

A refined version of Theorem 1.2 is Theorem 5.5.

Our algebras present themselves as the very first examples of non-commutative, non-cocommutative Hopf algebras for which the tensor product theorem is known. Examples of non-commutative non-cocommutative Hopf algebras for which the classification of thick tensor ideals is known are also scarce, although such a classification was recently given by Benson and the second author [8] using completely different methods for some Hopf algebras constructed from finite groups.

Due to difficulties that arise in working with a non-symmetric tensor product, there remain unanswered questions. We still do not know whether infinite dimensional modules satisfy the tensor product property for their varieties, even though we prove it for finite dimensional modules using a combination of homological and representation theoretic arguments. The original solution of the classification problem for finite groups (or finite group schemes) involved Rickard idempotent modules, the representation theoretic interpretation of the Bousfield localization technique in topology (see [4] and [13]). Without the tensor product theorem for infinite dimensional modules, we cannot apply Rickard idempotents to classify thick tensor ideals of stmod A. Instead, we employ ideas from the recent paper [11] of Carlson and Iyengar to circumvent the use of infinite dimensional modules and give a very simple proof of the classification.

The paper is organized as follows. In Section 2 we introduce notation and briefly recall definitions and basic properties of rank and cohomological support varieties for quantum elementary abelian groups. Most of the material in that section comes from [22]. In Section 3 we collect several elementary facts about A-modules which culminate in the equality of $V_A(M)$ and $V_A(M^\#)$, where $M^\#$ is an A-module dual to M. Even though the facts proven in that section might feel very familiar to the reader, one should be cautious when working with non-symmetric monoidal module categories in view of the recent results of Benson and the second author: There are examples in [8] of various anomalies in the behavior of support varieties for modules in such categories. In particular, it is shown that the varieties of M and its dual need not coincide in general. Motivated by these examples, we take extra care with proofs of "elementary" facts in Section 3.

In Section 4 we prove Theorem 1.1, the tensor product property. This proof naturally splits into two parts. We first prove the "easy" inclusion $V_A(M \otimes N) \subset V_A(M) \cap V_A(M)$ using exclusively the fact that the varieties of M and $M^\#$ coincide. The second part of this section is occupied by the proof of the other inclusion which uses the structural properties of A in an essential way. Finally, in the last section we classify the thick tensor ideals of stmod A.

Throughout this paper, k will denote an algebraically closed field containing a primitive ℓth root of unity q; in particular, ℓ is not divisible by the characteristic of k. All tensor products and dimensions will be over k, and all modules are finitely generated left modules, unless otherwise indicated.

2. Recollections

2.1. **Varieties for modules of a quantum elementary abelian group.** We recall some notation and results from [22]. Let n be a positive integer, let G be the abelian group $(\mathbb{Z}/\ell\mathbb{Z})^{\times n}$ with generators $\{g_1, \ldots, g_n\}$, and let $R = k[X_1, \ldots, X_n]/(X_1^\ell, \ldots, X_n^\ell)$ be a truncated polynomial ring. We let G act on R via the formula

$$g_i \cdot X_j = q^{\delta_{ij}} X_j$$

where δ_{ij} is the Kronecker delta. We set

$$A := R \rtimes G,$$

the semi-direct product taken with respect to the action defined above. That is, A is a free left R-module with basis G and multiplication determined by $(rg)(sh) = r(g \cdot s)gh$ for all $r, s \in R$, $g, h \in G$. In addition, A has a Hopf algebra structure defined as follows:

$$\Delta(X_i) = X_i \otimes 1 + g_i \otimes X_i, \quad \Delta(g_i) = g_i \otimes g_i,$$

$$\epsilon(X_i) = 0, \quad \epsilon(g_i) = 1, \quad S(X_i) = -g_i^{-1} X_i, \quad S(g_i) = -g_i^{-1}.$$

For $a \in A$, we use the Sweedler notation $\Delta(a) = \sum a_1 \otimes a_2$. For two A-modules M, N, the tensor product $M \otimes N$ is equipped with an A-module structure via the formula

$$a \cdot (m \otimes m') = \sum a_1 m \otimes a_2 m'$$

for all $a \in A$, $m \in M$, $m' \in N$. The A-module structure on a module dual to M, denoted $M^\# = \operatorname{Hom}_k(M, k)$, is defined by

$$(a \cdot f)(m) = f(S(a)m)$$

for all $a \in A$, $f \in M^\#$, $m \in M$. Equivalently, since S^2 is an inner automorphism (it is conjugation by $g_1^{-1} \cdots g_n^{-1}$), we may define the action by $(a \cdot f)(m) = f(S^{-1}(a)m)$.

Let $\{Y_1, \ldots, Y_n\}$ be non-commuting variables, and set

$$R_q = \frac{k\langle Y_1, \ldots, Y_n \rangle}{(Y_j Y_i - q Y_i Y_j, \ Y_i^\ell)}$$

where the commutator relations $Y_j Y_i - q Y_i Y_j$ are taken for all i, j, $1 \le i < j \le n$. The group G acts on R_q via the same formula as before: $g_i \cdot Y_j = q^{\delta_{ij}} Y_j$, and, hence, we can form a semi-direct product $R_q \rtimes G$. There is an algebra isomorphism

$$R_q \rtimes G \xrightarrow{\sim} R \rtimes G = A$$

given by

$$Y_i \mapsto X_i g_1 g_2 \ldots g_{i-1}, \quad g_i \mapsto g_i.$$

For $\underline{\lambda} = [\lambda_1 : \ldots : \lambda_n] \in \mathbb{P}^{n-1}$ we define an embedding of algebras

$$\tau_{\underline{\lambda}} : k[t]/(t^\ell) \to A \quad \text{via} \quad t \mapsto \lambda_1 Y_1 + \lambda_2 Y_2 + \ldots + \lambda_n Y_n.$$

Let $\langle \tau_{\underline{\lambda}}(t) \rangle \subset A$ denote the subalgebra generated by $\tau_{\underline{\lambda}}(t)$. We define the action of G on $\mathbb{P}^{n-1}$ by

$$g_i \cdot [\lambda_1 : \ldots : \lambda_n] := [\lambda_1 : \ldots : \lambda_{i-1} : q\lambda_i : \lambda_{i+1} : \ldots : \lambda_n].$$

For an A-module M, the restriction $M\downarrow_{\langle \tau_{\underline{\lambda}}(t) \rangle}$ of M to $\langle \tau_{\underline{\lambda}}(t) \rangle$ is projective if and only if $M\downarrow_{\langle \tau_{g \cdot \underline{\lambda}}(t) \rangle}$ is projective for any $g \in G$ (see [22, Lemma 2.5(ii)]). We define the *rank variety* of M, a closed subset of $\mathbb{P}^{n-1}$, denoted $V_A(M)$, as follows:

Definition 2.1. [22, Defn. 3.2]

$$V_A(M) := \{\underline{\lambda} \in \mathbb{P}^{n-1} \mid M\downarrow_{\langle \tau_{\underline{\lambda}}(t) \rangle} \text{ is not projective }\}/G.$$

Rank varieties have all the standard properties one expects (see [22, Section 3]).

