ELEMENTARY SUBALGEBRAS OF LIE ALGEBRAS

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Abstract. We initiate the investigation of the projective variety $\mathbb{E}(r, \mathfrak{g})$ of elementary subalgebras of dimension r of a (p-restricted) Lie algebra $\mathfrak{g}$ for some $r \geq 1$ and demonstrate that this variety encodes considerable information about the representations of $\mathfrak{g}$. For various choices of $\mathfrak{g}$ and r, we identify the geometric structure of $\mathbb{E}(r, \mathfrak{g})$. We show that special classes of (restricted) representations of $\mathfrak{g}$ lead to algebraic vector bundles on $\mathbb{E}(r, \mathfrak{g})$. For $\mathfrak{g} = \mathrm{Lie}(G)$ the Lie algebra of an algebraic group G, rational representations of G enable us to realize familiar algebraic vector bundles on G-orbits of $\mathbb{E}(r, \mathfrak{g})$.

Contents

0. Introduction

We say that a Lie subalgebra $\epsilon \subset \mathfrak{g}$ of a p-restricted Lie algebra $\mathfrak{g}$ over a field k of characteristic p is *elementary* if it is abelian with trivial p-restriction. Thus, if ϵ has dimension r, then $\epsilon \simeq \mathfrak{g}_a^{\oplus r}$ where $\mathfrak{g}_a$ is the one-dimensional Lie algebra of the additive group $\mathbb{G}_a$. This paper is dedicated to the study of the projective variety $\mathbb{E}(r, \mathfrak{g})$ of elementary subalgebras of $\mathfrak{g}$ for some positive integer r and its relationship to the representation theory of $\mathfrak{g}$.

For $r = 1$, $\mathbb{E}(1, \mathfrak{g})$ is the projectivization of the p-nilpotent cone $\mathcal{N}_p(\mathfrak{g})$; more generally, $\mathbb{E}(r, \mathfrak{g})$ is the orbit space under the evident GL_r-action on the variety of r-tuples of commuting, linearly independent, p-nilpotent elements of $\mathfrak{g}$. Our investigation of $\mathbb{E}(r, \mathfrak{g})$ and its close connections with the representation theory of $\mathfrak{g}$ can be traced back through the work of many authors to the fundamental papers of Daniel Quillen who established the important geometric role that elementary abelian p-subgroups play in the cohomology theory of finite groups [Q72].

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We have been led to the investigation of $\mathbb{E}(r,\mathfrak{g})$ through considerations of cohomology and modular representations of finite group schemes. Recall that the structure of a restricted representation of $\mathfrak{g}$ on a k vector space is equivalent to the structure of a module for the restricted enveloping algebra $\mathfrak{u}(\mathfrak{g})$ of $\mathfrak{g}$ (a cocommutative Hopf algebra over k of dimension $p^{\dim(\mathfrak{g})}$). A key precursor of this present work is the identification of the spectrum of the cohomology algebra $\mathrm{H}^*(\mathfrak{u}(\mathfrak{g}),k)$ with the p-nilpotent cone $\mathcal{N}_p(\mathfrak{g})$ achieved in [FP83], [Jan86], [AJ84], [SFB2]. It is interesting to observe that the theory of cohomological support varieties for restricted $\mathfrak{g}$-representations (i.e., $\mathfrak{u}(\mathfrak{g})$-modules) as considered first in [FP86] has evolved into the more geometric study of π-points as introduced by the second and third authors in [FP07]. This latter work closed a historical loop, relating cohomological considerations to earlier work on cyclic shifted subgroups as investigated by Everett Dade [D78] and the first author [C83].

For $r > 1$ and $\mathfrak{g}$ the Lie algebra of an algebraic group G, $\mathbb{E}(r,\mathfrak{g})$ is closely related to the spectrum of cohomology of the r-th Frobenius kernel $G_{(r)}$ of G (see [SFB1] for classical simple groups G; [M02], [S12] for more general types). Work of Alexander Premet concerning the variety of commuting, nilpotent pairs in $\mathfrak{g}$ [P03] gives considerable information about $\mathbb{E}(2,\mathfrak{g})$. Much less is known for larger r's, although work in progress indicates the usefulness of considering the representation theory of $\mathfrak{g}$ when investigating the topology of $\mathbb{E}(r,\mathfrak{g})$.

Although we postpone consideration of Lie algebras over fields of characteristic 0, we remark that much of the formalism of Sections 1, 3, 5, and 6 and many of the examples in Sections 2 and 6 are valid (and often easier) in characteristic 0. On the other hand, some of our results and examples, particularly in Section 4 and Section 7, require that k be of positive characteristic.

