REPRESENTATIONS OF ELEMENTARY ABELIAN p-GROUPS AND BUNDLES ON GRASSMANNIANS

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Abstract. We initiate the study of representations of elementary abelian p-groups via restrictions to truncated polynomial subalgebras of the group algebra generated by r nilpotent elements, $k[t_1, \ldots, t_r]/(t_1^p, \ldots, t_r^p)$. We introduce new geometric invariants based on the behavior of modules upon restrictions to such subalgebras. We also introduce modules of constant radical and socle type generalizing modules of constant Jordan type and provide several general constructions of modules with these properties. We show that modules of constant radical and socle type lead to families of algebraic vector bundles on Grassmannians and illustrate our theory with numerous examples.

Contents

Quillen's fundamental ideas on applying geometry to the study of group cohomology in positive characteristic [Quillen71] opened the door to many exciting developments in both cohomology and modular representation theory. Cyclic shifted subgroups, the prototypes of the rank r shifted subgroups studied in this paper, were introduced by Dade in [Dade78] and quickly became the subject of an intense study. In [AS82], Avrunin and Scott proved the conjecture of the first author tying the cohomological support variety originating from Quillen's approach with the variety of shifted subgroups (rank variety) introduced in [Car83].

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These ideas were successfully applied to restricted Lie algebras ([FPa86]) and, more generally, infinitesimal group schemes ([SFB1], [SFB2]) yielding many surprising geometric results which also underline the very different nature of infinitesimal group schemes and finite groups. Nonetheless, in [FP05], [FP07], the second and third authors found a unifying tool, called π-points, that allowed the generalization of cyclic shifted subgroups and the Avrunin-Scott's theorem to any finite group scheme.

In a surprising twist, the π-point approach has led to new discoveries even for elementary abelian p-groups, the context in which cyclic shifted subgroups were originally introduced. Among these, the most relevant to the present paper are modules of constant Jordan type ([CFP08]) and the connection between such modules and algebraic vector bundles on projective varieties ([FP11], see also [BP] and [Ben] for a treatment specific to elementary abelian p-groups).

Equipped with the understanding of the versatility as well as the limits of cyclic shifted subgroups, we set out on the quest of studying modular representations via their restrictions to rank r shifted subgroups. Following the original course of the development of the theory, we devote this paper entirely to modular representations of an elementary abelian p-group E over an algebraically closed field k of positive characteristic p. A *rank r shifted subgroup* of the group algebra kE is a subalgebra $C \subset kE$ isomorphic to a group algebra of an elementary abelian p-group of rank r, for $1 \leq r < n$, with the property that kE is free as a C–module. For an E-module M, we consider restrictions of M to such subalgebras C of kE. The concept of a "shifted subgroup" exists in the literature (see, e.g., [Ben91]) but no systematic study of such restrictions has been undertaken for $r > 1$.

Throughout the paper, we choose an n-dimensional linear subspace $\mathbb{V} \subset \operatorname{Rad}(kE)$ which gives a splitting of the projection $\operatorname{Rad}(kE) \to \operatorname{Rad}(kE)/\operatorname{Rad}^2(kE)$. Once such a $\mathbb{V}$ is fixed, we consider only the rank r shifted subgroups which are determined by a linear subspace of $\mathbb{V}$. Such shifted subgroups are naturally parametrized by the Grassmann variety $\operatorname{Grass}(r, \mathbb{V})$ of r-planes in $\mathbb{V}$. In Section 2 we prove a partial generalization of the main result in [FPS07] showing that some of the invariants we introduce do not depend on the choice of $\mathbb{V}$.

The paper naturally splits into two parts. In the first part which occupies Sections 1 through 5, we introduce new geometric and numerical invariants for modules arising from their restrictions to rank r shifted subgroups and then construct many examples to reveal some of the interesting behavior of these invariants. We show how to associate subvarieties of $\operatorname{Grass}(r, \mathbb{V})$ to a finite dimensional kE-module M; for $r = 1$, these subvarieties are refinements of the rank variety of M. In the second half of the paper we construct and study algebraic vector bundles on $\operatorname{Grass}(r, \mathbb{V})$ associated to certain kE–modules, extending the construction for $r = 1$ first introduced in [FP11].

Whereas the isomorphism type of a $k[t]/t^p$–module M is specified by a p–tuple of integers (the Jordan type of M), there is no such classification for $r > 1$. Indeed, except in the very special case in which $p = 2 = r$, the category of finite dimensional $C \simeq k[t_1, \ldots, t_r]/(t_1^p, \ldots, t_r^p)$–modules is wild. For $r \geq 1$, we consider dimensions of C–socles and C–radicals of a given kE–module M as C ranges over rank r shifted subgroups of kE. For $r = 1$, this numerical data is equivalent to the Jordan type of M. Although these ranks do not determine the isomorphism types of the restrictions of a given kE–module M for $r > 1$, they do provide intriguing new invariants for M.

Extending our earlier investigations of kE–modules of constant Jordan type, we formulate in (3.1) and then study the condition on a kE–module M that it have constant r–radical type or constant r–socle type. We introduce invariants for kE–modules which do not have constant r–radical type (or constant r–socle type). Our simplest invariant, a straightforward generalization of the rank variety of a kE–module M, is the r–rank variety $\mathrm{Grass}(r,\mathbb{V})_M \subset \mathrm{Grass}(r,\mathbb{V})$. More elaborate geometric invariants, also closed subvarieties of $\mathrm{Grass}(r,\mathbb{V})$, extend the generalized support varieties of [FP10].

The generalization to $r > 1$ raises many interesting questions for which we have only partial answers. For example, even though the rank r shifted subgroups are parametrized by $\mathrm{Grass}(r,\mathbb{V})$, for $r > 1$ the Zariski topology on this Grassmannian is not easily obtained from the representation theory of kE. This stands in stark contrast with the situation for $r = 1$ where the realization theorem asserts that any closed subvariety of the support variety of a finite group G is realized as a support (equivalently, rank) variety of some finite dimensional representation of G as proved in [Car84]. For $r = 1$, the Avrunin–Scott's theorem says that the rank variety of a kE-module M has an interpretation in terms of the action of $\mathrm{H}^*(kE,k)$ on $\mathrm{Ext}^*_{kE}(M,M)$; we know of no such cohomological interpretation for $r > 1$. Theorem 2.9 is a partial generalization to $r \geq 1$ of the fundamental theorem of [FPS07] concerning maximal Jordan type, yet we do not have the full generalization to all radical ranks.

We verify that the classes of kE–modules of constant r–radical type or constant r–socle type share some of the good properties of the class of modules of constant Jordan type. Informed by a variety of examples, we develop some sense of the complicated nature and independence of the condition of being of constant socle versus radical type. Many of our examples have very rich symmetries and, hence, have constant r–radical type and r–socle type for all r, $1 \leq r < n$. On the other hand, in Section 4 we introduce modules arising from quantum complete intersections which have much less symmetry and, therefore, much more intricate properties. In particular, we exhibit kE–modules which have constant 2–radical type but not constant 2–socle type. Using Carlson modules L_ζ in Section 5, we produce examples of modules which have constant r–radical type for a given r, $1 < r < n$, but not constant s–radical type for any s, $1 \leq s < r$. We also construct modules which have constant s–radical type all s, $1 \leq s < r < n$, but not constant r–radical type. Thanks to the duality of radicals of M and socles of $M^\#$, examples of constant radical types lead to examples of constant socle types.

The second part of the paper is dedicated to the construction of algebraic vector bundles on $\mathrm{Grass}(r,\mathbb{V})$ associated to kE-modules of constant r-radical type or constant r-socle type (and, more generally, to kE-modules of constant r–Rad^j rank or constant r–Soc^j rank for j, $1 \leq j \leq r(p-1)$, as defined in (3.1)). All are associated to images or kernels of the restrictions of the kE-module M to rank r shifted subgroups $C \subset kE$ indexed by points of $\mathrm{Grass}(r,\mathbb{V})$. We construct these bundles using various complementary techniques:

(1) patching images or kernels of local operators on standard affine open subsets of $\mathrm{Grass}(r,\mathbb{V})$ (Section 6.1);

(2) applying equivariant descent to images or kernels of global operators on Stiefel varieties over $\mathrm{Grass}(r, \mathbb{V})$ (Section 7);
(3) investigating explicit actions on graded modules over the homogeneous coordinate ring of $\mathrm{Grass}(r, \mathbb{V})$ generated by Plücker coordinates (Section 8).

We mention a few specific results of this paper. In Section 1, we investigate the generalization $\mathrm{Grass}(r, \mathbb{V})_M \subset \mathrm{Grass}(r, \mathbb{V})$ of the classical rank variety of a kE-module M; the choice of $\mathbb{V} \subset \mathrm{Rad}(kE)$ is less restrictive than the classical choice of a basis of $\mathrm{Rad}(kE)$ modulo $\mathrm{Rad}^2(kE)$. As shown in Corollary 2.5, $\mathrm{Grass}(r, \mathbb{V})_M \subset \mathrm{Grass}(r, \mathbb{V})$ and its refinements are closed subvarieties of $\mathrm{Grass}(r, \mathbb{V})$; moreover, in Corollary 2.10, we show that $\mathrm{Grass}(r, \mathbb{V})_M$ is essentially dependent only upon M and not upon a choice of $\mathbb{V} \subset \mathrm{Rad}(kE)$. In Section 3, we consider various classes of modules which have constant r–radical type and constant r–socle type for all r. The examples of Sections 4 and 5 reveal some of the subtle possibilities for restrictions of kE-modules to rank r shifted subgroups C of kE. The quantum complete intersections of Section 4 are perhaps new, and certainly not fully understood. The Carlson modules L_ζ of Section 5 show a surprising variability of behavior.

Section 6 contains two constructions of bundles arising from modules of constant socle or radical type. In Proposition 6.1, we show that kernels and images of some local operators defined via explicit equations on principal affine opens of $\mathrm{Grass}(r, \mathbb{V})$ patch together to give globally defined coherent sheaves associated to a given kE-module M, $\mathcal{K}er^{\ell}(M)$ and $\mathcal{I}m^{\ell}(M)$. Theorem 6.2 proves that starting with a kE–module of constant socle or radical type we get a locally free sheaf (equivalently, an algebraic vector bundle) on $\mathrm{Grass}(r, \mathbb{V})$. Finally, in Theorem 6.8, we prove that the local construction of bundles coincides with the construction by equivariant descent as described in §6.2.

In Section 7, we concentrate on algebraic vector bundles on $\mathrm{Grass}(r, \mathbb{V})$ associated to various GL_n-equivariant kE-modules introduced in Definition 3.5. For such kE-modules, Theorem 7.6 provides a useful method of determining their associated vector bundles on $\mathrm{Grass}(r, \mathbb{V})$ using a standard construction from the representation theory of reductive algebraic groups. We find that many familiar vector bundles on $\mathrm{Grass}(r, \mathbb{V})$ arise in this manner and fill the second half of Section 7 with examples. To demonstrate the explicit nature of our techniques, we show in Section 8 how to calculate (typically, with the aid of a computer) "generators" of kernel bundles arising from homogeneous elements of graded modules over the homogeneous coordinate algebra of $\mathrm{Grass}(r, \mathbb{V})$.

The appendix, written by the first author, shows how one can calculate explicitly generalized rank varieties for small examples using MAGMA. Any reader interested in obtaining the programs used for calculations should contact the first author.

Throughout this paper, k will denote an algebraically closed field of characteristic $p > 0$.

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1. The r-rank variety $\mathrm{Grass}(r, \mathbb{V})_M$

Throughout this section, E is an elementary abelian p group of rank $n \geq 1$ and r is a fixed integer satisfying $1 \leq r \leq n$. Recall that the group algebra kE is isomorphic to the truncated polynomial algebra $k[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$.

We choose a subspace $\mathbb{V} \subset \operatorname{Rad}(kE)$ of the radical of kE with the property that $\mathbb{V}$ is a choice of splitting of the projection $\operatorname{Rad}(kE) \to \operatorname{Rad}(kE)/\operatorname{Rad}^2(kE)$; in other words, the composition $\rho_{\mathbb{V}} : \mathbb{V} \to \operatorname{Rad}(kE) \to \operatorname{Rad}(kE)/\operatorname{Rad}^2(kE)$ is an isomorphism. Observe that if $\mathbb{W} \subset \operatorname{Rad}(kE)$ is another choice of splitting, then there is a unique map $\psi : \mathbb{V} \to \mathbb{W}$ such that $\rho_{\mathbb{W}} \circ \psi = \rho_{\mathbb{V}}$; that is, the following diagram commutes:

$$\begin{tikzcd} \mathbb{V} \arrow[rr, "\psi"] \arrow[dr, "\rho_{\mathbb{V}}"'] & & \mathbb{W} \arrow[dl, "\rho_{\mathbb{W}}"] \\ & \frac{\operatorname{Rad}(kE)}{\operatorname{Rad}^2(kE)} & \end{tikzcd}$$

Our choice of $\mathbb{V} \subset \operatorname{Rad}(kE)$ provides an identification

$$S^*(\mathbb{V})/\langle v^p, v \in \mathbb{V}\rangle \;\cong kE \tag{1.0.1}$$

which we employ throughout this paper.

For $r = 1$, rank varieties were originally defined in terms of a choice of $\mathbb{V} \subset \operatorname{Rad}(kE)$ together with a choice of ordered basis for $\mathbb{V}$; these $r = 1$ rank varieties have an interpretation in terms of cohomology, and thus are independent of such choices. More refined support varieties for $r = 1$ are also independent of such choices, thanks to results of [FPS07]. For $r > 1$, we do not have a cohomological interpretation of r-rank varieties, so that we take some care in establishing invariance properties. In particular, we consistently avoid specifying an ordered basis of $\mathbb{V}$.

We consider r-planes $U \subset \mathbb{V}$ (i.e., subspaces of the k-vector space $\mathbb{V}$ of dimension r). We recall the projective algebraic variety $\operatorname{Grass}(r, \mathbb{V})$ whose (closed) points are r-planes of $\mathbb{V}$. We construct this Grassmannian by fixing some r-plane $U_0 \subset \mathbb{V}$ and considering the set of k-linear maps of maximal rank

$$\operatorname{Hom}_k(U_0, \mathbb{V})^o \;\subset\; \operatorname{Hom}_k(U_0, \mathbb{V}); \tag{1.0.2}$$

then

$$\operatorname{Grass}(r, \mathbb{V}) \;\equiv\; \operatorname{GL}(\mathbb{V})/\operatorname{Stab}(U_0) \;\cong\; \operatorname{Hom}_k(U_0, \mathbb{V})^o/\operatorname{GL}(U_0).$$

In particular, we observe for later use that there is a natural transitive (left) action of $\operatorname{GL}(\mathbb{V})$ on $\operatorname{Grass}(r, \mathbb{V})$. We view $\operatorname{Hom}_k(U_0, \mathbb{V})^o \to \operatorname{Grass}(r, \mathbb{V})$ as the principal $\operatorname{GL}(U_0)$–bundle whose fiber above an r-plane $U \in \operatorname{Grass}(r, \mathbb{V})$ consists of vector space bases of U. Provide $\mathbb{V}$ with an ordered basis and choose U_0 to be the span of the first r basis elements, so that $\operatorname{Hom}_k(U_0, \mathbb{V})$ is identified with the affine space $\mathbb{A}^{nr}$. Then $\operatorname{Grass}(r, \mathbb{V})$ is given the identification

$$\operatorname{Grass}_{n,r} \;\equiv\; \operatorname{GL}_n/\operatorname{P}_{r,n-r} \cong\; M^o_{n,r}/\operatorname{GL}_r, \tag{1.0.3}$$

where $M_{n,r}$ is the affine space of $n \times r$-matrices, where $M^o_{n,r} \subset M_{n,r}$ consists of those matrices of rank r, and where $\operatorname{P}_{r,n-r} \simeq \operatorname{Stab}(U_0)$ is the standard parabolic subgroup stabilizing the vector $[\underbrace{1, \ldots, 1}_{r}, 0, \ldots, 0]$ in the standard representation of GL_n.

We employ the Plücker embedding $\mathfrak{p} : \operatorname{Grass}(r, \mathbb{V}) \;\hookrightarrow\; \mathbb{P}(\Lambda^r(\mathbb{V}))$ of $\operatorname{Grass}(r, \mathbb{V})$, providing $\operatorname{Grass}(r, \mathbb{V})$ with the structure of a closed subvariety of projective space. Once we choose an ordered basis for $\mathbb{V}$, this embedding can be described explicitly as follows. The inclusion (1.0.2) becomes $M^o_{n,r} \;\subset M_{n,r}$. For any subset $\Sigma \subset$

$\{1,\dots,n\}$ of cardinality r, the Σ–*submatrix* of an $n\times r$ matrix $A\in M_{n,r}$ is the $r\times r$ matrix obtained by removing all rows indexed by numbers not in Σ. The Plücker coordinates $\{\mathfrak{p}_\Sigma(U)\}$ of the r-plane $U\in \mathrm{GL}_n/\mathrm{P}_{r,n-r}$ are the entries of the ordered $\binom{n}{r}$-tuple (well defined up to scalar multiple) obtained by taking any matrix $A\in\mathrm{GL}_n$ representing U and setting $\mathfrak{p}_\Sigma(U)$ equal to the determinant of the Σ–submatrix of A. In these terms, the Plücker embedding becomes

$$\mathfrak{p}:\mathrm{Grass}_{n,r}\ \hookrightarrow \mathbb{P}^{\binom{n}{r}-1},\quad U\ \mapsto [\mathfrak{p}_\Sigma(A_U)]. \tag{1.0.4}$$

The homogeneous coordinate ring of the Grassmanian can be written as the quotient of the polynomial ring on $\binom{n}{r}$ variables $\{\mathfrak{p}_\Sigma\}$ by the homogeneous ideal generated by standard Plücker relations.

We investigate kE-modules by considering their restrictions along flat maps

$$k[t_1,\dots,t_r]/(t_i^p)\ \to\ kE,$$

where we use $k[t_1,\dots,t_r]/(t_i^p)$ to denote $k[t_1,\dots,t_r]/(t_1^p,\dots,t_r^p)$. To give such a map is to choose an ordered r-tuple of elements of $\mathrm{Rad}(kE)$ which are linearly independent modulo $\mathrm{Rad}^2(kE)$. We formulate our consideration so that our maps are parametrized by $U\in\mathrm{Grass}(r,\mathbb{V})$.

For any r-plane $U\in\mathrm{Grass}(r,\mathbb{V})$, we define the finite dimensional commutative k algebra

$$C(U)\ \equiv\ S^*(U)/\langle u^p,u\in U\rangle\ \simeq\ k[t_1,\dots,t_r]/(t_i^p)$$

to be the quotient of the symmetric algebra $S^*(U)$ by the ideal generated by p-th powers of elements of $U\subset S^*(U)$. We naturally associate to each $U\in\mathrm{Grass}(r,\mathbb{V})$ the map of k-algebras

$$\alpha_U: C(U)\ \to\ kE \tag{1.0.5}$$

induced by $S^*(U)\to S^*(\mathbb{V})$, the projection $S^*(\mathbb{V})\to S^*(\mathbb{V})/\langle v^p,v\in\mathbb{V}\rangle$, and the identification of (1.0.1).

The following characterization of flatness for certain maps of k-algebras applies in particular to show that α_U is flat. The essence of the proof of this fact (for $r=1$) is present in [Car83]. Recall that a finitely generated module over a commutative, local ring (such as C) is flat if and only if it is free. If $\alpha: C\to A$ is a homomorphism of k-algebras and M is a C-module, then we denote by $\alpha^*(M)$ the restriction of M along α.

Proposition 1.1. *Consider a k-algebra homomorphism*

$$\alpha: C\ \equiv\ k[t_1,\dots,t_r]/(t_1^p,\dots,t_r^p)\ \longrightarrow\ k[x_1,\dots,x_n]/(x_1^p,\dots,x_n^p)\equiv A.$$

The map α is flat as a map of C-modules if and only if the images of $\alpha(t_1),\dots,\alpha(t_r)$ in $\mathrm{Rad}(A)/\mathrm{Rad}^2(A)$ are linearly independent.

Proof. We first consider the case $r=1$ so that $C=k[t_1]/t_1^p$. Write

$$\alpha': C\ \to\ A,\qquad \alpha'(t_1)\overset{\mathrm{def}}{=} a_1x_1+\ldots+a_nx_n\equiv\alpha(t_1)\ \mathrm{mod}\ \mathrm{Rad}^2(A).$$

By [CTVZ03, 9.5.10] or [FP05, 2.2], A is a free C-module with respect to $\alpha: C\to A$ if and only if $\alpha(t_1)$ acts freely on A if and only if A is a free C-module with respect to $\alpha': C\to A$. Hence, we may replace α by α'. Applying a linear automorphism to A which maps $\alpha(t_1)$ to x_1, we may assume that $\alpha(t_1)=x_1$. For A, given the structure of a C-module through such a map α, it is clear that A is free as a C-module.

We now assume $r > 1$ and equip A with the structure of a C-module through the given k-algebra homomorphism $\alpha : C \to A$. By Dade's Lemma ([Dade78], [FP07, 5.3]), A is free as a C-module if and only if $\beta^*(A)$ is free as a $k[t]/t^p$-module for every non-zero k-algebra homomorphism

$$\beta : k[t]/t^p \ \to \ C, \quad \beta(t) = b_1 t_1 + \cdots + b_r t_r \neq 0.$$

Applying the case $r = 1$, A is free as a C-module if and only if $\alpha \circ \beta(t) \not\equiv 0 \bmod \operatorname{Rad}^2(A)$ for all (non-zero) β which is the case if and only if the images of $\alpha(t_1), \ldots, \alpha(t_r)$ in $\operatorname{Rad}(A)/\operatorname{Rad}^2(A)$ are linearly independent. $\square$

We now introduce the r-rank variety of a finite dimensional kE-module M.

Definition 1.2. For any finite dimensional kE-module M, we denote by

$$\operatorname{Grass}(r, \mathbb{V})_M \ \subset \ \operatorname{Grass}(r, \mathbb{V})$$

the set of those r-planes $U \in \operatorname{Grass}(r, \mathbb{V})$ with the property that $\alpha_U^*(M)$ is not a free $C(U)$-module (where α_U is given in (1.0.5)). We say that $\operatorname{Grass}(r, \mathbb{V})_M$ is the r-rank variety of M.

Remark 1.3. As shown in Corollary 2.10, $\operatorname{Grass}(r, \mathbb{V})_M$ is independent of the choice of $\mathbb{V}$ in the sense that if $\mathbb{W} \subset \operatorname{Rad}(A)$ is another choice of splitting for the projection $\operatorname{Rad}(A) \to \operatorname{Rad}(A)/\operatorname{Rad}^2(A)$, then the unique isomorphism $\psi : \mathbb{V} \to \mathbb{W}$ commuting with the projections to $\operatorname{Rad}(A)/\operatorname{Rad}^2(A)$ induces an isomorphism $\Psi : \operatorname{Grass}(r, \mathbb{V})_M \xrightarrow{\sim} \operatorname{Grass}_r(\mathbb{W})_M$.

The following interpretation of $\operatorname{Grass}(r, \mathbb{V})_M$ in terms of classical (i.e., $r = 1$) rank varieties follows immediately from Dade's Lemma asserting that a C-module N is free if and only $\beta^*(N)$ is a free $k[t]/t^p$-algebra for every $\beta : k[t]/t^p \ \to \ C$, with $\beta(t) = b_1 t_1 + \cdots + b_r t_r \neq 0$.

Proposition 1.4. *For any finite dimensional kE-module M and any r-plane $U \in \operatorname{Grass}(r, \mathbb{V})$,*

$$\operatorname{Grass}(r, \mathbb{V})_M \ = \ \{U \in \operatorname{Grass}(r, \mathbb{V}); \operatorname{Grass}(1, U)_{\alpha_U^*(M)} \ \neq \emptyset\}.$$

We employ the notation of (1.0.2) and (1.0.3). The projective variety $\operatorname{Grass}_{n,r}$ has an open covering by affine pieces $\mathcal{U}_\Sigma \simeq \mathbb{A}^{(n-r)r}$, the GL_r-orbits of matrices $A = (a_{ij})$ such that $\mathfrak{p}_\Sigma(A) \neq 0$,

$$\mathcal{U}_\Sigma \ \equiv \ \mathfrak{p}^{-1}(\mathbb{P}^{\binom{n}{r}-1} \backslash Z(\mathfrak{p}_\Sigma)) \ \subset \ \operatorname{Grass}_{n,r}.$$

We consider the section of $M^o_{n,r} \to \operatorname{Grass}_{n,r}$ over $\mathcal{U}_\Sigma$ defined by sending a GL_r-orbit to its unique representative such that the Σ–submatrix is the identify matrix.

Suppose that $\Sigma = \{i_1, \ldots, i_r\}$ with $i_1 < \cdots < i_r$. Our choice of section identifies $k[\mathcal{U}_\Sigma]$ with the quotient

$$k[M_{n,r}] = k[Y_{i,j}]_{1 \leq i \leq n, 1 \leq j \leq r} \ \longrightarrow \ k[Y^\Sigma_{i,j}]_{i \notin \Sigma, 1 \leq j \leq r} = k[\mathcal{U}_\Sigma] \tag{1.4.1}$$

sending $Y_{i,j}$ to 1, if $i = i_j \in \Sigma$; to 0 if $i = i_{j'} \in \Sigma$ and $j \neq j'$; and to $Y^\Sigma_{i,j}$ otherwise. For notational convenience, we set $Y^\Sigma_{i,j}$ equal to 1, if $i \in \Sigma$ and $i = i_j$, and we set $Y^\Sigma_{i,j} = 0$ if $i = i_{j'} \in \Sigma$ and $j \neq j'$.

Definition 1.5. For any $\Sigma = \{i_1, \dots, i_r\}$ with $i_1 < \cdots < i_r$, we define the map of $k[\mathcal{U}_\Sigma]$-algebras

$\alpha_\Sigma : C \otimes k[\mathcal{U}_\Sigma] = k[t_1, \dots, t_r]/(t_i^p) \otimes k[\mathcal{U}_\Sigma] \ \to k[x_1, \dots, x_n]/(x_i^p) \otimes k[\mathcal{U}_\Sigma] = kE \otimes k[\mathcal{U}_\Sigma]$

via

$$t_j \mapsto \sum_{i=1}^n x_i \otimes Y_{i,j}^\Sigma.$$

Pick a basis for $\mathbb{V}$ and choose U_0 to be the span of the first r basis elements. For any $U \in \mathcal{U}_\Sigma \subset \mathrm{Grass}(r, \mathbb{V})$, these choices enable us to identify $\alpha_U : C(U) \to kE$ with the result of specializing α_Σ by setting the variables $Y_{i,j}^\Sigma$ to values $a_{i,j} \in k$, where $A_U = (a_{i,j}) \in M_{n,r}$ is the unique representation of U whose Σ–submatrix is the identity.

Proposition 1.6. *For any finite dimensional A-module M,* $\mathrm{Grass}(r, \mathbb{V})_M \subset \mathrm{Grass}(r, \mathbb{V})$ *is a closed subvariety.*

Proof. It suffices to pick an ordered basis for $\mathbb{V}$ and thus work with $\mathrm{Grass}_{n,r}$. It further suffices to show that for any $\Sigma \subset \{1, \dots, n\}$ of cardinality r,

$$\mathcal{U}_\Sigma \cap (\mathrm{Grass}_{n,r})_M \ \subset \ \mathcal{U}_\Sigma$$

is closed. Having made a choice of ordered basis for $\mathbb{V}$ and a choice of Σ with $U \in \mathcal{U}_\Sigma$, we may identify $C(U)$ with $C = k[t_1, \dots, t_r]/(t_i^p)$ and thus identify α_U as a map of the form $\alpha_U : C \to kE$. The condition that the finite dimensional C-module $\alpha_U^*(M)$ is not free is equivalent to the condition that

$$\dim(\mathrm{Rad}(\alpha_U^*(M)) \ < \ \frac{p^r - 1}{p^r} \cdot \dim(M). \tag{1.6.1}$$

We consider the $k[\mathcal{U}_\Sigma]$-linear map of free $k[\mathcal{U}_\Sigma]$-modules

$$\sum_{i=1}^r \alpha_\Sigma(t_i) : (M \otimes k[\mathcal{U}_\Sigma])^{\oplus r} \ \to \ M \otimes k[\mathcal{U}_\Sigma]. \tag{1.6.2}$$

Denote by $\Phi(M) \in M_{m,rm}(k[\mathcal{U}_\Sigma])$ the associated matrix, where $m = \dim M$. The rank of the specialization of $\Phi(M)$ at some point of $U \in \mathcal{U}_\Sigma$ equals the dimension of $\mathrm{Rad}(C) \cdot \alpha_U^* M$,

$$\mathrm{rk}(\Phi(M) \otimes_{k[\mathcal{U}_\Sigma]} k) \ = \ \dim(\mathrm{Rad}(\alpha_U^*(M)), \tag{1.6.3}$$

where $k[\mathcal{U}_\Sigma] \to k$ is evaluation at U represented by $A_U \in M_{n,r}$ with Σ–submatrix equal to the identity.

The fact that (1.6.1) is a closed condition follows immediately from the lower semi–continuity of $\mathrm{rk}(\Phi(M))$ as a function on $\mathcal{U}_\Sigma$. □

Remark 1.7. Proposition 1.6 follows almost immediately from Prop. 1.4. Indeed, (1.4) states that $\mathrm{Grass}(r, \mathbb{V})_M$ is the locus of all r-planes in $\mathbb{P}^{n-1}$ meeting the projectivized support variety of M non-trivially. Hence, it is a closed subvariety in $\mathrm{Grass}(r, \mathbb{V})$ (see, for example, [Har77, 6.14]). We chose to give a self-contained proof since it will play a role in the proof of Theorem 2.4.

Example 1.8. Suppose that $n = 4$, and choose $[x_1, x_2, x_3, x_4]$ spanning $\mathbb{V} \subset \mathrm{Rad}(kE)$ determining an (ordered) basis for $\mathrm{Rad}(kE)/\mathrm{Rad}^2(kE)$. Take $r = 2$. Set $M = kE/(x_1, x_2)$. Then $(\mathrm{Grass}_{4,2})_M$ consists of all 2-planes which intersect non-trivially the plane $\langle x_1, x_2\rangle$ spanned by x_1 and x_2. Namely, $\alpha_U^* M$ is a free $C = k[t_1,t_2]/(t_1^p,t_2^p)$-module if and only if the 2-plane $U \subset \mathbb{V}$ does not intersect $\langle x_1, x_2\rangle$. Take $u_1 = \sum_{j=1}^4 u_{1,j}x_j$, $u_2 = \sum_{j=1}^4 u_{2,j}x_j$ spanning U. Then U does not intersect $\langle x_1,x_2\rangle$ if and only if the vectors $\{x_1,x_2,u_1,u_2\}$ span $\mathbb{V}$. This is equivalent to non-singularity of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & u_{22} & u_{23} & u_{24} \end{pmatrix}.$$

Hence, in Plücker coordinates, $(\mathrm{Grass}_{4,2})_M$ is the zero locus of $\mathfrak{p}_{\{3,4\}} = u_{13}u_{24} - u_{23}u_{14} = 0$.

For $r = 1$, $\mathrm{Grass}(1,\mathbb{V})_M \subset \mathrm{Grass}(1,\mathbb{V}) \simeq \mathbb{P}^{n-1}$ can be naturally identified with the projectivized support variety of the kE-module M (see [CTVZ03]). The following proposition extends to all $r \geq 1$ various familiar properties of support varieties. As usual, $\Omega^s(M)$ is the name of the s^{th} syzygy or s^{th} Heller shift of the kE-module M. (We also use this notation for the Heller shift of any C-module, where C is a commutative k-algebra of the form $k[t_1,\dots,t_r]/(t_i^p)$.) Recall that $\Omega(M)$ is the kernel of a projective cover $Q \to M$ of M, and $\Omega^{-1}(M)$ is the cokernel of an injective hull $M \to I$. Then inductively,

$$\Omega^s(M) \;=\; \Omega(\Omega^{s-1}(M)), \quad \Omega^{-s}(M) \;=\; \Omega^{-1}(\Omega^{-s+1}(M)), \quad s > 1. \tag{1.8.1}$$

Proposition 1.9. *Let M and N be finite dimensional kE-modules, and fix an integer $r \geq 1$.*

1. *M is projective as a kE-module if and only if* $\mathrm{Grass}(r,\mathbb{V})_M = \emptyset$.
2. $\mathrm{Grass}(r,\mathbb{V})_{M\oplus N} \;=\; \mathrm{Grass}(r,\mathbb{V})_M \cup \mathrm{Grass}(r,\mathbb{V})_N$.
3. $\mathrm{Grass}(r,\mathbb{V})_{\Omega^i(M)} = \mathrm{Grass}(r,\mathbb{V})_M$ *for any $i \in \mathbb{Z}$.*
4. *If $0 \to M_1 \to M_2 \to M_3 \to 0$ is an exact sequence of kE-modules, then*
$$\mathrm{Grass}(r,\mathbb{V})_{M_2} \;\subset\; \mathrm{Grass}(r,\mathbb{V})_{M_1} \cap \mathrm{Grass}(r,\mathbb{V})_{M_3}.$$
5. $\mathrm{Grass}(r,\mathbb{V})_{M\otimes N} \;\subset\; \mathrm{Grass}(r,\mathbb{V})_M \cap \mathrm{Grass}(r,\mathbb{V})_N$.