For an R-module M, we set

$$V_R(M) := V_A(M\uparrow^A),$$

where $M\uparrow^A = \operatorname{ind}_R^A M = A \otimes_R M$ is the (tensor) induced module. In this paper we follow standard conventions in finite group theory, referring to the left adjoint functor to restriction as *induction*, and the right adjoint, $\operatorname{coind}_R^A M = \operatorname{Hom}_R(A, M)$, as *coinduction*.
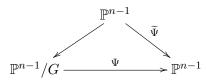
Recall that the cohomology of the quantum elementary abelian group A is

$$(2.2) \qquad \mathrm{H}^*(A, k) = \operatorname{Ext}_A^*(k, k) \simeq k[y_1, \ldots, y_n],$$

where $\deg y_i = 2$. For an A-module M, denote by $I_A(M)$ the annihilator of $\operatorname{Ext}_A^*(M, M)$ under the (left) action of $\mathrm{H}^*(A, k)$ by the cup product (equivalently, under the action given by tensoring an extension in $\operatorname{Ext}_A^*(k, k)$ with M *on the right* followed by Yoneda composition).

Definition 2.3. The cohomological *support variety* of M, $V_A^c(M)$, is the closed subset of $\mathbb{P}^{n-1} \simeq \operatorname{Proj} \mathrm{H}^*(A, k)$ defined by the homogeneous ideal $I_A(M)$, where $\operatorname{Proj} \mathrm{H}^*(A, k)$ denotes the space of homogeneous prime ideals of $\mathrm{H}^*(A, k)$ other than the ideal of all elements of positive degree.

We define a map $\widetilde{\Psi} : \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ as $\widetilde{\Psi}([\lambda_1 : \ldots : \lambda_n]) := [\lambda_1^\ell : \ldots : \lambda_n^\ell]$ and note that it factors through $\mathbb{P}^{n-1}/G$. The resulting map is denoted by Ψ:

$$
\begin{array}{ccc}
 & \mathbb{P}^{n-1} & \\
 \swarrow & & \searrow \widetilde{\Psi} \\
\mathbb{P}^{n-1}/G & \xrightarrow{\quad \Psi \quad} & \mathbb{P}^{n-1}
\end{array}
$$

Since we have $V_A = V_A(k) = \mathbb{P}^{n-1}/G$ and $V_A^c = \operatorname{Proj} \mathrm{H}^*(A, k) \simeq \mathbb{P}^{n-1}$, we get the map

$$\Psi : V_A \to V_A^c$$

induced by raising each coordinate of the equivalence class of $\underline{\lambda} \in V_A$ to the ℓth power. The main theorem of [22] states that $\Psi(V_A(M)) = V_A^c(M)$. Since Ψ is a homeomorphism, we identify the cohomological support variety $V_A^c(M)$ and rank variety $V_A(M)$ via the map Ψ.

We recall some standard properties of these varieties that we will need. The relative support variety $V_A^c(M, N)$ for A-modules M, N is defined analogously to $V_A^c(M)$ by using the action of $\mathrm{H}^*(A, k)$ on $\operatorname{Ext}_A^*(M, N)$ via $- \otimes M$ followed by Yoneda composition.

Proposition 2.4. [22, Prop. 4.3] *Let M, N, M_1, M_2, M_3 be A-modules.*

  (i) $V_A(M \oplus N) = V_A(M) \cup V_A(N)$.
 (ii) $V_A(\Omega(M)) = V_A(M)$, *where Ω is the Heller shift operator.*
(iii) *If $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of A-modules, then*

$$V_A(M_i) \subset V_A(M_j) \cup V_A(M_l)$$

  for any $\{i, j, l\} = \{1, 2, 3\}$.
(iv) *Let $S_1, \ldots, S_n$ be representatives of all isomorphism classes of simple A-modules. Then*

$$V_A(M) = \bigcup_{i=1}^{n} V_A(S_i, M) = \bigcup_{i=1}^{n} V_A(M, S_i).$$

Remark 2.5. We also claimed the inclusion $V_A(M \otimes N) \subset V_A(M) \cap V_A(N)$ in [22, Prop. 4.3]. This statement is true but the proof given was incomplete. In Section 4 we give a different proof of that inclusion as part of the proof of the "Tensor Product Theorem 1.1" which asserts the equality. As demonstrated with counterexamples in [8], one must be careful since the inclusion $V_A(M \otimes N) \subset V_A(M) \cap V_A(N)$ does not hold for an arbitrary Hopf algebra.

2.2. **Thick subcategories.** We recall a few basic definitions that we need and refer the reader to any of the multiple excellent sources on triangulated categories for the necessary background, such as [17] or [20].

Let T be a triangulated category. A *thick* subcategory $\mathcal{C} \subset T$ is a full triangulated subcategory that is closed under taking direct summands. If, in addition, T is tensor triangulated (that is, a monoidal triangulated category), then a *thick tensor ideal* of T is a thick subcategory $\mathcal{C}$ satisfying the property that for any $C \in \mathcal{C}$, and any $B \in T$, we have both $B \otimes C \in \mathcal{C}$ and $C \otimes B \in \mathcal{C}$.

Remark 2.6. We could modify the above definition of thick tensor ideal to define left and right tensor ideals. It is the notion of (two-sided) tensor ideal, as defined above, that naturally arises in the theory here due to the symmetry inherent in the tensor product property (Theorem 1.1) of varieties.