We consider numerous examples of restricted Lie algebras $\mathfrak{g}$ in Section 1, and give some explicit computations of $\mathbb{E}(r,\mathfrak{g})$. Influenced by the role of maximal elementary abelian p-subgroups in the study of the cohomology of finite groups, we are especially interested in examples of $\mathbb{E}(r,\mathfrak{g})$ considered in Section 2 for which r is maximal among the dimensions of elementary subalgebras of $\mathfrak{g}$. For simple Lie algebras over a field of characteristic 0, Anatoly Malcev determined this maximal dimension [Mal45] which is itself an interesting invariant of $\mathfrak{g}$. Our computations verify that $\mathbb{E}(n^2,\mathfrak{gl}_{2n})$ is isomorphic to the Grassmann variety of n planes in an $2n$-dimensional k-vector space and $\mathbb{E}\left(\frac{(n+1)n}{2},\mathfrak{sp}_{2n}\right)$ is isomorphic to the Lagrangian Grassmannian of isotropic n-planes in a $2n$-dimensional symplectic vector space. More generally, some computations are possible even for "non-classical" restricted Lie algebras not arising from algebraic groups.

We offer several explicit motivations for considering $\mathbb{E}(r,\mathfrak{g})$ in addition to the fact that these projective varieties are of intrinsic interest. These motivations are pursued in Sections 3 through 7 where (restricted) representations of $\mathfrak{g}$ come to the fore.

- The varieties $\mathbb{E}(r,\mathfrak{g})$ are the natural ambient varieties in which to define generalized support varieties for restricted representations of $\mathfrak{g}$ (as in [FP10]).

- Coherent sheaves on $\mathbb{E}(r,\mathfrak{g})$ are naturally associated to arbitrary (restricted) representations of $\mathfrak{g}$.

• For certain representations of $\mathfrak{g}$ with "constant properties", the associated coherent sheaves are algebraic vector bundles on $\mathbb{E}(r,\mathfrak{g})$. The classes of such special representations merit further study. They generalize modules of constant Jordan type and constant rank introduced in [CFP08], [FP10] and investigated in a recent series of papers by several authors (see, for example, [Ba11], [B10], [Ben2], [BP12], [CF09], [CFS11], [F09], and others).

• Since $\mathbb{E}(r,\mathfrak{g})$ is a projective variety with interesting geometry, the explicit construction of algebraic vector bundles on $\mathbb{E}(r,\mathfrak{g})$ from representations of $\mathfrak{g}$ should offer intriguing new examples.

• Calculations postponed to a forthcoming paper demonstrate how the investigation of the Zariski topology on $\mathbb{E}(r,\mathfrak{g})$ can be informed by the representation theory of $\mathfrak{g}$.

The isomorphism type of the restriction ϵ^*M of a $\mathfrak{u}(\mathfrak{g})$-module M to an elementary subalgebra ϵ of dimension 1 is given by its Jordan type, a partition of the dimension of M. On the other hand, the classification of indecomposable modules of an elementary subalgebra of dimension $r > 1$ is a wild problem (except in the special case in which $r = 2 = p$), so that the isomorphism types of ϵ^*M for $\epsilon \in \mathbb{E}(r,\mathfrak{g})$ do not form convenient invariants of a $\mathfrak{u}(\mathfrak{g})$-module M. Following the approach undertaken in [CFP12], we consider the dimensions of the radicals and socles of such restrictions, $\dim \operatorname{Rad}^j(\epsilon^*M)$ and $\dim \operatorname{Soc}^j(\epsilon^*M)$, for $\epsilon \in \mathbb{E}(r,\mathfrak{g})$ and any j with $1 \le j \le (p-1)r$. As we establish in Section 3, these dimensions give upper/lower semi-continuous functions on $\mathbb{E}(r,\mathfrak{g})$. In particular, they lead to "generalized rank varieties" refining those introduced in [FP10]. We achieve some computations of these generalized rank varieties $\mathbb{E}(r,\mathfrak{g})_M$ for $\mathfrak{u}(\mathfrak{g})$-modules M which are either L_ζ modules or induced modules.

One outgrowth of the authors' interpretation of cohomological support varieties in terms of π-points (as in [FP07]) is the identification of the interesting classes of modules of constant Jordan type and constant j-rank for $1 \le j < p$ (see [CFP08]). As already seen in [CFP12], this has a natural analogue in the context of elementary subalgebras of dimension $r > 1$. In Section 4, we give examples of $\mathfrak{u}(\mathfrak{g})$-modules of constant (r,j)-radical rank and of constant (r,j)-socle rank, typically adapting constructions for modules of constant Jordan type. We anticipate that the investigation of such modules which are not equipped with large groups of symmetries may provide algebraic vector bundles with interesting properties.