Proof. The assertion (1) follows from Proposition 1.4 together with Dade's Lemma. Assertion (2) is immediate. The assertion (3) follows from Proposition 1.4, the observation that the restriction of $\Omega^i(M)$ along some $\alpha_U : C(U) \to kE$ is stably isomorphic to the i-th Heller shift of the restriction of M along α_U, and the corresponding result for $r = 1$.

To prove (4), we first observe that if the restrictions along α_U of both M_1 and M_3 are free, then the pull-back along α_U of $0 \to M_1 \to M_2 \to M_3 \to 0$ splits and thus M_2 is also free.

Complicating the proof of (5) is the fact that, in general, the restriction functor along α_U does not commute with tensor products, for the tensor product operation depends upon on the choice of Hopf algebra structure. We use the fact proved in [Car83] (see also [FP07]), that

$$\mathrm{Grass}(1,\mathbb{V})_{M\otimes N} \;=\; \mathrm{Grass}(1,\mathbb{V})_M \cap \mathrm{Grass}(1,\mathbb{V})_N, \tag{1.9.1}$$

without regard to Hopf algebra structures. If U is in $\mathrm{Grass}(r,\mathbb{V})_{M\otimes N}$, then $\alpha_U^*(M \otimes N)$ is not a free module. So there exists $\beta : k[t]/(t^p) \to C(U)$ such that $\beta^*(\alpha_U^*(M \otimes$

$N)$) is not a free $k[t]/t^p$-module. Consequently, the line $W \in \mathrm{Grass}(1,\mathbb{V})$ generated by $\alpha(\beta(t))$ is in $\mathrm{Grass}(1,\mathbb{V})_{M\otimes N}$. Thus, the line $W' \subset U$ generated by $\beta(t)$ is in $\mathrm{Grass}(1,U)_{\alpha_U^*(M\otimes N)}$. By (1.9.1), (with $\mathbb{V}$ replaced by U), W' is in both $\mathrm{Grass}(1,U)_{\alpha_U^*(M)}$ and $\mathrm{Grass}(1,U)_{\alpha_U^*(N)}$. Therefore, neither $\alpha_U^*(M)$ nor $\alpha_U^*(N)$ is free, so that $U \in \mathrm{Grass}(r,\mathbb{V})_M \cap \mathrm{Grass}(r,\mathbb{V})_N$. □

Example 1.10. The reverse inclusion of Proposition 1.9(4) does not hold if $r \geq 2$. Retain the notation of Example 1.8. Let $U = \langle x_1, x_2\rangle \subset \mathbb{V}$, and let $M = kE/(x_1)$ and $N = kE/(x_2)$. Then $M \otimes N$ is a free kE-module so that $\mathrm{Grass}(2,\mathbb{V})_{M\otimes N} = \emptyset$; however, neither $\alpha_U^*(M)$ or $\alpha_U^*(N)$ is free as a $C(U)$-module, so that $U \in \mathrm{Grass}(2,\mathbb{V})_M \cap \mathrm{Grass}(2,\mathbb{V})_N$.

To end this section, we observe that it is not possible, in general, to realize all of the closed sets of $\mathrm{Grass}(2,\mathbb{V})$ as 2-support varieties of kE-modules. This contrasts with the case $r = 1$: every closed subvariety of the usual support variety $\mathrm{Grass}(1,\mathbb{V})$ is the support variety of a tensor product of Carlson modules L_ζ for suitably chosen cohomology classes $\zeta \in \mathrm{H}^*(kE,k)$ ([Car84]).

Example 1.11. Take $n = 3$, so that $\mathrm{Grass}(2,\mathbb{V}) \simeq \mathbb{P}^2$. Recall that the complexity of a kE-module M is the dimension of the affine support variety of M (whose projectivization is $\mathrm{Grass}(1,\mathbb{V})_M$).

- If M has complexity 0, then M is projective and $\mathrm{Grass}(2,\mathbb{V})_M = \emptyset$.
- If M has complexity 1, then the affine support variety of M is a finite union of lines. Under the identification $\mathrm{Grass}(2,\mathbb{V}) \simeq \mathbb{P}^2$, the subvariety of planes $U \in \mathrm{Grass}(2,\mathbb{V})$ containing a given line is a line in $\mathbb{P}^2$. By Proposition 1.4, $\mathrm{Grass}(2,\mathbb{V})_M$ consists of those $U \in \mathbb{V}$ such that U contains one of the lines whose union is the affine support variety of M. Hence, the subsets in $\mathrm{Grass}(2,\mathbb{V}) \simeq \mathbb{P}^2$ of the form $\mathrm{Grass}(2,\mathbb{V})_M$ for M of complexity 1 are finite unions of lines.
- If M has complexity 2 or 3, then there are no 2-planes in $\mathbb{V}$ which fail to intersect $\mathrm{Grass}(1,\mathbb{V})$. Consequently, $\mathrm{Grass}(2,\mathbb{V})_M = \mathrm{Grass}(2,\mathbb{V})$.

Hence, the closed subsets of $\mathrm{Grass}(2,\mathbb{V})$ of the form $\mathrm{Grass}(2,\mathbb{V})_M$ do not generate the Zariski topology of $\mathrm{Grass}(2,\mathbb{V})$.

2. Radicals and Socles

We retain the notation of Section 1: E is an elementary abelian p-group of rank n and $\mathbb{V} \subset \mathrm{Rad}(kE)$ is a splitting of the projection $\mathrm{Rad}(kE) \to \mathrm{Rad}(kE)/\mathrm{Rad}^2(kE)$. As in Definition 2.1, for a given kE-module M we consider radicals and socles with respect to rank r elementary subgroups parametrized by $U \in \mathrm{Grass}(r,\mathbb{V})$. The dimensions of these radicals and socles are numerical invariants which in some sense are the extension to $r > 1$ of the Jordan type of a kE-module at a cyclic shifted subgroup (or the Jordan type of a $\mathfrak{u}(\mathfrak{g})$-module at a 1-parameter subgroup of a p-restricted Lie algebra $\mathfrak{g}$).

Definition 2.1. Let M be a kE-module, $U \in \mathrm{Grass}(r,\mathbb{V})$ be an r-plane of $\mathbb{V}$, and take α_U as in (1.0.5). We define

$$\mathrm{Rad}_U(M) \equiv \mathrm{Rad}(\alpha_U^*(M)) = \sum_{u\in U} u \cdot M,$$

$$\mathrm{Soc}_U(M) \equiv \mathrm{Soc}(\alpha_U^*(M)) = \{m \in M \mid u \cdot m = 0 \ \forall u \in U\},$$

the radical and socle of M as a $C(U)$-module. For $j > 1$, we inductively define the kE-submodules of M

$$\mathrm{Rad}_U^j(N) \;=\; \mathrm{Rad}_U(\mathrm{Rad}_U^{j-1}(M))$$

and

$$\mathrm{Soc}_U^j(M) = \{m \in M \,|\overline{m} \in \mathrm{Soc}_U(M)/\,\mathrm{Soc}_U^{j-1}(M)\}.$$

Thus, if $\{u_1, \ldots, u_r\}$ spans U and if $S_j(u_1, \ldots, u_r) \subset \mathrm{Rad}(kE)$ denotes the subspace generated by all monomials on $\{u_1, \ldots, u_r\}$ of degree j, then

$$\mathrm{Rad}_U^j(M) \;=\; \sum_{s \in S_j(u_1,\ldots,u_r)} s \cdot M \tag{2.1.1}$$

and

$$\mathrm{Soc}_U^j(M) \;=\; \{m \in M \mid s \cdot m = 0 \text{ for all } s \in S_j(u_1, \ldots, u_r)\}. \tag{2.1.2}$$

The commutativity of E implies that each $\mathrm{Rad}_U^j(M)$ and each $\mathrm{Soc}_U^j(M)$ is a kE-submodule of M.

If A is a Hopf algebra and $f : L \subset M$ is an embedding of A-modules, then we denote by $f^\# : M^\# \to L^\#$ the induced map of A-modules and denote by $L^\perp \equiv \mathrm{Ker}\{f^\#\}$. Explicitly, the action of A on $M^\#$ is given by sending $a \in A$, $\phi : M \to k$ to

$$a \cdot \phi : M \to k, \quad (a \cdot \phi)(m) = \phi(\iota(a)(m)),$$

where $\iota : A \to A$ is the antipode of A; thus, the A-module structures on $M^\#$ and $L^\perp$ depend upon the Hopf algebra structure on A, not just the structure of A as an algebra.

Although we assume throughout this paper that kE is equipped with the Hopf algebra structure which is primitively generated (so that kE is viewed as a quotient of the primitively generated Hopf algebra $S^*(\mathbb{V})$), the following proposition is formulated to apply as well to the usual group-like Hopf algebra structure of kE.

For the automorphism $\iota : kE \to kE$ defined by the antipode of kE, and a kE–module M, we denote by $\iota(M)$ the kE–module M twisted by ι. That is, M coincides with $\iota(M)$ as a vector space but an element $x \in kE$ acts on $\iota(M)$ as $\iota(x)$ acts on M. We denote an element of $\iota(M)$ corresponding to $m \in M$ by $\iota(m)$.

Proposition 2.2. *Choose any Hopf algebra structure on kE, and let ι be the antipode of this structure. For any kE-module M, let $\iota(M)$ denote the kE-module which coincides with $\iota(M)$ as a k-vector space and such that $x \in kE$ acts on $m \in \iota(M)$ as $\iota(x) \cdot m$.*

For any $U \in \mathrm{Grass}(r, \mathbb{V})$ and any $j \geq 1$, there are natural isomorphisms of kE-modules

$$\mathrm{Soc}_U^j(\iota(M)^\#) \;\simeq\; (\mathrm{Rad}_U^j(M))^\perp, \quad \mathrm{Rad}_U^j(\iota(M)^\#) \;\simeq\; (\mathrm{Soc}_U^j(M))^\perp, \tag{2.2.1}$$

Proof. Choose a basis $\{u_1, \ldots, u_r\}$ for U. An element $\iota(f)$ in $\iota(M^\#)$ is in $\mathrm{Soc}_U^j(\iota(M^\#))$ if and only if for any monomial s of degree j in the elements $u_1, \ldots, u_r$, we have that $s \cdot \iota(f) = 0$. This happens if and only if for any such s and any m in M, $(s \cdot \iota(f))(m) = (\iota(s)f)(m) = f(sm) = 0$. In turn, this can happen if and only if f vanishes on $\mathrm{Rad}_U^j(M)$. This proves the first equality; the proof of the second is similar. $\square$

We introduce refinements of the r-rank variety $\mathrm{Grass}(r,\mathbb{V})_M$, thereby extending to $r>1$ the generalized support varieties of [FP10].

Definition 2.3. Let M be a finite dimensional kE-module, and let j be a positive integer. We define the *nonmaximal r–radical support variety* of M, $\mathrm{Rad}^j(r,\mathbb{V})_M \subset \mathrm{Grass}(r,\mathbb{V})$, to be

$$\mathrm{Rad}^j(r,\mathbb{V})_M \;\equiv\; \{U \in \mathrm{Grass}(r,\mathbb{V}) \mid \dim \mathrm{Rad}^j_U(M) < \max_{U'\in \mathrm{Grass}(r,\mathbb{V})} \dim \mathrm{Rad}^j_{U'}(M)\}.$$

Similarly, we define the *nonminimal r-socle support variety* of M, $\mathrm{Soc}^j(r,\mathbb{V})_M \subset \mathrm{Grass}(r,\mathbb{V})$, to be

$$\mathrm{Soc}^j(r,\mathbb{V})_M \;\equiv\; \{U \in \mathrm{Grass}(r,\mathbb{V}) \mid \dim \mathrm{Soc}^j_U(M) > \max_{U'\in \mathrm{Grass}(r,\mathbb{V})} \dim \mathrm{Soc}^j_{U'}(M)\}.$$

For $j=1$, we simplify this notation by writing

$$\mathrm{Rad}(r,\mathbb{V})_M \;=\; \mathrm{Rad}^1(r,\mathbb{V})_M, \quad \mathrm{Soc}(r,\mathbb{V})_M \;=\mathrm{Soc}^1(r,\mathbb{V})_M.$$

The proof of upper/lower semi-continuity in the next theorem is an extension of the proof of Proposition 1.6.

Theorem 2.4. *Let M be a finite dimensional kE-module. For any j, the function*

$$U \in \;\mathrm{Grass}(r,\mathbb{V}) \;\mapsto\; f_{M,j}(U) \equiv \dim \mathrm{Rad}^j_U(M)$$

is lower semi-continuous: in other words, there is a (Zariski) open subset $\mathcal{U} \subset$ $\mathrm{Grass}(r,\mathbb{V})$ of U such that $f_{M,j}(U) \le f_{M,j}(U')$ for all $U' \in \mathcal{U}$.

Simillarly, for any j the function

$$U \in \;\mathrm{Grass}(r,\mathbb{V}) \;\mapsto\; g_{M,j}(U) \equiv \dim \mathrm{Soc}^j_U(M)$$

is upper semi-continuous.

Proof. As in the proof of Proposition 1.6, we may equip $\mathbb{V}$ with an ordered basis, replacing $\mathrm{Grass}(r,\mathbb{V})$ by $\mathrm{Grass}_{n,r}$. It suffices to restrict to affine open subsets $\mathcal{U}_\Sigma \subset \mathrm{Grass}(r,\mathbb{V})$. Recall the notation $S_j(t_1,\dots,t_r) \subset C$ for the linear subspace generated by all monomials on $\{t_1,\dots,t_r\}$ of degree j, and let $d(j) = \dim S_j(t_1,\dots,t_r)$. We replace the map (1.6.2) by

$$\sum_{\substack{d_1+\ldots+d_r=j\\ 0\le d_i<p}} \alpha_\Sigma(t_1)^{d_1}\ldots\alpha_\Sigma(t_r)^{d_r} : (M\otimes k[\mathcal{U}_\Sigma])^{\oplus d(j)} \;\to\; M\otimes k[\mathcal{U}_\Sigma], \tag{2.4.1}$$

Let $\Phi^j(M) \in M_{m,d(j)m}(k[\mathcal{U}_\Sigma])$ denote the associated matrix, where $m=\dim M$. Then, as for (1.6.3) with the same notation, we have the equality

$$\mathrm{rk}(\Phi^j(M)\otimes_{k[\mathcal{U}_S]} k) \;=\; \dim(\mathrm{Rad}^j(C)\cdot\alpha^*_U(M)). \tag{2.4.2}$$

The lower semi-continuity of $U \mapsto f_{M,j}(U)$ now follows immediately from the lower semi-continuity of $\Phi^j(M)$ as a function on $\mathcal{U}_\Sigma$.

The upper semi-cotinuity for $U \mapsto g_{M,j}(U)$ is a consequence of lower semi-continuity for $U \mapsto f_{M^\#,j}(U)$ and Proposition 2.2. $\square$

As an immediate corollary of Theorem 2.4, we conclude that the subsets introduced in Definition 2.3 are Zariski closed subvarieties of $\mathrm{Grass}(r,\mathbb{V})$.

Corollary 2.5. *For any finite dimensional kE–module M, and any positive integer j, $\mathrm{Rad}^j(r,\mathbb{V})_M$ and $\mathrm{Soc}^j(r,\mathbb{V})_M$ are Zariski closed subsets of* $\mathrm{Grass}(r,\mathbb{V})$.

The reader should observe that the polynomial equations expressing the non-maximality of $f_{M,j}(U)$ must be expressible in terms of homogeneous polynomials in the Plücker coordinates. This fact is exploited in the appendix, where some computer calculations of nonminimal r-socle support varieties are presented.

Example 2.6. We return to Example 1.8, in which $n = 4$ and $[x_1, x_2, x_3, x_4]$ is an ordered basis of some $\mathbb{V} \subset \mathrm{Rad}(kE)$ splitting the projection $\mathrm{Rad}(kE) \to \mathrm{Rad}(kE)/\mathrm{Rad}^2(kE)$. As in Example 1.8, we take $M = kE/\langle x_1, x_2\rangle$.

If $r = 2$, then an argument similar to the one in Example 1.8 shows that $\mathrm{Rad}(2, \mathbb{V})_M = \mathrm{Grass}(2, \mathbb{V})_M$.

Now, set $r = 3$. We have $\mathrm{Grass}(3, \mathbb{V}) \simeq \mathbb{P}^3$. Let $\langle x_1, x_2\rangle \subset \mathbb{V}$ be the 2-plane spanned by x_1, x_2. Observe that the module M has dimension p^2. Let $U \subset \mathbb{V}$ be any 3-plane in $\mathbb{V}$. Then $\mathrm{Rad}_U M \subset M$ has codimension p if $\langle x_1, x_2\rangle \subset U$ and codimension 1 otherwise. Hence, $\mathrm{Rad}(3, \mathbb{V})_M \neq \emptyset$; indeed, $\mathrm{Rad}(3, \mathbb{V})_M$ consists of all 3-planes which contain $\langle x_1, x_2\rangle$. In Plücker coordinates $\mathrm{Rad}(3, \mathbb{V})_M$ is given as the zero locus of the equations $\mathfrak{p}_{\{1,3,4\}} = 0 = \mathfrak{p}_{\{2,3,4\}}$.

Our next example is more complicated and uses the identification of the rank variety $\mathrm{Grass}(1, \mathbb{V})$ with $\mathrm{Proj}\, \mathrm{H}^*(E, k)$.

Example 2.7. Choose some $\mathbb{V} \subset \mathrm{Rad}(kE)$ splitting the projection $\mathrm{Rad}(kE) \to \mathrm{Rad}(kE)/\mathrm{Rad}^2(kE)$, and assume that $p = 2$, $r = 2$. Let $\zeta \in \mathrm{H}^m(E, k)$ be a non-trivial homogeneous cohomology class of positive degree m. Let $\zeta : \Omega^m(k) \to k$ be the cocycle representing ζ and let L_ζ denote the kernel of the module map ζ (investigated in detail in Section 5). Recall that the support variety of L_ζ may be identified with the zero locus of ζ, $Z(\zeta) \subset \mathrm{Spec}\, \mathrm{H}^*(E, k)$ (see [Car84]).

There are two possibilities for the restriction of L_ζ along $\alpha_U : C \to kE$ for $U \in \mathrm{Grass}(2, \mathbb{V})$ (see Lemma 5.4):

$$\left[\begin{array}{ll} \alpha^*(M) \simeq L_{\alpha^*(\zeta)} \oplus C^s & \text{if } \alpha^*(\zeta) \neq 0 \\ \alpha^*(M) \simeq \Omega^m(k_C) \oplus \Omega(k_C) \oplus C^{s-1} & \text{if } \alpha^*(\zeta) = 0 \end{array}\right. ,$$

where $2m + 1 + 4s = \dim(\Omega^m(k))$. In particular, $\mathrm{Grass}(2, \mathbb{V})_{L_\zeta} = \mathrm{Grass}(2, \mathbb{V})$. Since $C \simeq k(\mathbb{Z}/2 \times \mathbb{Z}/2)$, we can compute $\dim \mathrm{Rad}(\Omega^m(k_C)) = \dim(L_{\alpha^*(\zeta)}) = m$ and $\dim \mathrm{Rad}(C) = 3$ (see [He61]). Hence, if $\alpha^*(\zeta) \neq 0$, $\dim \mathrm{Rad}_U(L_\zeta) = 3s + m$ while for $\alpha^*(\zeta) = 0$, $\dim \mathrm{Rad}_U(L_\zeta) = 3(s-1) + m + 1$. It follows that $\mathrm{Rad}(2, \mathbb{V})_{L_\zeta} \neq \emptyset$, with $\mathrm{Rad}(2, \mathbb{V})_{L_\zeta}$ consisting of exactly those 2–planes that are contained in $Z(\zeta)$. We can compute further that $\dim \mathrm{Rad}^2_U(L_\zeta) = s$ in the first case and $s - 1$ in the second. Hence,

$$\mathrm{Rad}^2(2, \mathbb{V})_{L_\zeta} = \mathrm{Rad}(2, \mathbb{V})_{L_\zeta} \neq \emptyset.$$

Finally, we find a curious thing happens when we consider socles. The point is that $\dim \mathrm{Soc}_U(L_\zeta) = s + m$ in both cases. Hence, $\mathrm{Soc}(2, \mathbb{V})_{L_\zeta} = \emptyset$, so that L_ζ has constant 2–Soc–rank in the terminology of Section 3. However, $\dim \mathrm{Soc}^2_U(L_\zeta) = 3s + 2m$ if $\alpha^*(\zeta) \neq 0$ and $3s + 2m + 1$ otherwise. Consequently, $\mathrm{Soc}^2(2, \mathbb{V})_{L_\zeta}$ is the same as the radical variety $\mathrm{Rad}(2, \mathbb{V})_{L_\zeta}$. Thus,

$$\mathrm{Soc}^2(2, \mathbb{V})_{L_\zeta} \neq \mathrm{Soc}(2, \mathbb{V})_{L_\zeta} = \emptyset.$$

By taking duals, we can get a module M with the property that $\mathrm{Rad}(2, \mathbb{V})_M = \emptyset$ and $\mathrm{Rad}^2(2, \mathbb{V})_M$ is a proper non-trivial subvariety of $\mathrm{Grass}(2, \mathbb{V})$.

We conclude this section with a consideration of the dependence of the dimension of radicals on the choice of $\mathbb{V}$, continuing the investigation of [FPS07]. Our statements are given for radicals, but using Proposition 2.2 one immediately gets similar statements for socles.

Definition 2.8. Fix a finite dimensional kE–module M. We say that M has *absolute maximal radical rank* at the r-plane $V \in \operatorname{Grass}(r, \mathbb{V})$ if

$$\dim \operatorname{Rad}_V(M) \geq \dim \operatorname{Rad}_W(M)$$

for any $\mathbb{W} \subset \operatorname{Rad}(kE)$ splitting the projection $\operatorname{Rad}(kE) \to \operatorname{Rad}(kE)/\operatorname{Rad}^2(kE)$ and any $W \in \operatorname{Grass}(r, \mathbb{W})$.

The following theorem is a generalization to $r > 1$ of [FPS07, 1.9].

Theorem 2.9. *Let $\mathbb{V}, \mathbb{W} \subset \operatorname{Rad}(kE)$ be splittings of the projection $\operatorname{Rad}(kE) \to \operatorname{Rad}(kE)/\operatorname{Rad}^2(kE)$ and let $\psi : \mathbb{V} \overset{\sim}{\to} \mathbb{W}$ be the unique isomorphism commuting with the projection isomorphisms to $\operatorname{Rad}(kE)/\operatorname{Rad}^2(kE)$. Denote by $\Psi : \operatorname{Grass}(r, \mathbb{V}) \overset{\sim}{\to} \operatorname{Grass}(r, \mathbb{W})$ the induced isomorphism of Grassmannians. Then*

(1) *Ψ restricts to an isomorphism*

$$\operatorname{Rad}(r, \mathbb{V})_M \overset{\sim}{\to} \operatorname{Rad}(r, \mathbb{W})_M.$$

(2) *For any $U \notin \operatorname{Rad}(r, \mathbb{V})_M$, $\dim \operatorname{Rad}_U(M) = \dim \operatorname{Rad}_{\Psi(U)}(M)$.*

(3)

$$\max_{V \in \operatorname{Grass}(r,\mathbb{V})} \dim \operatorname{Rad}_V(M) = \max_{W \in \operatorname{Grass}(r,\mathbb{W})} \dim \operatorname{Rad}_W(M).$$

Proof. We first assume that $\mathbb{V}$ satisfies the condition that there exists some r-plane $V \subset \mathbb{V}$ at which M has absolute radical rank. Since for any $U, U' \notin \operatorname{Rad}(r, \mathbb{V})_M$, we have an equality $\dim \operatorname{Rad}_U(M) = \dim \operatorname{Rad}_{U'}(M)$, we immediately conclude that any $U' \notin \operatorname{Rad}(r, \mathbb{V})_M$ satisfies the property that M has absolute maximal radical rank at U'. Hence, the validity of statements (1) and (3) will follow from the validity of statement (2) since Ψ is a bijection.

Let $U \in \operatorname{Grass}(r, \mathbb{V})$ satisfy the property that M has absolute maximal radical rank at U. Choose an ordered basis $[u_1, \dots, u_r]$ of U. For each m, $0 \leq m \leq r$, we consider $\alpha_m : C \equiv k[t_1, \dots, t_r]/(t_i^p) \to kE$ defined as follows:

$$\begin{cases} \alpha_m(t_i) = \psi(u_i) & i \leq m \\ \alpha_m(t_i) = u_i & m+1 \leq i \leq r. \end{cases}$$

Since ψ commutes with the projections to $\operatorname{Rad}(kE) \to \operatorname{Rad}(kE)/\operatorname{Rad}^2(kE)$, we conclude that

$$\psi(u_i) - u_i \in \operatorname{Rad}^2(kE), \ 1 \leq i \leq r.$$

Observe that

$$\operatorname{Rad}_U(M) = \operatorname{Rad}(\alpha_0^*(M)), \quad \operatorname{Rad}_{\psi(U)}(M) = \operatorname{Rad}(\alpha_r^*(M)). \tag{2.9.1}$$

Consider the kE-module $N = M/(u_2M + \ldots + u_rM)$. We have

$$\dim \operatorname{Rad}(\alpha_0^*(M)) = \dim \sum_{i=1}^r u_iM = \dim(u_1N) + \dim \sum_{i=2}^r u_iM \tag{2.9.2}$$

and

$$\dim \operatorname{Rad}(\alpha_1^*(M)) = \dim \sum_{i=1}^r \alpha_1(t_i)M = \dim(\psi(u_1)N) + \dim \sum_{i=2}^r u_iM. \tag{2.9.3}$$

Our assumption that $\operatorname{Rad}(\alpha_0^*(M)) = \operatorname{Rad}_U(M)$ has absolute maximal rank and equation (2.9.2) imply that

$$\dim(u_1 N) \geq \dim(uN), \quad \forall u \in \operatorname{Rad}(kE).$$

Together with the fact that $u_1 \equiv \psi(u_1) \bmod \operatorname{Rad}^2(kE)$, this implies the equality

$$\dim(u_1 \cdot N) = \dim(\psi(u_1) \cdot N) \tag{2.9.4}$$

by [FPS07, 1.9]. Equalities (2.9.2), (2.9.3), and (2.9.4) now imply

$$\dim \operatorname{Rad}(\alpha_0^*(M)) = \dim \operatorname{Rad}(\alpha_1^*(M)).$$

We proceed by induction on $m \geq 1$, replacing u_m by $\psi(u_m)$ as we just replaced u_1 by $\psi(u_1)$. We conclude that $\dim \operatorname{Rad}(\alpha_{m-1}^*(M)) = \dim \operatorname{Rad}(\alpha_m^*(M))$ for $1 \leq m \leq r$. Thus, by (2.9.1), we obtain

$$\dim \operatorname{Rad}(\alpha_U^*(M)) = \dim \operatorname{Rad}(\alpha_{\psi(U)}^*(M)).$$

To prove the theorem without the condition that $\mathbb{V}$ contains an r-plane at which M has absolute maximal radical rank, we consider two arbitrary $\mathbb{V}, \mathbb{W} \subset \operatorname{Rad}(A)$ subspaces which split the projection $\operatorname{Rad}(kE) \to \operatorname{Rad}(kE)/\operatorname{Rad}^2(kE)$ and choose some third $\mathbb{V}' \subset \operatorname{Rad}(kE)$ which also splits the projection $\operatorname{Rad}(kE) \to \operatorname{Rad}(kE)/\operatorname{Rad}^2(kE)$ and does contain an r-plane $V' \subset \mathbb{V}'$ at which M has absolute maximal rank. Then appealing to the above argument for the pairs $(\mathbb{V}', \mathbb{V})$ and $(\mathbb{V}', \mathbb{W})$, we conclude the theorem for the pair $(\mathbb{V}, \mathbb{W})$. □

Corollary 2.10. *Retain the notation of Theorem 2.9. Then Ψ restricts to an isomorphism*

$$\operatorname{Grass}(r, \mathbb{V})_M \xrightarrow{\sim} \operatorname{Grass}(r, \mathbb{W})_M.$$

Proof. For any $U \in \operatorname{Grass}(r, \mathbb{V})$, $\alpha_U^*(M)$ is free if and only if $\operatorname{Rad}_U(M)$ has dimension equal to $\frac{p^r-1}{p^r} \cdot \dim(M)$. For any $V \in \operatorname{Grass}(r, \mathbb{V})$ we have the inequality

$$\dim(\operatorname{Rad}_V(M)) \leq \frac{p^r - 1}{p^r} \cdot \dim(M).$$

The corollary now follows immediately from Theorem 2.9 (2). □

3. Modules of constant radical and socle rank

We continue our previous notation: E is an elementary abelian p-group of rank n and $\mathbb{V} \subset \operatorname{Rad}(kE)$ is a choice of splitting of the projection $\operatorname{Rad}(kE) \to \operatorname{Rad}(kE)/\operatorname{Rad}^2(kE)$ providing the identification $S^*(\mathbb{V})/\langle v^p, v \in \mathbb{V}\rangle \cong kE$ of (1.0.1). As in Theorem 2.4, we can associate to any finite dimensional kE–module M and any $j > 0$ the integer-valued functions

$$U \in \operatorname{Grass}(r, \mathbb{V}) \mapsto f_{M,j}(U) \equiv \dim \operatorname{Rad}_U^j(M)$$

and

$$U \in \operatorname{Grass}(r, \mathbb{V}) \mapsto g_{M,j}(U) \equiv \dim \operatorname{Soc}_U^j(M).$$

We view these functions as defining the *local radical ranks* and *local socle ranks* of M.

In this section we introduce kE–modules of constant r-radical (resp., r-socle) type and more generally of constant r-Rad^j-rank (resp., r-Soc^j-rank). By definition, these are the modules for which the functions $f_{M,j}$ (resp., $g_{M,j}$) whose value $f_{M,j}(U)$ in independent of U in $\operatorname{Grass}(r, \mathbb{V})$. These are natural analogues for $r > 1$ of modules of *constant Jordan type* (see [CFP08]) which have many good properties and lead to algebraic vector bundles (see [FP11]). In Section 6, we see how to associate vector bundles on $\operatorname{Grass}(r, \mathbb{V})$ to kE-modules of constant r-Rad^j-rank or constant r-Soc^j-rank.

Definition 3.1. We fix integers $r > 0$, j, $1 \leq j \leq (p-1)r$, and let M be a finite dimensional kE-module.

(1) The module M has constant r-Rad^j rank (respectively, r-Soc^j-rank) if the dimension of $\operatorname{Rad}^j_U(M)$ (resp., $\operatorname{Soc}^j_U(M)$) is independent of choice of $U \in \operatorname{Grass}(r, \mathbb{V})$.

(2) M has constant r-radical type (respectively, r-socle type) if it has constant r-Rad^j rank (resp. r-Soc^j rank) for all j, $1 \leq j \leq (p-1)r$.

To simplify notation, we refer to constant r-Rad^1 rank (respectively, r-Soc^1 rank) as constant r-Rad rank (respectively, r-Soc rank).

Remark 3.2. It is immediate from the definitions that M has constant r-Rad^j-rank (respectively, r-Soc^j-rank) if and only if $\operatorname{Rad}^j(r, \mathbb{V})_M = \emptyset$ (resp, $\operatorname{Soc}^j(r, \mathbb{V})_M = \emptyset$).