For a triangulated category T and an object $X \in T$, we denote by $\mathrm{Thick}_T(X)$ the thick subcategory generated by X. If T is tensor triangulated, we denote by $\mathrm{Thick}_T^{\otimes}(X)$ the thick tensor ideal generated by X.

Recall that the stable module category of A, stmod A, is the category where objects are finitely generated A-modules and morphisms are equivalence classes of morphisms in mod A, where we say that $f : M \to N$ is equivalent to $g : M \to N$ if $f - g$ factors through a projective module. Since A is a finite dimensional Hopf algebra, it is Frobenius ([18]), and, hence, projective modules coincide with injective modules. Therefore, the category stmod A is triangulated with exact triangles corresponding to short exact sequences in mod A and the shift functor given by the inverse Heller operator $M \mapsto \Omega^{-1}M$ (see, for example, [15]).

3. Preliminaries and the variety of $M^\#$

In this section, we obtain some needed elementary results about A-modules and their varieties. Let $\operatorname{Irr} G$ denote the set of irreducible characters of G. For $\chi \in \operatorname{Irr} G$, write S_χ for the one-dimensional A-module on which x_i acts as 0 and g_i acts as multiplication by $\chi(g_i)$. Note that $S_\chi^\# \simeq S_{\chi^{-1}}$.

Lemma 3.1. *Let M be an A-module and $\chi \in \operatorname{Irr} G$. Then there is an isomorphism of A-modules,*

$$S_\chi \otimes M \otimes S_\chi^\# \simeq M.$$

Proof. The simple modules S_χ form a group, under tensor product, that is generated by all S_{χ_i}, $1 \leq i \leq n$, where $\chi_i(g_j) = q^{\delta_{ij}}$. Thus it suffices to prove the statement for each character χ_i. Define a linear map

$$\phi : S_{\chi_i} \otimes M \otimes S_{\chi_i}^\# \to M$$

by $\phi(1 \otimes m \otimes 1) = g_i^{-1} m$ for all $m \in M$. A straightforward calculation shows that ϕ is A-linear with inverse given by the formula $\phi^{-1}(m) = 1 \otimes g_i m \otimes 1$ for all $m \in M$. Hence, it is an A-module isomorphism. $\square$

As a consequence of the lemma, simple modules commute with all modules under tensor product: $S_\chi \otimes M \simeq S_\chi \otimes M \otimes S_\chi^\# \otimes S_\chi \simeq M \otimes S_\chi$ for all $\chi \in \operatorname{Irr} G$.

Lemma 3.2. *Let M be an A-module. Then*
(i) $M{\downarrow_R}{\uparrow^A} \simeq \bigoplus_{\chi \in \operatorname{Irr} G}(M \otimes S_\chi)$.
(ii) $(M{\downarrow_R}{\uparrow^A})^\# \simeq \bigoplus_{\chi \in \operatorname{Irr} G}(M^\# \otimes S_\chi)$.

Proof. We will give explicitly two A-module homomorphisms

$$\phi \,:\, M{\downarrow_R}{\uparrow^A} \to \bigoplus_\chi (M \otimes S_\chi), \qquad \psi \,:\, \bigoplus_\chi (M \otimes S_\chi) \to M{\downarrow_R}{\uparrow^A}.$$

Let $g \in G$, $m \in M$, and

$$\phi(g \otimes m) = (\chi(g)gm \otimes 1)_{\chi \in \operatorname{Irr} G}.$$

For each $\chi \in \operatorname{Irr} G$, let $m_\chi \in M$, and

$$\psi((m_\chi \otimes 1)_{\chi \in \operatorname{Irr} G}) = \frac{1}{|G|} \sum_{\chi \in \operatorname{Irr} G} \sum_{g \in G} \chi(g^{-1}) g \otimes g^{-1} m_\chi.$$

It is straightforward to check that ϕ, ψ are mutually inverse A-module homomorphisms by applying the orthogonality relations for characters.

The second isomorphism follows immediately by dualizing the first (which reverses the order of tensor products) and applying Lemma 3.1. $\square$

The following lemma only requires A to be a finite dimensional Hopf algebra for which the cohomological support varieties are defined and the two notions of dual module (one using the antipode S and the other using its inverse S^{-1}) coincide. (If the two notions of dual are different, one must use S for (i) below and S^{-1} for (ii) below.) Since we work with left modules, we prefer the action given in (i), however, sometimes we will need to compare with the action in (ii).

Lemma 3.3. *There are isomorphisms of* $\mathrm{H}^*(A,k)$*-modules for all A-modules U, V, W:*

(i) $\mathrm{Ext}_A^*(U \otimes V, W) \simeq \mathrm{Ext}_A^*(U, W \otimes V^{\#})$, *where the action is* $- \otimes U \otimes V$ *(respectively* $- \otimes U$*) followed by Yoneda composition.*

(ii) $\mathrm{Ext}_A^*(U \otimes V, W) \simeq \mathrm{Ext}_A^*(V, U^{\#} \otimes W)$, *where the action is* $U \otimes V \otimes -$ *(respectively* $V \otimes -$*) followed by Yoneda composition.*

Proof. Let $P_{\bullet}$ be a projective resolution of k as an A-module, so that $P_{\bullet} \otimes U \otimes V$ and $U \otimes V \otimes P_{\bullet}$ are projective resolutions of $U \otimes V$, and there are similar projective resolutions of U and of V. One checks that the isomorphism of (i) is induced by the chain level isomorphism

$$\phi : \mathrm{Hom}_A(P_i \otimes U \otimes V, W) \xrightarrow{\sim} \mathrm{Hom}_A(P_i \otimes U, W \otimes V^{\#})$$

given by $\phi(f)(x \otimes u) = \sum_j f(x \otimes u \otimes v_j) \otimes v_j^*$, where $x \in P_i$, $u \in U$, and $\{v_j\}$, $\{v_j^*\}$ are dual bases for V. Similarly one checks that the isomorphism of (ii) is induced by the chain level isomorphism

$$\psi : \mathrm{Hom}_A(U \otimes V \otimes P_i, W) \xrightarrow{\sim} \mathrm{Hom}_A(V \otimes P_i, U^{\#} \otimes W)$$

given by $\psi(f)(v \otimes x) = \sum_j u_j^* \otimes f(u_j \otimes v \otimes x)$, where $v \in V$, $x \in P_i$, and $\{u_j\}$, $\{u_j^*\}$ are dual bases for U. $\qquad\square$