In Section 5, we consider locally closed subvarieties $X \subset \mathbb{E}(r,\mathfrak{g})$. We associate to any $\mathfrak{u}(\mathfrak{g})$-module M various coherent sheaves on X: for each $1 \le j \le (p-1)r$, we construct image and kernel sheaves $\mathcal{I}m^{j,X}(M)$ and $\mathcal{K}er^{j,X}(M)$. These coherent sheaves are presented in terms of local data in Theorem 5.8 and shown in Theorem 5.18 to be equivalent to sheaves arising from equivariant descent. The fibers of these sheaves are related in Proposition 5.10 to the radicals and socles of ϵ^*M for $\epsilon \in \mathbb{E}(r,\mathfrak{g})$. We use the notation $\mathcal{I}m^j(M)$ and $\mathcal{K}er^j(M)$ to denote $\mathcal{I}m^{j,X}(M)$ and $\mathcal{K}er^{j,X}(M)$ in the special case in which $X = \mathbb{E}(r,\mathfrak{g})$. If M has constant (r,j)-radical rank for some j, then $\mathcal{I}m^j(M)$ is an algebraic vector bundle on $\mathbb{E}(r,\mathfrak{g})$; similarly, if M has constant (r,j)-socle rank, then $\mathcal{K}er^j(M))$ is an algebraic vector bundle.

If $X = G \cdot \epsilon \subset \mathbb{E}(r,\mathfrak{g})$ is a G-orbit and M a rational G-module, these coherent sheaves are G-equivariant algebraic vector bundles on X. For such a G-orbit X, we identify the vector bundle $\mathcal{I}m^{j,X}(M)$ (respectively, $\mathcal{K}er^{j,X}(M)$) in Theorem 6.5 as the vector bundle obtained by inducing to G the representation of $H = \mathrm{Stab}_G(\epsilon) \subset G$ on $\mathrm{Rad}^j(\epsilon^* M)$ (respectively, $\mathrm{Soc}^j(\epsilon^* M)$). Using this identification and the examples discussed in earlier sections, we realize many familiar vector bundles as image and kernel bundles associated to rational G-modules.

The last Section 7 is devoted to the vector bundles which arise from the semi-direct product of an algebraic group H with a vector group associated to a rational representation W of H. We consider image and kernel bundles for (non-rational) representations of $\mathfrak{g}_{W,H} = \mathrm{Lie}(\mathbb{W} \rtimes H)$. Many of the examples of our recent paper [CFP12] are reinterpreted and extended using this construction. As we show in Theorem 7.9 and its corollary, all homogeneous bundles on H-orbits inside $\mathrm{Grass}(r, W) \subset \mathbb{E}(r, \mathfrak{g}_{W,H})$ are realized as image bundles in this manner.

Throughout, k is an algebraically closed field of characteristic $p > 0$. All Lie algebras $\mathfrak{g}$ considered in this paper are assumed to be finite dimensional over k and p-restricted; a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is assumed to be closed under p-restriction. Without explicit mention to the contrary, all $\mathfrak{u}(\mathfrak{g})$-modules are finite dimensional.

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1. The subvariety $\mathbb{E}(r, \mathfrak{g}) \subset \mathrm{Grass}(r, \mathfrak{g})$

We begin by formulating the definition of $\mathbb{E}(r, \mathfrak{g})$ of the variety of elementary subalgebras of $\mathfrak{g}$ and establishing the existence of a natural closed embedding of $\mathbb{E}(r, \mathfrak{g})$ into the projective variety $\mathrm{Grass}(r, \mathfrak{g})$ of r-planes of the underlying vector space of $\mathfrak{g}$. Once these preliminaries are complete, we introduce various examples which reappear frequently.

Let V be an n-dimensional vector space and $r < n$ a positive integer. We consider the projective variety $\mathrm{Grass}(r, V)$ of r-planes of V. We choose a basis for V, $\{v_1, \ldots, v_n\}$; a change of basis has the effect of changing the Plücker embedding (1.1.2) by a linear automorphism of $\mathbb{P}(\Lambda^r(V))$. We represent a choice of basis $\{u_1, \ldots, u_r\}$ for an r-plane $U \subset V$ by an $n \times r$-matrix $(a_{i,j})$, where $u_j = \sum_{i=1}^n a_{i,j} v_i$. Let $\mathbb{M}_{n,r}^o \subset \mathbb{M}_{n,r}$ denote the open subvariety of the affine space $\mathbb{M}_{n,r} \simeq \mathbb{A}^{nr}$ consisting of those $n \times r$ matrices of rank r and set $p : \mathbb{M}_{n,r}^o \to \mathrm{Grass}(r, V)$ equal to the map sending a rank r matrix $(a_{i,j})$ to the r-plane spanned by $\{\sum_{i=1}^n a_{i,1} v_i, \ldots, \sum_{i=1}^n a_{i,r} v_i\}$.