The following proposition, stating that the property of constant r-Rad and r-Soc-rank is independent of the choice of $\mathbb{V}$, is an immediate corollary of Theorem 2.9.

Proposition 3.3. *Let* $\mathbb{W} \subset \operatorname{Rad}(kE)$ *also provide a splitting of the projection* $\operatorname{Rad}(kE) \to \operatorname{Rad}(kE)/\operatorname{Rad}^2(kE)$. *Then for any kE-module M and any $r \geq 1$, M has constant r-radical rank (respectively, constant r-socle rank) as above if and only if* $\dim \operatorname{Rad}_W(M)$ *(resp.,* $\dim \operatorname{Soc}_W(M)$*) is independent of the choice of* $W \in \operatorname{Grass}(r, \mathbb{W})$.

The reader should observe that in the case that $r = 1$, either one of the set of 1-Rad^j ranks or the set of 1-Soc^j ranks, for all j, is sufficient to determine the Jordan type. Also the Jordan type determines all of the radical and socle ranks for $r = 1$. Consequently, a kE-module has constant 1-radical type if and only if it has constant 1-socle type. This is no longer true for $r \geq 2$ as we show in Examples 4.6, 4.7.

We begin with particularly easy examples of modules of constant radical and socle types. Since their identification does not depend upon the choice of $\mathbb{V} \subset \operatorname{Rad}(kE)$, we conclude that these examples are examples of constant radical and socle types for any choice of $\mathbb{V} \subset \operatorname{Rad}(kE)$.

Example 3.4. For any finite dimensional projective kE-module M, the r-radical type and the r-socle type of M are constant for every $r > 0$. Indeed, a projective module is free and its restriction along $\alpha_U : C(U) \to kE$ is a free module for any $U \in \operatorname{Grass}(r, \mathbb{V})$ whose rank is determined by r and the dimension of M.

Another evident family of examples of modules of constant radical and socle type arises from Heller shifts of the trivial module (see (1.8.1)). For any $s \in \mathbb{Z}$, if $M \simeq \Omega^s(k)$, then M has constant r-radical type and constant r-socle type for each $r > 0$. Indeed, for any $U \in \operatorname{Grass}(r, \mathbb{V})$, we have $\alpha_U^*(M) \simeq \Omega^s(k) \oplus Q$ as a $C(U)$-module, where Q is a free $C(U)$-module whose rank is determined by the dimension of M and the choice of r.

Recall that we identify kE with $S^*(\mathbb{V})/\langle v^p, v \in \mathbb{V}\rangle$; with this identification, any kE-module is equipped with the structure of an $S^*(\mathbb{V})$-module. Moreover, we get an action of $\mathrm{GL}_n \simeq \mathrm{GL}(\mathbb{V})$ on kE by algebra automorphisms induced by the standard representation of GL_n on $\mathbb{V}$. We view $S^*(\mathbb{V})$ as the coordinate algebra of the affine space $\mathbb{V}^{\#} = \mathbb{A}^n$. Thus, any kE-module M determines a quasi-coherent sheaf $\widetilde{M}$ of $\mathcal{O}_{\mathbb{V}^{\#}}$-modules. The natural action of $\mathrm{GL}_n = \mathrm{GL}(\mathbb{V})$ on $S^*(\mathbb{V})$ determines an action of $\mathrm{GL}_n = \mathrm{GL}(\mathbb{V})$ on the variety $\mathbb{V}^{\#}$. As recalled in Definition 6.4, there is a widely used concept of a GL_n- equivariant sheaf on a variety X which is provided with a GL_n-action. In the special case of $\mathrm{GL}_n = \mathrm{GL}(\mathbb{V})$ acting on $\mathbb{V}^{\#}$, this specializes to the following explicit definition of a GL_n*–equivariant* kE*–module*.

Definition 3.5. Let M be a kE-module, whose structure map is given by the k-linear pairing

$$S^*(\mathbb{V})/\langle v^p, v \in \mathbb{V}\rangle \otimes M \ \to \ M. \tag{3.5.1}$$

We say that M is GL_n-equivariant (or $\mathrm{GL}(\mathbb{V})$-equivariant) if it is provided with a second k-linear pairing

$$\mathrm{GL}(\mathbb{V}) \times M \ \to \ M, \quad (g, m) \mapsto gm \tag{3.5.2}$$

such that for any $g \in \mathrm{GL}(\mathbb{V}), x \in kE$, and $m \in M$, we have

$$g(xm) = (gx)(gm).$$

In other words, the $\mathrm{GL}(\mathbb{V})$-action on M of (3.5.2) is such that the pairing (3.5.1) is a map of $\mathrm{GL}(\mathbb{V})$-modules with $\mathrm{GL}(\mathbb{V})$ acting diagonally on the tensor product.

We employ the following notation: if M is a GL_n–equivariant kE-module and $N \subset M$ is a subset, then we denote by gN the image of N under the action of $g \in \mathrm{GL}_n$; if $U \in \mathrm{Grass}(r, \mathbb{V})$, then we denote by gU the image of U under the action of $g \in \mathrm{GL}(\mathbb{V})$.

As we see in the next proposition, the abundant symmetries of GL_n-equivariant kE-modules imply that they have constant radical and socle types.

Proposition 3.6. *Let M be a GL_n-equivariant kE-module. Then the following holds.*

(1) *M has constant r-radical and r-socle type for any $r > 0$.*
(2) *For any $U \in \mathrm{Grass}(r, \mathbb{V})$, any $g \in \mathrm{GL}_n$, and any ℓ, $1 \le \ell \le r(p-1)$,*

$$\mathrm{Rad}^{\ell}_{gU}(M) = g \,\mathrm{Rad}^{\ell}_{U}(M), \quad \mathrm{Soc}^{\ell}_{gU}(M) = g \,\mathrm{Soc}^{\ell}_{U}(M).$$

Proof. Clearly, (1) follows from (2). We prove (2) for $\mathrm{Rad}_U(M)$, the other statements are similar. Let $\{u_1, \ldots, u_r\}$ be a basis of U. We have

$$\mathrm{Rad}_{gU}(M) = \sum_{i=1}^{r} (gu_i)M = \sum_{i=1}^{r} g(u_i(g^{-1}M)) =$$

$$g \sum_{i=1}^{r} u_i(g^{-1}M) = g \,\mathrm{Rad}_U(g^{-1}M) = g \,\mathrm{Rad}_U(M)$$

where the second and last equality hold since M is GL_n-equivariant. ☐

Examples of GL_n-equivariant kE–modules arise as follows. The identification $kE \simeq S^*(\mathbb{V})/\langle v^p, v \in \mathbb{V}\rangle$ provides the kE-module

$$\mathrm{Rad}^i(kE)/\,\mathrm{Rad}^{i+j}(kE)$$

with a $\mathrm{GL}(\mathbb{V})$-structure. Thus, the subquotients $S^{*\geq i}(\mathbb{V})/S^{*\geq j}(\mathbb{V})$ for $i \leq j$, are naturally modules over $S^*(\mathbb{V})$ with a $\mathrm{GL}(\mathbb{V})$ action. If $j-i \leq p$, then the action of $S^*(\mathbb{V})$ on these subquotient factors through the quotient map $S^*(\mathbb{V}) \to kE$, so that

$$S^{*\geq i}(\mathbb{V})/S^{*\geq j}(\mathbb{V}) \quad \text{for } j-i \leq p \tag{3.6.1}$$

inherits a kE-module structure.

Let $kG = k[y_1,\ldots,y_n]/(y_1^{p^m},\ldots,y_n^{p^m}) \simeq k((\mathbb{Z}/p^m)^{\times n}) \simeq S^*(\mathbb{V})/(v^{p^m}, v \in \mathbb{V})$ for some $m > 0$. Arguing exactly as above, we give

$$\operatorname{Rad}^i(kG)/\operatorname{Rad}^j(kG) \tag{3.6.2}$$

the structure of a kE module for $j - i \leq p$.

If $\Lambda^*(\mathbb{V})$ denotes the exterior algebra on $\mathbb{V}$, then the $\mathrm{GL}(\mathbb{V})$-module

$$\operatorname{Rad}^i(\Lambda^*(\mathbb{V}))/\operatorname{Rad}^{i+2}(\Lambda^*(\mathbb{V})) \tag{3.6.3}$$

also inherits a kE-module structure. Note that the anticommutativity of $\Lambda^*(\mathbb{V})$ causes no problem in the definition of the action because it gives a relation in $\operatorname{Rad}^2(\Lambda^*(\mathbb{V}))$.

It is straightforward to check that the kE and GL_n-actions described above are compatible, so that the kE-modules of (3.6.1), (3.6.2), and (3.6.3) are GL_n-equivariant. Proposition 3.6 thus implies the following.

Proposition 3.7. *Each of the following kE-modules M is GL_n-equivariant. Consequently, each has constant r-radical type and constant r-socle type for every $r > 0$.*

1. $M = \operatorname{Rad}^i(kE)/\operatorname{Rad}^j(kE)$ *for any* $0 \leq i < j$,
2. $M = \operatorname{Rad}^i(\Lambda^*(\mathbb{V}))/\operatorname{Rad}^{i+2}(\Lambda^*(\mathbb{V}))$ *for any* $0 \leq i$,
3. $M = S^{*\geq i}(\mathbb{V})/S^{*\geq j}(\mathbb{V})$ *for any* $0 \leq i$, $1 \leq j-i \leq p$,
4. $M = \operatorname{Rad}^i(kG)/\operatorname{Rad}^j(kG)$ *for any* $0 \leq i$, $1 \leq j-i \leq p$.

We next see how to generate examples of modules of constant type arising from the consideration of (negative) Tate cohomology. Once again, their formulation does not depend upon a choice of $\mathbb{V}$ so that we conclude that these examples are modules of constant radical and socle type independent of the choice of $\mathbb{V} \subset \operatorname{Rad}(kE)$ splitting the projection $\operatorname{Rad}(kE) \to \operatorname{Rad}(kE)/\operatorname{Rad}^2(kE)$.

Proposition 3.8. *Consider the extension of kE-modules*

$$0 \to k \to M \to \Omega^{-t-1}(k) \to 0 \tag{3.8.1}$$

corresponding to a non-zero Tate cohomlogy class $\zeta \in \widehat{\mathrm{H}}^{-t}(E,k)$ for some $t > 0$. The kE-module M has constant r-radical type and constant r-socle type for every $0 < r < n$.

Proof. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of kE-modules with the property that for every $U \in \operatorname{Grass}(r,\mathbb{V})$ the restriction of this sequence along $\alpha_U : C(U) \to kE$ splits. If M_1 and M_3 have constant r-radical type (respectively, r-socle type), then so does M_2. Consequently, by Example 3.4 it suffices to prove that the sequence (3.8.1) splits along α_U for every $U \in \operatorname{Grass}(r,\mathbb{V})$. As shown in [BC90], the splitting of (3.8.1) is implied by

$$\alpha_U^*(\zeta) = 0 \in \widehat{\mathrm{H}}^{-t}(C,k), \quad \forall U \in \operatorname{Grass}(r,\mathbb{V}), \tag{3.8.2}$$

where $C \equiv k[t_1,\ldots,t_r]/(t_i^p) \simeq C(U)$.

To show that $\alpha^*(\zeta) = 0$, we employ the non-degenerate pairing of Tate duality (see [BC90]),

$$\widehat{\mathrm{H}}^{-t}(C,k) \otimes \widehat{\mathrm{H}}^{t-1}(C,k) \to \widehat{\mathrm{H}}^{-1}(C,k) = k. \tag{3.8.3}$$

Suppose that $\alpha^*(\zeta) \neq 0$. Then there exists $\eta' \in \widehat{\mathrm{H}}^{t-1}(C,k)$ such that $\alpha^*(\zeta)\eta' \neq 0$ Since $t-1 \geq 0$, $\alpha^* : \widehat{\mathrm{H}}^{t-1}(E,k) \longrightarrow \widehat{\mathrm{H}}^{t-1}(C,k)$ is surjective. Hence, there exists $\eta \in \widehat{\mathrm{H}}^{t-1}(E,k)$ such that $\eta' = \alpha^*(\eta)$. This implies, by the non-degeneracy of (3.8.3), that

$$\alpha^*(\zeta\eta) \;=\; \alpha^*(\zeta)\eta' \;\neq 0.$$

However, this is a contradiction, because we know that the map $\alpha^* : \widehat{\mathrm{H}}^{-1}(E,k) \to \widehat{\mathrm{H}}^{-1}(C,k)$ is the zero map [BC90]. Thus we conclude that $\alpha^*(\zeta) = 0$. $\square$

It certainly is not always the case that constant r-Rad-rank is preserved by Heller shifts. For a very easy example, let M be a 2-dimensional indecomposable kE-module where the rank n of E is at least 2. Then M does not have constant 1-Rad^1-rank, but $\Omega(M)$ does have constant 1-Rad^1-rank.

A more complicated example is the following. In this case, M is a kE-module with $\mathrm{Rad}^2(M) = 0$ such that the 2-Rad^1-rank of M is constant (hence, M has constant 2-radical type) but the Heller shifts of M do not have constant 2-radical type. Note that this also gives an example of a module with constant 2-radical type that does not have constant 1-radical type, that is, constant Jordan type.

Example 3.9. Assume that k is a field of characteristic 2. Suppose that $kE = k[w,x,y,z]/(w^2,x^2,y^2,z^2)$ is the group algebra of an elementary abelian group of order $2^4 = 16$. We consider the module $\mathrm{Rad}^2(kE)$ which is spanned as a subspace of kE by the monomials

$$wx,\ wy,\ wz,\ xy,\ xz,\ yz,\ wxy,\ wxz,\ wyz,\ xyz,\ wxyz.$$

Let L be the submodule generated by wx, which has k-basis $wx, wxy, wxz, wxyz$. Let M be the quotient $\mathrm{Rad}^2(kE)/L$. The reader can easily check that M has constant 2-radical type. In particular, for any $U \in \mathrm{Grass}(r,\mathbb{V})$, $\mathrm{Rad}_U(M) = \mathrm{Rad}(M)$ which is spanned by wyz and xyz. Because $\mathrm{Rad}^2(M) = \{0\}$, it also has 2-Rad^2-type.

In terms of diagrams, the restriction of M to $kF_1 = k[x,w]/[x^2,w^2]$ has the form

$$\begin{tikzcd} & yz \arrow[dl, "x"'] \arrow[dr, "w"] & & \oplus & wy & \oplus & wz & \oplus & xy & \oplus & xz \\ xyz & & wyz & & & & & & & & \end{tikzcd}$$

Thus we see that $M_{\downarrow kF_1} \cong \Omega^{-1}(k) \oplus k^{\oplus 4}$. On the other hand, the restriction to $kF_2 = k[y,z]/(y^2,z^2)$ has the form

$$\begin{tikzcd} wz \arrow[dr, "y"'] & & wy \arrow[dl, "z"] & \oplus & xz \arrow[dr, "y"'] & & xy \arrow[dl, "z"] & \oplus & yz \\ & wyz & & & & xyz & & & \end{tikzcd}$$

Thus we have that $M_{\downarrow kF_2} \cong (\Omega^1(k))^{\oplus 2} \oplus k$.

Now consider the modules $\Omega^t(M)$ with $t = 2j$ an even non-negative integer. First note that the dimension of M is 7, and so the dimension of $\Omega^{2n}(M)$ must be

$3+4d$ for some number d which depends on n. In what follows we use the facts that if $kF = k(\mathbb{Z}/2\times\mathbb{Z}/2)$ then for any $t>0$

$$\dim \Omega^t(k_F) = 2t+1, \quad \dim \mathrm{Rad}(\Omega^t(k_F)) = t, \quad \dim \mathrm{Rad}(kF) = 3$$

(see [He61]). The formula

$$\Omega^{2j}(M)_{\downarrow kF_1} = \Omega^{2j}(\Omega^{-1}(k)\oplus k^{\oplus 4})_{\downarrow kF_1} = \Omega^{2j-1}(k_{F_1})\oplus(\Omega^{2j}(k_{F_1}))^{\oplus 4}\oplus(kF_1)^{\oplus m_1}$$

for some m_1 yields the dimension formula

$$3+4d = (4j-1)+4(4j+1)+4m_1$$

which implies that $m_1 = d-5j$. When a similar thing is done for the restriction to kF_2, we get that $m_2 = d-3j-1$. We conclude that

$$\dim(\mathrm{Rad}(kF_1)\Omega^{2j}(M)) = 3d-5j-1 \neq 3d-3j-1 = \dim(\mathrm{Rad}(kF_2)\Omega^{2j}(M)).$$

Consequently, the 2-Rad^1-rank of $\Omega^{2j}(M)$ is constant if and only if $j=0$. A similar analysis can be performed on $\Omega^t(M)$, for t odd or negative with the same result.

In addition, $\dim(\mathrm{Rad}^2(kF_i)\Omega^{2j}(M)) = m_i$, the rank of the projective part of the restriction of $\Omega^{2j}(M)$ to kF_i. Thus, $\Omega^{2j}(M)$ has constant 2-Rad^2-rank if and only if $j=0$.

4. Modules from quantum complete intersections

In this section, we consider kE-modules constructed as subquotients of quantum complete intersection algebras. We demonstrate how by varying parameters, we get families of modules with interesting properties, such as modules of constant Jordan type or constant r-radical or r-socle type for $r>1$. We supplement our constructions with multiple specific examples.

Let E be an elementary abelian p-group of rank n, and let $kE = k[x_1,\dots,x_n]/(x_1^p,\dots,x_n^p)$. Let $\mathfrak{q} = (q_{ij})_{i,j=1}^n$ be the matrix of quantum parameters: choose non-zero $q_{ij}\in k$ for $1\le i<j\le n$, and set $q_{ij} = q_{ji}^{-1}$ and $q_{ii}=1$.

Let $k\langle z_1,\dots,z_n\rangle$ be the algebra generated by n (non-commuting) variables $z_1,\dots,z_n$, and let $s>1$ be an integer. Let

$$S = \frac{k\langle z_1,\dots,z_n\rangle}{(z_i^s, z_iz_j - q_{ij}z_jz_i)}$$

be a quotient of the quantum complete intersection algebra $k\langle z_1,\dots,z_n\rangle/(z_iz_j - q_{ij}z_jz_i)$ with respect to the ideal generated by $(z_1^s,\dots,z_n^s)$. Let $I = \mathrm{Rad}(S)$. When this causes no confusion, we denote the generators of the augmentation ideal I by the same letters z_i, $1\le i\le n$. For $0\le a\le n(s-1)-1$, we define

$$W_a(s,\mathfrak{q}) \;=\; I^a/I^{a+2}, \quad \text{frequently denoted by } W_a. \tag{4.0.1}$$

As a vector space, W_a is generated by the monomials $\{z_1^{a_1}\dots z_n^{a_n}\}$ where $a_1+\dots+a_n = a$ or $a+1$ and $a_i\le s-1$ for $1\le i\le n$. We define the structure of a kE-module on $W_a(s,\mathfrak{q})$ by letting x_i act via multiplication by z_i :

$$x_iw \overset{def}{=} z_iw \pmod{I^{a+2}}$$

for any $w\in W_a$. By construction, $\mathrm{Rad}^2(kE)W_a = 0$. We also note that for $a\le s-2$, W_a is independent of s.

Example 4.1. Let $n = 2$ and choose s and a such that $a < s - 1$. Let $q = q_{1,2}$ be the quantization parameter. In this case $kE = k[x, y]/(x^p, y^p)$ and $S = k\langle z, t\rangle/(z^s, t^s, zt - qtz)$. Then the module $W_a(s, q)$ looks as follows:

$$\begin{tikzcd}
 & z^a \arrow[dl, "x"'] \arrow[dr, "q^a y"] & & z^{a-1}t \arrow[dl, "x"'] \arrow[dr, "q^{a-1}y"] & & \dots & & t^a \arrow[dl, "x"'] \arrow[dr, "y"] & \\
z^{a+1} & & z^a t & & z^{a-1}t^2 & \dots & zt^a & & t^{a+1}
\end{tikzcd},$$

where, for example, an arrow $z^it^j \xrightarrow{q^iy} z^it^{j+1}$ indicates that the action of y on z^it^j is defined via $q^iy(z^it^j) = z^it^{j+1}$.

It is easy to see that this kE–module is isomorphic to the "zig-zag" module denoted $W_{a+1,2}$ in [CFS11]. That is, in the case $n = 2$ introducing the parameter q does not lead to new isomorphism classes of modules. For $n > 2$, though, the choice of the $q_{i,j}$ does make a difference as we demonstrate in this section.

We now show that if a is sufficiently large, then the module W_a has a very strong property of having equal r-images independently of the choice of q_{ij}. In particular, it has constant r-radical type.

Proposition 4.2. *Let $W_a = W_a(s, \mathfrak{q})$ for some fixed choice of $s \geq 1$ and elements $q_{ij} \in k$. If $a \geq (n-r)(s-1)$, then the module W_a has the equal r-images property, meaning that for any U in* $\mathrm{Grass}(r, \mathbb{V})$*, we have that* $\mathrm{Rad}_U(W_a) = \mathrm{Rad}(W_a) = \mathrm{Rad}(kE)W_a$*. Hence, W_a has constant r-*Rad*-type.*

Proof. Let $\mathbb{V} \subset \mathrm{Rad}(kE)$ be the subspace generated by $\{x_1, \dots, x_n\}$. Choose U in $\mathrm{Grass}(r, \mathbb{V})$. For the purposes of the argument we desire a basis for the subspace $U \subseteq \mathbb{V}$ that is chosen carefully as follows. Let $\underline{u} = [u_1, \dots, u_r]$ be an ordered basis for U and suppose that $u_i = \sum_{i=1}^n a_{i,j}x_j$ for $a_{i,j} \in k$. We may assume that the matrix $(a_{i,j})$ is in echelon form, so that there is some subset $\Sigma = \{i_1, \dots, i_r\}$ in $\{1, \dots, n\}$ such that the $r \times r$ submatrix having the columns indexed by Σ is the identity matrix. We claim that, without loss of generality, we may assume that that $\Sigma = \{1, \dots, r\}$. That is, if Σ is not of this form then we correct the situation by applying a suitable permutation to the basis $x_1, \dots, x_n$ of $\mathbb{V}$. The same permutation must be applied to the generators $z_1, \dots, z_n$ of the algebra S. Note that this changes the values of the q_{ij}, but because these are assumed to be non zero, the augmentation ideal $I \subset S$ is invariant under the permutation. Hence, W_a is unchanged.

Let
$$\alpha : \ kF \ = \ k[t_1, \dots, t_r]/(t_1^p, \dots, t_r^p) \quad \longrightarrow \quad kE$$
be given by $\alpha(t_i) = u_i$ for $\underline{u} = [u_1, \dots, u_r]$ chosen as above. For $i \in \{1, \dots, r\}$, let $u_i = \sum a_{i,j}x_j$, and set $w_i = \sum a_{i,j}z_j \in S$, so that $\alpha(t_i)$ acts on W_a by multiplication by w_i (mod I^{a+2}). Because of the way that the basis was chosen, we have that for each i, $1 \leq i \leq r$, $w_i = z_i + \sum_{j=r+1}^n a_{i,j}z_j$.

The module W_a has a basis consisting of the monomials
$$Z_{s_1,\dots,s_n} \ = \ z_1^{s_1} \dots z_n^{s_n}$$
where $s_1 + \cdots + s_n = a$ or $a + 1$ and $0 \leq s_i < s$ for all i, taken modulo $(z_1^s, \dots, z_n^s)$ and I^{a+2}. Since $\alpha(t_i)$ acts on W_a via w_i which is a linear conbination of z_i, we have
$$\alpha(t_i)I^a \subset I^{a+1}.$$

Therefore,

$$\mathrm{Rad}_U(W_a) \subset I^{a+1}/I^{a+2} = \mathrm{Rad}(kE)W_a$$

Hence, we need to show that

$$I^{a+1}/I^{a+2} \subset \mathrm{Rad}_U(W_a),$$

or, equivalently, that every monomial $z_1^{s_1} \cdots z_n^{s_n}$ with $s_1 + \cdots + s_n = a+1$ is in $\Sigma_{i=1}^r w_i W_a$. We accomplish this by an induction on the number $N = s_1 + \cdots + s_r$.

Because we assume that $a \geq (n-r)(s-1)$, the minimum value that N can have is $a+1-(n-r)(s-1) > 0$ and that occurs when the monomial has the form $z_1^{s_1} \ldots z_r^{s_r} z_{r+1}^{s-1} \ldots z_n^{s-1}$ for $s_1 + \cdots + s_r = a+1-(n-r)(s-1)$. Let i be the least integer such that $s_i > 0$. Since $z_j^s = 0$ in S and $w_i = z_i + \sum\limits_{j=r+1}^{n} a_{i,j} z_j$, we have

$$\text{(4.2.1)} \qquad z_i^{s_i} \cdots z_r^{s_r} z_{r+1}^{s-1} \ldots z_n^{s-1} \; - \; w_i z_i^{s_i-1} \ldots z_r^{s_r} z_{r+1}^{s-1} \ldots z_n^{s-1} \; = \\ - \sum_{j=r+1}^{n} a_{i,j} z_j z_i^{s_i-1} \ldots z_r^{s_r} z_{r+1}^{s-1} \ldots z_n^{s-1} = 0$$

Hence, the class of $z_i^{s_i} \ldots z_r^{s_r} z_{r+1}^{s-1} \ldots z_n^{s-1}$ is in $\mathrm{Rad}_U(W_a)$.

For the induction step, let $Z = z_1^{s_1} \ldots z_n^{s_n}$ with $N = s_1 + \cdots + s_r > a+1-(n-r)(s-1)$. If i is the least integer with $s_i > 0$, then we get the exact same formula as in 4.2.1. By induction, the classes of the elements on the right hand side are all in $\mathrm{Rad}_U(W_a)$. Hence, so too is the class of Z.

We conclude that $\mathrm{Rad}_U(W_a) = I^{\alpha+1}/I^{a+2}$ is independent of U. On the other hand, for r-planes $U, V \subset \mathbb{V}$

$$\mathrm{Rad}_U(I^2) \;=\; \mathrm{Rad}_U(\mathrm{Rad}_V(M_a)) \;=\; \mathrm{Rad}_V(\mathrm{Rad}_U(M_a)) \;=\; \mathrm{Rad}_V(I^2).$$

Continuing by induction on j, we conclude that W_a has constant r-Rad^j-rank for all $j, 1 \leq j < p$. □

The following lemma (whose proof we leave to the reader) is proved by induction using the q-binomial formula: suppose X, Y are q-commuting variables, that is $YX = qXY$. Then

$$(X+Y)^n = \sum_{i=0}^{n} \binom{n}{i}_q X^i Y^{n-i},$$

where $\binom{n}{i}_q = \frac{(n)_q!}{(i)_q!(n-i)_q!}$, $(i)_q = 1 + q + \cdots + q^{i-1}$, and $(i)_q! = (i)_q(i-1)_q \ldots (1)_q$.

Lemma 4.3. *Let $s > 1$ be an integer prime to p, and let $\zeta \in k$ be a primitive s^{th} root of unity. Let $z_1, \ldots, z_n$ be ζ-commuting variables, that is $z_i z_j = \zeta z_j z_i$ for $1 \leq i < j \leq n$. Then for any $a_1, \ldots, a_n \in k$,*

$$(a_1 z_1 + \cdots + a_n z_n)^s = a_1^s z_1^s + \cdots + a_n^s z_n^s.$$

This lemma enables us to show that the modules $W_{a,\mathfrak{q}}$ of (4.0.1) are of constant Jordan type provided that our quantum parameters $\mathfrak{q}$ are given by a single s^{th} root of unity.

Proposition 4.4. *Let $s > 1$ be an integer. Assume that one of the following holds*

I. $a < s-1$ *or*

II. $(s,p) = 1$ *and* $q_{i,j} = \zeta$ *for* $1 \leq i < j \leq n$ *where* ζ *be a primitive* s^{th} *root of unity in* k.

Then the module $W_a = W_a(s, \mathfrak{q})$ has constant Jordan type.

Proof. To prove that W_a has constant Jordan type, we need to show that for every non-trivial n-tuple $(a_1, \ldots, a_n) \in k^n$, the Jordan type of the element $u = a_1x_1 + \cdots + a_nx_n$ as an operator on W_a is the same. Since u^2 acts trivially by construction of W_a, we just need to show that the rank of u is constant.

Let $\ell = \sum\limits_{i=1}^{n} a_iz_i \in S$. Choose some i so that $a_i \neq 0$. Then $I = \operatorname{Rad}(S)$ is generated by the elements $\ell, z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n$. By an argument as in the proof of Proposition 4.2, we have that W_a has a basis as a k-vector space consisting of the classes modulo I^{a+2} of the monomials

$$\ell^v z_1^{v_1} \ldots z_{i-1}^{v_{i-1}} z_{i+1}^{v_{i+1}} \ldots z_n^{v_n}$$

for $0 \leq v, v_i \leq s-1$, and $(v + \sum_{j \neq i} v_i) \in \{a, a+1\}$ under either one of our two assumptions. By the definition of the action, u acts on W_a via multiplication by ℓ. We compute the kernel of the action of u on W_a in our two cases.

I. Assume $a < s-1$. In this case, the kernel of u is precisely $\operatorname{Rad}(W_a)$ since multiplication by ℓ does not annihilate any linear combination of the monomials $\ell^v z_1^{v_1} \cdots z_{i-1}^{v_{i-1}} z_{i+1}^{v_{i+1}} \cdots z_n^{v_n}$ with $v + \sum_{j \neq i} v_i = a$. Hence, W_a has constant Jordan type.

II. Now suppose $q_{ij} = \zeta$ for $1 \leq i < j \leq n$. Since we also assume $(s, p) = 1$, Lemma 4.3 implies that $\ell^s = 0$ in this case.

The kernel of multiplication by ℓ on W_a is precisely the space spanned by those monomials $\ell^v z_1^{v_1} \cdots z_{i-1}^{v_{i-1}} z_{i+1}^{v_{i+1}} \cdots z_n^{v_n}$ for which either $v + \sum_{j \neq i} v_i = a+1$ or $v = s-1$. Since the number of such monomials is again independent of the choice of ℓ we conclude that W_a has constant Jordan type. $\square$

The next example illustrates that the condition of Proposition 4.4 requiring that ζ is the s-th root of unity is crucial.

Example 4.5. Let $n = 3$, $s = 2$, and $a = 1$. Let $kE = k[x, y, z]/(x^p, y^p, z^p)$. Pick $q \in k^*$ and let $q_{ij} = q$ for any $i < j$. Let $\tilde{x}, \tilde{y}, \tilde{z}$ be the algebraic generators of S, that is, $S = k\langle \tilde{x}, \tilde{y}, \tilde{z} \rangle / (\tilde{x}^2, \tilde{y}^2, \tilde{z}^2, \tilde{x}\tilde{y} - q\tilde{y}\tilde{x}, \tilde{x}\tilde{z} - q\tilde{z}\tilde{x}, \tilde{y}\tilde{z} - q\tilde{z}\tilde{y})$. Then $W_1(2, q)$ can be depicted as follows:

$$\begin{tikzcd}
 & \tilde{z} \arrow[dl, "y"'] \arrow[dr, "x"] & \\
\tilde{y}\tilde{z} & & \tilde{x}\tilde{z} \\
\tilde{y} \arrow[u, "qz"] \arrow[dr, "x"'] & & \tilde{x} \arrow[u, "qz"'] \arrow[dl, "qy"] \\
 & \tilde{x}\tilde{y} &
\end{tikzcd}$$

For $q = -1$, this module is a special case of modules in Proposition 3.7(2). In particular, it has constant Jordan type. We show that for $q \neq -1$, $W_1(2, q)$ fails to have constant Jordan type. To achieve this, we compute the non-maximal support variety of $W_1(2, q)$ for a generic q.