Lemma 3.4. *Let M be an A-module. Then*

$$V_A^c(M\!\downarrow_R\!\uparrow^A) = V_A^c(M).$$

Proof. By Lemma 3.3(i), there is an isomorphism of $\mathrm{H}^*(A,k)$-modules,

$$(3.5) \qquad \mathrm{Ext}_A^*(M \otimes S_\chi, M \otimes S_\chi) \simeq \mathrm{Ext}_A^*(M, M \otimes S_\chi \otimes S_\chi^{\#}) \simeq \mathrm{Ext}_A^*(M, M),$$

with the action of $\mathrm{H}^*(A,k)$ given by tensoring on the right (by $M \otimes S_\chi$, respectively, by M) followed by Yoneda composition. The isomorphism (3.5) implies that $V_A^c(M) = V_A^c(M \otimes S_\chi)$ for each $\chi \in \mathrm{Irr}\, G$. Applying Lemma 3.2(i) and Proposition 2.4(i), we thus have $V_A^c(M\!\downarrow_R\!\uparrow^A) = \cup_{\chi \in \mathrm{Irr}\, G} V_A^c(M \otimes S_\chi) = V_A^c(M)$. $\qquad\square$

Since $V_R(M\!\downarrow_R) = V_A(M\!\downarrow_R\!\uparrow^A)$ by definition, Lemma 3.4 immediately implies the following.

Corollary 3.6. *Let M be an A-module. Then*

$$V_R(M\!\downarrow_R) = V_A(M).$$

Theorem 3.7. *Let M be an A-module. Then*

$$V_A^c(M) = V_A^c(M^{\#}).$$

Proof. By Lemma 3.3, there are isomorphisms of $\mathrm{H}^*(A,k)$-modules: $\mathrm{Ext}_A^*(M^{\#}, k) \simeq \mathrm{Ext}_A^*(k, M^{\#\#}) \simeq \mathrm{Ext}_A^*(k, M)$ and, hence, $V_A^c(k, M) = V_A^c(M^{\#}, k)$.

Now let M be any A-module, and $N = M\!\downarrow_R\!\uparrow^A$. By Proposition 2.4(iv) and Lemma 3.2(i),

$$V_A^c(N) = \bigcup_{\chi \in \mathrm{Irr}\, G} V_A^c(S_\chi, N) = \bigcup_{\chi \in \mathrm{Irr}\, G} V_A^c(k, N \otimes S_\chi^{\#}) = V_A^c(k, N),$$

since $N \otimes S_\chi^{\#} \simeq N$ for all $\chi \in \mathrm{Irr}\, G$ (as $N \simeq \oplus_{\chi \in \mathrm{Irr}\, G} M \otimes S_\chi$). Similarly we find that $V_A^c(N) = V_A^c(N, k)$. It follows, from the previous observation, that $V_A^c(N) = V_A^c(k, N) =$

$V_A^c(N^\#, k)$. By Lemma 3.2(ii), there is an isomorphism $N^\# \otimes S_\chi^\# \simeq N^\#$ for each $\chi \in \mathrm{Irr}\, G$, and so by a similar argument to the above, we have $V_A^c(N^\#) = V_A^c(N^\#, k)$. Thus we have shown that $V_A^c(N) = V_A^c(N^\#)$.

By the above observations and Lemmas 3.2(ii) and 3.4, we now have

$$
\begin{aligned}
V_A^c(M) \;=\; V_A^c(M{\downarrow_R}{\uparrow^A}) \;&=\; V_A^c((M{\downarrow_R}{\uparrow^A})^\#) \\
&=\; V_A^c\left(\bigoplus_{\chi \in \mathrm{Irr}\, G} (M^\# \otimes S_\chi) \right) \\
&=\; V_A^c(M^\#{\downarrow_R}{\uparrow^A}) \;=\; V_A^c(M^\#).
\end{aligned}
$$

$\square$

4. Tensor product property

In this section, we prove that for a quantum elementary abelian group A, the variety of a tensor product of A-modules is the intersection of their varieties, as stated in Theorem 1.1. First, in Corollary 4.2, we fix the proof of Proposition 4.3(vi) from [22], establishing the inclusion $V_A^c(M \otimes N) \subset V_A^c(M) \cap V_A^c(N)$. Since the proof is entirely cohomological and applies more generally than for a quantum elementary abelian group, we stay with the notation V_A^c for this part of the proof. The inclusion $V_A^c(M \otimes N) \subset V_A^c(M)$ is a formal consequence of the definition of the cohomological support variety and the action of $\mathrm{H}^*(A, k)$ on $\mathrm{Ext}_A^*(M, M)$. The inclusion for the second factor, $V_A^c(M \otimes N) \subset V_A^c(N)$, requires more careful consideration since the tensor product is not symmetric. Our approach is based on Theorem 3.7 stating that $V_A^c(M) = V_A^c(M^\#)$.

The proof of the opposite inclusion $V_A(M) \cap V_A(N) \subset V_A(M \otimes N)$ uses both the cohomological and the non-cohomological description of the variety. We observe that the tensor product $A \otimes A$ is again a quantum elementary abelian group and check in Lemma 4.6 that the rank variety behaves well with respect to restriction along the map induced by the coproduct $A \to A \otimes A$. Using the naturality of the isomorphism between rank and support varieties with respect to the coproduct as described in (4.5), we deduce the "restriction" property for the cohomological support varieties $V_A^c(M)$. The rest of the proof goes via standard arguments using the Künneth formula, and employs only the cohomological support.

4.1. **The inclusion $V_A^c(M \otimes N) \subset V_A^c(M) \cap V_A^c(N)$.** Recall that we defined the action of $\mathrm{H}^*(A, k)$ on $\mathrm{Ext}^*(M, M)$ via tensoring an extension with M on the right followed by Yoneda composition. In the following theorem we verify that defining the action by $M \otimes -$ followed by Yoneda composition will lead to the same result under the assumption that $V_A^c(M) = V_A^c(M^\#)$.