We summarize a few useful, well known facts about $\mathrm{Grass}(r, V)$. Note that there is a natural (left) action of GL_r on $\mathbb{M}_{n,r}$ via multiplication by the inverse on the right.

Proposition 1.1. *For any subset $\Sigma \subset \{1, \ldots, n\}$ of cardinality r, set $U_\Sigma \subset \mathrm{Grass}(r, V)$ to be the subset of those r-planes $U \subset V$ with a representing $n \times r$ matrix A_U whose $r \times r$ minor indexed by Σ (denoted by $\mathfrak{p}_\Sigma(A_U)$) is non-zero.*

- *$p : \mathbb{M}_{n,r}^o \to \mathrm{Grass}(r, V)$ is a principal GL_r-torsor, locally trivial in the Zariski topology.*
- *Sending an r-plane $U \in U_\Sigma$ to the unique $n \times r$-matrix A_U^Σ whose Σ-submatrix (i.e., the $r \times r$-submatrix whose rows are those of A_U^Σ indexed by elements of Σ) is the identity determines a section of p over U_Σ:*

$$s_\Sigma : U_\Sigma \to \mathbb{M}_{r,n}^o \tag{1.1.1}$$

- *The Plücker embedding*

$$\mathfrak{p} : \mathrm{Grass}(r, V) \hookrightarrow \mathbb{P}(\Lambda^r(\mathbb{V})), \quad U \mapsto [\mathfrak{p}_\Sigma(A_U)] \tag{1.1.2}$$

  sending $U \in U_\Sigma$ to the $\binom{n}{r}$-tuple of $r \times r$-minors of A_U^Σ is a closed immersion of algebraic varieties.
- *$U_\Sigma \subset \mathrm{Grass}(r, V)$ is a Zariski open subset, the complement of the zero locus of $\mathfrak{p}_\Sigma$, and is isomorphic to $\mathbb{A}^{r(n-r)}$.*

Elementary subalgebras as defined below play the central role in what follows.

Definition 1.2. An elementary subalgebra $\epsilon \subset \mathfrak{g}$ of dimension r is a Lie subalgebra of dimension r which is commutative and has p-restriction equal to 0. We define

$$\mathbb{E}(r, \mathfrak{g}) \;=\; \{\epsilon \subset \mathfrak{g} : \; \epsilon \text{ elementary subalgebra of dimension } r\}$$

We denote by $\mathcal{N}_p(\mathfrak{g}) \subset \mathfrak{g}$ the closed subvariety of p-nilpotent elements (i.e., $x \in \mathfrak{g}$ with $x^{[p]} = 0$), by $\mathcal{N}_p^r(\mathfrak{g}) \subset (\mathcal{N}_p(\mathfrak{g}))^{\times r}$ the variety of r-tuples of p-nilpotent, pairwise commuting elements of $\mathfrak{g}$, and by $\mathcal{N}_p^r(\mathfrak{g})^o \subset \mathcal{N}_p^r(\mathfrak{g})$ the open subvariety of linearly independent r-tuples of p-nilpotent, pairwise commuting elements of $\mathfrak{g}$.

Proposition 1.3. *Let $\mathfrak{g}$ be a Lie algebra of dimension n. Forgetting the Lie algebra structure of $\mathfrak{g}$ and viewing $\mathfrak{g}$ as a vector space, we consider the projective variety* $\mathrm{Grass}(r, \mathfrak{g})$ *of r-planes of $\mathfrak{g}$ for some r, $1 \le r \le n$. There exists a natural cartesian square*

$$\begin{tikzcd} \mathcal{N}_p^r(\mathfrak{g})^o \arrow[r, hook] \arrow[d] & \mathbb{M}_{n,r}(\mathfrak{g})^o \arrow[d, "p"] \\ \mathbb{E}(r, \mathfrak{g}) \arrow[r, hook] & \mathrm{Grass}(r, \mathfrak{g}) \end{tikzcd} \tag{1.3.1}$$

whose vertical maps are GL_r-torsors locally trivial for the Zariski topology and whose horizontal maps are closed immersions. In particular, $\mathbb{E}(r, \mathfrak{g})$ has a natural structure as a projective algebraic variety.

