# GENERIC JORDAN TYPE OF MODULAR REPRESENTATIONS

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Elementary abelian $p$-subgroups of a finite group $G$ capture significant aspects of the cohomology and representation theory of $G$. For example, if $k$ is a field of characteristic $p > 0$, then a theorem of D. Quillen [9] asserts that the Krull dimension of the cohomology algebra $H^\bullet(G, k)$ is equal to the maximum of the ranks of elementary abelian $p$-subgroups of $G$ and a theorem of L. Chouinard [3] asserts that a $kG$-module is projective if and only if its restrictions to all elementary abelian $p$-subgroups of $G$ are projective. Quillen's geometric description of $\operatorname{Spec} H^\bullet(G, k)$ [9] provides the basis for interesting invariants of $kG$-modules, most notably the cohomological support variety $|G|_M$ of a $kG$-module $M$.

The geometric methods developed by Quillen were further applied to study representations of other algebraic structures, such as restricted Lie algebras. E. Friedlander and B. Parshall developed a theory of support varieties for finite dimensional $p$-restricted Lie algebras $g$ over a field $k$ of characteristic $p > 0$ (e.g., [4]). For restricted Lie algebras, the role of the group algebra $kG$ of the finite group $G$ is played by the restricted enveloping algebra $u(g)$. Indeed, restricted Lie algebras lead one to more interesting geometrical structure than do finite groups, and seemingly lead to stronger results. For example, the theorem of G. Avrunin and L. Scott [1] identifying the cohomological support variety $|E|_M$ of a finite dimensional $kE$ module $M$ for an elementary abelian $p$-group $E$ with the rank variety of J. Carlson [2] admits a formulation in the case of a restricted Lie algebras $g$ in terms of closed subvarieties of the $p$-nilpotent cone of $g$ (cf. [4], [8], [11]).

A uniform approach to the study of the cohomology and related representation theory of all finite group schemes was presented in [5], [6]. This approach involves the use of $\pi$-points of $G$, which are finite flat maps of $K$-algebras $K[t]/t^p \to KG$ for field extensions $K/k$; these play the role of "cyclic shifted subgroups" in the case that $G$ is an elementary abelian $p$-group and the role of 1-parameter subgroups in the case that $G$ is an infinitesimal group scheme. In [6], the space $\Pi(G)$ of equivalence classes $[\alpha_K]$ of $\pi$-points $\alpha_K : K[t]/t^p \to KG$ of $G$ is given a scheme structure without reference to cohomology such that $\Pi(G)$ is isomorphic as a scheme to $\operatorname{Proj} H^\bullet(G, k)$. In particular, there is a natural bijection between such equivalence classes of $\pi$-points and homogeneous prime ideals of $H^\bullet(G, k)$.

The purpose of the talk, though, is to demonstrate that one can go further in the search of information about modules encoded geometrically.

For a finite group scheme $G$ over a field $k$ of characteristic $p > 0$, we associate new invariants to a finite dimensional $kG$-module $M$. Namely, for each generic point of the projectivized cohomological variety $\operatorname{Proj} H^\bullet(G, k)$ we exhibit a "generic Jordan type" of $M$. We prove the following theorem which both describes the invariant and demonstrates that it is well-defined.

**Theorem 1.** *Let $G$ be a finite group scheme, let $M$ be a finite dimensional $G$-module and let $\alpha_K : K\mathbb{Z}/p \to KG$ be a $\pi$-point of $G$ which represents a generic point $[\alpha_K] \in \Pi(G)$. Then the Jordan type of $\alpha_K(t)$ viewed as a nilpotent operator on $M_K$ depends only upon $[\alpha_K]$ and not the choice of $\alpha_K$ representing $[\alpha_K]$.*

The Jordan type of such $\alpha_K(t)$ is called the *generic Jordan type* of $M$. In a special case when $G = E$ is an elementary abelian p-group, the theorem specializes to the non-trivial observation that the Jordan type obtained by restricting $M$ via a generic cyclic shifted subgroup does not depend upon a choice of generators for $E$.

We verify that sending a module $M$ to its generic Jordan type $[\alpha_K]^*(M_K)$ for generic $[\alpha_K] \in \operatorname{Proj} H^\bullet(G,k)$ determines a well-defined tensor triangulated functor on stable module categories

$$[\alpha_K]^* : stmod(kG) \to stmod(K[t]/t^p).$$

The second invariant we present is the *non-maximal support variety* of a finite-dimensional $kG$-module $M$, $\Gamma(G)_M \subset \operatorname{Proj} H^\bullet(G,k)$. The non-maximal support variety coincides with the "classical" support when the module is generically projective but gives a new non-tautological invariant in the case when the "classical" support variety of $M$ is the entire cohomological spectrum: for example, when the dimension of $M$ is not divisible by $p$.

This is a joint work with E. Friedlander and A. Suslin.

## References


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