Theorem 4.1. *Let M be an A-module for which $V_A^c(M) = V_A^c(M^\#)$. Then the closed subvariety of $\mathrm{Proj}\, \mathrm{H}^*(A, k)$, defined by the annihilator ideal of the action of $\mathrm{H}^*(A, k)$ on $\mathrm{Ext}^*(M, M)$ given by $M \otimes -$ followed by Yoneda composition, coincides with $V_A^c(M)$.*

Proof. By Lemma 3.3, the action of $\mathrm{H}^*(A, k)$ on $\mathrm{Ext}_A^*(M, M)$ given by $M \otimes -$ followed by Yoneda composition corresponds to that on $\mathrm{Ext}_A^*(k, M^\# \otimes M)$ given by Yoneda

composition, under the isomorphism $\mathrm{Ext}_A^*(M, M) \simeq \mathrm{Ext}_A^*(k, M^\# \otimes M)$. There is a further isomorphism $\mathrm{Ext}_A^*(k, M^\# \otimes M) \simeq \mathrm{Ext}_A^*(k, M^\# \otimes M^{\#\#}) \simeq \mathrm{Ext}_A^*(M^\#, M^\#)$, under which the action of $\mathrm{H}^*(A, k)$ by Yoneda composition on the former corresponds to that given on the latter by $- \otimes M^\#$ followed by Yoneda composition. The variety $V_A^c(M^\#)$ is defined by the annihilator of this action, and since $V_A^c(M) = V_A^c(M^\#)$, the variety $V_A^c(M)$ is defined by the annihilator of the first action given above. $\qquad\square$

Corollary 4.2. *If $V_A^c(M) = V_A^c(M^\#)$ for all A-modules M, then*

$$V_A^c(M \otimes N) \subset V_A^c(M) \cap V_A^c(N)$$

for all A-modules M, N.

Proof. Since dualizing reverses the order of tensor product,

$$V_A^c(M \otimes N) = V_A^c((M \otimes N)^\#) = V_A^c(N^\# \otimes M^\#).$$

By the definition of support variety, $V_A^c(M \otimes N) \subset V_A^c(M)$ and $V_A^c(N^\# \otimes M^\#) \subset V_A^c(N^\#)$, and under our hypothesis, $V_A^c(N^\#) = V_A^c(N)$. $\qquad\square$

Now one inclusion in the tensor product theorem is an immediate consequence of Theorem 3.7 and Corollary 4.2.

Corollary 4.3. *Let M, N be two A-modules. Then $V_A^c(M \otimes N) \subset V_A^c(M) \cap V_A^c(N)$.*

4.2. **The equality.** We choose the following ordered generating set for $A \otimes A$:

$$X_1 \otimes 1, \ 1 \otimes X_1, \ X_2 \otimes 1, \ 1 \otimes X_2, \ldots, \ X_n \otimes 1, \ 1 \otimes X_n,$$
$$g_1 \otimes 1, \ 1 \otimes g_1, \ g_2 \otimes 1, \ 1 \otimes g_2, \ldots, \ g_n \otimes 1, \ 1 \otimes g_n.$$

Under this choice, letting $\tau_{\underline{\lambda}}(t) \in A$ as defined in Section 2, we find that in $A \otimes A$,

$\tau_{(\lambda_1, \lambda_1, \ldots, \lambda_n, \lambda_n)}(t)$

$$\begin{aligned}
&= \ \lambda_1 X_1 \otimes 1 + \lambda_1 g_1 \otimes X_1 + \lambda_2 X_2 g_1 \otimes g_1 + \lambda_2 g_1 g_2 \otimes X_2 g_1 + \cdots \\
&\qquad\qquad + \lambda_n X_n g_1 \cdots g_{n-1} \otimes g_1 \cdots g_{n-1} + \lambda_n g_1 \cdots g_n \otimes X_n g_1 \cdots g_{n-1} \\
&= \ \Delta(\tau_{(\lambda_1, \ldots, \lambda_n)}(t)).
\end{aligned}$$

Accordingly we define a map on rank varieties, $\Delta^r : V_A \to V_{A \otimes A}$, by

$$\Delta^r[\lambda_1 : \lambda_2 : \cdots : \lambda_n] = [\lambda_1 : \lambda_1 : \lambda_2 : \lambda_2 : \cdots : \lambda_n : \lambda_n].$$

We denote the map on cohomology induced by the coproduct $\Delta : A \to A \otimes A$ by

$$\Delta^* : \mathrm{H}^*(A, k) \otimes \mathrm{H}^*(A, k) \simeq \mathrm{H}^*(A \otimes A, k) \to \mathrm{H}^*(A, k)$$

(where the first isomorphism is given by the Künneth theorem), and the corresponding map on support varieties by

$$\Delta_* : V_A^c \to V_{A \otimes A}^c \simeq V_A^c \times V_A^c.$$

This map arises from the cup product on the graded commutative algebra $\mathrm{H}^*(A, k)$, and so is the diagonal map.

Lemma 4.4. *The following diagram commutes:*

(4.5)
$$
\begin{array}{ccc}
V_A & \xrightarrow{\Delta^r} & V_{A\otimes A} \\
\downarrow{\psi_A} & & \downarrow{\psi_{A\otimes A}} \\
V_A^c & \xrightarrow{\Delta_*} & V_{A\otimes A}^c
\end{array}
$$

Proof. This is an immediate consequence of the definitions of the diagonal maps Δ_* and of Δ^r. (One must order the basis of $\mathrm{H}^2(A\otimes A, k)$ in accordance with the ordering of the generating set for $A \otimes A$ used in the definition of Δ^r.) $\qquad\square$

Lemma 4.6. *Let M be an $A \otimes A$-module, considered to be an A-module via $\Delta : A \to A \otimes A$. Then*
  (i) $(\Delta^r)^{-1} V_{A\otimes A}(M) = V_A(M)$.
  (ii) $(\Delta_*)^{-1} V_{A\otimes A}^c(M) = V_A^c(M)$.