If G is a linear algebraic group with $\mathfrak{g} = \mathrm{Lie}(G)$, then $\mathbb{E}(r, \mathfrak{g}) \hookrightarrow \mathrm{Grass}(r, \mathfrak{g})$ is a G-stable embedding.

Proof. The horizontal maps of (1.3.1) are the evident inclusions, the left vertical map is the restriction of p. Clearly, (1.3.1) is cartesian; in particular, $\mathcal{N}_p^r(\mathfrak{g})^o \subset \mathbb{M}_{n,r}^o$ is stable under the action of GL_r.

To prove that $\mathbb{E}(r, \mathfrak{g}) \subset \mathrm{Grass}_r(\mathfrak{g})$ is closed, it suffices to verify for each Σ that $(\mathbb{E}(r, \mathfrak{g}) \cap U_\Sigma) \subset U_\Sigma$ is a closed embedding. The restriction of (1.3.1) above U_Σ takes the form

$$\begin{tikzcd} \mathcal{N}_p^r(\mathfrak{g})^o \cap p^{-1}(U_\Sigma) \arrow[r] \arrow[d] & p^{-1}(U_\Sigma) \arrow[r, "\sim"] \arrow[d, "p"] & U_\Sigma \times GL_r \arrow[d, "pr"] \\ \mathbb{E}(r, \mathfrak{g}) \cap U_\Sigma \arrow[r] & U_\Sigma \arrow[r, equal] & U_\Sigma \end{tikzcd} \tag{1.3.2}$$

Consequently, to prove that $\mathbb{E}(r, \mathfrak{g}) \subset \mathrm{Grass}_r(\mathfrak{g})$ is closed and that $\mathcal{N}_p^r(\mathfrak{g})^o \to \mathbb{E}(r, \mathfrak{g})$ is a GL_r-torsor which is locally trivial for the Zariski topology it suffices to prove that $\mathcal{N}_p^r(\mathfrak{g})^o \subset \mathbb{M}_{n,r}^o$ is closed.

It is clear that $\mathcal{N}_p^r(\mathfrak{g}) \subset \mathbb{M}_{n,r}$ is a closed subvariety since it is defined by the vanishing of the Lie bracket and the p-operator $(-)^{[p]}$ both of which can be expressed as polynomial equations on the matrix coefficients. Hence, $\mathcal{N}_p^r(\mathfrak{g})^o = \mathcal{N}_p^r(\mathfrak{g}) \cap \mathbb{M}_{n,r}^o$ is closed in $\mathbb{M}_{n,r}^o$.

If $\mathfrak{g} = \mathrm{Lie}(G)$, then the (diagonal) adjoint action of G on $n \times r$-matrices $\mathfrak{g}^{\oplus r}$ sends a matrix whose columns pair-wise commute and which satisfies the condition that $(-)^{[p]}$ vanishes on these columns to another matrix satisfying the same conditions (since $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ preserves both the Lie bracket and the p^{th}-power). Thus, $\mathbb{E}(r, \mathfrak{g})$ is G-stable. □

Remark 1.4. Let V be a k-vector space of dimension n. Consider $\mathbb{V} \equiv \mathrm{Spec}\, S^*(V^\#) \simeq \mathbb{G}_a^{\times n}$, the vector group on the (based) vector space V. Then $\mathrm{Lie}(\mathbb{V}) \simeq \mathfrak{g}_a^{\oplus n}$ and we have an isomorphism of algebras

$$\mathfrak{u}(\mathrm{Lie}\,\mathbb{V}) \;\simeq\; \mathfrak{u}(\mathfrak{g}_a^{\oplus n}) \;\simeq\; k[t_1, \ldots, t_n]/(t_1^p, \ldots, t_n^p).$$

Let $E = (\mathbb{Z}/p)^{\times n}$ be an elementary abelian p-group of rank n and choose an embedding of V into the radical $\mathrm{Rad}(kE)$ of the group algebra of E such that the composition with the projection to $\mathrm{Rad}(kE)/\mathrm{Rad}^2(kE)$ is an isomorphism. This choice determines an isomorphism

$$\mathfrak{u}(\mathrm{Lie}(\mathbb{V})) \xrightarrow{\sim} kE.$$

With this identification, the investigations of [CFP12] will be seen to be a special case of considerations of this paper.

