Invariants in modular representation theory

Julia Pevtsova

(joint work with Eric Friedlander, Jon Carlson, and Dave Benson)

It is well known that the category of modular representations of a finite group scheme is almost always wild. Hence, there is no hope to classify indecomposable representations making them fit into a nice, concise picture. In this talk, we present several new constructions of invariants of modular representations with an idea that they will serve as navigational devices in this wild jungle of representations. Some of our invariants are defined locally and generalize the well-known notion of support variety. They allow for a finer distinction between modules than support varieties; in addition, they lead to distinguished families of modules characterized by vanishing of one of those invariants. For these distinguished classes of modules we define new global invariants which we prove to be algebraic vector bundles on the Proj of cohomology of our group scheme.

For the purposes of this talk, we shall concentrate the case of a restricted Lie algebra. Nonetheless, many of the results and new invariants we introduce exist for infinitesimal group schemes and sometimes more generally for any finite group scheme.

This is an outline of the talk:

(1) Local Jordan type and non-maximal rank varieties;
(2) Global p-nilpotent operator and vector bundles associated to a representation;
(3) π^r-points, constant Rad and Soc-type, and bundles on Grassmanians.

Let k be an algebraically closed field of characteristic $p > 0$, and let $\mathfrak{g}$ be a restricted Lie algebra defined over k. We denote by $[p] : \mathfrak{g} \to \mathfrak{g}$ the $[p]$-th power map that defines the restricted structure. A standard example of a restricted Lie algebra is gl_n over $\overline{\mathbb{F}}_p$.

Let $u(\mathfrak{g})$ be the restricted enveloping algebra of $\mathfrak{g}$, a finite dimensional cocommutative Lie algebra. We have an equivalence of categories:

$$\text{restricted } \mathfrak{g} - \text{mod} \ \sim \ u(\mathfrak{g}) - \text{mod}$$

Comment. Let $\mathfrak{g}_a = \operatorname{Lie} \mathbb{G}_a$ be a one-dimensional abelian Lie algebra with trivial $[p]$-restriction. Let $E = (\mathbb{Z}/p)^{\times n}$ be an elementary abelian p-group of rank n. Then

$$kE \simeq u(\mathfrak{g}_a^{\oplus n})$$

Hence, the theory that will be described applies to elementary abelian p-groups. We just need to think of them as abelian Lie algebras.

I. Local invariants. Let $\mathcal{N}^{[p]} = \{x \in \mathfrak{g} \mid x^{[p]} = 0\}$, the $[p]$-restricted nullcone of $\mathfrak{g}$. Recall that if $\mathfrak{g}$ is a classical Lie algebra and $h > p$ then $\mathcal{N}^{[p]} = \mathcal{N}$, the nullcone of $\mathfrak{g}$. The following theorem connects our work with cohomology.

Theorem 1 (Friedlander-Parshall, Andersen-Jantzen, Suslin-Friedlander-Bendel)**.** *There is an isomorphism of varieties*

$$\operatorname{Spec} \mathrm{H}^\bullet(u(g), k) \simeq \mathcal{N}^{[p]}(\mathfrak{g})$$

where $\mathrm{H}^\bullet$ *denotes the even dimensional cohomology unless* $p = 2$.

Let M be a finite-dimensional (restricted) $\mathfrak{g}$-module. We study M "locally" on $\mathcal{N}^{[p]}$. Let $x \in \mathcal{N}^{[p]}$. The isomorphism class of the restriction of M to the abelian Lie algebra generated by x, $M\downarrow_{\langle x\rangle}$, is determined by the Jordan type of x considered as an operator on M. We denote this Jordan type by $\mathbf{Jtype(x, M)}$. For each number j, $1 \leq j \leq p-1$, we denote by $\operatorname{rk}(x^j, M)$ the rank of x^j as an operator on M. Since $x^{[p]} = 0$, the Jordan type $\operatorname{Jtype}(x, M)$ is completely determined by the sequence $\{\operatorname{rk}(x, M), \operatorname{rk}(x^2, M), \ldots, \operatorname{rk}(x^{p-1}, M), \dim M\}$. We generalize the notion of *support variety* to *non-maximal rank varieties*, geometric invariants defined "locally":

Definition 2. Let M be a finite-dimensional $\mathfrak{g}$-module.
(1) $\Gamma^j_{\mathfrak{g}}(M) = \{x \in \mathcal{N}^{[p]} \,:\, \operatorname{rk}(x^j, M) \text{ is not maximal } \} \cup \{0\}$, $1 \leq j \leq p-1$
(2) $\Gamma_{\mathfrak{g}}(M) = \{x \in \mathcal{N}^{[p]} \,:\, \operatorname{Jtype}(x, M) \text{ is not maximal } \} \cup \{0\}$.