Proof. (i) This follows from the definitions since
$$
\tau_{(\lambda_1, \lambda_1, \ldots, \lambda_m, \lambda_m)}(t) = \Delta(\tau_{(\lambda_1, \ldots, \lambda_m)}(t)).
$$
Thus $M\!\downarrow^{A\otimes A}_{k\langle\tau_{(\lambda_1, \lambda_1, \ldots, \lambda_m, \lambda_m)}(t)\rangle}$ is projective if, and only if, $M\!\downarrow^{A}_{k\langle\tau_{(\lambda_1, \ldots, \lambda_m)}(t)\rangle}$ is projective.
  (ii) This follows from (i) and Lemma 4.4, since ψ_A and $\psi_{A\otimes A}$ are homeomorphisms. $\qquad\square$

The following result is a straightforward consequence of the Künneth Theorem, valid for any finite dimensional Hopf algebra A satisfying the hypothesis of Theorem 4.1, that is the varieties of a module and of its dual are the same.

Lemma 4.7. *Let M, N be A-modules. Then*
$$
V_A^c(M) \times V_A^c(N) \simeq V_{A\otimes A}^c(M \otimes N).
$$

Proof. The following diagram induces isomorphisms on varieties, as we explain below.

$$
\begin{array}{ccc}
\mathrm{Ext}_A^*(k,k) \otimes \mathrm{Ext}_A^*(k,k) & \xrightarrow{(M\otimes-)\otimes(-\otimes N)} & \mathrm{Ext}_A^*(M,M) \otimes \mathrm{Ext}_A^*(N,N) \\
\downarrow{\simeq} & & \downarrow{\simeq} \\
\mathrm{Ext}_{A\otimes A}^*(k,k) \xrightarrow[-\otimes M\otimes k]{M\otimes k\otimes-} \mathrm{Ext}_{A\otimes A}^*(M\otimes k, M\otimes k) & \xrightarrow{-\otimes k\otimes N} & \mathrm{Ext}_{A\otimes A}^*(M\otimes N, M\otimes N)
\end{array}
$$

The vertical maps are given by the Künneth Theorem. The variety $V_A^c(M) \times V_A^c(N)$ is defined by the kernel of the top horizontal map, since $M \otimes -$ gives rise to the same variety as $- \otimes M$ by Theorem 4.1. The variety $V_{A\otimes A}^c(M \otimes N)$ is defined by the kernel of the bottom horizontal map. The middle horizontal map is part of a commuting diagram together with the top horizontal map, by definition of the Künneth isomorphisms: These are simply different ways of viewing the same tensor product of complexes, since k is the trivial module. Now $M \otimes k \otimes -$ gives rise to the same variety as $- \otimes M \otimes k$, again by Theorem 4.1. Therefore the bottom horizontal map gives rise to the same variety as the middle horizontal map. $\qquad\square$

Proof of Theorem 1.1. It is more convenient to work with affine varieties for this proof. Since $H^*(A, k)$ is graded, connected, and generated in degree 2, the ideal $I_A(M)$ for any module M is homogeneous, and the field k is algebraically closed, we have

(4.8) $$V_A^c(M) = [\text{Spec } H^*(A, k) - \{0\}] / k^*$$

where $\{0\}$ corresponds to the irrelevant ideal $H^{>0}(A, k)$. Hence, it suffices to show that $\text{Spec } H^*(A, k)/I_A(M \otimes N) = \text{Spec } H^*(A, k)/I_A(M) \cap \text{Spec } H^*(A, k)/I_A(N)$.

Recall that the map $\Delta^* : H^*(A, k) \otimes H^*(A, k) \to H^*(A, k)$ induces the diagonal map on Spec. Hence,

$$\Delta_*^{-1}(\text{Spec } H^*(A, k)/I_A(M) \times \text{Spec } H^*(A, k)/I_A(N))$$
$$= \text{Spec } H^*(A, k)/I_A(M) \cap \text{Spec } H^*(A, k)/I_A(N).$$

By Lemma 4.6(ii) (which applies to affine varieties thanks to the observation (4.8)),

$$\text{Spec } H^*(A, k)/I_A(M \otimes N) = (\Delta_*)^{-1}(\text{Spec } H^*(A \otimes A, k)/I_{A \otimes A}(M \otimes N)).$$

By Lemma 4.7, and a reordering of the basis of $H^2(A \otimes A, k)$, the latter equals

$$\Delta_*^{-1}(\text{Spec } H^*(A, k)/I_A(M) \times \text{Spec } H^*(A, k)/I_A(N)).$$

Combining with the first equation, we conclude

$$\text{Spec } H^*(A, k)/I_A(M \otimes N) = \text{Spec } H^*(A, k)/I_A(M) \cap \text{Spec } H^*(A, k)/I_A(N).$$

Passing to projective varieties by (4.8), we get the desired property:

$$V_A^c(M \otimes N) = V_A^c(M) \cap V_A^c(N).$$

$\square$

Remark 4.9. One consequence of the theorem involves modules of the form $V(\underline{\lambda})$, defined in [22, §2] for each $\underline{\lambda} \in \mathbb{P}^{n-1}$: It is shown there that the variety of $V(\underline{\lambda})$ is simply $\{\underline{\lambda}\}$. Let M be an A-module. Then $\underline{\lambda} \in V_A(M)$ if and only if $V(\underline{\lambda}) \otimes M$ is not projective. To see this, note that by Theorem 1.1, since $V_A(V(\underline{\lambda})) = \{\underline{\lambda}\}$, we have $\underline{\lambda} \in V_A(M)$ if and only if

$$V_A(V(\underline{\lambda}) \otimes M) = V_A(V(\underline{\lambda})) \cap V_A(M) = \{\underline{\lambda}\} \neq \emptyset,$$

which happens if and only if $V(\underline{\lambda}) \otimes M$ is not projective.

5. CLASSIFICATION OF THICK TENSOR IDEALS OF stmod A

The proof of the classification is based on the following general strategy (which can be traced to [16], see also [21]) used recently in [11]. Let A be any finite dimensional Hopf algebra such that finitely generated A-modules have a support variety theory, denoted $V_A(M)$, satisfying the usual properties (such as in [22, Prop. 4.3]) plus two additional hypotheses:

(i) *(Realization)* For every closed subset V in $V_A(k)$, there exists $M \in \text{mod } A$ such that $V_A(M) = V$.

(ii) *(Thick subcategory lemma)* Let M, N be finitely generated A-modules. If $V_A(M) \subset V_A(N)$, then $M \in \text{Thick}_A^{\otimes}(N)$.