Fix the following order of the linear generators of $W_1(2,q)$: $\tilde{x}, \tilde{y}, \tilde{z}, \tilde{x}\tilde{y}, \tilde{x}\tilde{z}, \tilde{y}\tilde{z}$. Let $[a:b:c] \in \mathbb{P}^2$ and let $\ell = ax + by + cz \in \mathbb{V}$ be a generator of the corresponding line in $\operatorname{Rad} kE$. The matrix of ℓ as an endomorphism of $W_1(2,q)$ with respect to our fixed basis has the form

$$\ell \leftrightarrow \begin{pmatrix} 0 & 0 \\ A_\ell & 0 \end{pmatrix}$$

where for x, y and z we have

$$A_x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_y = \begin{pmatrix} q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_z = \begin{pmatrix} 0 & 0 & 0 \\ q & 0 & 0 \\ 0 & q & 0 \end{pmatrix}.$$

For the general element $\ell = ax + by + cz$ we get

$$A_\ell = \begin{pmatrix} qb & a & 0 \\ qc & 0 & a \\ 0 & qc & b \end{pmatrix}.$$

The determinant of A_ℓ is $-q(q+1)abc$. Hence, for $q \neq -1$ the nonmaximal support variety is a union of three lines: $a = 0, b = 0, c = 0$. In particular, $W_1(2,q)$ has constant Jordan type if and only if $q = -1$.

We finish this example recording the properties of radicals and socles of $W_1(2,q)$. First, since the condition $a \geq (n-r)(s-1)$ is satisfied for $r = 2$ (we get $1 \geq (3-2)(2-1)$), Proposition 4.2 implies that $W_1(2,q)$ has constant 2-radical type. Since the module $W_1(2,q)$ is self-dual, it also has constant 2-socle type. So, in particular, we conclude that for $q \neq -1$, $W_1(2,q)$ does not have constant Jordan type but has constant 2-radical and 2-socle type.

In the following example, we construct a module of the form $W_a(s,q)$ that has constant Jordan type and constant 2-socle type but fails to have constant 2-radical type. It follows that the dual of such $W_a(s,q)$ has constant Jordan type, constant 2-radical type, but not constant 2-socle type.

Example 4.6. Let $n = 3$, $s \geq 3$, $a = 1$. Let $q \neq 0$ be a quantum parameter, and set $q_{ij} = q$ for $1 \leq i < j \leq 3$. Let $M_q = W_1(s, \mathfrak{q})$. Let $kE = k[x_1, x_2, x_3]/(x_1^p, x_2^p, x_3^p)$ and $S = k\langle z_1, z_2, z_3\rangle/(z_i^s = 0, z_i z_j = q_{ij} z_j z_i)$. Here is a depiction of M_q:

$$\begin{tikzcd}
z_1^2 & & z_1z_2 & & z_2^2 \\
 & z_1 \arrow[ul, "x_1"'] \arrow[ur, "x_2", "(q)"'] \arrow[d, "x_3"', "(q)"] & & z_2 \arrow[ul, "x_1"'] \arrow[ur, "x_2"] \arrow[d, "(q)"', "x_3"] & \\
 & z_1z_3 & & z_2z_3 & \\
 & & z_3 \arrow[ul, "x_1"] \arrow[ur, "x_2"'] \arrow[d, "x_3"'] & & \\
 & & z_3^2 & &
\end{tikzcd}$$

Here, an arrow marked with (q) means that the action is twisted by q. For example, $x_3 \circ z_1 = q^{-1} z_1 z_3$.

We make several observations about M_q.

I. By Proposition 4.4, M_q has constant Jordan type.

II. The module M_q has constant 2-Socle type. Indeed, let $a = 3s-5$ and consider the module $W_a(s, \mathfrak{q})$ for arbitrary non-zero parameters $\mathfrak{q} = (q_{ij})$:

$$
\begin{tikzcd}
z_1^{s-3}z_2^{s-1}z_3^{s-1} \arrow[dr, "x_1"] & & z_1^{s-2}z_2^{s-2}z_3^{s-1} \arrow[dl, "(?)"', "x_2"] \arrow[dr, "x_1"] & & z_1^{s-1}z_2^{s-3}z_3^{s-1} \arrow[dl, "x_2"'] \\
& z_1^{s-2}z_2^{s-1}z_3^{s-1} & & z_1^{s-1}z_2^{s-2}z_3^{s-1} & \\
& z_1^{s-2}z_2^{s-1}z_3^{s-2} \arrow[u, "x_3", "(?)"'] \arrow[dr, "x_1"] & & z_1^{s-1}z_2^{s-2}z_3^{s-2} \arrow[u, "x_3", "(?)"'] \arrow[dl, "(?)"', "x_2"] & \\
& & z_1^{s-1}z_2^{s-1}z_3^{s-2} & & \\
& & z_1^{s-1}z_2^{s-1}z_3^{s-3} \arrow[u, "x_3"] & &
\end{tikzcd}
$$

The action along the arrows marked with (?) is twisted by some monomials in q_{ij}. By choosing the parameters q_{12}, q_{23} and q_{13} appropriately, we can arrange the twists so that

$$W_{3s-5}(s, \mathfrak{q}) \simeq M_q^{\#}.$$

Since $3s-5 > (3-2)(s-1)$ for $s \geq 3$, Proposition 4.2 implies that $W_{3s-5}(s, \mathfrak{q})$ has constant 2-radical type. By duality, M_q has constant 2-socle type.

III. Proposition 4.2 does not apply to 2-images of M_q since the parameters $n = 3$, $r = 2$, $s \geq 3$, and $a = 1$ fail to satisfy the condition $a \geq (n-r)(s-1)$. In fact, we proceed to show that M_q does not have constant 2-radical type *unless* $q = 1$.

Let $U \in \operatorname{Grass}(2, \mathbb{V})$ be a 2–plane in the three-dimensional space $\mathbb{V}$. Let

$$\begin{aligned} u_1 &= a_1x_1 + a_2x_2 + a_3x_3 \\ u_2 &= b_1x_1 + b_2x_2 + b_3x_3 \end{aligned}$$

be a basis of U, and let

$$\begin{aligned} \ell_1 &= a_1z_1 + a_2z_2 + a_3z_3 \\ \ell_2 &= b_1z_1 + b_2z_2 + b_3z_3 \end{aligned}$$

be the corresponding elements in $S = k\langle z_1, z_2, z_3\rangle/(z_i^s, z_iz_j - q_{ij}z_jz_i)$.

We fix the following order of the basis of M_q: z_1, z_2, z_3 for $M_q/\operatorname{Rad}(M_q)$ and $z_1z_2, z_1z_3, z_2z_3, z_1^2, z_2^2, z_3^2$ for $\operatorname{Rad}(M_q)$. Since $\operatorname{Rad}_U(M_q) \subset \operatorname{Rad}(M_q)$, we work inside $\operatorname{Rad}(M_q)$. We have

$$\begin{aligned} \ell_1z_1 &= a_1z_1^2 + qa_2z_1z_2 + qa_3z_1z_3 \\ \ell_1z_2 &= a_1z_1z_2 + a_2z_2^2 + qa_3z_2z_3 \\ \ell_1z_3 &= a_1z_1z_3 + a_2z_2z_3 + a_3z_3^2 \end{aligned}$$

and similarly for ℓ_2. Hence, with respect to our fixed basis, $\mathrm{Rad}_U(M_q)$ is generated by the following six vectors:

$$R = \begin{pmatrix} qa_2 & a_1 & 0 & qb_2 & b_1 & 0 \\ qa_3 & 0 & a_1 & qb_3 & 0 & b_1 \\ 0 & qa_3 & a_2 & 0 & qb_3 & b_2 \\ a_1 & 0 & 0 & b_1 & 0 & 0 \\ 0 & a_2 & 0 & 0 & b_2 & 0 \\ 0 & 0 & a_3 & 0 & 0 & b_3 \end{pmatrix}$$

To compute the nonmaximal 2-radical support variety of M_q, one would need to calculate the rank of this matrix for different parameters a_i, b_i. We leave such calculations to the Appendix and just show here that the rank of this matrix is not constant.

First, take $u_1 = x_1$ and $u_2 = x_2$. In this case we see from the picture that $\alpha_U^*(M_q)$ for $U = \langle x_1, x_2 \rangle$ splits as a direct sum of three "zig-zag" modules:

$$\begin{tikzcd} \bullet & \oplus & \bullet \arrow[dl] \arrow[dr] & \oplus & & \bullet \arrow[dl] \arrow[dr] & & \bullet \arrow[dl] \arrow[dr] & & . \\ & \bullet & & \bullet & \bullet & & \bullet & & \bullet & \end{tikzcd}$$

Hence, $\dim \mathrm{Rad}_U(M_q) = 5$.

Second, take $u_1 = x_1 + x_2, u_2 = x_2 + x_3$. Hence, $a_1 = b_1 = 1$, $b_2 = c_2 = 1$ and $c_1 = a_1 = 0$. In this case,

$$R = \begin{pmatrix} q & 1 & 0 & q & 0 & 0 \\ 0 & 0 & 1 & q & 0 & 0 \\ 0 & 0 & 1 & 0 & q & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We have $\det R = q(1-q)$. Hence, if $q \neq 1$, the rank of R is 6, and, therefore, for the 2-plane U spanned by u_1, u_2, we have $\dim \mathrm{Rad}_U(M_q) = 6$. We conclude that M_q does not have constant 2-radical rank.

We give another example of the same phenomenon. This time, we construct a module which has constant Jordan type, constant 2-radical type but does not have constant 2-socle type.

Example 4.7. Assume that $p > 3$. Let $n = 4$, $s = 3$ and $q_{i,j} = \zeta_3$ for all $1 \leq i < j \leq r$, where $\zeta = \zeta_3 \in k$ is a primitive third root of unity. Consider the module

$$M = W_6(3, \zeta_3) = I^6/I^8.$$

By Proposition 4.4, M has constant Jordan type. Since $6 > (4-2)(3-1)$, Proposition 4.2 implies that M has constant 2-radical type. We wish to show that M fails to have constant 2-socle type.

The module M has dimension 14, and has a basis consisting of the classes of the monomials of the form $z_1^a z_2^b z_3^c z_4^d$ with $0 \leq a, b, c, d \leq 2$ and where $a+b+c+d$ is either 6 or 7. The radical of M, which is spanned by the monomials with $a+b+c+d = 7$, has dimension 4. Because the module has the equal 1–images property by 4.2, the image of multiplication by any nonzero $u = a_1x_1 + \cdots + a_4x_4$ is the entire radical.

Consequently the Jordan type of any such u consists of 4 blocks of size 2, and 6 blocks of size 1. Also, the dimension of the kernel of multiplication by u is 10.

Assume first that $U \subseteq \mathbb{V}$ is the subspace spanned by x_1 and x_2. Then $\mathrm{Soc}_U(M)$ is the set of all elements annihilated by multiplication by both x_1 and x_2. Clearly, the monomials $z_1^2 z_2^2 z_3^2$, $z_1^2 z_2^2 z_3 z_4$, and $z_1^2 z_2^2 z_4^2$ are in $\mathrm{Soc}_U(M)$. Moreover, $\mathrm{Rad}(M) \in \mathrm{Soc}_U(M)$. From this we see that $\mathrm{Soc}_U(M)$ has dimension at least 7, and further investigation shows that the dimension is exactly 7.

Next suppose that U is the subspace spanned by the elements $u_1 = x_1 + x_2$ and $u_2 = x_1 + x_3$. We claim that the dimension of $\mathrm{Soc}_U(M)$ is 6. Let K_i denote the kernel of multiplication by u_i on M. Then $\mathrm{Rad}(M)$ is in both K_1 and K_2. In addition, the elements

$$z_1^2 z_2^2 z_3^2, \quad z_1^2 z_2^2 z_3 z_4, \quad z_1^2 z_2^2 z_4^2, \quad z_1^2 z_3^2 z_4^2, \quad z_1^1 z_2^2 z_3^2 z_4 - \zeta z_1^2 z_2^1 z_3^2 z_4, \quad z_1^2 z_2^1 z_3^2 z_4,$$
$$z_1^1 z_2^2 z_3^1 z_4^2 - \zeta z_1^2 z_2^1 z_3^1 z_4^2, \qquad z_1^2 z_3^2 z_4^2 + z_2^2 z_3^2 z_4^2 - \zeta^2 z_1^2 z_2^1 z_3^1 z_4^2,$$
$$z_1^1 z_2^1 z_3^2 z_4^2 - z_1^2 z_2^1 z_3^1 z_4^2, \qquad z_1^1 z_2^2 z_3^1 z_4^2 - z_2^2 z_3^2 z_4^2 - \zeta^2 z_1^2 z_1^2 z_3^2 z_4^2$$

are in $K_1 + K_2$. That is, the reader may check that each of the above elements is annihilated by either u_1 or by u_2. Moreover, it is straightforward to check that these elements are linearly independent and independent of $\mathrm{Rad}(M)$. Therefore, $K_1 + K_2$ has dimension 14, whereas each K_i has dimension 10. Hence $\dim \mathrm{Soc}_U(M) = \dim(K_1 \cap K_2) = 6$.

In the appendix, we calculate some nonminimal r-socle support varieties for modules of the form W_a. Whereas calculations in the two examples above were simple enough to do by hand, the calculations left in the appendix use computational software.

5. Radicals of L_ζ-modules

As in previous sections, E is an elementary abelian p–group of rank n and $\mathbb{V} \subset \mathrm{Rad}(kE)$ is chosen as in (1.0.1). For a homogeneous cohomology class $\zeta \in \mathrm{H}^m(E, k)$, we recall that the module L_ζ is defined to be

$$L_\zeta \;\equiv\; \mathrm{Ker}\{\zeta : \Omega^m(k) \to k\} \tag{5.0.1}$$

Here, we have abused notation by using $\zeta : \Omega^m(k) \to k$ also to denote the map representing $\zeta \in \mathrm{H}^m(G, k)$. As we see in this section, the L_ζ-modules give good examples of behavior of radical and socle ranks.

If $\alpha : C = k[t_1, \ldots, t_r]/(t_i^p) \to kE$ is a flat map, we write $\Omega^m(k_C)$ for the m^{th} Heller translate of the trivial C-module, thereby distinguishing this Heller translate from the restriction $\alpha^*(\Omega^m(k))$ of the kE-module (which is stably equivalent to $\Omega^m(k_C)$).

We employ the following notation:

$$\mathrm{H}^\bullet(E, k) \;= \begin{cases} \mathrm{H}^*(E, k) & \text{if } p = 2, \\ \mathrm{H}^{\mathrm{even}}(E, k) & \text{otherwise.} \end{cases} \tag{5.0.2}$$

Thus, $\mathrm{H}^\bullet(E, k)$ is a commutative algebra, and $\mathrm{Proj}\, \mathrm{H}^\bullet(E, k) \simeq \mathbb{P}^{n-1} \simeq \mathrm{Grass}(1, \mathbb{V})$.

For our analysis of the behavior of radicals of L_ζ, we need to exploit a somewhat finer structure of the cohomology ring of $kE = k[x_1, \ldots, x_n]/(x_i^p)$ and of the restriction map on cohomology.

Let $f : k \xrightarrow{a\mapsto a^p} k$ be the Frobenius map. For a k-vector space V we use the standard notation $V^{(1)}$ for the *Frobenius twist* of V, a vector space obtained via base change $f : k \to k$

$$V^{(1)} = V \otimes_f k$$

If R is a (finitely generated commutative) k-algebra, then we have a map of k-algebras

$$R^{(1)} \to R$$

which sends $x \otimes a$ to $a^p x$. Hence, there is an induced map of k-varieties

$$F : \operatorname{Spec} A \to (\operatorname{Spec} A)^{(1)} \stackrel{def}{=} \operatorname{Spec} A^{(1)}.$$

The same construction applies globally. If X is any k-variety, we obtain a Frobenius twist $X^{(1)}$ and a map of k-varieties

$$F : X \to X^{(1)}$$

Moreover, if X is defined over $\mathbb{F}_p$, then we have a natural identification $X^{(1)} \simeq X$ and the Frobenius map becomes a self-map

$$F_X : X \to X.$$

We direct the reader to [Jan03, I.2] and [FS97, §1] for a detailed discussion of the properties of the Frobenius twist.

We apply the above discussion to the algebra $S^*(V^\#)$, so that the k-points of $\operatorname{Spec} S^*(V^\#)$ constitute the vector space V. Using the natural k-algebra isomorphisms $S^*((V^{(1)})^\#) \simeq (S^*(V^\#))^{(1)} = S^*(V^\#) \otimes_f k$ (see [FS97, §1]), we get a map of varieties over k

$$F : V \to V^{(1)}.$$

Suppose that V is given an $\mathbb{F}_p$-structure; in other words, V is identified with $V_0 \otimes_{\mathbb{F}_p} k$ where V_0 is an $\mathbb{F}_p$-vector space. Then we have a natural identification $V^{(1)} \simeq V$, and the Frobenius map becomes a self-map

$$F = F_V : V \to V$$

If we pick a basis $\{e_1, \dots, e_n\}$ of V_0, then the Frobenius map is given explicitly via the formula

$$F_V : V \to V, \quad a_1e_1 + \cdots + a_ne_n \mapsto a_1^pe_1 + \cdots + a_n^pe_n.$$

Since k is assumed to be algebraically closed (hence, perfect), the Frobenius map is a bijection on V.

The following description of the cohomology of $A = k[x_1, \dots, x_n]/(x_i^p)$ can be found in [Jan03, I.4.27]. Note that $V^{(1)}$ has a natural structure of a GL_n–module given by pulling back the standard representation of $\mathrm{GL}_n = \mathrm{GL}(V)$ on V via the Frobenius map $F : \mathrm{GL}_n \to \mathrm{GL}_n$.

Proposition 5.1. *Let V be an n-dimensional k-vector space with a basis $\{x_1, \dots, x_n\}$, and let $A = S^*(V)/(v^p, v \in V)$. There is an isomorphism of graded GL_n-algebras*

$$\begin{aligned} \mathrm{H}^*(A, k) &\simeq S^*(V^\#) && \text{for } p = 2, \\ \mathrm{H}^*(A, k) &\simeq S^*((V^{(1)})^\#[2]) \otimes \Lambda^*(V^\#) && \text{for } p > 2, \end{aligned}$$

where $(V^{(1)})^\#[2]$ is the vector space $(V^{(1)})^\#$ placed in degree 2.

Identifying kE with $S^*(\mathbb{V})/(v^p, v \in \mathbb{V})$, we conclude that

$$\mathrm{H}^*(E,k) = \begin{cases} k[\zeta_1,\dots,\zeta_n] & \text{if } p=2, \\ k[\zeta_1,\dots,\zeta_n]\otimes\Lambda(\eta_1,\dots,\eta_n) & \text{otherwise,} \end{cases} \tag{5.1.1}$$

where $\deg(\zeta_i) = 1$ if $p = 2$ and $\deg(\zeta_i) = 2$ for $p > 2$. Hence, $k[\zeta_1,\dots,\zeta_n]$ is the homogeneous coordinate ring of $\operatorname{Proj}\mathrm{H}^\bullet(E,k) = \operatorname{Proj}S^*(\mathbb{V}^\#) \simeq \mathbb{P}^{n-1}$ for $p = 2$ and $\operatorname{Proj}(\mathrm{H}^\bullet(E,k)_{red}) = \operatorname{Proj}S^*((\mathbb{V}^{(1)})^\#) \simeq \mathbb{P}^{n-1}$ (with GL_n action twisted by Frobenius) for $p > 2$.

The functoriality of the identifications of Proposition 5.1 immediately implies the following corollary.

Corollary 5.2. *Let $U \subset V$ be an r-dimensional subspace with ordered basis $u_1,\dots,u_r$, let $C = k[t_1,\dots,t_r]/(t_1^p,\dots,t_r^p)$, and let $\alpha : C \to A$ be a k-algebra map such that $\{\alpha(t_1) = u_1,\dots,\alpha(t_r) = u_r\}$ is a basis for U. Then there is a commutative diagram of k-algebras*

$$\begin{tikzcd} \mathrm{H}^\bullet(A,k)_{\mathrm{red}} \arrow[r, "\sim"] \arrow[d, "\alpha^*"] & S^*((V^{(1)})^\#) \arrow[d] \\ \mathrm{H}^\bullet(C,k)_{\mathrm{red}} \arrow[r, "\sim"] & S^*((U^{(1)})^\#) \end{tikzcd}$$

for $p > 2$ with the right vertical map induced by the Frobenius twist of the embedding $U \subset V$, and

$$\begin{tikzcd} \mathrm{H}^*(A,k) \arrow[r, "\sim"] \arrow[d, "\alpha^*"] & S^*(V^\#) \arrow[d] \\ \mathrm{H}^*(C,k) \arrow[r, "\sim"] & S^*(U^\#). \end{tikzcd}$$

for $p = 2$.

Let

$$\alpha_* : \operatorname{Spec}(\mathrm{H}^\bullet(C,k)_{red}) \to \operatorname{Spec}(\mathrm{H}^\bullet(A,k)_{red})$$

be the map of k-varieties induced by α. Then we have a commutative diagram of k-varieties

$$\begin{tikzcd} (\operatorname{Spec}\mathrm{H}^\bullet(C,k)_{red}) \arrow[r, "\alpha_*"] \arrow[d, "\simeq"] & (\operatorname{Spec}\mathrm{H}^\bullet(A,k)_{red}) \arrow[d, "\simeq"] \\ U^{(1)} \arrow[r, hook] & V^{(1)} \end{tikzcd}$$

for $p > 2$ and

$$\begin{tikzcd} \operatorname{Spec}\mathrm{H}^\bullet(C,k) \arrow[r, "\alpha_*"] \arrow[d, "\simeq"] & \operatorname{Spec}\mathrm{H}^\bullet(A,k) \arrow[d, "\simeq"] \\ U \arrow[r, hook] & V \end{tikzcd}$$

for $p = 2$.

The following proposition is our key tool in determining whether the modules L_ζ have constant r-Rad^j-rank. In contrast to most of the results of this paper, this proposition is proved for a general finite group scheme.

Proposition 5.3. *Let G be a finite group scheme, and let ζ be a non-zero cohomology class of degree m. Then*

$$\dim \operatorname{Rad}(\Omega^m(k)) - \dim \operatorname{Rad}(L_\zeta) \;=\; \dim \operatorname{Ker}\{\cdot\zeta : \mathrm{H}^1(G,k) \to \mathrm{H}^{m+1}(G,k)\}.$$

In particular, if $\zeta : \mathrm{H}^1(G,k) \to \mathrm{H}^{m+1}(G,k)$ is injective, then

$$\operatorname{Rad}^j(L_\zeta) = \operatorname{Rad}^j(\Omega^m(k))$$

for any $j > 0$.

Proof. To prove the proposition, we construct a linear isomorphism

$$\Psi : (\operatorname{Rad}(\Omega^m(k))/\operatorname{Rad}(L_\zeta))^\# \to \operatorname{Ker}\{\cdot\zeta : \mathrm{H}^1(G,k) \to \mathrm{H}^{m+1}(G,k)\}.$$

Let

$$0 \longrightarrow L_\zeta \longrightarrow \Omega^m(k) \longrightarrow k \longrightarrow 0$$

be the defining sequence for L_ζ, and let

$$\gamma :\; 0 \longrightarrow L_\zeta/\operatorname{Rad}(L_\zeta) \longrightarrow \Omega^m(k)/\operatorname{Rad}(L_\zeta) \longrightarrow k \longrightarrow 0$$

be the induced sequence. For a non-trivial map $f : L_\zeta/\operatorname{Rad}(L_\zeta) \to k$, we let

$$\gamma_f :\; 0 \longrightarrow k \longrightarrow M \longrightarrow k \longrightarrow 0$$

be the pushout of the sequence γ along the map f. In other words, we have a commutative diagram with exact rows:

(5.3.1)

$$\begin{tikzcd}
 & \operatorname{Rad}(\Omega^m(k))/\operatorname{Rad}(L_\zeta) \arrow[r, equal] \arrow[d, hook, "i"'] & \operatorname{Rad}(\Omega^m(k))/\operatorname{Rad}(L_\zeta) \arrow[d, hook] & & \\
0 \arrow[r] & L_\zeta/\operatorname{Rad}(L_\zeta) \arrow[r] \arrow[d, "f"'] & \Omega^m(k)/\operatorname{Rad}(L_\zeta) \arrow[r, "\zeta"] \arrow[d, "f'"'] & k \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & k \arrow[r] & M \arrow[r] & k \arrow[r] & 0
\end{tikzcd}$$

The cohomology class $\gamma_f\zeta \in \mathrm{H}^{m+1}(G,k)$ is represented by the composition

(5.3.2)

$$\begin{tikzcd}
 & \Omega^m(k) \arrow[d] \arrow[dr, "\zeta"] & & \\
k \arrow[r] & M \arrow[r] & k \arrow[r, "\gamma_f"] & \Omega^{-1}(k)
\end{tikzcd}$$

Because the composition $\gamma_f\zeta : \Omega^m(k) \to \Omega^{-1}(k)$ factors through M and the bottom row of (5.3.2) is a distinguished triangle in the stable category $\mathbf{stmod}(G)$, $\gamma_f\zeta$ must be zero.

Since $L_\zeta/\operatorname{Rad}(L_\zeta)$ is semi-simple, we have a splitting

$$\rho : L_\zeta/\operatorname{Rad}(L_\zeta) \to \operatorname{Rad}(\Omega^m(k))/\operatorname{Rad}(L_\zeta)$$

of the map i. For any linear map $\phi : \operatorname{Rad}(\Omega^m(k))/\operatorname{Rad}(L_\zeta) \to k$ we thus have a map $\phi\circ\rho : L_\zeta/\operatorname{Rad}(L_\zeta) \to k$, and therefore an extension $\gamma_{\phi\circ\rho}$ such that $\gamma_{\phi\circ\rho}\zeta = 0$. We define

$$\Psi : (\operatorname{Rad}(\Omega^m(k))/\operatorname{Rad}(L_\zeta))^\# \to \operatorname{Ker}\{\cdot\zeta : \mathrm{H}^1(G,k) \to \mathrm{H}^{m+1}(G,k)\}, \quad \phi \mapsto \gamma_{\phi\circ\rho}.$$

To show that Ψ is injective, let $\phi \in (\operatorname{Rad}(\Omega^m(k))/\operatorname{Rad}(L_\zeta))^\#$, and set $f = \phi \circ \rho$. Observe that the extension γ_f (the bottom row of (5.3.1)) is not split if and only if the map $f' : \Omega^m(k)/\operatorname{Rad}(L_\zeta) \to M$ does not factor through $\Omega^m(k)/\operatorname{Rad}(\Omega^m(k))$ which happens if and only if $f \circ i \neq 0$. Since $(\phi \circ \rho) \circ i = \phi$, we conclude that Ψ is injective.

To verify that Ψ is surjective, consider some $\eta \in \mathrm{H}^1(G,k)$ such that $\eta\zeta = 0$. Then $\zeta : \Omega^m(k) \to \Omega^m(k)/\operatorname{Rad}(\Omega^m(k)) \to k$ must factor through the extension $k \to M \to k$ corresponding to η. Let $f' : \Omega^m(k) \to M$ be the factorization map, and denote by $f : L_\zeta/\operatorname{Rad}(L_\zeta) \to k$ the restriction to $L_\zeta/\operatorname{Rad}(L_\zeta)$. Then by construction $\eta = \gamma_f$.

Finally, if $\dim \operatorname{Ker}\{\cdot\zeta : \mathrm{H}^1(G,k) \to \mathrm{H}^{m+1}(G,k)\} = 0$, we conclude that $\operatorname{Rad}^j(L_\zeta) = \operatorname{Rad}^j(\Omega^m(k))$ for all j. □

To apply Proposition 5.3, we require the follow facts about restrictions of L_ζ modules.

Lemma 5.4. *Suppose that $U \in \operatorname{Grass}(r, \mathbb{V})$ is an r-plane in $\mathbb{V}$. Let $\alpha : C = k[t_1, \dots, t_r]/(t_1^p, \dots, t_r^p) \to kE$ be a flat map such that $\alpha(t_1), \dots, \alpha(t_r)$ is a basis for U. Suppose that $\zeta \in \mathrm{H}^m(E,k)$ is a non-zero homogeneous cohomology element of degree $m > 0$. There exists a number γ_m independent of α such that*

$$\begin{array}{llll} \alpha^*(L_\zeta) \simeq C^{\oplus \gamma_m - 1} \oplus \Omega(k_C) \oplus \Omega^m(k_C) & \text{if } \alpha^*(\zeta) = 0 \\ \alpha^*(L_\zeta) \simeq C^{\oplus \gamma_m} \oplus L_{\alpha^*(\zeta)} & \text{if } \alpha^*(\zeta) \neq 0. \end{array}$$

Consequently,

$$\begin{array}{ll} \dim \operatorname{Rad}(\alpha^*(L_\zeta)) = \gamma_m(p^r - 1) - r + \dim \operatorname{Rad}(\Omega^m(k_C)) & \text{if } \alpha^*(\zeta) = 0 \\ \dim \operatorname{Rad}(\alpha^*(L_\zeta)) = \gamma_m(p^r - 1) + \dim \operatorname{Rad}(L_{\alpha^*(\zeta)}) & \text{if } \alpha^*(\zeta) \neq 0. \end{array}$$

Proof. We have an exact sequence

$$0 \longrightarrow L_\zeta \longrightarrow \Omega^m(k) \xrightarrow{\zeta} k \longrightarrow 0$$

defining L_ζ. Restricting along α, we get

$$\alpha^*(\Omega^m(k)) \simeq C^{\oplus \gamma_m} \oplus \Omega^m(k_C),$$

where the rank γ_m of the free summand is determined entirely by the dimensions of the other two modules. Explicitly, $\gamma_m = (\dim \Omega^m(k_E) - \dim \Omega^m(k_C))/p^r$, which depends only on m and r. The case that $\alpha^*(\zeta) \neq 0$ is now clear from the restriction of the sequence. In the case that $\alpha^*(\zeta) = 0$, we have that the map ζ in the sequence vanishes on the C-summand $\Omega^m(k_C)$. It is an easy exercise to show that the restriction of the kernel of ζ in the sequence is as indicated (see also [Ben91, II, §5.9]).

For the computations of the dimensions of $\operatorname{Rad}(\alpha^*(L_\zeta))$, we recall that $\Omega(k_C) \simeq \operatorname{Rad}(C)$ and, hence, $\dim \operatorname{Rad}(\Omega(k_C)) = p^r - 1 - r$. □

The relevance of Proposition 5.3 to radical types of L_ζ modules is made explicit in the following theorem.

Theorem 5.5. *Suppose that $\zeta \in \mathrm{H}^m(E,k)$ is a non-nilpotent cohomology class satisfying the condition that the hypersurface*

$$Z(\zeta) \subset \operatorname{Proj} \mathrm{H}^\bullet(E,k)$$

does not contain a linear hyperplane of dimension $r - 1$. Then L_ζ has constant r-radical type.

Proof. For any $U \in \mathrm{Grass}(r, \mathbb{V})$, let $\alpha : C = k[t_1, \dots, t_r]/(t_1^p, \dots, t_r^p) \to kE$ be a homomorphism with $\{\alpha(t_1), \dots, \alpha(t_r)\}$ a basis for U. By Corollary 5.2, we may identify $\alpha_* : \mathrm{Proj}\,\mathrm{H}^\bullet(C, k) \to \mathrm{Proj}\,\mathrm{H}^\bullet(E, k)$ with the linear embedding of projective spaces associated to the embedding $U^{(1)} \subset \mathbb{V}^{(1)}$. Hence, the image of α_* is a linear subspace of dimension $r - 1$. Our hypothesis implies that the image of α_* can not be in the zero set of ζ and, therefore, the restriction $\alpha^*(\zeta) \in \mathrm{H}^*(C, k)$ is not nilpotent.

Since $\mathrm{H}^*(C, k)$ is a product of a symmetric algebra and an exterior algebra, this implies that $\alpha^*(\zeta)$ is not a zero divisor. Hence, $\mathrm{Ker}\{\cdot\alpha^*(\zeta) : \mathrm{H}^1(C, k) \to \mathrm{H}^{m+1}(C, k)\} = 0$. By Proposition 5.3, we get that $\mathrm{Rad}^i(L_{\alpha^*(\zeta)}) = \mathrm{Rad}^i(\Omega^m k_C)$ for $i \geq 1$. Lemma 5.4 now implies that $\alpha^*(L_\zeta)$ has constant r-radical type. $\square$

We now see how L_ζ-modules give us examples of modules of constant r-radical type but not constant s-radical type for any s with $1 \leq s < r$.

Proposition 5.6. *Suppose that $\zeta \in k[\zeta_1, \dots, \zeta_n]$ is a homogeneous polynomial of degree m such that the zero locus of ζ inside $\mathrm{Proj}\, k[\zeta_1, \dots, \zeta_n] \simeq \mathrm{Proj}(\mathrm{H}^\bullet(E, k)_{red})$ contains a linear subspace of dimension $r - 2$ but not of dimension $r - 1$. Then the kE-module L_ζ has constant r-radical type but not constant s-radical type for any $s, 1 \leq s < r$.*

Proof. We view ζ as a homogeneous polynomial function on $\mathbb{V}^{(1)}$ of degree m. Theorem 5.5 implies that L_ζ has constant r-radical type.