These sets are, indeed, closed subvarieties of $\mathcal{N}^{[p]}(\mathfrak{g})$. Unlike support varieties, they are always proper subvarieties. Using these invariants, we can define modules of constant Jordan type (compare to D. Benson's talk), and, more generally, modules of constant j-rank.

Definition 3. Let M be a finite-dimensional $\mathfrak{g}$-module.
(1) M is a module of constant j-rank if $\Gamma^j_{\mathfrak{g}}(M) = \{0\}$.
(2) M is a module of constant Jordan type if $\Gamma_{\mathfrak{g}}(M) = \{0\}$.

II. Global invariants. Let $k[\mathcal{N}^{[p]}]$ be the coordinate algebra of $\mathcal{N}^{[p]}$, let $\{x_1, \ldots, x_n\}$ be a basis of $\mathfrak{g}$, and let $\{Y_1, \ldots, Y_n\}$ be the dual basis of $\mathfrak{g}^{\#}$. Denote by $y_1, \ldots, y_n$ the images of $Y_1, \ldots, Y_n$ under the surjetive map $S^*(\mathfrak{g}^{\#}) \to k[\mathcal{N}^{[p]}]$. We define the *universal p-nilpotent element* $\Theta \in u(\mathfrak{g}) \otimes k[\mathcal{N}^{[p]}]$ via the explicit formula

$$\Theta = \sum x_i \otimes y_i.$$

For any $\mathfrak{g}$-module M, the element Θ induces the *global p-nilpotent operator*

$$\Theta_M : M \otimes k[\mathcal{N}^{[p]}] \to M \otimes k[\mathcal{N}^{[p]}]$$

defined via

$$\Theta_M : m \otimes f \mapsto \sum x_i(m) \otimes y_i f$$

This is a $k[\mathcal{N}^{[p]}]$-linear, homogeneous operator of degree one. If $\mathbb{P}(g) = \operatorname{Proj} k[\mathcal{N}^{[p]}]$ and $\mathcal{O} = \mathcal{O}_{\mathbb{P}(\mathfrak{g})}$, then Θ induces a map of $\mathcal{O}$-modules

$$\Theta_M : M \otimes \mathcal{O} \to M \otimes \mathcal{O}(1)$$

The operator Θ_M allows us to associate an algebraic vector bundle to a module of constant j-rank thanks to the following theorem.

Theorem 4 (Friedlander-P)**.** *Let M be a $\mathfrak{g}$-module of constant j-rank. Then*

(1) $\operatorname{Im}\Theta^j_M$ *and* $\operatorname{Ker}\Theta^j_M$ *are algebraic vector bundles on* $\mathbb{P}(\mathfrak{g})$.
(2) *For any point* $\bar{x} \in \mathbb{P}(\mathfrak{g})$*, the fiber of* $\operatorname{Im}\Theta^j_M$ *at the point* $\bar{x}$ *is isomorphic to* $\operatorname{im}\{x^j : M \to M\}$*, and similarly for kernel.*
(3) $\operatorname{rk}\operatorname{Im}\Theta^j_M = \operatorname{rk}(x^j, M)$*, and* $\operatorname{rk}\operatorname{Ker}\Theta^j_M = \dim\ker(x^j, M)$

Remark 1. This theorem is true for infinitesimal group schemes but we do not know the appropriate analogue for finite groups.

Various quotients of the bundles constructed above are again algebraic vector bundles. It turns out that using one such quotient we can recover all vector bundles on a projective space (up to a Frobenius twist) from modules of constant Jordan type for elementary abelian p-groups. Let M be a module of constant Jordan type. Then

$$\mathcal{F}_1(M) = \frac{\operatorname{Ker}\Theta_M}{\operatorname{Ker}\Theta_M \cap \operatorname{Im}\Theta_M}$$

is an algebraic vector bundle on $\mathbb{P}(\mathfrak{g})$.