In this case, the classification theorem holds. We implement this strategy in the proof of Theorem 1.2. The realization property for our theory of support varieties holds by [22, Cor. 4.5], and the "Thick subcategory lemma" is proved below as Proposition 5.4.

The following lemma is straightforward from the definitions (see also [7, Lemma 2.2]).

Lemma 5.1. *Let* $\mathfrak{F} : \mathcal{A} \to \mathcal{B}$ *be an exact functor between small triangulated categories* $\mathcal{A}$ *and* $\mathcal{B}$, *and let* X *be an object in* $\mathcal{A}$. *Then for any* $Y \in \mathrm{Thick}_{\mathcal{A}}(X)$, $\mathfrak{F}(Y) \in \mathrm{Thick}_{\mathcal{B}}(\mathfrak{F}(X))$.

Lemma 5.2. *Let* $M \in \mathrm{stmod}\, A$. *Then* $\mathrm{Thick}_A^{\otimes}(M) = \mathrm{Thick}_A^{\otimes}(M{\downarrow}_R {\uparrow}^A)$.

Proof. By Lemma 3.2, $M{\downarrow}_R{\uparrow}^A \simeq \oplus_{\chi \in \mathrm{Irr}\, G}(M \otimes S_\chi)$. Since M is a direct summand of $M{\downarrow}_R{\uparrow}^A$ (corresponding to the trivial character of G), $M \in \mathrm{Thick}_A^{\otimes}(M{\downarrow}_R{\uparrow}^A)$. Conversely, since $M \otimes S_\chi \in \mathrm{Thick}_A^{\otimes}(M)$ for each $\chi \in \mathrm{Irr}\, G$, we find that $M{\downarrow}_R{\uparrow}^A \in \mathrm{Thick}_A^{\otimes}(M)$. Hence, $\mathrm{Thick}_A^{\otimes}(M) = \mathrm{Thick}_A^{\otimes}(M{\downarrow}_R {\uparrow}^A)$. $\square$

The following proposition is a version of the "thick subcategory lemma" for $V_R(M)$. It is proven in [11] but the definition of a variety used there is different from ours. So the proof is based on identifying the variety of [11] with $V_R(M)$ as defined in this paper.

Proposition 5.3. *Let* M, N *be* R-modules. *If* $V_R(M) \subset V_R(N)$, *then*

$$M \in \mathrm{Thick}_R(N).$$

Proof. The varieties in [11] are the same as we have defined up to projectivization: Let M be an R-module, and identify it with a chain complex concentrated in degree 0. In [11], $V_R(M)$ is defined to be the support, in $\mathrm{Spec}\, k[\theta]$, of $\mathrm{Ext}_R^*(k, M)$, where $k[\theta]$ is the subalgebra of $\mathrm{Ext}_R^*(k,k)$ generated in degree 2 by the usual generators. The action of $\mathrm{Ext}_R^*(k,k)$ is the Yoneda action. This is the same as the definition in [22], where the support variety $V_R^c(M)$ (with $R = \Lambda$ in that paper) is the closed subset of $V_R^c(k)$ determined by the annihilator of $\mathrm{Ext}_R^*(k, M)$ (see [22, Definition 2, p. 587]). Since the nilpotent generators (in degree 1) are contained in any prime ideal, this is equivalent to [11, Definition 4.3]. See also the remark on equivalent definitions, [22, p. 587].

The result now follows from [11, Theorem 5.6]. $\square$

Proposition 5.4. *Let* M, N *be* A-modules. *If* $V_A(M) \subset V_A(N)$, *then*

$$M \in \mathrm{Thick}_A^{\otimes}(N).$$

Proof. By Corollary 3.6, the inclusion $V_A(M) \subset V_A(N)$ implies $V_R(M{\downarrow}_R) \subset V_R(N{\downarrow}_R)$. Therefore, $M{\downarrow}_R \in \mathrm{Thick}_R(N{\downarrow}_R)$ by Proposition 5.3. Since induction $(-){\uparrow}^A :$ $\mathrm{stmod}\, R \to \mathrm{stmod}\, A$ is exact, we conclude that $M{\downarrow}_R{\uparrow}^A \in \mathrm{Thick}_A(N{\downarrow}_R{\uparrow}^A)$ by Lemma 5.1. By Lemma 3.2, M is a direct summand of $M{\downarrow}_R{\uparrow}^A$, and so $M \in \mathrm{Thick}_A(M{\downarrow}_R{\uparrow}^A)$. Finally, $\mathrm{Thick}_A(N{\downarrow}_R{\uparrow}^A) \subset \mathrm{Thick}_A^{\otimes}(N{\downarrow}_R{\uparrow}^A) = \mathrm{Thick}_A^{\otimes}(N)$ by Lemma 5.2. Putting these together, we conclude that $M \in \mathrm{Thick}_A^{\otimes}(N)$. $\square$

Recall that a subset V of a topological space X is called *specialization closed* if for any $W \subset V$, the closure of W belongs to V. Equivalently, a specialization closed subset is a union of closed subsets.

Theorem 5.5. *There is an inclusion-preserving one-to-one correspondence*

$$\left\{ \begin{array}{c} \textit{thick tensor ideals } \mathcal{C} \\ \textit{in } \operatorname{stmod} A \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{specialization closed subsets } V \\ \textit{of } V_A(k) = \mathbb{P}^{n-1}/G \end{array} \right\}$$

given by maps ϕ, ψ where

$$\phi(\mathcal{C}) = \bigcup_{M \in \mathcal{C}} V_A(M) \quad \textit{and} \quad \psi(V) = \{M : V_A(M) \subset V\}.$$

Proof. First note that these maps have images as expected: If $\mathcal{C}$ is a thick tensor ideal, then by definition $\phi(\mathcal{C})$ is specialization closed, since $V_A(M)$ is closed for each M. If V is a specialization closed subset, then by the properties of support varieties (Proposition 2.4), $\psi(V)$ is a thick tensor ideal.