For $s < r$, we proceed to find s-planes $U, V \in \mathrm{Grass}(s, \mathbb{V})$ such that $\dim \mathrm{Rad}_U(L_\zeta) \neq \dim \mathrm{Rad}_V(L_\zeta)$. By assumption, we can find a linear s-subspace $\widetilde{U} \subset \mathbb{V} \simeq \mathbb{V}^{(1)}$ such that ζ vanishes on $\widetilde{U}$. Let $F_\mathbb{V} : \mathbb{V} \to \mathbb{V}^{(1)}$ be the Frobenius map on $\mathbb{V}$, and let $U = F^{-1}(\widetilde{U})$. Note that U is again a linear subspace of $\mathbb{V}$, and by construction we have

$$U^{(1)} = F_\mathbb{V}(U) = \widetilde{U}$$

Choose an ordered basis $\underline{u} = [u_1, \dots, u_s]$ of U, and define

$$\alpha : C = k[t_1, \dots, t_s]/(t_1^p, \dots, t_s^p) \to kE$$

to be the flat k-algebra homomorphism defined by $\alpha(t_i) = u_i$. Corollary 5.2 enables us to identify

$$\alpha_* : \mathrm{Spec}(\mathrm{H}^\bullet(C, k)_{red}) \to \mathrm{Spec}(\mathrm{H}^\bullet(kE, k)_{red})$$

with the inclusion $U^{(1)} \subset \mathbb{V}^{(1)} \simeq \mathbb{V}$ obtained by applying the Frobenius twist to $U \subset \mathbb{V}$. Since $U^{(1)} = \widetilde{U}$, we conclude that $\alpha^*(\zeta) = 0$. Applying Lemma 5.4, we get

$$\dim \mathrm{Rad}_U(L_\zeta) = \dim \mathrm{Rad}(\alpha^*(L_\zeta)) = \gamma_m(p^r - 1) - r + \dim \mathrm{Rad}(\Omega^m(k_C)).$$

Now let $\widetilde{W}$ be a linear s-subspace in $\mathbb{V}$ such that ζ does not vanish on $\widetilde{W}$, and let $W = F_\mathbb{V}^{-1}(\widetilde{W})$, so that $\widetilde{W} = W^{(1)}$. Let $\underline{w} = [w_1, \dots, w_s]$ be a basis of W, and let

$$\beta : C = k[t_1, \dots, t_s]/(t_1^p, \dots, t_s^p) \to kE$$

be the flat k-algebra homomorphism defined by $\beta(t_i) = w_i$. Then $\beta^*(\zeta)$ is not nilpotent, and, in particular,

$$\dim \mathrm{Rad}_W(L_\zeta) = \dim \mathrm{Rad}(\beta^*(L_\zeta)) = \gamma_m(p^r - 1) + \dim \mathrm{Rad}(L_{\beta^*(\zeta)})$$

by Lemma 5.4. Since $\beta^*(\zeta)$ is not nilpotent, we conclude that $\mathrm{Ker}\{\beta^*(\zeta) : \mathrm{H}^1(C,k) \to \mathrm{H}^{m+1}(C,k)\} = 0$. Hence, by Prop. 5.3,

$$\dim \mathrm{Rad}_W(L_{\beta^*(\zeta)}) = \dim \mathrm{Rad}(\Omega^m(k_C)).$$

Therefore,

$$\dim \mathrm{Rad}_W(L_\zeta) = \dim \mathrm{Rad}_U(L_\zeta) + r$$

which implies the desired inequality. □

The following proposition provides examples of homogeneous polynomials which satisfy the condition of Proposition 5.6. We are grateful to Sándor Kovács for suggesting the argument in the proof that follows.

Proposition 5.7. *Let $n > r$ be positive integers. There exists a homogeneous polynomial $f \in k[X_1, \dots, X_n]$ such that the zero locus of f, $Z(f) \subset \mathbb{P}^{n-1}$, contains a linear subspace of dimension $r-1$ ($\mathbb{P}^{r-1}$) but not of dimension r.*

Proof. Fix $L = \mathbb{P}^{r-1}$ to be the projective subspace which is the zero set of the ideal

$$\mathcal{I}_L = (X_{r+1}, \dots, X_n).$$

Fix a positive degree d. Then the set of polynomials f of degree d such that $L \subset Z(f)$ is the set of global sections of $\mathcal{I}_L(d)$ on $\mathbb{P}^{n-1}$, that is, $\mathrm{H}^0(\mathbb{P}^n, \mathcal{I}_L(d))$. The exact sequence

$$0 \longrightarrow \mathrm{H}^0(\mathbb{P}^{n-1}, \mathcal{I}_L(d)) \longrightarrow \mathrm{H}^0(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(d)) \longrightarrow \mathrm{H}^0(\mathbb{P}^{n-1}, \mathcal{O}_L(d)) \longrightarrow 0$$

implies the equality of dimensions:

$$\dim \mathrm{H}^0(\mathbb{P}^{n-1}, \mathcal{I}_L(d)) = \dim \mathrm{H}^0(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(d)) - \dim \mathrm{H}^0(\mathbb{P}^{n-1}, \mathcal{O}_L(d)) =$$
$$\dim k[X_1, \dots, X_n]_{(d)} - \dim k[X_1, \dots, X_{r-1}]_{(d)} = \binom{n+d-1}{d} - \binom{r+d-1}{d}.$$

Now, let $L' = \mathbb{P}^r$ be any linear subspace of dimension r. Such subspaces are parametrized by $\mathrm{Grass}(r+1, n)$. For each one, the corresponding space of homogeneous functions of degree d that vanish on L' has dimension $\dim \mathrm{H}^0(\mathbb{P}^{n-1}, \mathcal{I}_{L'}(d)) = \binom{n+d-1}{d} - \binom{r+d}{d}$. Let $T \subset \mathrm{Grass}(r+1, n) \times \mathrm{H}^0(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(d))$ be a subspace defined as follows:

$$T = \{(L', f), \mathbb{P}^r = L' \subset \mathbb{P}^{n-1}, f \in \mathrm{H}^0(\mathbb{P}^{n-1}, \mathcal{I}_{L'}(d))\}.$$

This is a vector bundle with the fiber of dimension $\dim \mathrm{H}^0(\mathbb{P}^{n-1}, \mathcal{I}_{L'}(d))$ and the base $\mathrm{Grass}(r+1, n)$, and it is precisely the space of functions we need to avoid. Hence, altogether we need to avoid a total space of dimension

$$\dim T = \binom{n+d-1}{d} - \binom{r+d}{d} + (r+1)(n-r-1).$$

Therefore, to prove the claim, we need to establish that for a large enough d, we have an inequality

$$\binom{n+d-1}{d} - \binom{r+d-1}{d} > \binom{n+d-1}{d} - \binom{r+d}{d} + (r+1)(n-r-1).$$

Equivalently,

$$\binom{r+d-1}{d-1} > (r+1)(n-r-1).$$

Since r and n are fixed but d can be chosen arbitrarily large, this is now evident. □

The following corollary is immediate from Prop. 5.6 and 5.7.

Corollary 5.8. *Let E be an elementary abelian p-group of rank n. For any integer r, $1 < r < n$, there exists a module of constant r-radical type but not of constant s-radical type for $s < r$.*

We next construct examples of a kE-modules which have constant r-radical type for small r, but not for large r.

Proposition 5.9. *Assume that $p > 2$. As before we write $\mathrm{H}^*(E, k) = k[\zeta_1, \dots, \zeta_n] \otimes \Lambda^*(\eta_1, \dots, \eta_n)$. Let $\zeta = \eta_1 \dots \eta_s$ for some s with $1 < s < n$. Then L_ζ satisfies the following properties:*

(1) *L_ζ has constant r-radical type for any r, $r < s$.*
(2) *L_ζ has constant s-*Rad*-rank, but not constant s-radical type.*
(3) *L_ζ does not have constant r-*Rad*-rank for any r such that $s < r < n$.*

Proof. Let U be an r–plane in $\mathbb{V}$, and let $\alpha : C = k[t_1, \dots, t_r]/(t_1^p, \dots, t_r^p) \to kE$ be a map such that $\alpha(t_1), \dots, \alpha(t_r)$ is a basis for U.

For $r < s$, the product of any s elements of degree one is necessarily zero in $\mathrm{H}^s(C, k)$. Hence, $\alpha^*(\zeta) = 0$. By Lemma 5.4, $\alpha^*(L_\zeta) \simeq C^{\oplus \gamma_s - 1} \oplus \Omega(k_C) \oplus \Omega^s(k_C)$ for some γ_s which does not depend on the choice of U. Consequently, L_ζ has constant r-radical type.

Assume that $s \le r \le n$. If U is the subspace such as the one spanned by $x_1, \dots, x_r$, then $\alpha^*(\zeta) \neq 0$. Since $\alpha^*(\zeta)$ is a product of s degree 1 classes, it annihilates a subspace of dimension s of $\mathrm{H}^1(C, k)$. Hence, Lemma 5.4 and Proposition 5.3 imply that

$$
\begin{aligned}
\dim \mathrm{Rad}_U(L_\zeta) &= \gamma_s(p^r - 1) + \dim \mathrm{Rad}(L_{\alpha^*(\zeta)}) \\
&= \gamma_s(p^r - 1) + \dim \mathrm{Rad}(\Omega^r(k_C)) - s
\end{aligned}
\tag{5.9.1}
$$

If U is the subspace spanned by $x_2, \dots, x_{r+1}$, then $\alpha^*(\zeta) = 0$ and

$$
\dim \mathrm{Rad}_U(L_\zeta) = \gamma_s(p^r - 1) - r + \dim \mathrm{Rad}(\Omega^r(k_C)). \tag{5.9.2}
$$

It follows that L_ζ has constant r-Rad–rank if and only if $r = s$. This proves (3) and the first part of (2).

For the remainder of part (2), notice that the dimension of $\mathrm{Rad}^{r(p-1)}(M)$ of a C-module M counts the number of direct summands of C in a decomposition of the module into indecomposable submodules. In the case $r \ge s$, we can get two different values for $\dim \mathrm{Rad}^{r(p-1)}(\alpha^*(L_\zeta))$ depending on whether $\alpha^*(\zeta)$ is zero or not, by Lemma 5.4. Therefore L_ζ does not have constant r-$\mathrm{Rad}^{r(p-1)}$-rank for any $r \ge s$. In particular, it does not have constant s-radical type. □

Corollary 5.10. *Let $p > 2$, and let $\zeta \in \mathrm{H}^s(E, k)$ be a product of s degree one cohomology classes. For any $r > s$ the nonmaximal radical support variety $\mathrm{Rad}(r, \mathbb{V})_{L_\zeta}$ consists of exactly those r-planes U for which $\alpha^*(\zeta) = 0$, where $\alpha : k[t_1, \dots, t_r]/(t_1^p, \dots, t_r^p) \to kE$ is a map such that $\alpha(t_1), \dots, \alpha(t_r)$ form a basis for U.*

Proof. This follows by comparing equalities (5.9.1) and (5.9.2) of the proof of Prop. 5.9. □

In a similar way, we get the following statement about nonmaximal radical support varieties.

Corollary 5.11. *Let $\zeta \in \mathrm{H}^{2m}(E,k)$. If $r = 1,2,3$ or if ζ is a product of one-dimensional cohomology classes, then*

$$\mathrm{Rad}(r,\mathbb{V})_{L_\zeta} = \{U \in \mathrm{Grass}(r,\mathbb{V}) \,|\, \alpha^*(\zeta) = 0 \;\; in \;\; \mathrm{H}^*(C,k)_{\mathrm{red}}\}$$

where $\alpha : C = k[t_1,\dots,t_r]/(t_1^p,\dots,t_r^p) \to KE$ is a map such that $\alpha(t_1),\dots,\alpha(t_r)$ form a basis for U.

On the other hand, for $r > 3$, there exists a homogeneous cohomology class ζ for which this equality is not valid.

Proof. If $\zeta \in \mathrm{H}^{2m}(E,k)$ satisfies the hypothesis of the corollary, then the condition that $\mathrm{Ker}\{\cdot\alpha_{\underline{u}}^*(\zeta) : \mathrm{H}^1(C,k) \to \mathrm{H}^{2m+1}(C,k)\}$ be zero is equivalent to a simpler condition that $\alpha_{\underline{u}}^*(\zeta)$ is not nilpotent. Hence, Prop. 5.3 implies the desired equality.

On the other hand, suppose that $r > 3$ and let $\eta_1,\dots,\eta_r$ span $\mathrm{H}^1(E,k)$. Then $\eta = \eta_1\eta_2 + \eta_3\eta_4$ is a nilpotent element in $\mathrm{H}^*(C,k)$ which does not annihilate any non-zero class of degree 1. $\square$

We finish this section with a simple observation about the socle series of $\alpha^*(L_\zeta)$.

Proposition 5.12. *Suppose that $\zeta \in \mathrm{H}^m(E,k)$ is a non-zero cohomology class. If $r > 1$, then for any U in $\mathrm{Grass}(r,\mathbb{V})$ we have that $\mathrm{Soc}_U(L_\zeta) = \mathrm{Soc}_U(\Omega^m(k))$.*

Consequently, L_ζ has constant r-Soc-rank for any $r > 1$.

Proof. Choose U in $\mathrm{Grass}(r,\mathbb{V})$. Let $\alpha : C = k[t_1,\dots,t_r]/(t_1^p,\dots,t_r^p) \to kE$ be a k-algebra homomorphism such that $\alpha(t_1),\dots,\alpha(t_r)$ is a basis for U. Suppose there is a simple submodule in $\alpha^*(\Omega^m k)$ which does not map to 0 under $\alpha^*(\zeta)$ and, hence, is not a submodule in $\mathrm{Soc}(\alpha^*(L_\zeta))$. Then it maps isomorphically onto k. This implies that the sequence $0 \to \alpha^*(L_\zeta) \to \alpha^*(\Omega^m(k)) \to k \to 0$ splits. But if $r > 1$, then this is not possible because $\alpha^*(\Omega^m(k))$ has no summand that is isomorphic to k. $\square$

6. Construction of Bundles on $\mathrm{Grass}(r,\mathbb{V})$

This section opens the second part of the paper in which we discuss algebraic vector bundles on Grassmannians arising from finite dimensional kE-modules having either constant r-Rad^j-rank or constant r-Soc^j-rank for some j. We begin by developing two approaches of constructing vector bundles on $\mathrm{Grass}(r,\mathbb{V})$ which we then show determine isomorphic algebraic vector bundles. The first approach uses a local analysis on standard affine open subsets of the Grassmannian, while the second is a global process defining the bundles by equivariant descent. In the next section we show that for the class of GL_n-equivariant kE-modules discussed in Section 3, our construction can be recognized as a familiar functor widely used for algebraic groups and homogeneous spaces. Our first series of examples appears in the same section. Finally, in Section 8 we introduce a formula that constructs a graded module over the homogeneous coordinate ring of the Grassmannian whose associated coherent sheaf is the kernel bundle associated to a module of constant r-socle rank.

We use notations and conventions for the Grassmannian discussed in detail in Section 1.

6.1. **A local construction of bundles.** Let $x_1, \ldots, x_n$ be a basis for the space $\mathbb{V} \subset \operatorname{Rad}(kE)$ splitting the projection $\operatorname{Rad}(kE) \to \operatorname{Rad}(kE)/\operatorname{Rad}^2(kE)$. Let $C = k[t_1, \ldots, t_r]/(t_1^p, \ldots, t_r^p)$. For

$$\alpha_\Sigma \,:\, C \otimes k[\mathcal{U}_\Sigma] \longrightarrow kE \otimes k[\mathcal{U}_\Sigma]$$

as in Definition 1.5, we denote by θ_j^Σ, $1 \le j \le r$, the $k[\mathcal{U}_\Sigma]$–linear p-nilpotent operator on $M \otimes k[\mathcal{U}_\Sigma]$ given by multiplication by $\alpha_\Sigma(t_j)$:

$$M \otimes k[\mathcal{U}_\Sigma] \xrightarrow{\theta_j^\Sigma} M \otimes k[\mathcal{U}_\Sigma] \tag{6.0.1}$$

$$m \otimes f \longmapsto \sum_{i=1}^{n} x_i m \otimes Y_{i,j}^\Sigma f.$$

For any r-subset $\Sigma \subset \{1, \ldots, n\}$, and any ℓ, $1 \le \ell \le r(p-1)$, we define $k[\mathcal{U}_\Sigma]$–modules

$$\operatorname{Ker}^\ell(M)_{U_\Sigma} = \bigcap_{1 \le j_1, \ldots, j_\ell \le r} \operatorname{Ker}\{\theta_{j_1}^\Sigma \cdots \theta_{j_\ell}^\Sigma : M \otimes k[\mathcal{U}_\Sigma] \to M \otimes k[\mathcal{U}_\Sigma]\} \tag{6.0.2}$$

$$\operatorname{Im}^\ell(M)_{U_\Sigma} = \sum_{1 \le j_1, \ldots, j_\ell \le r} \operatorname{Im}\{\theta_{j_1}^\Sigma \cdots \theta_{j_\ell}^\Sigma : M \otimes k[\mathcal{U}_\Sigma] \to M \otimes k[\mathcal{U}_\Sigma]\} \tag{6.0.3}$$

We denote by $\mathcal{O}_{Gr}$ the structure sheaf of $\operatorname{Grass}(r, \mathbb{V})$. For any finite dimensional kE-module M, the coherent sheaf $M \otimes \mathcal{O}_{Gr}$ is a free $\mathcal{O}_{Gr}$-module of rank equal to the dimension of M. In the next proposition, we define the ℓ^{th} kernel and image sheaves,

$$\mathcal{K}er^\ell(M) \quad \text{and} \quad \mathcal{I}m^\ell(M), \tag{6.0.4}$$

associated to a kE–module M.

Proposition 6.1. *Let M be a finite-dimensional kE–module. There is a unique subsheaf $\mathcal{K}er^\ell(M) \subset M \otimes \mathcal{O}_{Gr}$ whose restriction to U_Σ equals $\operatorname{Ker}^\ell(M)_{U_\Sigma}$ for each subset $\Sigma \subset \{1, \ldots, n\}$ of cardinality r. We refer to $\mathcal{K}er^\ell(M)$ as the ℓ^{th} kernel sheaf.*

Similarly, there is a unique subsheaf $\mathcal{I}m^\ell(M) \subset M \otimes \mathcal{O}_{Gr}$ whose restriction to U_Σ equals $\operatorname{Im}^\ell(M)_{U_\Sigma}$ for each subset $\Sigma \subset \{1, \ldots, n\}$ of cardinality r. We refer to $\mathcal{I}m^\ell(M)$ as the ℓ^{th} image sheaf.

Proof. Let $\Sigma, \Sigma' \subset \{1, \ldots, n\}$ be two r-subsets and let

$$\tau_{\Sigma,\Sigma'} : k[Y_{i,j}^\Sigma, \mathfrak{p}_{\Sigma'}^{-1}] \;\simeq k[\mathcal{U}_\Sigma \cap \mathcal{U}_{\Sigma'}] \;\simeq k[Y_{i,j}^{\Sigma'}, \mathfrak{p}_\Sigma^{-1}]$$

denote the evident transition function. Observe that on $\mathcal{U}_\Sigma \cap \mathcal{U}_{\Sigma'}$, each θ_j^Σ can be written using the transition functions $\tau_{\Sigma,\Sigma'}$ as a $k[Y_{i,j}^{\Sigma'}, \mathfrak{p}_\Sigma^{-1}]$-linear combination of the $\theta_j'^{\Sigma}$'s:

$$\theta_j^{\Sigma'} = \tau_{\Sigma,\Sigma'}(\theta_j^\Sigma) : M \otimes k[Y_{a,b}^{\Sigma'}, \mathfrak{p}_\Sigma^{-1}] \to M \otimes k[Y_{a,b}^{\Sigma'}, \mathfrak{p}_\Sigma^{-1}]. \tag{6.1.1}$$

This enables us to identify $\operatorname{Ker}^\ell(M)_{U_\Sigma}$ and $\operatorname{Im}^\ell(M)_{U_\Sigma}$ when restricted to $\mathcal{U}_\Sigma \cap \mathcal{U}_{\Sigma'}$ as submodules of $M \otimes k[Y_{a,b}^{\Sigma'}, \mathfrak{p}_\Sigma^{-1}]$. It can be verified that the kernels and images of the products of θ_j^Σ, $\theta_j^{\Sigma'}$ acting on $M \otimes k[Y_{a,b}^{\Sigma'}, \mathfrak{p}_\Sigma^{-1}]$ are equal by specializing to each point $x \in \mathcal{U}_\Sigma \cap \mathcal{U}_{\Sigma'}$ and using the relationship (6.1.1). □

For $\ell = 1$, we write $\mathcal{K}er(M)$ for $\mathcal{K}er^1(M)$, and we write $\mathcal{I}m(M)$ for $\mathcal{I}m^1(M)$.

Theorem 6.2. *Let M be a finite dimensional kE–module, and $U \subset \mathbb{V}$ an r-plane. Let ℓ be an integer, $1 \leq \ell \leq (p-1)r$.*

(1) *If M has constant r–Soc^ℓ–rank, then*
  - $\mathcal{K}er^\ell(M)$ *is an algebraic vector bundle on* $\mathrm{Grass}(r, \mathbb{V})$,
  - $\mathrm{rk}\,\mathcal{K}er^\ell(M) = \dim \mathrm{Soc}^\ell_U(M)$.

(2) *If M has constant r-Rad^ℓ-rank, then*
  - $\mathcal{I}m^\ell(M)$ *is an algebraic vector bundle on* $\mathrm{Grass}(r, \mathbb{V})$,
  - $\mathrm{rk}\,\mathcal{I}m^\ell(M) = \dim \mathrm{Rad}^\ell_U(M)$.

Proof. First assume that $\ell = 1$.
(1). Let Σ be an r-subset of $\{1, \ldots, n\}$. We proceed to define a map

$$\Theta^\Sigma : M \otimes k[\mathcal{U}_\Sigma] \xrightarrow{[\theta^\Sigma_1, \ldots, \theta^\Sigma_r]} (M \otimes k[\mathcal{U}_\Sigma])^{\oplus r} \tag{6.2.1}$$

such that $\mathrm{Ker}(M)_{\mathcal{U}_\Sigma} = \mathrm{Ker}\,\Theta^\Sigma$. Let $U \in \mathcal{U}_\Sigma \subset \mathrm{Grass}(r, \mathbb{V})$ and let $\{u_1, \ldots, u_r\}$ be the unique choice of ordered basis for U such that the Σ–submatrix of $A_U = (a_{i,j})$ equals $[u_1, \ldots, u_r]$ (expressed with respect to the fixed basis $\{x_1, \ldots, x_n\}$ of $\mathbb{V}$) is the identity matrix. Then $\alpha_U : C \to kE$, defined by $\alpha_U(t_i) = u_i$, equals the result of specializing $\alpha_\Sigma : C \otimes k[\mathcal{U}_\Sigma] \to kE \otimes k[\mathcal{U}_\Sigma]$ by setting the variables $Y^\Sigma_{i,j}$ to values $a_{i,j} \in k$. Hence, the specialization of the map Θ^Σ at the point $U \in \mathcal{U}_\Sigma$ gives the k-linear map $[\alpha_U(t_1), \ldots, \alpha_U(t_r)] : M \to M^{\oplus r}$. In other words,

$$\Theta^\Sigma \otimes_{k[\mathcal{U}_\Sigma]} k = [\alpha_U(t_1), \ldots, \alpha_U(t_r)]$$

where the tensor is taken over the map $k[\mathcal{U}_\Sigma] \to k$ corresponding to the point $U \in \mathcal{U}_\Sigma$. Since specialization is right exact, we have an equality

$$\mathrm{Coker}\{\Theta^\Sigma\} \otimes_{k[\mathcal{U}_\Sigma]} k = \mathrm{Coker}\{\Theta^\Sigma \otimes_{k[\mathcal{U}_\Sigma]} k\} = \mathrm{Coker}\{[\alpha_U(t_1), \ldots, \alpha_U(t_r)] : M \to M^{\oplus r}\}.$$

Let $f : W \to W'$ be a linear map of k-vector spaces. Then $\dim \mathrm{Coker}\, f = \dim \mathrm{Ker}\, f - \dim W + \dim W'$. Using this observation, we further conclude that

$$\dim \mathrm{Coker}\{[\alpha_U(t_1), \ldots, \alpha_U(t_r)] : M \to M^{\oplus r}\} =$$
$$\dim \mathrm{Ker}\{[\alpha_U(t_1), \ldots, \alpha_U(t_r)] : M \to M^{\oplus r}\} + (r-1)\dim M =$$
$$\dim \mathrm{Soc}_U(M) + (r-1)\dim M.$$

Therefore, all specializations of the $k[\mathcal{U}_\Sigma]$–module $\mathrm{Coker}\,\Theta^\Sigma$ have the same dimension. By [FP11, 4.11] (see also [Har77, 5 ex.5.8]), $\mathrm{Coker}\,\Theta^\Sigma$ is a projective module over $k[\mathcal{U}_\Sigma]$. Now the exact sequence

$$0 \longrightarrow \mathrm{Ker}\,\Theta^\Sigma \longrightarrow M \otimes k[\mathcal{U}_\Sigma] \xrightarrow{\Theta^\Sigma} (M \otimes k[\mathcal{U}_\Sigma])^{\oplus r} \longrightarrow \mathrm{Coker}\,\Theta^\Sigma \longrightarrow 0$$

implies that $\mathrm{Ker}(M)_{\mathcal{U}_\Sigma} = \mathrm{Ker}\,\Theta^\Sigma$ is also projective. Since this holds for any r-subset $\Sigma \subset \{1, \ldots, n\}$, we conclude that $\mathcal{K}er(M)$ is locally free.

(2). For an r-subset $\Sigma \subset \{1, \ldots, n\}$, define a map $\Theta^\Sigma : (M \otimes k[\mathcal{U}_\Sigma])^{\oplus r} \to M \otimes k[\mathcal{U}_\Sigma]$ as the composition

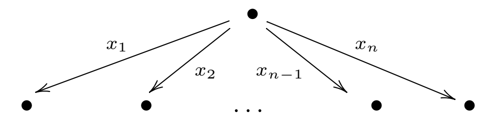
$$\Theta^\Sigma : (M \otimes k[\mathcal{U}_\Sigma])^{\oplus r} \xrightarrow{\mathrm{diag}[\theta^\Sigma_1, \ldots, \theta^\Sigma_r]} (M \otimes k[\mathcal{U}_\Sigma])^{\oplus r} \xrightarrow{\Sigma} M \otimes k[\mathcal{U}_\Sigma]$$

where the second map is the sum over all r coordinates. Arguing as in (1) and using that $\dim \mathrm{Coker}\, f = \dim W' - \dim \mathrm{Im}\, f$ for a map of k-vector spaces $f : W \to W'$, we conclude (2) for $\ell = 1$.

Finally, the proof for $\ell > 1$ is very similar with the map Θ^{Σ} replaced by its ℓ-th iterate. $\square$

The two basic examples we give below can be justified directly from the local construction just described; indeed, both are defined in terms of moving frames inside trivial bundles of appropriate ranks on the Grassmannian. Formal verifications are given in Examples 7.4 and 7.8.

Example 6.3. (1) [Tautological/universal subbundle γ_r]. Let $kE = k[x_1, \dots, x_r]/(x_1^p, \dots, x_n^p)$, and let $M = kE/\operatorname{Rad}^2(kE)$. We can represent M pictorially as follows:



Then $\operatorname{Rad}_U(M) \subset \operatorname{Rad}(M)$ can be naturally identified with the plane $U \subset \mathbb{V}$ under our fixed isomorphism $\operatorname{Rad}(M) = \operatorname{Rad}(kE)/\operatorname{Rad}^2(kE) \simeq \mathbb{V}$. Thus,

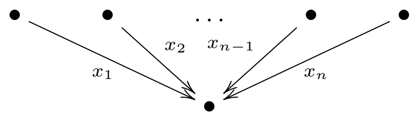
$$\mathcal{I}m(M) \quad = \quad \gamma_r,$$

where $\gamma_r \subset \ \mathcal{O}_{Gr}^{\oplus n} \simeq \operatorname{Rad}(M) \otimes \mathcal{O}_{Gr}$ is the *tautological* (or universal) rank r subbundle of the rank n trivial bundle on $\operatorname{Grass}_{n,r}$.

(2) [Universal subbundle δ_{n-r}]. Let δ_{n-r} be the universal rank $n-r$ subbundle of the trivial bundle of rank n on $\operatorname{Grass}_{n,r}$, that is, the subbundle whose dual, $\delta_{n-r}^{\vee}$, fits into a short exact sequence

$$0 \longrightarrow \gamma_r \longrightarrow \mathcal{O}_{Gr}^{\oplus n} \longrightarrow \delta_{n-r}^{\vee} \longrightarrow 0 \ .$$

Let $M = kE/\operatorname{Rad}^2(kE)$. Note that $M^{\#}$ can be represented pictorially as follows:



We have

$$\{\mathcal{K}er(M^{\#}) \ \subset \ M^{\#} \otimes \mathcal{O}_{Gr}\} \quad = \quad \{\delta_{n-r} \oplus \mathcal{O}_{Gr} \ \subset \ \mathcal{O}_{Gr}^{\oplus n+1}\}.$$

6.2. **A construction by equivariant descent.** Our second construction has the advantage of producing bundles on $\operatorname{Grass}(r, \mathbb{V})$ by a "global" process rather than as a patching of locally defined kernels or images. In this sense, it resembles the global operator Θ in the case $r = 1$ employed in [FP11] to construct bundles on cohomological support varieties of infinitesimal group schemes. However, the reader should be alert to the fact that the kernels (or images) are not produced as kernels (or images) of a map of bundles on $\operatorname{Grass}(r, \mathbb{V})$ but rather by a descent process.

We begin by recalling the definition of a G-equivariant sheaf followed by a general lemma. We refer the reader to [CG97, 5] or [BL94, I.0] for a detailed discussion of equivariant sheaves.

Definition 6.4. Let G be a linear algebraic group and let Y be a G-variety; in other words, Y is a variety equipped with an algebraic G–action $\mu : G \times Y \to Y$. Denote by $p : G \times Y \to Y$ the projection map, and by $m : G \times G \to G$ multiplication in G. A sheaf $\mathcal{F}$ of $\mathcal{O}_Y$–modules is G-equivariant if there is an isomorphism $f : \mu^*\mathcal{F} \simeq p^*\mathcal{F}$ satisfying the natural cocycle condition. Explicitly, for

$$\begin{aligned} p_1 &= \mathrm{id}_G \times \mu : G \times G \times Y \to G \times Y \\ p_2 &= m \times \mathrm{id}_Y : G \times G \times Y \to G \times Y \\ p_3 &= \mathrm{proj}_{G\times Y} : G \times G \times Y \to G \times Y \end{aligned}$$

(where p_3 is the projection along the first factor), $\mathcal{F}$ satisfies the condition

$$p_1^*(f) \circ p_3^*(f) \ = \ p_2^*(f). \tag{6.4.1}$$

The following fact is well known although usually mentioned without proof (e.g., [BL94, 0.3] or [CG97, 5.2.15]). We provide a straightforward proof for completeness.

Lemma 6.5. *Let G be a linear algebraic group and let $p : Y \to X$ be a principal homogeneous space for G locally trivial in the étale topology. There is an equivalence of categories given by the pull-back functor*

$$p^* : \mathrm{Coh}(X) \xrightarrow{\sim} \mathrm{Coh}^G(Y)$$

between coherent sheaves of $\mathcal{O}_X$–modules and G-equivariant coherent sheaves of $\mathcal{O}_Y$–modules.