Theorem 5 (Benson-P)**.** *Let E be an elementary abelian p-group of rank n.*

(1) *If $p = 2$ then for any algebraic vector bundle $\mathcal{F}$ on $\mathbb{P}^{n-1}_k$ there exists an E-module M of constant Jordan type such that $\mathcal{F}_1(M) \simeq \mathcal{F}$.*
(2) *If $p > 2$ then for any algebraic vector bundle $\mathcal{F}$ on $\mathbb{P}^{n-1}_k$ there exists an E-module M of constant Jordan type such that $\mathcal{F}_1(M) \simeq F^*(\mathcal{F})$ where $F : \mathbb{P}^{n-1} \to \mathbb{P}^{n-1}$ is the Frobenius map.*

Another quotient bundle relates endo-trivial modules which were discussed in detail in N. Mazza's talk to line bundles:

Theorem 6 (Frieldnader-P)**.** *Let M be a module of constant Jordan type, and let*

$$\mathcal{H}^{[1]}(M) = \frac{\operatorname{Ker}\Theta_M}{\operatorname{Im}\Theta^{p-1}_M}.$$

Then $\mathcal{H}^{[1]}(M)$ is a line bundle on $\mathbb{P}(\mathfrak{g})$ if and only if M is an endo-trivial module.

III. Bundles on Grassmanians. By generalizing the notion of a π-point introduced in [6], [7], we can extend our bundle construction to produce bundles on Grassmanians.

Definition 7. Let E be an elementary abelian p-group of rank n, and let $r < n$.
(1). A π^r-point of kE is a flat map of algebras

$$\alpha : k[t_1, \ldots, t_r]/(t_1^p, \ldots, t_r^p) \to kE$$

(2). Let α, β be π^r-points. We say that $\alpha \sim \beta$ if for any finite dimensional kE-module M, $\alpha^*(M)$ is free if and only if $\beta^*(M)$ is free.

Proposition 8 (Carlson-Friedlander-P)**.** *Let $\Pi^r(E)$ be the set of equivalence classes of π^r-points. Then $\Pi^r(E) \simeq \operatorname{Grass}_{r,n}$, the Grassmanian of r-planes in n-space.*

Definition 9. An E-module M has *constant* $r-\mathrm{Rad}$ *type* if $\dim \mathrm{Rad}\, \alpha^*(M)$ is constant for all π^r-points $\alpha : k[t_1, \dots, t_r]/(t_1^p, \dots, t_r^p) \to kE$.

Similarly, M has *constant* $r - \mathrm{Soc}$ *type* if $\dim \mathrm{Soc}\, \alpha^*(M)$ is constant for all π^r-points α.

With a bit more work than in the case of π-points, one can define sheaves $\mathrm{Ker}\{\Theta, M\}$ and $\mathrm{Im}\{\Theta, M\}$. The following theorem asserts that this construction gives a way to obtain bundles on Grassmanians starting with modules of constant socle or radical type.

Theorem 10 (Carlson-Friedlander-P)**.** *(1) Let E be an elementary abelian p-group or rank n, and let M be a module of constant $r - \mathrm{Rad}$ type. Then $\mathrm{Im}\{\Theta, M\}$ is an algebraic vector bundle on $\mathrm{Grass}_{r,n}$.*
(2) Let M be an E-module of constant $r - \mathrm{Soc}$ type. Then $\mathrm{Ker}\{\Theta, M\}$ is an algebraic vector bundle on $\mathrm{Grass}_{r,n}$.

References

[1] H. Anderson, J. Jantzen, Cohomology of induced representations for algebraic groups, *Math. Ann.* **269** (1984), no. 4, 487–525.

[2] D. Benson, J Pevtsova, Realization theorem for modules of constant Jordan type and vector bundles, preprint (2010).

[3] J. Carlson, *The varieties and the cohomology ring of a module*, J. Algebra **85** (1983) 104-143.

[4] J. Carlson, E. Friedlander, J. Pevtsova, Modules of Constant Jordan Type, *J. für die Reine und Ang. Math. (Crelle)* **614** (2008), 191-234.

[5] E. Friedlander, B. Parshall. *Support varieties for restricted Lie algebras*, Inventiones Math. **86** (1986), 553–562.

[6] E. Friedlander, J. Pevtsova, Representation-theoretic support spaces for finite group schemes, *American Journal of Math.* **127** (2005), 379-420.

[7] E. Friedlander, J. Pevtsova, Π-supports for modules for finite group schemes, *Duke. Math. J.* **139** (2007), 317–368.

[8] E. Friedlander, J. Pevtsova, Constructions for infinitesimal group schemes, to appear in *Transactions of the AMS.*

[9] E. Friedlander, J. Pevtsova, Generalized support varieties for finite group schemes, *Documenta Mathematica*, Special volume Suslin (2010), 197–222.

[10] A. Suslin, E. Friedlander, C. Bendel, Support varieties for infinitesimal group schemes, *J. Amer. Math. Soc.* **10** (1997), 729–759.