Specialization finite group schemes with applications to (co)-stratification and local duality

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Let R be a commutative Noetherian ring, and consider the derived category of ModR, $T = \mathsf{D}(R)$. The compact objects of T form the bounded derived category of perfect complex over R, $T^c = \mathsf{D}^{\text{perf}}(R)$. For a complex M in $\mathsf{D}(R)$, and a prime $\mathfrak{p} \in \mathsf{Spec}\, R$, we can consider the localization $M_{\mathfrak{p}}$ of M at p and the specialization $M \otimes^{\mathbb{L}}_R k(\mathfrak{p})$ at $\mathfrak{p}$, invariants which provide "local information" about the complex M at the point $\mathfrak{p}$. One invariant we can use this local information for is the support.

Definition 1.

$$\operatorname{supp} M = \{\mathfrak{p} \in \mathsf{Spec}\, R \mid M \otimes^{\mathbb{L}}_R k() \not\cong 0\}$$

This geometric invariant is faithful in the sense that it detects vanishing of the object M; it also behaves nicely with respect to the standard operations in $\mathsf{D}(R)$: completing triangles, shifts, direct sums and tensor products.

One result in commutative algebra which motivates some of our considerations in modular representation theory is Neeman's classification of colocalising subcategories in $\mathsf{D}(R)$ [16]: namely, there is one-to-one correspondence

$$\left\{ \begin{array}{c} \text{Colocalising} \\ \text{subcategories of } \mathsf{D}(R) \end{array} \right\} \sim \left\{ \begin{array}{c} \text{subsets of} \\ \mathsf{Spec}\, R \end{array} \right\}$$

Combined with Neeman's classification of localizing subcategories, this gives one-to-one correspondence

$$\left\{ \begin{array}{c} \text{Colocalising} \\ \text{subcategories of } \mathsf{D}(R) \end{array} \right\} \sim \left\{ \begin{array}{c} \text{Localising} \\ \text{subcategories of } \mathsf{D}(R) \end{array} \right\}$$

given by taking a subcategory C to $C^{\perp}$

Finally, restricted to $\mathsf{D}^{\text{perf}}(R)$ this gives one-to-one correspondence between the thick subcategories of $\mathsf{D}^{\text{perf}}(R)$ and specialization closed subsets of $\mathsf{Spec}\, R$. The latter can be expressed in the language of *triangular geometry* introduced by P. Balmer [1]:

$$\mathsf{Spec}_{\text{Bal}}\, \mathsf{D}^{\text{perf}}(R) \cong \mathsf{Spec}\, R \tag{1}$$

where the left hand side is the *Balmer spectrum* of the tensor triangulated category $\mathsf{D}^{\text{perf}}(R)$.

We see that for schemes the notion of a "point" can be realized on the level of categories: namely, a scheme theoretic point $\mathsf{Spec}\, K \to \mathsf{Spec}\, R$ corresponds to a ring map $R \to K$ which gives rise to a triangulated functor which is the specialization: $\mathsf{D}^{\text{perf}}\, R \to \mathsf{D}^{\text{perf}}\, K$. Finally, applying the Balmer spectrum, we get back the original point $\mathsf{Spec}\, K \to \mathsf{Spec}\, R$. Hence, the category $\mathsf{D}^{\text{perf}}\, K$ together with the triangulated functor $\mathsf{D}^{\text{perf}}\, R \to \mathsf{D}^{\text{perf}}\, K$ "realizes" the point on the spectrum on the categorical level. It is this construction that we would like to mimic in modular representation theory.

Now let k be an algebraically closed field of positive characteristic, and let G be a finite group scheme defined over k. The coordinate algebra of G, $k[G]$, is a finite dimensional commutative Hopf algebra. We denote its linear dual, $\mathsf{Hom}_k(k[G], k)$, by kG. This is a finite dimensional cocommutative Hopf algebra whose category of modules is equivalent to the category of rational representations of G over k. Hence, we may identify representations of G with kG-modules; for the rest of this note we shall refer to representations of G as *G-modules*. Examples of finite group schemes include finite groups, restricted Lie algebras and Frobenius kernels of algebraic groups.