Proof. Note that our assumption implies that $Y \to X$ is faithfully flat and quasi-compact. Hence, we can use faithfully flat descent ([SGAI], VIII, §.1). Therefore, we have an equivalence between the category of coherent sheaves of $\mathcal{O}_X$–modules and the category of coherent sheaves of $\mathcal{O}_Y$–modules with descent data. Consider the diagram

$$\begin{tikzcd} Y \times_X Y \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & Y \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

Recall that the descent data for an $\mathcal{O}_Y$–module $\mathcal{F}$ is an isomorphism $\phi : \pi_1^*(\mathcal{F}) \simeq \pi_2^*(\mathcal{F})$ such that

$$\pi_{23}^*(\phi)\pi_{12}^*(\phi) \ = \ \pi_{13}^*(\phi), \tag{6.5.1}$$

where $\pi_{ij} : Y \times_X Y \times_X Y \to Y \times_X Y$ is the projection on the (i, j) component. Since $p : Y \to X$ is a principal homogeneous space for G (i.e., a G-torsor for $G \times X$ over X), $G \times Y \to Y \times_X Y$ defined by sending (g, y) to (gy, y) is an isomorphism. With this isomorphism, the Cartesian square above becomes

$$\begin{tikzcd} G \times Y \arrow[r, "\mu"] \arrow[d, "p"] & Y \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

and the maps $\pi_{i,j} : Y \times_X Y \times_X Y \to Y \times_X Y$ become precisely the maps in Definition 6.4 with $\pi_{i,j}$ going to p_ℓ for $\ell \neq i, j$

$$\begin{aligned} p_1 &: G \times G \times Y \to G \times Y \\ p_2 &: G \times G \times Y \to G \times Y \\ p_3 &: G \times G \times Y \to G \times Y. \end{aligned}$$

Consequently, the descent data (6.5.1) is transformed into the condition (6.4.1) for G-equivariance. ☐

Remark 6.6. Suppose $p : Y \to X$ is a trivial G-fiber bundle, that is, there is a section $s : X \to Y$ such that $Y = s(X) \times G \simeq X \times G$. In this special case, p^* is given simply by tensoring with the structure sheaf of G: for $\mathcal{F} \in \mathrm{Coh}(X)$,

$$p^*(\mathcal{F}) = s_*(\mathcal{F}) \otimes \mathcal{O}_G \simeq \mathcal{F} \otimes \mathcal{O}_G.$$

We fix an ordered basis of $\mathbb{V}$ and an r-plane $U_0 \subset \mathbb{V}$. As in (1.0.3), we identify $\mathbb{M} = M_{n,r} \simeq \mathrm{Hom}_k(U_0, \mathbb{V})$ with the affine variety of $n \times r$ matrices, and we set $\mathbb{M}^o \subset \mathbb{M}$ equal to the open quasi-affine subvariety of matrices of maximal rank. Then $\mathrm{Grass}_{n,r} \simeq \mathbb{M}^o/\,\mathrm{GL}_r$ and, moreover, $\mathbb{M}^o \to \mathrm{Grass}_{n,r}$ is a principal GL_r-equivariant bundle. Hence, we have an equivalence of categories

$$\mathrm{Coh}(\mathrm{Grass}_{n,r}) \simeq \mathrm{Coh}^{\mathrm{GL}_r}(\mathbb{M}^o). \tag{6.6.1}$$

Moreover, using the action of GL_n on $\mathbb{M}$ via multiplication on the left which commutes with the action by GL_r (via multiplication by the inverse on the right), we get an equivalence between $(\mathrm{GL}_n, \mathrm{GL}_r)$-equivariant sheaves on $\mathbb{M}^o$ (with GL_n acting on the left and GL_r on the right) and GL_n-equivariant sheaves on $\mathrm{Grass}_{n,r}$ (with GL_n acting on $\mathrm{Grass}_{n,r} \simeq \mathrm{GL}_n \,/\, \mathrm{GL}_r$ via multiplication on the left).

We denote by

$$\mathcal{R} : \mathrm{Coh}^{\mathrm{GL}_r}(\mathbb{M}) \longrightarrow \mathrm{Coh}(\mathrm{Grass}_{n,r}) \tag{6.6.2}$$

the functor defined as a composition

$$\mathcal{R} : \mathrm{Coh}^{\mathrm{GL}_r}(\mathbb{M}) \xrightarrow{\mathrm{res}} \mathrm{Coh}^{\mathrm{GL}_r}(\mathbb{M}^o) \xrightarrow{\sim} \mathrm{Coh}(\mathrm{Grass}_{n,r})$$

of the restriction functor and the inverse to the pull-back functor which defines the equivalence of categories in Lemma 6.5. Since $\mathbb{M}$ is an affine scheme, the category of GL_r-equivariant coherent $\mathcal{O}_{\mathbb{M}}$–modules is equivalent to the category of GL_r–equivariant $k[\mathbb{M}]$–modules. Using this equivalence, we apply the functor $\mathcal{R}$ to GL_r–equivariant $k[\mathbb{M}]$–modules. Finally, recall that the choice of basis for $\mathbb{V}$ determines the choice of the dual basis of $(\mathbb{V}^{\oplus r})^\#$ which we denoted by $\{Y_{i,j}\}_{\substack{1 \leq i \leq n \\ 1 \leq j \leq r}}$ in Section 1. Since $k[\mathbb{M}] = S^*(M_{n,r}^\#) = S^*((\mathbb{V}^{\oplus r})^\#)$, we get that $\{Y_{i,j}\}$ are algebraic generators of $k[\mathbb{M}]$. We use the identification $k[\mathbb{M}] \simeq k[Y_{i,j}]$.

Let M be a finite dimensional kE-module, and let $\widetilde{M} = M \otimes k[Y_{i,j}]$ be a free module of rank $\dim M$ over $k[Y_{i,j}]$. We define a $k[Y_{i,j}]$-linear map

$$\widetilde{\Theta} = [\widetilde{\theta}_1, \dots, \widetilde{\theta}_r] : \widetilde{M} \to (\widetilde{M})^{\oplus r}$$

by

$$\widetilde{\theta}_j(m \otimes f) = \sum_{i=1}^n x_i m \otimes Y_{i,j} f$$

for all j, $1 \leq j \leq r$. We further define

$$\mathrm{Ker}\{\widetilde{\Theta}, M\} = \mathrm{Ker}\{\widetilde{\Theta} : \widetilde{M} \to (\widetilde{M})^{\oplus r}\} \tag{6.6.3}$$

to be the $k[Y_{i,j}]$–submodule of $\widetilde{M}$ which is the kernel of the map $\widetilde{\Theta}$. Letting

$$\widetilde{\Theta}^\ell = [\widetilde{\theta}_1^\ell,\ \widetilde{\theta}_1^{\ell-1}\widetilde{\theta}_2,\ \dots,\ \widetilde{\theta}_r^\ell],$$

(all monomials of degree ℓ in $\widetilde{\theta}_1, \dots, \widetilde{\theta}_r$) we similarly define

$$\mathrm{Ker}\{\widetilde{\Theta}^\ell, M\} = \mathrm{Ker}\{\widetilde{\Theta}^\ell : \widetilde{M} \to (\widetilde{M})^{\oplus\binom{r+\ell-1}{\ell}}\} \tag{6.6.4}$$

for any ℓ, $1 \le \ell \le (p-1)r$.

An analogous construction is applied to the image. Let
(6.6.5)

$$\operatorname{Im}\{\widetilde{\Theta}, M\} = \operatorname{Im}\{\ (M \otimes k[Y_{i,j}])^r \xrightarrow{\operatorname{diag}[\widetilde{\theta}_1,\ldots,\widetilde{\theta}_r]} (M \otimes k[Y_{i,j}])^r \xrightarrow{\ \Sigma\ } M \otimes k[Y_{i,j}]\ \}$$

Replacing $\widetilde{\Theta}$ with $\widetilde{\Theta}^\ell$, we obtain $k[Y_{i,j}]$–modules $\operatorname{Im}\{\widetilde{\Theta}^\ell, M\}$ for any ℓ, $1 \le \ell \le (p-1)r$.

Lemma 6.7. *Let M be a kE-module. Then $\operatorname{Ker}\{\widetilde{\Theta}^\ell, M\}$, $\operatorname{Im}\{\widetilde{\Theta}^\ell, M\}$ are GL_r-equivariant $k[\mathbb{M}]$–submodules of $M \otimes k[\mathbb{M}]$ for any ℓ, $1 \le \ell \le r(p-1)$, where the action of GL_r is trivial on M and is given by the multiplication by the inverse on the right on $\mathbb{M}$.*

Proof. We prove the statement for $\operatorname{Ker}\{\widetilde{\Theta}, M\}$, other cases are similar.

Let $g \in \mathrm{GL}(U_0) \simeq \mathrm{GL}_r$ and denote the action of g on $f \in k[\mathbb{M}] \simeq k[Y_{i.j}]$ by $f \mapsto f^g$. Let $[A_g] \in \mathrm{GL}_r$ be the matrix that gives the action of g on $M^{\#}_{n,r}$ with respect to the basis $\{Y_{i,j}\}$. Consider the diagram (which is not commutative!)

$$\begin{tikzcd}
M \otimes k[\mathbb{M}] \arrow[r, "\widetilde{\Theta}"] \arrow[d, "g"] & (M \otimes k[\mathbb{M}])^{\oplus r} \arrow[d, "g"] \\
M \otimes k[\mathbb{M}] \arrow[r, "\widetilde{\Theta}"] & (M \otimes k[\mathbb{M}])^{\oplus r}
\end{tikzcd}$$

Going to the right and then down, we get

$$(\widetilde{\Theta}(m \otimes f))^g = \left(\begin{pmatrix} x_1 & \ldots & x_n \end{pmatrix} \otimes \begin{pmatrix} Y_{1,1} & \ldots & Y_{1,r} \\ \vdots & \ddots & \vdots \\ \\ \vdots & \ddots & \vdots \\ Y_{n,1} & \ldots & Y_{n,r} \end{pmatrix}^g \right) (m \otimes f^g) =$$

$$\left(\begin{pmatrix} x_1 & \ldots & x_n \end{pmatrix} \otimes \begin{pmatrix} Y_{1,1} & \ldots & Y_{1,r} \\ \vdots & \ddots & \vdots \\ \\ \vdots & \ddots & \vdots \\ Y_{n,1} & \ldots & Y_{n,r} \end{pmatrix} \right) [A_g](m \otimes f^g) = [\widetilde{\theta}_1(m \otimes f^g), \ldots, \widetilde{\theta}_r(m \otimes f^g)][A_g].$$

Going down and to the left, we get

$$\widetilde{\Theta}(m \otimes f^g) = \left(\begin{pmatrix} x_1 & \ldots & x_n \end{pmatrix} \otimes \begin{pmatrix} Y_{1,1} & \ldots & Y_{1,r} \\ \vdots & \ddots & \vdots \\ \\ \vdots & \ddots & \vdots \\ Y_{n,1} & \ldots & Y_{n,r} \end{pmatrix} \right) (m \otimes f^g) =$$

$$[\widetilde{\theta}_1(m \otimes f^g), \ldots, \widetilde{\theta}_r(m \otimes f^g)].$$

Since the results differ by multiplication by an invertible matrix, we conclude that $\operatorname{Ker}\{\widetilde{\Theta}, M\}$ is a GL_r-invariant submodule of $M \otimes k[\mathbb{M}]$. $\square$

Lemmas 6.7 and 6.5 imply that the GL_r-equivariant sheaf $\mathrm{Ker}\{\widetilde{\Theta}^\ell, M\}$ (resp., $\mathrm{Im}\{\widetilde{\Theta}^\ell, M\}$) descends to a coherent sheaf on $\mathrm{Grass}_{n,r}$ via the functor $\mathcal{R}$. We denote the resulting sheaf by $\mathcal{R}(\mathrm{Ker}\{\widetilde{\Theta}^\ell, M\})$ (resp, $\mathcal{R}(\mathrm{Ker}\{\widetilde{\Theta}^\ell, M\})$).

Note that $\mathcal{K}er^\ell(M)$ (resp., $\mathcal{I}m^\ell(M)$) is a subsheaf of $M \otimes \mathcal{O}_{Gr}$ by construction. The equality $\mathcal{R}(\widetilde{M}) = M \otimes \mathcal{O}_{Gr}$ and the naturality of $\mathcal{R}$ imply that $\mathcal{R}(\mathrm{Ker}\{\widetilde{\Theta}^\ell, M\})$ (resp., $\mathcal{R}(\mathrm{Im}\{\widetilde{\Theta}^\ell, M\})$) is also a subsheaf of $M \otimes \mathcal{O}_{Gr}$. We now show that the subsheaves $\mathcal{K}er^\ell(M)$ and $\mathcal{R}(\mathrm{Ker}\{\widetilde{\Theta}^\ell, M\})$ (resp., $\mathcal{I}m^\ell(M)$ and $\mathcal{R}(\mathrm{Im}\{\widetilde{\Theta}^\ell, M\})$) of $M \otimes \mathcal{O}_{Gr}$ are equal.

Theorem 6.8. *For any finite dimensional kE-module M and any integer ℓ, $1 \leq \ell \leq (p-1)r$, we have equalities of coherent $\mathcal{O}_{Gr}$–modules*

$$\mathcal{K}er^\ell(M) = \mathcal{R}(\mathrm{Ker}\{\widetilde{\Theta}^\ell, M\}),$$

$$\mathcal{I}m^\ell(M) = \mathcal{R}(\mathrm{Im}\{\widetilde{\Theta}^\ell, M\}).$$

Proof. We establish the equality $\mathcal{K}er(M) \simeq \mathcal{R}(\mathrm{Ker}\{\widetilde{\Theta}, M\})$, other cases are similar.

Let $\{x_1, \ldots, x_n\}$ be the fixed basis of $\mathbb{V}$ so that $kE \simeq k[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$. Globally on $\mathbb{M}$ the operator

$$\widetilde{\Theta} = [\widetilde{\theta}_1, \ldots, \widetilde{\theta}_r]^T : \widetilde{M} \to (\widetilde{M})^{\oplus r}$$

is given as a product

$$\widetilde{\Theta} = \begin{pmatrix} x_1 & \ldots & x_n \end{pmatrix} \otimes \begin{pmatrix} Y_{1,1} & \ldots & Y_{1,r} \\ \vdots & \ddots & \vdots \\ & & \\ \vdots & \ddots & \vdots \\ Y_{n,1} & \ldots & Y_{n,r} \end{pmatrix}. \tag{6.8.1}$$

Let $\Sigma = \{i_1, \ldots, i_r\}$, $i_1 < \cdots < i_r$, be a subset of $\{1, \ldots, n\}$, and let $\mathcal{U}_\Sigma \subset \mathrm{Grass}_{n,r}$ be the corresponding principal open. Let $\widetilde{\mathcal{U}}_\Sigma \subset \mathbb{M}^o \subset \mathbb{M}$ be the principal open subset defined by the non-vanishing of the minor corresponding to the columns numbered by Σ. Hence, $k[\widetilde{\mathcal{U}}_\Sigma]$ is the localization of $k[\mathbb{M}]$ at the determinant of the matrix $[Y_{i_t,j}]_{1 \leq t,j \leq r}$. Note that $\widetilde{\mathcal{U}}_\Sigma$ is GL_r-invariant subset of $\mathbb{M}$ and that $\widetilde{\mathcal{U}}_\Sigma \to \mathcal{U}_\Sigma$ is a trivial GL_r-bundle. Denote by

$$\eta_{\mathcal{U}_\Sigma} : \mathrm{Coh}^{\mathrm{GL}_r}(\widetilde{\mathcal{U}}_\Sigma) \simeq \mathrm{Coh}(\mathcal{U}_\Sigma)$$

the corresponding equivalence of categories as in Lemma 6.5. As in Section 1 (prior to Defn. 1.5), we choose a section of $\mathbb{M}^0 \to \mathrm{Grass}_{n,r}$ over $\mathcal{U}_\Sigma$ defined by sending a GL_r-orbit to its unique representative such that the Σ-matrix is the identiy matrix. This section splits the trivial bundle $\widetilde{\mathcal{U}}_\Sigma \to \mathcal{U}_\Sigma$ giving an isomorphism $\widetilde{\mathcal{U}}_\Sigma \simeq \mathcal{U}_\Sigma \times \mathrm{GL}_r$ and, hence,

$$k[\widetilde{\mathcal{U}}_\Sigma] \simeq k[\mathcal{U}_\Sigma] \otimes k[\mathrm{GL}_r] = k[\mathrm{Y}_{i,j}^\Sigma] \otimes k[Y_{i_t,j}] \left[\frac{1}{\det(Y_{i_t,j})}\right]$$

where $Y^{\Sigma}_{i,j}$ are as defined in (1.4.1). Using the identification of $k[\widetilde{\mathcal{U}}_\Sigma]$ as $k[\mathcal{U}_\Sigma] \otimes k[\mathrm{GL}_r]$, we can write

$$\begin{pmatrix} Y_{1,1} & \dots & Y_{1,r} \\ \vdots & \ddots & \vdots \\ & & \\ \vdots & \ddots & \vdots \\ Y_{n,1} & \dots & Y_{n,r} \end{pmatrix} = \begin{pmatrix} Y^{\Sigma}_{1,1} & \dots & Y^{\Sigma}_{1,r} \\ \vdots & \ddots & \vdots \\ & & \\ \vdots & \ddots & \vdots \\ Y^{\Sigma}_{n,1} & \dots & Y^{\Sigma}_{n,r} \end{pmatrix} \otimes \begin{pmatrix} Y_{i_1,1} & \dots & Y_{i_1,r} \\ \vdots & \ddots & \vdots \\ Y_{i_r,1} & \dots & Y_{i_r,r} \end{pmatrix}^{-1}$$

Hence, we can decompose the operator $\widetilde{\Theta}\downarrow_{\widetilde{\mathcal{U}}_\Sigma}$ on $M \otimes k[\widetilde{\mathcal{U}}_\Sigma] \simeq M \otimes k[\mathcal{U}_\Sigma] \otimes k[\mathrm{GL}_r]$ as follows:

$$\widetilde{\Theta}\downarrow_{\widetilde{\mathcal{U}}_\Sigma} = \Theta^{\Sigma} \otimes [Y_{i_t,j}]^{-1},$$

where Θ^Σ is as defined in (6.2.1). Since the last factor is invertible, we conclude that

$$\mathrm{Ker}\{\widetilde{\Theta}\downarrow_{\widetilde{\mathcal{U}}_\Sigma}\} = \mathrm{Ker}\,\Theta^\Sigma \otimes k[\mathrm{GL}_r] = \eta^{-1}_{\mathcal{U}_\Sigma}(\mathrm{Ker}\,\Theta^\Sigma),$$

where the last equality holds by the triviality of the bundle $\widetilde{\mathcal{U}}_\Sigma \to \mathcal{U}_\Sigma$ and Remark 6.6. Since localization is exact, we have $\mathrm{Ker}\{\widetilde{\Theta}, M\}\downarrow_{\widetilde{\mathcal{U}}_\Sigma} = \mathrm{Ker}\{\widetilde{\Theta}\downarrow_{\widetilde{\mathcal{U}}_\Sigma}\}$. Hence,

$$\eta_{\mathcal{U}_\Sigma}(\mathrm{Ker}\{\widetilde{\Theta}, M\}\downarrow_{\widetilde{\mathcal{U}}_\Sigma}) = \mathrm{Ker}\,\Theta^\Sigma. \tag{6.8.2}$$

The Cartesian square

$$\begin{tikzcd} \widetilde{\mathcal{U}}_\Sigma \arrow[r, hook] \arrow[d] & \mathbb{M}^o \arrow[d] \\ \mathcal{U}_\Sigma \arrow[r, hook] & \mathrm{Grass}_{n,r} \end{tikzcd}$$

gives rise to a commutative diagram where the vertical arrows are equivalences of categories as in Lemma 6.5

$$\begin{tikzcd} \mathrm{Coh}^{\mathrm{GL}_r}(\mathbb{M}^o) \arrow[r, "\mathrm{Res}"] \arrow[d, "\simeq"] & \mathrm{Coh}^{\mathrm{GL}_r}(\widetilde{\mathcal{U}}_\Sigma) \arrow[d, "\simeq"] \\ \mathrm{Coh}(\mathrm{Grass}_{n,r}) \arrow[r, "\mathrm{Res}"] & \mathrm{Coh}(\mathcal{U}_\Sigma) \end{tikzcd}$$

Therefore, $\mathcal{R}(\mathrm{Ker}\{\widetilde{\Theta}, M\})\downarrow_{\mathcal{U}_\Sigma} = \eta_{U_\Sigma}(\mathrm{Ker}\{\widetilde{\Theta}, M\}\downarrow_{\widetilde{\mathcal{U}}_\Sigma})$. Combining this observation with the equality (6.8.2), we conclude

$$\mathcal{R}(\mathrm{Ker}\{\widetilde{\Theta}, M\})\downarrow_{\mathcal{U}_\Sigma} = \eta_{\mathcal{U}_\Sigma}(\mathrm{Ker}\{\widetilde{\Theta}, M\}\downarrow_{\widetilde{\mathcal{U}}_\Sigma}) = \mathrm{Ker}\,\Theta^\Sigma = \mathcal{K}er(M)\downarrow_{\mathcal{U}_\Sigma}, \tag{6.8.3}$$

where the last equality holds by the definition of $\mathcal{K}er(M)$. Since this holds for any r-subset $\Sigma \subset \{1, \dots, n\}$, we conclude that $\mathcal{K}er(M) = \mathcal{R}(\mathrm{Ker}\{\widetilde{\Theta}, M\})$. $\square$

7. Bundles for GL_n-equivariant modules.

For the special class of GL_n-equivariant kE–modules (see Definition 3.5), the constructions from the previous section can be shown to coincide with a well known construction of algebraic vector bundles arising in representation theory of algebraic groups. This enables us to identify various algebraic vector bundles on Grassmannians associated to such GL_n-equivariant kE–modules. We give many examples of the applicability of this approach: Examples 7.7, 7.8, 7.11, 7.12, 7.13, 7.14, 7.15, and 7.16.

We start by recalling some generalities. Let G be an algebraic group and $H \subset G$ be a closed subgroup. For any rational H-module V, we consider the flat map of varieties

$$\pi : G \times^H V \ \to G/H$$

with fiber V. We recall the functor ([Jan03, I.5])

$$\mathcal{L} : H\text{-}\mathrm{mod} \longrightarrow \mathcal{O}_{G/H}\text{-}\mathrm{mod}$$

which sends a rational H-module V to a quasi–coherent sheaf of $\mathcal{O}_{G/H}$–modules which is the sheaf of sections of $G \times^H V$. That is, for $U \subset G/H$ we have

$$\mathcal{L}(V)(U) = \Gamma(U, G \times^H V).$$

We summarize properties of the functor $\mathcal{L}$ in the following proposition.

Proposition 7.1. *Let G be an algebraic group and $H \subset G$ be a closed subgroup.*
(1) [Jan03, II.4.1]. *The functor $\mathcal{L}$ is exact and commutes with tensor products, duals, symmetric and exterior powers, and Frobenius twists.*
(2) [Jan03, I.5.14]. *Let V be a rational G–module. Then $\mathcal{L}(V\downarrow_H) \simeq \mathcal{O}_{G/H} \otimes V$ is a trivial bundle.*

We say that an algebraic vector bundle $\mathcal{E}$ on G/H (i.e., a locally free, coherent sheaf on G/H) is G-equivariant if G acts on $\mathcal{E}$ compatibly with the action of G on the base G/H (via multiplication on the left). That is, for all Zariski open subsets $U \subset G/H$ and each $g \in G$, there is an isomorphism

$$g^* : \mathcal{E}(U) \longrightarrow \mathcal{E}(g^{-1} \cdot U) \tag{7.1.1}$$

such that

$$g^*(fs) = g^*(f)g^*(s), \ s \in \mathcal{E}(U), \ f \in \mathcal{O}_{G/H}(U).$$

In other words, the algebraic vector bundle $\mathcal{E}$ on G/H is G-equivariant in the sense of Definition 6.4.

Proposition 7.2. *(1)* [CG97, 5.1.8]. *Let G be a linear algebraic group, H be a closed subgroup of G, and V a rational H-module. Then the sheaf of sections of $\pi : G \times^H V \to V$ (a quasi-coherent sheaf of $\mathcal{O}_{G/H}$-modules) is G-equivariant.*
(2) [Jan03, II.4.1]. *Let G be a reductive linear algebraic group, $P \subset G$ be a parabolic subgroup, and V be a rational P-module. Then $G \times^P V \to G/P$ is locally trivial for the Zariski topology of G/P. Hence, $\mathcal{L}(V)$ is an algebraic vector bundle on G/P.*

The following result complements the preceding recollections.

Proposition 7.3. *Let G be an algebraic group and $H \subset G$ be a closed subgroup such that $p : G \to G/H$ is locally trivial with respect to the Zariski topology on G/H. Consider a G-equivariant algebraic vector bundle $\mathcal{E}$ on G/H. Then there is an isomorphism $\mathcal{L}(V) \xrightarrow{\sim} \mathcal{E}$ of G-equivariant vector bundles on G/H,*

$$\mathcal{L}(V)(U) = \Gamma(U, G \times^H V) \ \xrightarrow{\sim} \ \mathcal{E}(U), \quad U \subset G/H,$$

where V is the fiber of $\mathcal{E}$ over $eH \in G/H$ provided with the structure of a rational H-module by the restriction of the G-action on $\mathcal{E}$.

Proof. Let $U \subset G/H$ be a Zariski open neighborhood of $eH \in G/H$ such that $p_{|U} : p^{-1}(U) \to U$ is isomorphic to the product projection $U \times V \to U$ and $\mathcal{E}_{|U} \simeq V \otimes_k \mathcal{O}_U$ is trivial. Choices of trivialization of p_U and $\mathcal{E}_{|U}$ determine an isomorphism $\phi : \mathcal{L}(V)(U) \xrightarrow{\sim} \mathcal{E}(U)$. Some finite collection of subsets $g_i \cdot U \subset G/H$ is a finite open covering of G/H. For each g_i, we define $\phi_i : \mathcal{L}(g_i \cdot U) \to \mathcal{E}(g_i \cdot U)$ by sending $g_i s \in \mathcal{L}(V)(g_i \cdot U)$ for any $s \in \mathcal{L}(U)$ to $g_i \phi(s)$; this is well defined, for each $s' \in \mathcal{L}(g_i \cdot U)$ is uniquely of the form $g_i s$ for some $s \in \mathcal{L}(U)$. We readily check that each ϕ_i induces an isomorphism on fibers, and that $(\phi_i)_{|U_i \cap U_j} = (\phi_j)_{|U_i \cap U_j}$. □

Let $U_0 \subset \mathbb{V}$ be a fixed r-dimensional subspace, and let $\mathrm{P}_0 = \mathrm{Stab}(U_0)$. With U_0 chosen, we may identify G as GL_n and P_0 as the standard parabolic subgroup of type $(r, n-r)$ of GL_n. We consider the above construction of the functor $\mathcal{L}$ with $G = \mathrm{GL}(\mathbb{V})$ and $H = \mathrm{P}_0$. Since $\mathrm{GL}(\mathbb{V})/\mathrm{P}_0 \simeq \mathrm{Grass}(r, \mathbb{V})$, we get a functor

$$\mathcal{L} : \mathrm{P}_0\text{-}\mathrm{mod} \longrightarrow \text{locally free}\, \mathcal{O}_{Gr}\text{-}\mathrm{mod}$$

where we denote by $\mathcal{O}_{Gr}$ the structure sheaf on $\mathrm{Grass}(r, \mathbb{V})$.

Example 7.4. We revisit and supplement the examples of Example 6.3.

(1) Let γ_r be the universal subbundle (of $\mathcal{O}_{Gr}^{\oplus n}$) of rank r on $\mathrm{Grass}_{n,r}$. Then

$$\mathcal{L}(U_0) = \gamma_r. \tag{7.4.1}$$

(2) Let δ_{n-r} be the universal subbundle (of $\mathcal{O}_{Gr}^{\oplus n}$) of rank $n-r$ on $\mathrm{Grass}_{n,r}$. Then

$$\mathcal{L}(W_0) \ = \ \delta_{n-r}, \quad \text{where} \quad W_0 = \mathrm{Ker}\{\mathbb{V}^\# \to U_0^\#\}, \tag{7.4.2}$$

as can be verified using Proposition 7.1 and the short exact sequence

$$0 \to \gamma_r \to \mathcal{O}_{Gr}^{\oplus n} \to \delta_{n-r}^\vee \to 0.$$

(3) By Proposition 7.1,

$$\mathcal{L}(\Lambda^r(U_0)) \ = \ \Lambda^r(\gamma_r).$$

Let $\mathfrak{p} : \mathrm{Grass}(r, \mathbb{V}) \to \mathbb{P}(\Lambda^r(\mathbb{V}))$ be the Plücker embedding, and let $\mathcal{O}_{\mathbb{P}(\Lambda^r(\mathbb{V}))}(-1)$ be the tautological line bundle on $\mathbb{P}(\Lambda^r(\mathbb{V}))$. Then by definition

$$\mathcal{O}_{Gr}(-1) = \mathfrak{p}^*(\mathcal{O}_{\mathbb{P}(\Lambda^r(\mathbb{V}))}(-1)).$$

The fiber of $\mathcal{O}_{\mathbb{P}(\Lambda^r(\mathbb{V}))}(-1)$ over a point $v_1 \wedge \ldots \wedge v_r \in \mathbb{P}^r(\Lambda^r(\mathbb{V}))$ equals $k(v_1 \wedge \ldots \wedge v_r)$. Pulling back via $\mathfrak{p}$, we get that the fiber of $\mathcal{O}_{Gr}(-1)$ over the r-plane $U = kv_1 + \cdots + kv_r = \mathfrak{p}^{-1}(v_1 \wedge \ldots \wedge v_r)$ equals $\Lambda^r(U)$. Thus,

$$\mathcal{L}(\Lambda^r(U_0)) = \Lambda^r(\gamma_r) \simeq \mathcal{O}_{Gr}(-1). \tag{7.4.3}$$

Proposition 7.5. *Let M be a GL_n–equivariant kE–module. Then $\mathcal{I}m^\ell(M)$, $\mathcal{K}er^\ell(M)$ are GL_n-equivariant $\mathcal{O}_{Gr}$-modules for any ℓ, $1 \le \ell \le (p-1)r$.*

Proof. We first observe that Proposition 3.6 implies that $\mathrm{Soc}^\ell_{U_0}(M)$ and $\mathrm{Rad}^\ell_{U_0}(M)$ are stable under the action of the standard parabolic subgroup $\mathrm{P}_0 \subset \mathrm{GL}(\mathbb{V})$ on M for any ℓ, $1 \le \ell \le (p-1)r$.

We consider only $\mathcal{I}m$; verification of the proposition for $\mathcal{I}m^\ell(M)$, $\mathcal{K}er^\ell(M)$ with ℓ, $1 \le \ell \le (p-1)r$ is similar.

Let $\mathbb{M} = M_{n,r}$ be the affine variety of $n \times r$-matrices. We identify

$$\mathbb{M} = \mathbb{V}^{\oplus r}$$

as k-linear space and note that both GL_n and GL_r act on $\mathbb{M}$: GL_n via multiplcation on the left and GL_r via multiplication on the right. Moreover, these actions obviously commute. Hence, the coordinate ring $k[\mathbb{M}]$ is a $(\mathrm{GL}_n, \mathrm{GL}_r)$-bimodule.

Recall the GL_r-invariant submodule $\mathrm{Im}\{\widetilde{\Theta}, M\}$ of $M \otimes k[\mathbb{M}]$ defined in (6.6.5). The GL_r-action on $M \otimes k[\mathbb{M}]$ is given via the trivial action on M and the action on $k[\mathbb{M}]$ induced by multiplication on $\mathbb{M}$ on the right. There is also a GL_n-action on $M \otimes k[\mathbb{M}]$ which is diagonal: as given on the GL_n-equivariant module M and via the left multiplication on $\mathbb{M}$. We first show that $\mathrm{Im}\{\widetilde{\Theta}, M\}$ is a GL_n-invariant submodule of $M \otimes k[\mathbb{M}]$ (and, hence, a $(\mathrm{GL}_n, \mathrm{GL}_r)$-submodule).

Recall $\widetilde{\Theta} = [\widetilde{\theta}_1, \ldots, \widetilde{\theta}_r] : \widetilde{M} \to (\widetilde{M})^{\oplus r}$ where for each j, $1 \leq j \leq r$,

$$\widetilde{\theta}_j(m \otimes f) = \sum_i x_i m \otimes Y_{i,j} f.$$

Fix an element $g \in \mathrm{GL}_n$. We proceed to compute the effect of the action of g on $\widetilde{\theta}_j(m \otimes f)$.