Example 1.5. For any (finite dimensional, p-restricted) Lie algebra,

$$\mathbb{E}(1, \mathfrak{g}) \;\simeq\; \mathrm{Proj}\, k[\mathcal{N}_p(\mathfrak{g})]$$

as shown in [SFB2], where $k[\mathcal{N}_p(\mathfrak{g})]$ is the (graded) coordinate algebra of the p-null cone of $\mathfrak{g}$. If G is reductive with $\mathfrak{g} = \mathrm{Lie}(G)$ and if p is good for G, then $\mathcal{N}_p(\mathfrak{g})$ is irreducible and equals the G-orbit $G \cdot \mathfrak{u}$ of the nilpotent radical of a specific parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ (see [NPV02, 6.3.1]).

Example 1.6. Let G be a connected reductive algebraic group, let $\mathfrak{g} = \mathrm{Lie}\, G$, and assume that p is good for G. As shown by A. Premet in [P03], $\mathcal{N}_p^2(\mathfrak{g})$ is equidimensional with irreducible components enumerated by the distinguished nilpotent orbits of $\mathfrak{g}$; in particular, $\mathcal{N}_p^2(\mathfrak{gl}_n)$ is irreducible. This easily implies that $\mathbb{E}(2, \mathfrak{g})$ is an equidimensional variety, irreducible in the special case $\mathfrak{g} = \mathfrak{gl}_n$. Since $\dim \mathbb{E}(2, \mathfrak{g}) = \dim \mathcal{N}_p^2(\mathfrak{g}) - \dim \mathrm{GL}_2$, $\dim \mathbb{E}(2, \mathfrak{g}) = \dim[G, G] - 4$. In particular, $\mathbb{E}(2, \mathfrak{gl}_n)$ has dimension $n^2 - 5$ for $p > n$.

Example 1.7. Let $\mathfrak{u}_3 \subset \mathfrak{gl}_3$ denote the Lie subalgebra of strictly upper triangular matrices and take $r = 2$. Then a 2-dimensional elementary Lie subalgebra $\epsilon \subset \mathfrak{u}_3$ is spanned by $E_{1,3}$ and another element $X \in \mathfrak{u}_3$ not a scalar multiple of $E_{1,3}$. We can further normalize the basis of ϵ by subtracting a multiple of $E_{1,3}$ from X, so that $X = a_{1,2}E_{1,2} + a_{2,3}E_{2,3}$. Thus, 2-dimensional elementary Lie subalgebras $\epsilon \subset \mathfrak{u}$ are in parametrized by points $\langle a_{1,2}, a_{2,3} \rangle \in \mathbb{P}^1$, so that $\mathbb{E}(2, \mathfrak{u}_3) \simeq \mathbb{P}^1$.

In this case, $\mathfrak{u}_3$ is the Lie algebra of the unipotent radical of the Borel subgroup $B_3 \subset \mathrm{GL}_3$ of upper triangular matrices. With respect to the action of B_3 on $\mathbb{E}(2, \mathfrak{u}_3)$, $\mathbb{E}(2, \mathfrak{u}_3)$ is the union of an open dense orbit consisting of regular nilpotent elements of the form $a_{1,2}E_{1,2} + a_{2,3}E_{2,3}$, with $a_{1,2} \neq 0 \neq a_{2,3}$; and two closed orbits. The open orbit is isomorphic to the 1-dimensional torus $\mathbb{G}_m \subset \mathbb{P}^1$ and the two closed orbits are single points $\{0\}, \{\infty\}$.

Remark 1.8. The action of B_3 on $\epsilon \in \mathbb{E}(2, \mathfrak{u}_3)$ is implicitly taken to be the restriction of the adjoint action of GL_3 on $\mathfrak{gl}_3$ (which has the property that B_3 stabilizes $\mathfrak{u}_3$). In the examples that follow, we consistently use the adjoint action of an algebraic group G on $\mathbb{E}(r, \mathfrak{g})$ where $\mathfrak{g} = \mathrm{Lie}(G)$.

Example 1.9. We consider the algebraic group $G = \mathrm{GL}_n$ and some $r,\ 1 \leq r < n$. Let $\mathfrak{u}_{r,n-r} \subset \mathfrak{gl}_n$ denote the Lie subalgebra of $n \times n$ matrices $(a_{i,j})$ with $a_{i,j} = 0$ unless $1 \leq i \leq r,\ r+1 \leq j \leq n$. Then $\mathfrak{u}_{r,n-r} \subset \mathfrak{gl}_n$ is an elementary subalgebra of dimension $r(n-r)$. The argument given in [MP87, §5] applies in our situation to show that $\mathfrak{u}_{r,n-r}$ is a maximal elementary subalgebra (that is, not contained in any other elementary subalgebra).