The tensor triangulated category associated to G is the stable module category $\mathsf{StMod}\,G$. Recall that the objects of $\mathsf{StMod}\,G$ are G-modules, whereas the Hom-sets are defined as follows:

$$\underline{\mathrm{Hom}}(M, N) := \frac{\mathrm{Hom}_G(M, N)}{\mathrm{PHom}_G(M, N)}$$

with $\mathrm{PHom}_G(M, N)$ being the subset of all G-maps between M and N which factor through a projective G-module. The category $\mathsf{StMod}\,G$ is a compactly generated tensor triangulated category with the compact objects being the finite dimensional G-modules. This subcategory is denoted $\mathsf{stmod}\,G$.

Let $R = H^*(G, k)$ be the cohomology algebra of G. It is finitely generated as a k-algebra by a celebrated theorem of Friedlander and Suslin [15]. We have R acting on $\underline{\mathsf{Hom}}^*(M, N) = \bigoplus_{n\in\mathbb{Z}} \underline{\mathsf{Hom}}^n(M, N)$ via Yoneda product. The question motivated by the classical construction in commutative algebra is then the following:

Let M be a G-module, and $\mathfrak{p} \in \mathsf{Proj}\,R$ be a homogenenous prime ideal strictly contained in the irrelevant ideal. How can we "specialize" or "localize" M at $\mathfrak{p}$?

We give two answers to this question: one involves a representation theoretic construction and the notion of a π-point and the other uses the local cohomology functors introduced by Benson, Iyengar and Krause.

Definition 2 ([13], [14])**.** *A* π-point *of G, defined over a field extension K of k, is a flat map of K-algebras*

$$\alpha : K[t]/(t^p) \to KG_K$$

which factors through the group algebra of a unipotent abelian subgroup scheme C of G_K.

Given a π-point $\alpha : K[t]/(t^p) \to KG_K$, we can construct a point $\mathfrak{p} = \Psi(\alpha) \in \mathsf{Proj}\,R$ as follows. Let $H^*(\alpha)$ be the map on cohomology induced by α:

$$H^*(\alpha) : H^*(G, k) \xrightarrow{-\otimes_k K} H^*(G, K) \xrightarrow{\alpha^*} H^*(K[t]/t^p, K)\ ,$$

and define $\Psi(\alpha) := \sqrt{\mathsf{Ker}\, H^*(\alpha)}$. This determines a surjective correspondence:

Theorem 3 ([13], [14])**.** *For any $\mathfrak{p} \in \mathsf{Proj}\,R$ there exists a π-point α of G such that*

$$\Psi(\alpha) = \mathfrak{p}.$$

In this way, we can "realize" points on $\mathsf{Proj}\, R$ as π-points. This, in turn, gives a way to specialize G-modules at prime ideals on $\mathsf{Spec}\, R$ and to define supports (and cosupports).

For M a G-module, and $\mathfrak{p} \in \mathsf{Proj}\, R$ a homogenenous prime ideal, we consider a π-point $\alpha_\mathfrak{p}$ such that $\Psi(\alpha_\mathfrak{p}) = \mathfrak{p}$. Then the pull-back $\alpha_\mathfrak{p}^*(K \otimes_k M)$ which is a $K[t]/t^p$-module plays the role of a "specialization" of M at $\mathfrak{p}$.

Definition 4. *The* π-support *of M is the subset of* $\mathsf{Proj}\, H^*(G,k)$ *defined by*

$$\pi\text{-}\operatorname{supp}(M) := \{\mathfrak{p} \in \mathsf{Proj}\, H^*(G,k) \mid \alpha_\mathfrak{p}^*(K \otimes_k M) \textit{ is not projective}\}.$$

The π-cosupport *of M is the subset of* $\mathsf{Proj}\, H^*(G,k)$ *defined by*

$$\pi\text{-}\operatorname{cosupp}(M) := \{\mathfrak{p} \in \mathsf{Proj}\, H^*(G,k) \mid \alpha_\mathfrak{p}^*(\mathsf{Hom}_k(K,M)) \textit{ is not projective}\}.$$

It was proved in [13] that these invariants are well-defined (that is, independent of the choice of $\alpha_\mathfrak{p}$ corresponding to $\mathfrak{p}$.

The usefulness of π-support and π-cosupport is postulated in the following theorem.