Let $(y_{ij})_{1 \leq i \leq n, 1 \leq j \leq r}$ be linear generators of $\mathbb{M} = \mathbb{V}^{\oplus r}$ chosen in such a way that y_{ij} is simply the generator x_i of $\mathbb{V}$ put in the j^{th} column. Suppose the action of g on $\mathbb{V}$ with respect to the fixed basis $\{x_1, \ldots, x_n\}$ is given by a matrix $A = (a_{st})$. The action of g on y_{ij} is then given by $gy_{ij} = \sum_\ell a_{\ell i} y_{\ell j}$, the same action on each factor $\mathbb{V}$ in $\mathbb{M}$.

We identify the coordinate algebra $k[\mathbb{M}]$ as $S^*(\mathbb{M}^\#) \simeq k[Y_{i,j}]$ with the coordinate functions $Y_{i,j}$ defined as the linear duals of $y_{i,j}$. For $f \in k[\mathbb{M}]$, we have $g \circ f(-) = f(g^{-1}-)$. Consequently, the action of g on $\mathbb{M}^\#$ with respect to the basis $\{Y_{i,j}\}_{1 \leq i \leq n, 1 \leq j \leq r}$ is given by multiplication on the right by A^{-1}. We compute

$$g(\sum_i x_i \otimes Y_{i,j}) = g([x_1, \ldots, x_n] \otimes [Y_{1,j}, \ldots, Y_{n,j}]^T) =$$

$$g([x_1, \ldots, x_n]) \otimes g([Y_{1,j}, \ldots, Y_{n,j}]^T) = [x_1, \ldots, x_n] \cdot A^T \otimes ([Y_{1,j}, \ldots, Y_{n,j}] \cdot A^{-1})^T =$$

$$[x_1, \ldots, x_n] \cdot A^T \otimes (A^T)^{-1} \cdot [Y_{1,j}, \ldots, Y_{n,j}]^T = \sum_i x_i \otimes Y_{i,j}.$$

Hence,

$$g(\widetilde{\theta}_j(m \otimes f)) = g(\sum_i x_i \otimes Y_{i,j}) g(m \otimes f) = (\sum_i x_i \otimes Y_{i,j})(gm \otimes gf) = \widetilde{\theta}_j(gm \otimes gf).$$

With the given GL_n-actions on $k[\mathbb{M}]$ and M, we have $gm \otimes gf \in M \otimes k[\mathbb{M}]$. Hence,

$$g(\widetilde{\theta}_j(m \otimes f)) = \widetilde{\theta}_j(gm \otimes gf) \in \mathrm{Im}\, \widetilde{\theta}_j.$$

Since this holds for all j, we conclude that $\mathrm{Im}\{\widetilde{\Theta}, M\} = \sum\limits_{j=1}^{r} \mathrm{Im}\, \theta_j \subset M \otimes k[\mathbb{M}]$ is invariant under the GL_n-action. Hence, $\mathrm{Im}(\widetilde{\Theta}, M)$ determines a $(\mathrm{GL}_n, \mathrm{GL}_r)$-equivariant sheaf on $\mathbb{M}$. Moreover, since $\mathbb{M}^0 \subset \mathbb{M}$ is a $(\mathrm{GL}_n, \mathrm{GL}_r)$-stable subvariety, the restriction of $\mathrm{Im}\{\widetilde{\Theta}, M\}$ to $\mathbb{M}^0$ is a $(\mathrm{GL}_n, \mathrm{GL}_r)$-equivariant sheaf on $\mathbb{M}^0$. Since the actions of GL_n and GL_r commute, the equivalence

$$\mathrm{Coh}^{\mathrm{GL}_r}(\mathbb{M}^0) \simeq \mathrm{Coh}(\mathrm{Grass}_{n,r})$$

of Lemma 6.5 restricts to an equivalence of GL_n-equivariant sheaves. Consequently, $\mathcal{I}m(M) = \mathcal{R}(\mathrm{Im}(\widetilde{\Theta}, M))$ (by Theorem 6.8) is a GL_n-equivariant $\mathcal{O}_{Gr}$–module. $\square$

The following theorem enables us to indentify kernel and image bundles as in (6.1) with bundles obtained via the functor $\mathcal{L}$.

Theorem 7.6. *Let M be a GL_n-equivariant kE-module, and let $U_0 = kx_0 + \cdots + kx_r \subset \mathbb{V}$. Then for any ℓ, $1 \leq \ell \leq r(p-1)$, we have an isomorphism of GL_n-equivariant algebraic vector bundles on $\mathrm{Grass}_{n,r} = \mathrm{Grass}(r, \mathbb{V})$*

$$\mathcal{K}er^\ell(M) \simeq \mathcal{L}(\mathrm{Soc}^\ell_{U_0}(M)), \quad \mathcal{I}m^\ell(M) \simeq \mathcal{L}(\mathrm{Rad}^\ell_{U_0}(M)).$$

Proof. By Proposition 7.5, $\mathcal{K}er^\ell(M)$ is a GL_n-equivariant vector bundle on $\mathrm{Grass}(r, \mathbb{V})$. The fiber of $\mathcal{K}er^\ell(M)$ above the base point of $\mathrm{Grass}(r, \mathbb{V})$ equals $\mathrm{Soc}^\ell_{U_0}(M)$. We now apply Proposition 7.3 to conclude that $\mathcal{K}er^\ell(M) \simeq \mathcal{L}(\mathrm{Soc}^\ell_{U_0}(M))$.

The proof that $\mathcal{I}m^\ell(M) \simeq \mathcal{L}(\mathrm{Rad}^\ell_{U_0}(M))$ is strictly analogous. $\square$

In the following examples, we show how to realize various "standard" bundles on $\mathrm{Grass}(r, \mathbb{V})$ as kernel and image bundles associated to GL_n-equivariant kE-modules. For convenience, we fix a basis $\{x_1, \ldots, x_n\}$ of $\mathbb{V}$ and choose U_0 to be the subspace generated by $\{x_1, \ldots, x_r\}$. As before, the action of $\mathrm{GL}_n \simeq \mathrm{GL}(\mathbb{V})$ on kE is given via the identification $kE \simeq S^*(\mathbb{V})/\langle v^p, v \in \mathbb{V}\rangle \simeq k[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$.

Example 7.7 (*Universal subbundle of rank r*)**.** Let $M = kE/\mathrm{Rad}^2(kE)$. As a P_0-module, $\mathrm{Rad}_{U_0}(M) \simeq U_0$. Hence, $\mathcal{I}m(M) \simeq \gamma_r$ by Example 7.4(1).

Example 7.8 (*Universal subbundle of rank $n-r$*)**.** Let $M = \mathrm{Rad}^{n-1}(\Lambda^*(\mathbb{V}))$. Then

$$\mathrm{Soc}_{U_0}(M) \simeq (\sum_{j=r+1}^{n} kx_1 \wedge \ldots \wedge x_{j-1} \wedge x_{j+1} \wedge \ldots \wedge x_n) \oplus \Lambda^n(\mathbb{V})$$

as a P_0–module. Moreover, the second direct summand is a GL_n–module. The first direct summand can be naturally identified with the P_0-module

$$W_0 = \mathrm{Ker}\{\mathbb{V}^\# \to U_0^\#\}$$

as in Example 7.4(2). We get

$$\mathcal{K}er(M) = \mathcal{L}(W_0) \oplus \mathcal{L}(\Lambda^n(\mathbb{V})) = \delta_{n-r} \oplus \mathcal{O}_{Gr}.$$

It is straightforward to see that $\mathcal{I}m(M)$ is a trivial bundle of rank one. Hence,

$$\mathcal{K}er(M)/\mathcal{I}m(M) \simeq \delta_{n-r}.$$

We also note that we have an isomorphism of kE–modules: $\mathrm{Rad}^{n-1}(\Lambda^*(\mathbb{V})) \simeq \big(kE/\mathrm{Rad}^2(kE)\big)^\#$. Hence, we have also justified the second part of Example 6.3.

The previous two examples are connected by a certain "duality" which we now state formally. As before, we fix the basis $\{x_1, \ldots, x_n\}$ of $\mathbb{V}$. We give $kE \simeq k[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$ the Hopf algebra structure of the truncated polynomial algebra. That is, the elements x_i are primitive with respect to the coproduct, and the antipode sends x_i to $-x_i$. In particular, $\mathbb{V} \subset \mathrm{Rad}(kE)$ is stable under the antipode. We emphasize that the kE-module structure of the dual $M^\#$ of a kE–module M utilizes this Hopf algebra structure.

Proposition 7.9. *Let M be a GL_n-equivariant $kE \simeq k[x_1,\dots,x_n]/(x_1^p,\dots,x_n^p)$–module. Then $M^\#$ is also a GL_n-equivariant kE-module (with the standard GL_n-action on the dual) and for any j, $1 \le j \le p-1$, we have a short exact sequence of algebraic vector bundles on* $\mathrm{Grass}(r,\mathbb{V})$*:*

$$0 \longrightarrow \mathcal{K}er^j(M^\#) \longrightarrow M^\# \otimes \mathcal{O}_{Gr} \longrightarrow \mathcal{I}m^j(M)^\vee \longrightarrow 0.$$

Proof. Let $U_0 \subset \mathbb{V}$ be the r-plane spanned by $\{x_1,\dots,x_r\}$. Proposition 2.2 implies that the following sequence of $P_0 = \mathrm{Stab}(U_0)$–modules

$$0 \longrightarrow \mathrm{Soc}^j_{U_0}(M^\#) \longrightarrow M^\# \longrightarrow \mathrm{Rad}^j_{U_0}(M)^\# \longrightarrow 0 \tag{7.9.1}$$

is exact. Applying the functor $\mathcal{L}$ to the short exact sequence (7.9.1), using the properties of $\mathcal{L}$ given in Proposition 7.1, and appealing to Theorem 7.6, we conclude the desired short exact sequence of bundles. □

Remark 7.10. Let $M = kE/\mathrm{Rad}^2(kE)$ as in Examples 6.3(1) and 7.7. Then the short exact sequence of Proposition 7.9 (with $j = 1$) takes the form

$$0 \longrightarrow \delta_{n-r} \oplus \mathcal{O}_{Gr} \longrightarrow \mathcal{O}_{Gr}^{\oplus n+1} \longrightarrow \gamma_r^\vee \longrightarrow 0\ .$$

Example 7.11 (*The Serre twist bundle $\mathcal{O}_{Gr}(-1)$*)**.** Let

$$M = \mathrm{Rad}^r(\Lambda^*(\mathbb{V}))/\mathrm{Rad}^{r+2}(\Lambda^*(\mathbb{V})).$$

Then $\mathrm{Soc}_{U_0}(M) = \Lambda^r(U_0) \oplus \mathrm{Rad}^{r+1}(\Lambda^*(\mathbb{V}))$ as a P_0–module. Hence,

$$\mathcal{K}er(M) \simeq \mathcal{L}(\Lambda^r(U_0)) \oplus \mathcal{L}(\mathrm{Rad}^{r+1}(\Lambda^*(\mathbb{V}))).$$

Since the structure of P_0 on $\mathrm{Rad}^{r+1}(\Lambda^*(\mathbb{V}))$ is the restriction of GL_n-structure, Prop. 7.1.2 implies that $\mathcal{L}(\mathrm{Rad}^{r+1}(\Lambda^*(\mathbb{V})))$ is a trivial bundle. Hence, Prop. 7.1 and Example 7.4 imply that

$$\mathcal{K}er(M) \simeq \Lambda^r(\gamma_r) \oplus (\mathcal{O}_{Gr} \otimes \Lambda^{r+1}(\mathbb{V})) \ \simeq\ \mathcal{O}_{Gr}(-1) \oplus \mathcal{O}_{Gr}^{\binom{n}{r+1}}.$$

Example 7.12. [*Symmetric powers*] Let j be a positive integer, $j \le p-1$, and let

$$M = S^*(\mathbb{V})/S^{*\ge j+1}(\mathbb{V}).$$

Then $\mathrm{Rad}^j_{U_0}(M)$ is isomorphic to $S^j(U_0)$ as a P_0–module. Hence, by Prop. 7.1 and Example 7.4,

$$\mathcal{I}m^j(M) = S^j(\gamma_r).$$

More generally, let $M = S^{*\ge i}(\mathbb{V})/S^{*\ge i+j+1}(\mathbb{V})$. Consider the multiplication map

$$\mu : S^j(U_0) \otimes S^i(\mathbb{V}) \to S^{i+j}(\mathbb{V}).$$

and the corresponding exact sequence of P_0–modules

$$0 \longrightarrow \mathrm{Ker}\,\mu \longrightarrow S^j(U_0) \otimes S^i(\mathbb{V}) \xrightarrow{\ \mu\ } S^{i+j}(\mathbb{V}) \longrightarrow \mathrm{Coker}\,\mu \longrightarrow 0.$$

The image of the multiplication map μ is spanned by all monomials divisible by a monomial in $x_1,\dots,x_r$ of degree j. Hence, $\mathrm{Rad}^j_{U_0}(M) \simeq \mathrm{Im}\,\mu$. Applying the functor $\mathcal{L}$ to the exact sequence above, we conclude that

$$\mathcal{I}m^j(M) \simeq \mathrm{Im}\{\mathcal{L}(\mu)\}$$

where $\mathcal{L}(\mu) : S^j(\gamma_r) \otimes S^i(\mathbb{V}) \subset S^j(\mathbb{V}) \otimes S^i(\mathbb{V}) \otimes \mathcal{O}_{Gr} \to S^{i+j}(\mathbb{V}) \otimes \mathcal{O}_{Gr}$ is the multiplication map.

We now specialize to the case $j = 1$. Then,

$$M = S^{*\geq i}(\mathbb{V})/S^{*\geq i+2}(\mathbb{V}).$$

In this case, the image of the multiplication map $\mu : U_0 \otimes S^i(\mathbb{V}) \to S^{i+1}(\mathbb{V})$ is spanned by all monomials divisible by one of the variables $x_1, \dots, x_r$. Therefore, we have a short exact sequence of P_0-modules

$$(7.12.1) \qquad 0 \longrightarrow \operatorname{Rad}(M) = \operatorname{Im}\mu \longrightarrow S^{i+1}(\mathbb{V}) \longrightarrow S^{i+1}(\mathbb{V}/U_0) \longrightarrow 0\ .$$

In the notation of Example 7.4(2), $\mathbb{V}/U_0 \simeq W_0^{\#}$. Hence, Proposition 7.1 and Example 7.4(2) imply that

$$\mathcal{L}(S^{i+1}(\mathbb{V}/U_0)) \ \simeq\ \mathcal{L}(S^{i+1}(W_0^{\#})) \ =\ S^{i+1}(\delta_{n-r}^{\vee}).$$

Applying the exact functor $\mathcal{L}$ to (7.12.1), we conclude that $\mathcal{I}m(M)$ fits into the following short exact sequence of vector bundles

$$(7.12.2) \qquad 0 \longrightarrow \mathcal{I}m(M) \longrightarrow S^{i+1}(\mathbb{V}) \otimes \mathcal{O}_{Gr} \longrightarrow S^{i+1}(\delta_{n-r}^{\vee}) \longrightarrow 0\ .$$

Example 7.13. Let i be a positive integer such that $i \leq p-1$, and let

$$M = \frac{\operatorname{Rad}^{n(p-1)-i-1}(kE)}{\operatorname{Rad}^{n(p-1)-i+1}(kE)}.$$

Note that as a kE-module,

$$M^{\#} \simeq \operatorname{Rad}^i(kE)/\operatorname{Rad}^{i+2}(kE).$$

Moreover, the restriction on i implies that

$$\operatorname{Rad}^i(kE)/\operatorname{Rad}^{i+2}(kE) \simeq S^{*\geq i}(\mathbb{V})/S^{*\geq i+2}(\mathbb{V}).$$

Applying Proposition 7.9, we get a short exact sequence of bundles

$$0 \longrightarrow \mathcal{K}er(M) \longrightarrow M \otimes \mathcal{O}_{Gr} \longrightarrow \mathcal{I}m(M^{\#})^{\vee} \longrightarrow 0.$$

Since the bottom radical layer of M is in the socle for any $U \subset \mathbb{V}$, the kernel bundle $\mathcal{K}er(M)$ has a trivial subbundle $\operatorname{Rad}(M) \otimes \mathcal{O}_{Gr} \simeq S^i(\mathbb{V}^{\#}) \otimes \mathcal{O}_{Gr}$ as a direct summand. Hence, we can rewrite the exact sequence above as

$$0 \longrightarrow \tfrac{\mathcal{K}er(M)}{\operatorname{Rad}(M)\otimes\mathcal{O}_{Gr}} \oplus (\operatorname{Rad}(M) \otimes \mathcal{O}_{Gr}) \longrightarrow (S^i(\mathbb{V}^{\#}) \oplus S^{i+1}(\mathbb{V}^{\#})) \otimes \mathcal{O}_{Gr}$$

$$\longrightarrow \mathcal{I}m(M^{\#})^{\vee} \longrightarrow 0.$$

Discarding the direct summand $\operatorname{Rad}(M) \otimes \mathcal{O}_{Gr}$ which splits off, we get

$$0 \longrightarrow \tfrac{\mathcal{K}er(M)}{\operatorname{Rad}(M)\otimes\mathcal{O}_{Gr}} \longrightarrow S^{i+1}(\mathbb{V}^{\#}) \otimes \mathcal{O}_{Gr} \longrightarrow \mathcal{I}m(M^{\#})^{\vee} \longrightarrow 0.$$

Dualizing, we further get

$$0 \longrightarrow \mathcal{I}m(M^{\#}) \longrightarrow S^{i+1}(\mathbb{V}) \otimes \mathcal{O}_{Gr} \longrightarrow \left(\tfrac{\mathcal{K}er(M)}{\operatorname{Rad}(M)\otimes\mathcal{O}_{Gr}}\right)^{\vee} \longrightarrow 0\ .$$

It follows from the construction that the embedding $\mathcal{I}m(M^{\#}) \hookrightarrow S^{i+1}(\mathbb{V}) \otimes \mathcal{O}_{Gr}$ in this short exact sequence coincides with the corresponding map in (7.12.2) which was induced by the multiplication map $\mu : \gamma_r \otimes S^i(\mathbb{V}) \to S^{i+1}(\mathbb{V}) \otimes \mathcal{O}_{Gr}$. Hence,

$$\frac{\mathcal{K}er(M)}{\mathcal{I}m(M)} = \frac{\mathcal{K}er(M)}{\operatorname{Rad}(M) \otimes \mathcal{O}_{Gr}} \simeq S^i(\delta_{n-r}).$$

Example 7.14 (*The Serre twist bundle* $\mathcal{O}_{Gr}(1-p)$)**.** Let

$$M = \operatorname{Rad}^{r(p-1)}(kE)/\operatorname{Rad}^{r(p-1)+2}(kE).$$

Then

$$\operatorname{Soc}_{U_0}(M) = kx_1^{p-1}\dots x_r^{p-1} \oplus \operatorname{Rad}(M)$$

We have an obvious isomorphism of one-dimensional P_0-modules

$$\underbrace{\Lambda^r(U_0) \otimes \dots \otimes \Lambda^r(U_0)}_{p-1} \simeq kx_1^{p-1}\dots x_r^{p-1}$$

given by sending $x_1 \wedge \dots \wedge x_r \otimes \dots \otimes x_1 \wedge \dots \wedge x_r$ to $x_1^{p-1}\dots x_r^{p-1}$. Hence,

$$\mathcal{K}er(M) \simeq \mathcal{L}((\Lambda^r(U_0)^{\otimes p-1}) \oplus \mathcal{L}(\operatorname{Rad}(M)) \simeq \mathcal{O}_{Gr}(1-p) \oplus \operatorname{Rad}(M) \otimes \mathcal{O}_{Gr}$$

where the last equality follows from Example 7.11 and Proposition 7.1.

Example 7.15 ($\delta_{n-r}^{\vee}$ *via cokernel*)**.** Let $\mathcal{C}oker(M) \stackrel{def}{=} (M \otimes \mathcal{O}_{Gr})/\mathcal{I}m(M)$. Let $M = kE/\operatorname{Rad}^2(kE)$. The exactness of $\mathcal{L}$ together with Example 7.4 imply that

$$\mathcal{C}oker(M) \simeq \delta_{n-r}^{\vee}.$$

In the following example we study a bundle that comes not from a GL_n-equivariant kE-module but from the cohomology of E considered as a GL_n-module. For the coherence of notation, assume that $p > 2$. Recall that $\mathrm{H}^*(E,k)$ has a GL_n-structure and, moreover, we have an isomorphism of GL_n–modules

$$\mathrm{H}^*(kE,k) \simeq \Lambda^*(V^{\#}) \otimes S^*((V^{(1)}[2])^{\#})$$

as stated in Proposition 5.1.

Example 7.16. Let $\alpha_0 : C = k[t_1,\dots,t_r]/(t_i^p) \to kE$ be the map defined by $\alpha_0(t_i) = x_i$ for $1 \le i \le r$, and let

$$\alpha_0^* : \mathrm{H}^{2m}(kE,k) \to \mathrm{H}^{2m}(C,k)$$

be the induced map on cohomology for some positive integer m. Reducing modulo nilpotents, we get a map

$$\alpha_0^* : \mathrm{H}^{2m}(kE,k)_{\mathrm{red}} \simeq S^m((V^{(1)})^{\#}) \to \mathrm{H}^{2m}(C,k)_{\mathrm{red}} \simeq S^m((U_0^{(1)})^{\#})$$

which is induced by $(\alpha_0^{(1)})^{\#} : (V^{(1)})^{\#} \to (U_0^{(1)})^{\#}$ by Proposition 5.1. This implies that the kernel of α_0^* is stable under the action of the standard parabolic P_0. Hence, we can apply the functor $\mathcal{L}$ to $\operatorname{Ker}\alpha_0^*$. Since $\mathcal{L}$ is exact and commutes with Frobenius twist we obtain a short exact sequence of bundles

$$0 \longrightarrow \mathcal{L}(\operatorname{Ker}\alpha_0^*) \longrightarrow \mathcal{O}_{Gr} \otimes S^m((V^{(1)})^{\#}) \longrightarrow S^m(F^*(\gamma_r^{\vee})) \longrightarrow 0$$

where $F : \operatorname{Grass}(r,\mathbb{V}) \to \operatorname{Grass}(r,\mathbb{V})$ is the Frobenius map.

8. A construction using the Plücker embedding

We present another construction of bundles from modules of constant r-socle rank, one that applies only to kernel bundles. This construction provides "generators" for graded modules for the coordinate algebra of the Grassmannian whose associated coherent sheaf is the kernel bundle of Theorem 6.8.

We denote the homogeneous coordinate ring of $\mathrm{Grass}_{n,r}$ by $\mathcal{A}$ and identify it with a quotient of $k[p_\Sigma]$ via the Plücker embedding $\mathfrak{p} : \mathrm{Grass}_{r,n} \to \mathbb{P}^{\binom{n}{r}-1}$. As before, $\mathcal{O}_{Gr}$ denotes the structure sheaf of $\mathrm{Grass}_{n,r}$. Since $\mathcal{A}$ is generated in degree one, we have an equivalence of categories (the Serre correspondence)

$$\mathrm{Coh}(\mathrm{Grass}_{n,r}) \simeq \frac{\text{Fin. gen. graded } \mathcal{A}-\text{mod}}{\text{fin. dim. graded } \mathcal{A}-\text{mod}} \tag{8.0.1}$$

between the category of coherent $\mathcal{O}_{Gr}$–modules and the quotient category of finitely generated graded $\mathcal{A}$–modules modulo the finite dimensional graded $\mathcal{A}$–modules. The equivalence is given explicitly by sending an $\mathcal{O}_{Gr}$–module $\mathcal{F}$ to $\bigoplus_{i\in\mathbb{Z}} \Gamma(\mathrm{Grass}_{n,r}, \mathcal{F}(i))$ (see [Har77, II.5]).

Starting with a module of constant r-socle rank, we construct a graded $\mathcal{A}$–module $\mathrm{Ker}\{\Theta_\mathcal{A}, M\}$ which is in the equivalence class of the kernel bundle $\mathcal{K}er(M)$ via the Serre correspondence (Theorem 8.2). We then develop an algorithm that can be used to construct a collection of generators $w_1, \dots, w_t$, in degrees $d_1, \dots, d_t$, of the graded module $\mathrm{Ker}\{\Theta_\mathcal{A}, M\}$, up to a finite dimensional quotient. Applying the Serre correspondence again, we obtain a surjective map of vector bundles

$$\bigoplus_{i=1}^{t} \mathcal{O}_{Gr}(-d_i) \longrightarrow \mathcal{K}er(M)\ .$$

Definition 8.1. Let M be a $kE \simeq k[x_1, \dots, x_r]/(x_1^p, \dots, x_n^p)$–module. We define the map

$$\Theta_\mathcal{A} : \quad M \otimes \mathcal{A} \quad \longrightarrow \quad (M \otimes \mathcal{A})^{\binom{n}{r-1}}$$

by components $\Theta_\mathcal{A} = \{\vartheta_W\}$ where the index is over the subsets $W \subset \{1 \dots n\}$ having $r-1$ elements. For any such W, and any $m \in M$, let

$$\vartheta_W(m \otimes 1) \quad = \quad \sum_{i \notin W} (-1)^{u(W,i)}\ x_i m \otimes \mathfrak{p}_{W\cup\{i\}}$$

where $u(W, i) = \#\{j \in W \mid j < i\}$.

Since the operator $\Theta_\mathcal{A}$ is graded of degree one (with respect to the standard grading of the homogeneous coordinate algebra $k[p_\Sigma]$ of $\mathbb{P}^{\binom{n}{r}-1}$ where the Plücker coordinates $\mathfrak{p}_\Sigma$ have degree 1), the kernel of $\Theta_\mathcal{A}$, denoted $\mathrm{Ker}\{\Theta_\mathcal{A}, M\}$, is a graded $\mathcal{A}$–module.

Theorem 8.2. *For any finite–dimensional kE–module M, the graded $\mathcal{A}$-module $\mathrm{Ker}\{\Theta_\mathcal{A}, M\}$ corresponds to the coherent sheaf $\mathcal{K}er(M)$ as defined in* (6.1) *via the equivalence of categories* (8.0.1).

Proof. Let $\mathcal{U}_\Sigma \subset \mathrm{Grass}_{n,r}$ be a principal open subset indexed by some subset $\Sigma \subset \{1, \dots, n\}$ of cardinality r. Then

$$k[\mathcal{U}_\Sigma] \;=\; (\mathcal{A}[1/\mathfrak{p}_\Sigma])_0, \quad \mathrm{Ker}\{\Theta_\mathcal{A}, M\}_{\mathcal{U}_\Sigma} = (\mathrm{Ker}\{\Theta_\mathcal{A}, M\} \otimes \mathcal{A}[1/\mathfrak{p}_\Sigma])_0,$$

where $\mathrm{Ker}(M)_{\mathcal{U}_\Sigma} = \mathcal{K}er(M){\downarrow_{\mathcal{U}_\Sigma}}$.

We show that for any r-subset $\Sigma \subset \{1, \ldots, n\}$,

$$\mathrm{Ker}\{\Theta_{\mathcal{A}}, M\}_{\mathcal{U}_\Sigma} = \mathrm{Ker}(M)_{\mathcal{U}_\Sigma}$$

as submodules of $M \otimes k[\mathcal{U}_\Sigma]$ which is sufficient to prove the theorem.

Let $\mathcal{I}_{r-1}$ be the set of all subsets W of $\{1, \ldots, n\}$ of cardinality $r-1$. Recall that $\mathrm{Ker}(M)_{\mathcal{U}_\Sigma}$ is given as the kernel of the operator

$$[\theta_1^\Sigma, \ldots, \theta_r^\Sigma] = [x_1, \ldots, x_n] \otimes \begin{pmatrix} Y_{1,1}^\Sigma & \ldots & Y_{1,r}^\Sigma \\ \vdots & \ddots & \vdots \\ & & \\ \vdots & \ddots & \vdots \\ Y_{n,1}^\Sigma & \ldots & Y_{n,r}^\Sigma \end{pmatrix} : M \otimes k[\mathcal{U}_\Sigma] \to (M \otimes k[\mathcal{U}_\Sigma])^{\oplus r}.$$

On the other hand, the operator $\Theta_{\mathcal{A}} : M \otimes k[\mathcal{U}_\Sigma] \to (M \otimes k[\mathcal{U}_\Sigma])^{\binom{n}{r-1}}$ is given by localizing $[\vartheta_W]_{W \in \mathcal{I}_{r-1}}$ as defined in (8.1) to $k[\mathcal{U}_\Sigma]$. We show that the operators $[\theta_1^\Sigma, \ldots, \theta_r^\Sigma]$ and $[\vartheta_W]_{W \in \mathcal{I}_{r-1}}$ are related by multiplication by a matrix B (of size $\binom{n}{r-1} \times r$) which does not change the kernel.

To simplify notation, assume that $\Sigma = \{1, \ldots, r\}$. We define the matrix B with columns indexed by subsets $W = \{i_1, \ldots, i_{r-1}\}$ of $\{1, \ldots, n\}$ and rows indexed by j, $1 \le j \le r$. Let $B_{W,j}$ be the $(-1)^j$ times the determinant of the $(r-1) \times (r-1)$ submatrix obtained from $[Y_{i,j}^\Sigma]$ by taking the rows indexed by W and deleting the j^{th} column. That is,

$$B_{j,W} = (-1)^j \ \mathrm{Det} \begin{bmatrix} Y_{i_1,1}^\Sigma & \ldots & Y_{i_1,j-1}^\Sigma & Y_{i_1,j+1}^\Sigma & \ldots & Y_{i_1,r}^\Sigma \\ Y_{i_2,1}^\Sigma & \ldots & Y_{i_2,j-1}^\Sigma & Y_{i_2,j+1}^\Sigma & \ldots & Y_{i_2,r}^\Sigma \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ Y_{i_{r-1},1}^\Sigma & \ldots & Y_{i_{r-1},j-1}^\Sigma & Y_{i_{r-1},j+1}^\Sigma & \ldots & Y_{i_{r-1},r}^\Sigma \end{bmatrix}$$

We pick a special order on the subsets $W \in \mathcal{I}_{r-1}$, so that the first r columns of B are indexed by $\{1, \ldots, r-1\}, \{1, \ldots, r-2, r\}, \ldots, \{1, 3, \ldots, r\}, \{2, \ldots, r\}$. With this assumption, the first r columns of B form an identity matrix. Indeed, since the first r rows of $[Y_{i,j}^\Sigma]$ form an identity matrix, we have

$$B_{j,\{1,\ldots,j-1,j+1,\ldots r\}} = 1 \quad \text{and } B_{j',\{1,\ldots,j-1,j+1,\ldots r\}} = 0 \text{ for } j' \neq j.$$

We rewrite

$$B = \begin{pmatrix} I_r & | & B' \end{pmatrix} \tag{8.2.1}$$

Next we compute the $n \times \binom{n}{r-1}$ – matrix $[Y_{i,j}^\Sigma] \cdot B$. We have that

$$([Y_{i,j}^\Sigma] \cdot B)_{i,W} = Y_{i,1}^\Sigma B_{1,W} + Y_{i,2}^\Sigma B_{2,W} + Y_{i,r}^\Sigma B_{r,W}$$

which is the determinant of the matrix

$$\begin{pmatrix} Y^{\Sigma}_{i,1} & Y^{\Sigma}_{i,2} & \dots & Y^{\Sigma}_{i,r} \\ Y^{\Sigma}_{i_1,1} & Y^{\Sigma}_{i_1,2} & \dots & Y^{\Sigma}_{i_1,r} \\ \vdots & & \ddots & \vdots \\ Y^{\Sigma}_{i_{r-1},1} & Y^{\Sigma}_{i_{r-1},2} & \dots & Y^{\Sigma}_{i_{r-1},r} \end{pmatrix},$$

where $W = \{i_1, \dots, i_{r-1}\}$. If i is in W then the matrix has two identical columns and its determinant is zero. If i is not in W then the determinant is precisely $(-1)^{u(W,i)}\mathfrak{p}_{W\cup\{i\}}$. That is, the only difference between the above matrix and the matrix whose determinant is $\mathfrak{p}_{W\cup\{i\}}$ is that the first row must be moved to the proper position so that the elements $i, i_1, \dots, i_{r-1}$ are rearranged to be consecutive. This requires $u(W,i)$ moves. We conclude that the matrix B has an entry $(-1)^{u(W,i)}\mathfrak{p}_{W\cup\{i\}}$ at the place $\{W, i\}$ (where we assume for convenience that $\mathfrak{p}_{W\cup\{i\}} = 0$ if $i \in W$). Hence,

$$([x_i] \cdot [Y^{\Sigma}_{i,j}] \cdot B)_W = \vartheta_W.$$

The formula $[\theta^{\Sigma}_1, \dots, \theta^{\Sigma}_r] = [x_i] \cdot [Y^{\Sigma}_{i,j}]$ now implies the equality

$$[\theta^{\Sigma}_1, \dots, \theta^{\Sigma}_r] \cdot B = [\vartheta_W]_{W\in\mathcal{I}_{r-1}}. \tag{8.2.2}$$

Since $B = [I \mid B']$ has maximal rank, we conclude that

$$\mathrm{Ker}\{[\theta^{\Sigma}_1, \dots, \theta^{\Sigma}_r] : M \otimes k[\mathcal{U}_\Sigma] \to (M \otimes k[\mathcal{U}_\Sigma])^{\oplus r}\} =$$
$$\mathrm{Ker}\{[\vartheta_W] : M \otimes k[\mathcal{U}_\Sigma] \to (M \otimes k[\mathcal{U}_\Sigma])^{\binom{n}{r-1}}\}.$$

Hence, $\mathrm{Ker}\{\Theta_{\mathcal{A}}, M\}_{\mathcal{U}_\Sigma} = \mathrm{Ker}(M)_{\mathcal{U}_\Sigma}$. □

Combining Theorems 8.2 and 6.2, we get the following Corollary.