Let $X \subset \mathbb{E}(r(n-r), \mathfrak{gl}_n)$ denote the GL_n-orbit of $\mathfrak{u}_{r,n-r}$. Let P_r be the standard parabolic subgroup of GL_n defined by the equations $a_{i,j} = 0$ for $i > r, j \leq n-r$. Since P_r is the stabilizer of $\mathfrak{u}_{r,n-r}$ under the adjoint action of GL_n, $X = G \cdot \mathfrak{u}_{r,n-r} \simeq \mathrm{GL}_n / P_r \simeq \mathrm{Grass}_{n,r}$. Since X is projective, it is a closed GL_n-stable subvariety of $\mathbb{E}(r(n-r), \mathfrak{gl}_n)$.

We next give examples of restricted Lie algebras which are not the Lie algebras of algebraic groups.

Example 1.10. Let $\phi : \mathfrak{gl}_{2n} \to k$ be a semi-linear map (so that $\phi(av) = a^p \phi(v)$), and consider the extension of p-restricted Lie algebras, split as an extension of Lie algebras (see [FP83, 3.11]):

$$0 \to k \to \widetilde{\mathfrak{gl}}_{2n} \to \mathfrak{gl}_{2n} \to 0, \quad (b, x)^{[p]} = (\phi(x), x^{[p]}). \tag{1.10.1}$$

Then $\mathbb{E}(n^2+1, \widetilde{\mathfrak{gl}}_{2n})$ can be identified with the subvariety of $\mathrm{Grass}_{2n,n}$ consisting of those elementary subalgebras $\epsilon \subset \mathfrak{gl}_{2n}$ of dimension n^2 such that the restriction of ϕ to ϵ is 0 (or, equivalently, such that ϵ is contained in the kernel of ϕ).

In Section 7, we shall see that the following semi-direct product construction leads to many of the examples of algebraic vector bundles obtained in [CFP12].

Example 1.11. (1). Consider the general linear group GL_n and let V be the defining representation. Let $\mathbb{V}$ be the vector group associated to V as in Remark 1.4. We set

$$G_{1,n} \stackrel{\mathrm{def}}{=\!=} \mathbb{V} \rtimes \mathrm{GL}_n, \qquad g_{1,n} \stackrel{\mathrm{def}}{=\!=} \mathrm{Lie}\, G_{1,n} \tag{1.11.1}$$

Any subspace $\epsilon \subset V$ of dimension $r < n$ can be considered as an elementary subalgebra of $g_{1,n}$. Moreover, the $G_{1,n}$-orbit of $\epsilon \in \mathbb{E}(r, \mathfrak{g}_{1,n})$ can be identified with the Grassmannian $\mathrm{Grass}(r, V)$ of all r-planes in V.

(2). More generally, let H be an algebraic group, W be a rational representation of H, and $\mathbb{W}$ be the vector group associated to W. Let $G \equiv \mathbb{W} \rtimes H$, and let $\mathfrak{h} = \mathrm{Lie}\, H$. A subspace $\epsilon \subset W$ of dimension $r < \dim W$ can be viewed as an elementary subalgebra of $\mathfrak{h}$. Moreover, the G-orbit of $\epsilon \in \mathbb{E}(r, \mathfrak{h})$ can be identified with the H-orbit of ϵ in $\mathrm{Grass}_r(W)$.

We conclude this section by giving a straightforward way to obtain additional computations from known computations of $\mathbb{E}(r, \mathfrak{g})$. The proof is immediate.

Proposition 1.12. *Let $\mathfrak{g}_1, \mathfrak{g}_2, \ldots, \mathfrak{g}_s$ be finite dimensional p-restricted Lie algebras and let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s$. Then there is a natural morphism of projective varieties*

$$\mathbb{E}(r_1, \mathfrak{g}_1) \times \cdots \times \mathbb{E}(r_s, \mathfrak{g}_s) \to \mathbb{E}(r, \mathfrak{g}), \quad r = \sum r_i, \tag{1.12.1}$$

sending $(\epsilon_1 \subset \mathfrak{g}_1, \ldots, \epsilon_s \subset \mathfrak{g}_s)$ to $\epsilon_1 \oplus \cdots \oplus \epsilon_s \subset \mathfrak{g}$. Moreover, if r_i is the maximum of the dimensions of the elementary subalgebras of $\mathfrak{g}_i$ for each $i, 1 \leq i \leq s$, then this morphism is an isomorphism.