Theorem 5. i *[Detection] Let M be a G-module. Then $\pi\text{-}\operatorname{supp} M = \emptyset$ if and only if M is a projective G-module.*

ii *[Tensor and Hom formulae] Let M and N be G-modules. Then there are equalities*

$$\pi\text{-}\operatorname{supp}(M \otimes_k N) = \pi\text{-}\operatorname{supp}(M) \cap \pi\text{-}\operatorname{supp}(N),$$
$$\pi\text{-}\operatorname{cosupp}(\mathsf{Hom}_k(M,N)) = \pi\text{-}\operatorname{supp}(M) \cap \pi\text{-}\operatorname{cosupp}(N).$$

The detection property is an ultimate generalization of the famous Dade's lemma [12]. It builds on the work of many authors, see [3], [2], [17], [18]. In this generality it is proved in [7].

From the triangular geometry point of view, a π-point α gives rise to a restriction functor

$$\mathsf{StMod}\, G \to \mathsf{StMod}\, K[t]/t^p$$

which, once we apply Balmer's Spec construction, realizes the corresponding point $\Psi(\alpha) \in \mathsf{Spec}\, H^*(G,k)$.

A different approach to localization is given by Benson-Iyngar-Krause local cohomology functors. To any homogeneous prime ideal $\mathfrak{p} \in \mathsf{Proj}\, R$ one associates a universal local cohomology module $\Gamma_\mathfrak{p}$ (see [4]). Then the *cohomological* support and cosupport are defined as follows:

Definition 6 ([4], [5])**.**

$$\operatorname{supp}(M) := \{\mathfrak{p} \in \mathsf{Proj}\, H^*(G,k) \mid \Gamma_\mathfrak{p}(k) \otimes_k M \textit{ is not projective}\}.$$

$$\operatorname{cosupp}(M) := \{\mathfrak{p} \in \mathsf{Proj}\, H^*(G,k) \mid \mathsf{Hom}_k(\Gamma_\mathfrak{p}(k), M) \textit{ is not projective}\}.$$

One important property of universal local cohomology modules developed in [7], [9] is that they satisfy the "reduction to closed points principle":

Theorem 7. *Let $\mathfrak{p}$ be a point on $\mathsf{Proj}\, H^*(G,k)$, and let $d = \dim H^*(G,k)/\mathfrak{p}$. There exists a field extension K/k of transcendence degree d, and a maximal ideal $\mathfrak{m} \in \mathsf{Proj}\, H^*(G_K, K)$ lying over $\mathfrak{p}$, such that there is an isomorphism*

$$\Gamma_{\mathfrak{p}} \cong \mathsf{Res}_G^{G_K}(\Gamma_{\mathfrak{m}} K \otimes K//b)$$

Here, $K//b$ is a *Koszul object* associated to the prime ideal $\mathfrak{p}$.

The point of this theorem is that it allows to reduce questions at prime ideal $\mathfrak{p} \in \mathsf{Proj}\, R$ to closed point, that is, to maximal homogeneous prime ideals $\mathfrak{m} \in \mathsf{Spec}\, R_K$ where they become more approachable.

Corollary 8. *In the notation of the theorem, the restriction functor*

$$\Gamma_{\mathfrak{m}}(\mathsf{StMod}\, G_K) \to \Gamma_{\mathfrak{p}}(\mathsf{StMod}\, G)$$

is full and dense.

The Detection theorem 5 is the key step in identifying the two support theories: the π- supp and the local cohomology support of Benson-Iyengar-Krause. That unified support theory, combined with the powerful reduction to closed points principle and one more new construction, that of a point module associated to a π-point α, allows us to prove the ultimate analogue of Neeman's classification for finite groups schemes:

Theorem 9 ([8])**.** *For any finite group scheme G, there is a one-to-one correspondence*

$$\left\{ \begin{array}{c} \textit{Colocalizing Hom-closed} \\ \textit{subcategories of } \mathsf{StMod}\, G \end{array} \right\} \sim \left\{ \begin{array}{c} \textit{subsets of} \\ \mathsf{Proj}\, H^*(G,k) \end{array} \right\}$$

given by cosupport.

This classification implies in the usual manner the classification for localising tensor ideal subcategories in $\mathsf{StMod}\, G$ and the tensor ideal subcategories in $\mathsf{stmod}\, G$ but it will be misleading given the historical development of the subject to state these classifications as corollaries.