Corollary 8.3. *Assume that M is a kE-module of constant r-Soc^1 rank. Then the Serre correspondent (via the equivalence* (8.0.1)*) of the graded $\mathcal{A}$-module* $\mathrm{Ker}\{\Theta_{\mathcal{A}}, M\}$ *is an algebraic vector bundle on* $\mathrm{Grass}_{n,r}$.

In some of our calculations, we use the following variation of the operator $\Theta_{\mathcal{A}}$ given in Definition 8.1. Let $\mathcal{I}_{r-1}$ be the set of all subsets of $\{1, \dots, n\}$ having $r-1$ elements. Let M be a kE-module of constant socle type with the property that $\mathrm{Rad}^2(M) = \{0\}$. Note that the assumption $\mathrm{Rad}^2(M) = 0$ implies that constant r-socle type is equivalent to constant r-Soc^1 rank.

We define the map

$$\overline{\Theta}_{\mathcal{A}} : \quad M/\mathrm{Rad}(M) \otimes \mathcal{A} \quad \longrightarrow \quad (\mathrm{Rad}(M) \otimes \mathcal{A})^{\binom{n}{r-1}}$$

by its components $\overline{\Theta}_{\mathcal{A}} = \{\overline{\vartheta}_W\}$ where the index is over $W \in \mathcal{I}_{r-1}$. For any such W, and any $m \in M$, let

$$\overline{\vartheta}_W((m + \mathrm{Rad}(M)) \otimes 1) \quad = \quad \sum_{i\notin W} (-1)^{u(W,i)}\, x_i m \otimes \mathfrak{p}_{W\cup\{i\}} \tag{8.3.1}$$

where $u(W,i) = \#\{j \in W \mid j < i\}$.

Corollary 8.4. *Let M be a kE-module of constant socle type with the property that* $\mathrm{Rad}^2(M) = \{0\}$. *Then the graded $\mathcal{A}$–module* $\mathrm{Ker}\{\overline{\Theta}_{\mathcal{A}}, M\}$ *corresponds to an algebraic vector bundle on* $\mathrm{Grass}_{n,r}$ *via the equivalence* (8.0.1).

Proof. Because $\mathrm{Rad}^2(M) = \{0\}$, the free $\mathcal{A}$–module $\mathrm{Rad}(M) \otimes \mathcal{A}$ is a submodule of $\mathrm{Ker}\{\Theta_{\mathcal{A}}, M\}$ with quotient $\mathrm{Ker}\{\overline{\Theta}_{\mathcal{A}}, M\}$. □

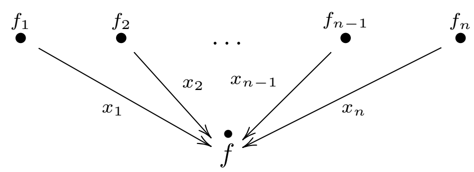
Remark 8.5. For certain kE-modules M of constant r-socle rank, Corollary 8.4 can be used to determine a graded $\mathcal{A}$-submodule of $M \otimes \mathcal{A}$ with Serre correspondent $\mathcal{K}er(M) \subset M \otimes \mathcal{O}_{Gr}$. The process goes in two steps. First, a set of elements of the kernel is calculated. This can be done using a computer seaching through the degrees. That is, we use (8.3.1) to calculate a matrix of the map $\overline{\Theta}_{\mathcal{A}}$ on the degree one grading of $M/\mathrm{Rad}(M) \otimes \mathcal{A}$ to the degree two grading of $\mathrm{Rad}(M) \otimes \mathcal{A}$. A spanning set of elements of the null space of this matrix constitutes part of our set of "generators". We continue, next looking for a spanning set of the null space of our matrix for $\overline{\Theta}_{\mathcal{A}}$ on the degree two grading of $M/\mathrm{Rad}(M) \otimes \mathcal{A}$. We proceed to higher and higher gradings.

The next step is to verify that we have found sufficiently many elements in the kernel to generate a graded module with Serre correspondent $\mathcal{K}er(M)$. In certain examples, it is possible to show that the elements obtained by considering gradings less than or equal to a given degree generate a graded submodule $N \subseteq \mathrm{Ker}\{\overline{\Theta}_{\mathcal{A}}, M\}$ with Serre correspondent $\mathcal{K}er(M)$. We start with the information that the Serre correspondent of N should have rank equal to $d = \dim \mathrm{Soc}_U(M) - \dim \mathrm{Rad}(M)$ (which is independent of r-plane U since M has constant r-socle rank).

Because the module M has constant r-socle rank, for any extension K of k and any specialization $\mathcal{A} \to K$ at a homogeneous prime ideal of $\mathcal{A}$, the induced inclusion map $N \otimes_{\mathcal{A}} K \to M/\mathrm{Rad}(M) \otimes_{\mathcal{A}} K$ can not have rank more than d by Corollary 8.4. If it can be shown that the rank of any such specialization is exactly d, then we have that N is a graded module corresponding to a vector bundle of rank d that is contained in $\mathrm{Ker}\{\overline{\Theta}_{\mathcal{A}}, M\}$ which also has rank d. Consequently, the Serre correspondents of N and $\mathrm{Ker}\{\overline{\Theta}_{\mathcal{A}}, M\}$ are equal.

We revisit some of the examples of Section 7 to illustrate how this method works.

Example 8.6 (Universal subbundle δ_{n-r})**.** Let $M \simeq \mathrm{Rad}^{(p-1)n-1}(kE)$. Then $\mathrm{Rad}^2(M) = \mathrm{Rad}^{(p-1)n+1}(kE) = \{0\}$ and, hence, M satisfies the hypothesis of Corollary 8.4. Pictorially, we can represent M as follows:



It is then evident that $M \simeq \mathrm{Rad}^{n-1}(\Lambda^*(\mathbb{V}))$. By (7.8), $\mathcal{K}er(M)/(\mathrm{Rad}(M) \otimes \mathcal{O}_{Gr}) \simeq \delta_{n-r}$, the universal subbundle of $\mathcal{O}_{Gr}^{\oplus n}$ or rank $n-r$.

We proceed to write down explicit generators for the kernel $\mathrm{Ker}\{\overline{\Theta}_{\mathcal{A}}, M\}$ as a submodule of $M/\mathrm{Rad}(M) \otimes \mathcal{A}$. Let $\{f, f_1, \dots, f_n\}$ be linear generators of M as indicated on the diagram above. Let $\mathcal{I}_{r+1}$ be the set of subsets of $\{1, \dots, n\}$ having exactly $r+1$ elements. For each $S \in \mathcal{I}_{r+1}$ let w_S be the element of $M/\mathrm{Rad}(M) \otimes \mathcal{A}$ given as

$$w_S \; = \; \sum_{j \in S} (-1)^{u(S,j)} f_j \otimes \mathfrak{p}_{S \backslash \{j\}}$$

where $u(S,j) = \#\{i \in S | i \leq j\}$. These w_S, all of grading one, generate δ_{n-r} as we verify in the next proposition.

Proposition 8.7. *The elements w_S generate a graded $\mathcal{A}$-module corresponding to the algebraic vector bundle δ_{n-r} via* (8.0.1).

Proof. A proof proceeds as follows. We should note that the elements were generated by computer in special cases, but it is a straightforward exercise to check that these elements are in the kernel of $\overline{\Theta}_{\mathcal{A}}$. We leave this exercise to the reader.

The defining equations for the elements w_S can be written as a matrix equation

$$\mathbf{w} = \mathbf{f} \otimes \mathbf{P}$$

where $\mathbf{w} = [w_S]_{S \in \mathcal{I}_{r+1}}$, $\mathbf{P} = (p_{j,S})$ is the $n \times \binom{n}{r+1}$ matrix with entries $p_{j,S} = (-1)^{u(S,j)}\mathfrak{p}_{S\setminus\{j\}}$ if $j \in S$ and $p_{j,S} = 0$ otherwise, and $\mathbf{f} = [f_1, \ldots, f_n]$. Because the elements $f_1, \ldots, f_n$ are linearly independent, the dimension of the image depends entirely on the rank of the matrix $\mathbf{P}$. As was noted Remark 8.5, at any specialization $\phi : \mathcal{A} \to K$, K an extension of k, the rank of $\phi(\mathbf{P})$ can not be greater than $n - r$ which is $\dim(\mathrm{Soc}_U(M)) - 1$ for any U. So the task is to show that the rank of the matrix $\mathbf{P}$ at any specialization is at least $n - r$.

In any specialization, one of the Plücker coordinates, call it $\mathfrak{p}_\Sigma$, must be nonzero. Consider the $(n-r) \times (n-r)$ submatrix of $\mathbf{P}$ determined by the columns indexed by subsets $T \in \mathcal{I}_{r+1}$ that contain a fixed $\Sigma \in \mathcal{I}_r$ and the rows indexed by all j such that $j \notin \Sigma$. The (i,T) entry in this matrix is $(-1)^{u(T,j)}\mathfrak{p}_\Sigma$ if $i = j$, and is 0 if $i \neq j$. Consequently, the determinant of this submatrix is $\pm\mathfrak{p}_\Sigma^{n-r}$ which is not zero. So we have proved that the elements w_S generate a locally free graded module whose corresponding bundle is the kernel bundle $\mathcal{K}er(M)/(\mathrm{Rad}(M) \otimes \mathcal{O}_{Gr})$. $\square$

Remark 8.8. In the notation of the Example 7.8, it is not difficult to see that the kernel $\mathrm{Ker}\{\overline{\Theta}_{\mathcal{A}}, \mathrm{Rad}^{n-1}(\Lambda^*(\mathbb{V}))\}$ is generated by the element

$$v \;=\; \sum_{i \notin \Sigma} (-1)^{u(\Sigma,i)}(y_\Sigma \otimes \mathfrak{p}_\Sigma),$$

where $u(\Sigma, i)$ is the number of elements in $\Sigma \in \mathcal{I}_r$ that are less than i, and the sum is over all subsets of $\{1, \ldots, n\}$ having exactly r elements. Here $y_\Sigma = x_{i_1} \wedge \cdots \wedge x_{i_r}$ where $\Sigma = \{i_1, \ldots, i_r\}$. Hence, in this case we have a graded $\mathcal{A}$-module corresponding to the universal bundle δ_{n-r} generated by only one element.

Example 8.9. Set $p = 3, n = 4, r = 2$ and consider $M = \mathrm{Rad}^4 kE/\mathrm{Rad}^6 kE$. By Example 7.14, $\mathcal{K}er(M)/(\mathrm{Rad}(M) \otimes \mathcal{O}_{Gr}) \simeq \mathcal{O}_{Gr}(-2)$. The following generator of the graded module $\mathrm{Ker}\{\overline{\Theta}_{\mathcal{A}}, M\}$ whose associated bundle is $\mathcal{K}er(M)/(\mathrm{Rad}(M) \otimes \mathcal{O}_{Gr})$, was constructed with the aid of the computational algebra package Magma [BoC].

$$\begin{aligned} v = {} & x_1^2x_2^2 \otimes \mathfrak{p}_{12}^2 - x_1^2x_2x_3 \otimes \mathfrak{p}_{12}\mathfrak{p}_{13} - x_1^2x_2x_4 \otimes \mathfrak{p}_{12}\mathfrak{p}_{14} + x_1x_2^2x_3 \otimes \mathfrak{p}_{12}\mathfrak{p}_{23} + \\ & x_1x_2^2x_4 \otimes \mathfrak{p}_{12}\mathfrak{p}_{24} + x_1^2x_3^2 \otimes \mathfrak{p}_{13}^2 - x_1^2x_3x_4 \otimes \mathfrak{p}_{13}\mathfrak{p}_{14} - x_1x_2x_3^2 \otimes \mathfrak{p}_{13}\mathfrak{p}_{23} - \\ & x_1x_2x_3x_4 \otimes \mathfrak{p}_{13}\mathfrak{p}_{24} + x_1x_3^2x_4 \otimes \mathfrak{p}_{13}\mathfrak{p}_{34} + x_1^2x_4^2 \otimes \mathfrak{p}_{14}^2 - x_1x_2x_3x_4 \otimes \mathfrak{p}_{14}\mathfrak{p}_{23} - \\ & x_1x_2x_4^2 \otimes \mathfrak{p}_{14}\mathfrak{p}_{24} - x_1x_3x_4^2 \otimes \mathfrak{p}_{14}\mathfrak{p}_{34} + x_2^2x_3^2 \otimes \mathfrak{p}_{23}^2 - x_2^2x_3x_4 \otimes \mathfrak{p}_{23}\mathfrak{p}_{24} + \\ & x_2x_3^2x_4 \otimes \mathfrak{p}_{23}\mathfrak{p}_{34} + x_2^2x_4^2 \otimes \mathfrak{p}_{24}^2 - x_2x_3x_4^2 \otimes \mathfrak{p}_{24}\mathfrak{p}_{34} + x_3^2x_4^2 \otimes \mathfrak{p}_{34}^2. \end{aligned}$$

Note that the degree of this generator is 2, which is consistent with the fact that the associated bundle is $\mathcal{O}_{Gr}(-2)$.

We end this section with nontrivial computation of the graded module of a vector bundle of rank 3 over $\mathrm{Grass}(2, \mathbb{V})$. It confirms the intuition that modules become more complicated as the rank and degree increase. The generators in this example were calculated using Magma [BoC] for specific fields, but were checked for general fields by hand.

Example 8.10. Assume that $r = 2$ and $n = 4$. We consider the module

$$M = \mathrm{Rad}^{n(p-1)-2}(kE)/\,\mathrm{Rad}^{n(p-1)}(kE).$$

and look at the kernel of the operator

$$\overline{\Theta}_{\mathcal{A}} : M/\,\mathrm{Rad}(M) \otimes \mathcal{A} \longrightarrow \mathrm{Rad}(M)^4 \otimes \mathcal{A}.$$

as in 8.4. Taking $i = 1$ in Example 7.13, we see that

$$\frac{\mathcal{K}er(M)}{\mathrm{Rad}(M) \otimes \mathcal{O}_{Gr}} \simeq S^2(\delta_{n-r}) = S^2(\delta_2).$$

This gives us a rank 3 vector bundle on $\mathrm{Grass}(2, \mathbb{V})$. For a plane spanned by vectors $u_i = \sum_{j=1}^4 a_{i,j} x_j \in \mathbb{V} = k^4$, $i = 1, 2$, we have that $\mathrm{Soc}_U(M)$ is spanned by a basis for $\mathrm{Rad}(M)$ together with the classes of the elements

$$u_1^{p-1}u_2^{p-1}u_3^{p-3}u_4^{p-1}, \quad u_1^{p-1}u_2^{p-1}u_3^{p-2}u_4^{p-2}, \quad u_1^{p-1}u_2^{p-1}u_3^{p-1}u_4^{p-3},$$

where u_3 and u_4 are two elements of $\mathbb{V}$ which together with u_1 and u_2 span $\mathbb{V} \simeq \mathrm{Rad}(kE)/\,\mathrm{Rad}^2(kE)$.

We proceed to write down generators of the graded $\mathcal{A}$–module $\mathrm{Ker}\{\overline{\Theta}_{\mathcal{A}}, M\}$. They come in two types. We note that neither of the collections of all generators of a single type generates a subbundle. That is to say, if we specialize the Plücker coordinates to a random point, then (in general) the subspace of k^{10} spanned by the specialized generators of each type has dimension 3 and hence is equal to the subspace spanned by all of the specialized generators. The generators are described as follows.

Generators of Type 1. These are indexed by the set $\{1, 2, 3, 4\}$. For each $\ell \in \{1, 2, 3, 4\}$, let i, j and k denote the other three elements. In what follows, we are not assuming that i, j, k are in any particular order. The generator corresponding to the choice of ℓ has the form

$$\gamma_\ell = \sum \mu_{a,b,c} \otimes x_i^{p-1-a} x_j^{p-1-b} x_k^{p-1-c} x_\ell^{p-1}$$

where the index is over all tuples (a, b, c) such that a, b, c are in $\{0, 1, 2\}$ and $a+b+c = 2$. The coefficient $\mu_{a,b,c}$ is determined by the following rule. First, $\mu_{2,0,0} = \mathfrak{p}_{j,k}^2$. In the other cases, $\mu_{1,1,0} = \beta\mathfrak{p}_{i,k}\mathfrak{p}_{j,k}$, where β is 1 if k is between i and j and -1 otherwise. The other coefficients are obtained by permuting i, j and k. The notational convention is that $\mathfrak{p}_{i,j} = \mathfrak{p}_{j,i}$ in the event that $i > j$. So in the case that $\ell = 2$, the generator has the form

$$\begin{aligned}
\gamma_2 = \mathfrak{p}_{1,3}^2 \otimes x_1^{p-1}x_2^{p-1}x_3^{p-1}x_4^{p-3} &\quad + \mathfrak{p}_{1,4}^2 \otimes x_1^{p-1}x_2^{p-1}x_3^{p-3}x_4^{p-1} \\
+ \mathfrak{p}_{3,4}^2 \otimes x_1^{p-3}x_2^{p-1}x_3^{p-1}x_4^{p-1} &\quad - \mathfrak{p}_{1,3}\mathfrak{p}_{1,4} \otimes x_1^{p-1}x_2^{p-1}x_3^{p-2}x_4^{p-2} \\
+ \mathfrak{p}_{1,3}\mathfrak{p}_{3,4} \otimes x_1^{p-2}x_2^{p-1}x_3^{p-1}x_4^{p-2} &\quad - \mathfrak{p}_{1,4}\mathfrak{p}_{3,4} \otimes x_1^{p-2}x_2^{p-1}x_3^{p-2}x_4^{p-1}
\end{aligned}$$

Generators of Type 2. The generators of the second type are indexed by subsets $S = \{i, j\}$ with two elements in $I = \{1, 2, 3, 4\}$. Let k, ℓ denote the other two elements in I. Again, we are not assuming any ordering on i, j, k and ℓ. The generator corresponding to S has the form

$$\gamma_S = \sum \mu_{a,b,c,d} \otimes x_i^{p-1-a} x_j^{p-1-b} x_k^{p-1-c} x_\ell^{p-1-d}$$

where the sum is over the set of all tuples (a, b, c, d) such that $\{a, b\} \subset \{0, 1, 2\}$, $\{c, d\} \subset \{0, 1\}$, and $a + b + c + d = 2$. The coefficients $\mu_{a,b,c,d}$ are determined by the following rules.

(1) Let $\mu_{0,0,1,1} = \mathfrak{p}_{i,j}^2$.
(2) Let $\mu_{0,1,1,0} = \beta\mathfrak{p}_{i,j}\mathfrak{p}_{i,k}$, where $\beta = 1$ if i is between j and k (*i. e.* $j < i < k$ or $k < i < j$) and $\beta = -1$ otherwise.
(3) Assume that $i < j$ then $\mu_{1,1,0,0} = \beta_1\mathfrak{p}_{i,k}\mathfrak{p}_{j,\ell} + \beta_2\mathfrak{p}_{i,\ell}\mathfrak{p}_{j,k}$ where $\beta_1 = \gamma_1\delta_1$ for

$$\gamma_1 = \begin{cases} 1 & \text{if } j < \ell \\ -1 & \text{otherwise} \end{cases}, \qquad \delta_1 = \begin{cases} 1 & \text{if } i < k \\ -1 & \text{otherwise,} \end{cases}$$

and β_2 is given by the same formula with k and ℓ interchanged.
(4) Let $\mu_{0,2,0,0} = \beta\mathfrak{p}_{j,k}\mathfrak{p}_{j,\ell}$ where β is 2 if j is between k and ℓ and is -2 otherwise.

So, for example, if $S = \{2, 4\}$, then

$$\begin{aligned}\gamma_S = {} & \mathfrak{p}_{2,4}^2 \otimes x_1^{p-2}x_2^{p-1}x_3^{p-2}x_4^{p-1} \quad - 2\mathfrak{p}_{1,2}\mathfrak{p}_{2,3} \otimes x_1^{p-1}x_2^{p-1}x_3^{p-1}x_4^{p-3} \\ & \mathfrak{p}_{1,2}\mathfrak{p}_{2,4} \otimes x_1^{p-1}x_2^{p-1}x_3^{p-2}x_4^{p-2} \quad - \mathfrak{p}_{1,4}\mathfrak{p}_{2,4} \otimes x_1^{p-1}x_2^{p-2}x_3^{p-2}x_4^{p-1} \\ & 2\mathfrak{p}_{1,4}\mathfrak{p}_{3,4} \otimes x_1^{p-1}x_2^{p-3}x_3^{p-1}x_4^{p-1} \quad - \mathfrak{p}_{2,3}\mathfrak{p}_{2,4} \otimes x_1^{p-2}x_2^{p-1}x_3^{p-1}x_4^{p-2} \\ & - \mathfrak{p}_{2,4}\mathfrak{p}_{3,4} \otimes x_1^{p-2}x_2^{p-2}x_3^{p-1}x_4^{p-1} + (-\mathfrak{p}_{1,2}\mathfrak{p}_{3,4} + \mathfrak{p}_{1.4}\mathfrak{p}_{2,3}) \otimes x_1^{p-1}x_2^{p-1}x_3^{p-1}x_4^{p-4}.\end{aligned}$$

9. APPENDIX (BY J. CARLSON).
COMPUTING NONMINIMAL 2-SOCLE SUPPORT VARIETIES USING MAGMA

We reveal the results of computer calculations of the nonminimal 2-socle support variety of some modules. Our aim is to illustrate the computational method and to show some examples using modules that have been discussed in this paper. All of the calculations were made using the computer algebra system Magma [BoC].

Our first interest is the module $M = W_6 = I^6/I^8$ of Example 4.7. In that example, we showed that the module has constant 2-radical type, but not constant 2-socle type. The collection of all $U \in \operatorname{Grass}(2, \mathbb{V})$ for which the dimension of $\operatorname{Soc}_U(M)$ is more than minimal form a closed subvariety of $\operatorname{Grass}(2, \mathbb{V})$, $\operatorname{Soc}(2, \mathbb{V})_M$.

Example 9.1. Assume that $p > 3$. We recall the situation in Example 4.7. Let ζ be a primitive third root of unity in k. Let $q_{i,i} = 1$, $q_{i,j} = \zeta$, and $q_{j,i} = \zeta^{-1}$ for $1 \leq i < j \leq 4$. Then

$$S = k\langle z_1, \ldots, z_4\rangle/J$$

where J is the ideal generated by z_i^3 and by all $z_jz_i - q_{i,j}z_iz_j$ for $i, j \in 1, 2, 3, 4$. Let I be the ideal generated by the classes of $z_1, \ldots, z_4$. Let the generator x_i of kE act on $M = I^6/I^8$ by multiplication by z_i. This is a module with constant 2-radical rank but not constant 2-socle rank. Recall from the proof of 4.7 that M has dimension 14, and $\operatorname{Rad}(M)$ has dimension 4, so $M/\operatorname{Rad}(M)$ has dimension 10. The matrix of multiplication by any x_i has rank 4.

If $U \in \operatorname{Grass}(2, \mathbb{V})$, then U is spanned by two elements which we can denote $u_1 = ax_1 + bx_2 + cx_3 + dx_4$ and that $u_2 = Ax_1 + Bx_2 + Cx_3 + Dx_4$ where a, b, c, d

and A, B, C, D are elements of k. In the generic case we consider them to be indeterminants. We are interested in the maps

$$u_i : M/\operatorname{Rad}(M) \to \operatorname{Rad}(M)$$

of multiplication by u_i for $i = 1, 2$. If Y_1 is the 4×10 matrix of u_1 for this map (which is computed by taking the indicated linear combination of the matrices for $x_1, \dots, x_4$) and Y_2 is the matrix for u_2, then the intersection of the kernels of multiplication by u_1 and u_2 is the null space of the 8×10-matrix Y obtained by stacking Y_1 on top of Y_2. (Note here that we are taking a vertical join of the matrices rather than a horizontal join as we would everywhere else in the paper because the computer algebra system takes right modules rather than left modules.) Generically, this matrix has rank 8. That is, when U has minimal socle type on M, then $\operatorname{Rad}_U(M)$ has dimension 6, which counts 4 for the dimension of $\operatorname{Rad}(M)$ and another 2 for the dimension of the intersection of the kernels of u_1 and u_2 on $M/\operatorname{Rad}(M)$. The dimension of $\operatorname{Soc}_U(M)$ is more than minimal precisely when the rank of Y is less than 8.

Hence, the exercise of finding the nonminimal 2-socle support variety of M is reduced to that of finding all 8×8 minors of the matrix U. These are polynomials in a, b, c, d, A, B, C, D and to make sense of them in terms of the Grassmannian, they should be converted to Plücker coordinates. The variety is the zero locus of the converted polynomials. The Plücker coordinates are $\mathfrak{p}_{12}, \mathfrak{p}_{13}, \mathfrak{p}_{14}, \mathfrak{p}_{23}, \mathfrak{p}_{24}, \mathfrak{p}_{34}$ which are the determinants of the 2×2 minors of the basis matrix of the plane. So, for example, $\mathfrak{p}_{14} = aD - dA$. One example is the following.

Proposition 9.2. *Suppose that $p = 7$ and that M is the module given above. Then the nonminimal 2-socle support variety of M is the zero locus of the ideal generated by the elements*

$$\mathfrak{p}_{12}\mathfrak{p}_{14}\mathfrak{p}_{24},\ \mathfrak{p}_{12}\mathfrak{p}_{13}\mathfrak{p}_{23},\ \mathfrak{p}_{12}\mathfrak{p}_{14}\mathfrak{p}_{34},\ \mathfrak{p}_{23}\mathfrak{p}_{24}\mathfrak{p}_{34},\ \mathfrak{p}_{13}\mathfrak{p}_{14}\mathfrak{p}_{34},$$
$$\mathfrak{p}_{12}\mathfrak{p}_{14}\mathfrak{p}_{23},\ \mathfrak{p}_{13}\mathfrak{p}_{14}\mathfrak{p}_{34},\ \mathfrak{p}_{14}\mathfrak{p}_{23}\mathfrak{p}_{34},\ \mathfrak{p}_{12}\mathfrak{p}_{23}\mathfrak{p}_{34}.$$

With a little work we can interpret the zero locus in terms of the geometric model for the Grassmannian. Thinking of a point in the zero locus as a plane in four space we get that it consists of planes satisfying any one of the conditions below. For notation, let V_{ij} be the two dimensional subspace of k^4 spanned by the i^{th} and j^{th} coordinate vectors. So V_{23} consists of all vectors of the form $(0, a, b, 0)$ for $a, b \in k$. Then a closed point (plane defined over k) is in the variety of the proposition if and only if it satisfies one of the following:

- it contains one of the coordinate vectors, or
- it has a basis u_1, u_2 where $u_1 \in V_{12}$ and $u_2 \in V_{34}$, or
- it has a basis u_1, u_2 where $u_1 \in V_{14}$ and $u_2 \in V_{23}$.

At first it may seem surprising that the description is not symmetric. That is, it does not include the case that $u_1 \in V_{13}$ and $u_2 \in V_{24}$. However, we should recall that the algebra S is not symmetric. There is no automorphism that interchanges the variables.

Some similar calculations have been made in other cases. The identical result was obtained when $p = 13$. We conjecture that Proposition 9.2 is true for all primes $p > 3$.

We also got a very similar outcome in the case that $p = 3$, $s = 4$ (That is where relations satisfied by the variables of S consist of $z_i^4 = 0$ and $z_i z_j = q z_j z_i$

for $i > j$ and q a primitive 4^{th} root of 1) and we consider the module $M = \operatorname{Rad}^{10}(S)/\operatorname{Rad}^{12}(S)$. For the case that E is an elementary abelian group of rank 5, $p = 7$ and $M = \operatorname{Rad}^{8}(S)/\operatorname{Rad}^{10}(S)$, the variety again appears to be generated by monomials which are the products of three distinct Plücker coordinates. This case was not fully completed in that not all of the relations were converted to Plücker coordinates. However, the experimental evidence suggests that the closed points in the variety consist of planes which contain a coordinate vector or have a basis u_1, u_2 where u_1 is in the subspace V_{ij} for $\{i, j\}$ one of the sets $\{1, 2\}, \{1, 5\}, \{2, 3\}, \{3, 4\}$ or $\{4, 5\}$ and in each case u_2 is the subspace spanned by the other three coordinate vectors.

Finally, we can also experiment with changing the commutativity relations in the ring S defined as above. These are the relations with the form $z_j z_i = q_{ij} z_i z_j$ for $j > i$. If $q_{ij} = 1$ for all i and j, so that S is commutative, then the module M has constant 2-socle type. In another experiment, we made random choices of the elements q_{ij} in the field $k = \mathbb{F}_7$. For one such choice the module M has a nonminimal 2-socle support variety which is the zero locus of the ideal generated by the polynomials $\mathfrak{p}_{12}\mathfrak{p}_{13}\mathfrak{p}_{23}$, $\mathfrak{p}_{12}\mathfrak{p}_{13}\mathfrak{p}_{24}$, $\mathfrak{p}_{12}\mathfrak{p}_{14}\mathfrak{p}_{24}$, $\mathfrak{p}_{12}\mathfrak{p}_{23}\mathfrak{p}_{24}$, $\mathfrak{p}_{12}\mathfrak{p}_{23}\mathfrak{p}_{34}$, $\mathfrak{p}_{12}\mathfrak{p}_{24}\mathfrak{p}_{34}$, $\mathfrak{p}_{13}\mathfrak{p}_{23}\mathfrak{p}_{24}$, $\mathfrak{p}_{14}\mathfrak{p}_{23}$, $\mathfrak{p}_{23}\mathfrak{p}_{24}\mathfrak{p}_{34}$ and $\mathfrak{p}_{13}\mathfrak{p}_{24}$. This variety includes all planes that contain a coordinate vector (except that if it is the second coordinate vector, then the other spanning vector must have zero in one of its other coordinates). It also included all planes contained in the subspace V_{134}.

We end with the remark that several other examples similar to Example 4.7 were checked for constant 2-socle rank. In every experiment 100 random planes $U \in \operatorname{Grass}(2, \mathbb{V})$ were chosen and the value of $d = \dim \operatorname{Soc}_U(M) - \dim \operatorname{Rad}(M)$ was calculated for each. Here $M = W_a(s, \{q_{i,j}\})$, with $q_{i,j} = \zeta_s$, a primitive s^{th} root of unity. For example, for $k = \mathbb{F}_7$, the value of d was calculated in the cases for which $n = 4$, $s = 3$, $a = 4, 5, 6$ and $n = 5$, $s = 3$ and $a = 6, 7, 8$. For $k = \mathbb{F}_5$, d was calculated for $n = 4$, $s = 4$, $a = 6, 7, 8, 9$. In all of these and in other cases, the module $M = W_a = I^a/I^{a+2}$, failed to have constant 2-socle type, even though it has constant Jordan type and constant 2-radical type. With this evidence in hand, we conjecture that M never has constant 2-socle type for $(n - r)(s - 1) \leq a \leq n(s - 1) - 2$.

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