We wish to show that ϕ, ψ are inverse maps. We first show that $\phi \circ \psi(V) = V$. By definition,

$$\phi \circ \psi(V) = \bigcup_{M \in \psi(V)} V_A(M),$$

where the union is taken over all M for which $V_A(M) \subset V$. Hence, $\phi \circ \psi(V) \subset V$. For the other containment, note that any point x in V is contained in a closed set in V, which is realized as the variety of some module M (see [22, Corollary 4.5]). Thus, M is in $\psi(V)$ by definition, and, therefore, $x \in \phi \circ \psi(V)$.

It remains to establish the equality $\psi \circ \phi(\mathcal{C}) = \mathcal{C}$. The easy part of this is $\mathcal{C} \subset \psi \circ \phi(\mathcal{C})$, which follows from the definitions. So we only need to show the inclusion

$$\psi \circ \phi(\mathcal{C}) \subset \mathcal{C}$$

for some thick tensor ideal $\mathcal{C}$ in $\operatorname{stmod} A$. Let $N \in \psi \circ \phi(\mathcal{C})$, that is, $V_A(N) \subset \bigcup_{M \in \mathcal{C}} V_A(M)$. Since $\mathrm{H}^*(A, k)$ is Noetherian, there exist finitely many $M_i \in \mathcal{C}$ such that

$$V_A(N) \subset \bigcup_{i=1}^{n} V_A(M_i)$$

(see [21, Lemma 1.3]). Hence, $V_A(N) \subset V_A(M_1 \oplus M_2 \oplus \cdots \oplus M_n)$. By Proposition 5.4, $N \in \operatorname{Thick}_A^\otimes(M_1 \oplus M_2 \oplus \ldots \oplus M_n)$. Since $M_1 \oplus M_2 \oplus \cdots \oplus M_n \in \mathcal{C}$ and $\mathcal{C}$ is a thick tensor ideal, this implies $N \in \mathcal{C}$. $\square$

Remark 5.6. As an immediate corollary to the classification, there is a one-to-one correspondence

$$\{ \text{ thick tensor ideals in } \operatorname{stmod} A \} \longleftrightarrow \{ \text{ thick subcategories in } \operatorname{stmod} R \}.$$

We briefly discuss the situation for the quantum complete intersection algebra R_q. We note that R_q is a Frobenius algebra, which follows from the fact that $R_q \rtimes G \simeq A$ (see also [12, Cor. 5.8]). We consider the stable module category $\operatorname{stmod} R_q$. For an R_q-module M, and $g \in G$, denote by ${}^g M$ the R_q-module which is the "twist" of M by g: ${}^g M = M$ as a vector space, and the action is given by $Y_i(m) := (g^{-1} \cdot Y_i)m = \chi_i(g^{-1})Y_i m$. (Recall that $\chi_i(g_j) = q^{\delta_{ij}}$.) Sending M to ${}^g M$ induces an action of G on $\operatorname{stmod} R_q$. We say that a subcategory $\mathcal{C}$ of $\operatorname{stmod} R_q$ is *G-stable* if $M \in \mathcal{C}$ implies ${}^g M \in \mathcal{C}$ for all $g \in G$.

For an R_q-module M, we define

$$V_{R_q}(M) := V_A(M\uparrow^A).$$

Corollary 5.7. *There is an inclusion-preserving one-to-one correspondence*

$$\left\{ \begin{array}{c} \textit{G-stable thick subcategories} \\ \textit{in } \operatorname{stmod} R_q \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \textit{specialization closed subsets} \\ \textit{of } \mathbb{P}^{n-1}/G \end{array} \right\}$$

induced by the maps ϕ, ψ defined as in Theorem 5.5 with V_A replaced by V_{R_q}.

Proof. The proof is very similar to the proof of Theorem 5.5. We discuss two points where differences occur.

First, for any R_q-module M, we have $M\uparrow^A \simeq ({}^g M)\uparrow^A$, and, therefore, $V_{R_q}(M) = V_{R_q}({}^g M)$ for any $g \in G$. The definition of ψ now implies that for $V \subset \mathbb{P}^{n-1}/G$ a subset closed under specialization, $\psi(V)$ is a G-stable thick subcategory of $\operatorname{stmod} R_q$.

As in the proof of Theorem 5.5, it now suffices to show that for any G-stable thick subcategory $\mathcal{C} \subset \operatorname{stmod} R_q$, we have $\psi(\phi(\mathcal{C})) \subset \mathcal{C}$. Let M be an R_q-module such that $V_{R_q}(M) \subset \phi(\mathcal{C})$. By the Noetherian argument as in the proof of Theorem 5.5 and the definition of the variety $V_{R_q}(-)$, there is a module $N \in \mathcal{C}$ such that $V_A(M\uparrow^A) \subseteq V_A(N\uparrow^A)$, which implies that $M\uparrow^A \in \operatorname{Thick}_A^{\otimes}(N\uparrow^A)$ by Proposition 5.4. Since the S_χ's are all simple A-modules up to isomorphism, we have $\operatorname{Thick}_A^{\otimes}(N\uparrow^A) = \operatorname{Thick}_A(\oplus_\chi(N\uparrow^A \otimes S_\chi))$. This implies that $M\uparrow^A\downarrow_{R_q} \in \operatorname{Thick}_{R_q}(\oplus_\chi(N\uparrow^A \otimes S_\chi)\downarrow_{R_q})$, however $(N\uparrow^A \otimes S_\chi)\downarrow_{R_q} \simeq \oplus_{g \in G} {}^g N$. Since $N \in \mathcal{C}$ and $\mathcal{C}$ is G-stable, we conclude that $M\uparrow^A\downarrow_{R_q} \in \mathcal{C}$. Therefore, $M \in \mathcal{C}$ since M is a direct summand of $M\uparrow^A\downarrow_{R_q}$. $\square$

Remark 5.8. Do there exist non-G-stable thick subcategories in $\operatorname{stmod} R_q$? Or do N, ${}^g N$ always generate the same thick subcategory of $\operatorname{stmod} R_q$, for any N? In general, one does not expect this, however in this case, N and ${}^g N$ have the same variety, and in case $q = 1$, it is true. This is related to a question that Benson and Green asked in [6]: The parameters were a bit different, but in our case their question would translate as follows. Do there exist modules M, N, neither of which is free on restriction to the subalgebra generated by Y_1, and for which $\operatorname{Ext}_{R_q}^n(M, N) = 0$ for all $n > 0$?

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