Another application of the local techniques we develop, including the reduction to closed points principle, is the local Serre duality for the category $\Gamma_{\mathfrak{p}}(\mathsf{StMod}\, G)$. In the theorem below, $I_{\mathfrak{p}}$ is the universal injective cohomology object in $\mathsf{StMod}\, G$ introduced in [10], and δ_G is the one dimensional modular character of G which in a sense measures how far kG is from being symmetric. The functor $\Omega^d \delta_G \otimes_k -$ plays the role of the local Serre functor in the sense of Bondal-Kapranov [11]

Theorem 10 ([9])**.** *Let $C = (\Gamma_{\mathfrak{p}}\, \mathsf{StMod}\, G)^c$ be the category of compact objects in $\Gamma_{\mathfrak{p}}\, \mathsf{StMod}\, G$, and let $M, N \in \mathcal{C}$. There is a natural isomorphism:*

$$\mathsf{Hom}_R(\mathsf{Hom}^*_{\mathcal{C}}(M,N), I_{\mathfrak{p}}) \simeq \mathsf{Hom}_{\mathcal{C}}(N, \Omega^d \delta_G \otimes M)$$

where $d = \dim R/\mathfrak{p}$.

References

[1] P. Balmer, The spectrum of prime ideals in tensor triangulated categories, *J. für die Reine und Ang. Math. (Crelle)*, 588, (2005), 149–168.

[2] C. Bendel, *Cohomology and projectivity of modules for finite group schemes*, Math. Proc. Camb. Phil. Soc. **131** (2001), 405–425.

[3] D. J. Benson, J. F. Carlson, and J. Rickard, *Thick subcategories of the stable module category*, Fundamenta Mathematicae **153** (1997), 59–80.

[4] D. J. Benson, S. B. Iyengar, and H. Krause, *Local cohomology and support for triangulated categories*, Ann. Scient. Éc. Norm. Sup. (4) **41** (2008), 575–621.

[5] D. J. Benson, S. B. Iyengar, and H. Krause, *Colocalising subcategories and cosupport*, J. Reine & Angew. Math. **673** (2012), 161–207.

[6] D. J. Benson, S. B. Iyengar, H. Krause, and J. Pevtsova, *Stratification and π-cosupport: Finite groups*, arXiv:1505.06628, 2015.

[7] D. J. Benson, S. B. Iyengar, H. Krause, and J. Pevtsova, *Stratification and π-cosupport: Finite groups schemes*, in preparation.

[8] Colocolising subcategories of modules over finite groups schemes, to appear in the Annals of K-theory.

[9] Local duality for representations of finite group schemes, preprint 2016.

[10] D. J. Benson and H. Krause, *Pure injectives and the spectrum of the cohomology ring of a finite group*, J. Reine Angew. Math. **542** (2002), 23–51.

[11] A. I. Bondal and M. M. Kapranov, *Representable functors, Serre functors, and mutations*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 6, 1183–1205, 1337; translation in Math. USSR-Izv. **35** (1990), no. 3, 519–541.

[12] E. C. Dade, *Endo-permutation modules over p-groups, II*, Ann. of Math **108** (1978), 317–346.

[13] E. M. Friedlander and J. Pevtsova, *Representation theoretic support spaces for finite group schemes*, Amer. J. Math. **127** (2005), 379–420, correction: AJM **128** (2006), 1067–1068.

[14] E. M. Friedlander and J. Pevtsova, Π-*supports for modules for finite groups schemes*, Duke Math. J. **139** (2007), 317–368.

[15] E. M. Friedlander and A. Suslin, *Cohomology of finite group schemes over a field*, Invent. Math. **127** (1997), 209–270.

[16] A. Neeman, *Colocalizing subcategories of $D(R)$*, J. Reine Angew. Math. **653** (2011), 221–243.

[17] J. Pevtsova, *Infinite dimensional modules for Frobenius kernels*, J. Pure & Applied Algebra **173** (2002), 59–86.

[18] J. Pevtsova, *Support cones for infinitesimal group schemes*, Hopf Algebras, Lect. Notes in Pure & Appl. Math., **237**, Dekker, New York, (2004), 203–213.