Corollary 1.13. *In the special case of Proposition 1.12 in which each $\mathfrak{g}_i \simeq \mathfrak{sl}_2$, $r_1 = \cdots = r_s = 1$, (1.12.1) specializes to*

$$(\mathbb{P}^1)^{\times r} \;\simeq\; \mathbb{E}(r, \mathfrak{sl}_2^{\oplus r}).$$

Proof. This follows from the fact that $\mathbb{E}(1, \mathfrak{sl}_2) = \operatorname{Proj} k[\mathcal{N}(\mathfrak{sl}_2)] \simeq \mathbb{P}^1$ (see, for example, [FP11]). $\square$

2. Elementary subalgebras of maximal dimension

In this section, we explicitly determine $\mathbb{E}(r, \mathfrak{g})$ for several families of p-restricted Lie algebras $\mathfrak{g}$ and r the maximal dimension of an elementary subalgebra of $\mathfrak{g}$. In Proposition 2.3 we establish that $\mathbb{E}(n, \mathfrak{g})$ for an extraspecial Lie algebra of dimension $2n-1$ is the Lagrangian Grassmannian $\mathrm{LG}_{n-1,n-1}$. In Theorem 2.13, we get a similar answer for the variety of elementary subalgebras of maximal dimension for $\mathfrak{sp}_{2n}$. Similarly, in Theorems 2.9 and 2.10 we identify these varieties for $\mathfrak{sl}_n$ with Grassmannians corresponding to maximal parabolics. In the last example of this section we extend the calculation for the special linear Lie algebra to its maximal parabolic. As an immediate application, we compute that $\mathbb{E}(r, \mathfrak{g}_{1,n})$ where $\mathfrak{g}_{1,n}$ is as defined in Example 1.11(1) is the disjoint union of two Grassmannians of different dimension for $r = n(n+1)/2$. We also establish some general - and well-known to the experts - results on cominuscule parabolics that will be used in Section 6.

As the study of maximal abelian subalgebras in complex semi-simple Lie algebras has a long history we feel that we owe the reader a few comments connecting some of the existing literature to our own investigations in the modular case. The dimensions of maximal abelian subalgebras of a complex simple Lie algebra are known thanks to the classical work of Malcev [Mal45]. It appears that the general linear case was first considered by Schur at the turn of last century [Sch05]. Determination of the varieties of abelian subalgebras of maximal dimension also has a large footprint in the literature although we were unable to find a reference that would pin down precisely the calculation of $\mathbb{E}(r, \mathfrak{g})$. The work that possibly comes closest to our interests is the one of Barry [B79] who considered a similar problem in the context of Chevalley groups. It was pointed out to us by S. Mitchell that the idea of the proof of Theorems 2.9, 2.10, and 2.13 is very similar to the one used in [B79] for the Chevalley group case (and also present in [MP87] for the general linear group case).

Definition 2.1. We call a restricted Lie algebra $\mathfrak{g}$ *extraspecial* if the center $\mathfrak{z}$ of $\mathfrak{g}$ is one-dimensional and $\mathfrak{g}/\mathfrak{z}$ is an elementary Lie algebra.

Lemma 2.2. *Let $\mathfrak{g}$ be an extraspecial Lie algebra.*

1. *The dimension of $\mathfrak{g}$ is odd.*
2. *There exists a basis*

$$\{x_1, \ldots x_{n-1}, y_1, \ldots y_{n-1}, y_n\} \tag{2.2.1}$$

of $\mathfrak{g}$ such that y_n generates the one-dimensional center $\mathfrak{z}$ of $\mathfrak{g}$ and the following equations are satisfied for any i, j, $1 \leq i, j \leq n-1$:

(a) $[x_i, x_j] = [y_i, y_j] = 0$,

(b) $[x_i, y_j] = \delta_{i,j} y_n$.

Proof. Let $\varphi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{z} = W$ be the natural quotient map. Because W is commutative, the commutator algebra $[\mathfrak{g}, \mathfrak{g}]$ is in $\mathfrak{z}$. Because $\mathfrak{g}$ is not commutative, we have that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{z}$.

Let y_n be a generator of $\mathfrak{z}$, and fix the corresponding linear isomorphism $\mathfrak{z} \simeq k$. Now we define a skew-symmetric bilinear form $B(-,-) : W \times W \to \mathfrak{z} \simeq k$ by $B(x,y) = [\sigma(x), \sigma(y)]$, where $\sigma : W \to \mathfrak{g}$ is a k-linear right splitting of φ. It is easy to check that this form is well defined (does not depend on the choice of the splitting σ) and that it is bilinear and skew-symmetric. In addition, the form is nondegenerate as otherwise the center of $\mathfrak{g}$ would have dimension greater than one. Hence, B defines a symplectic form on W. Therefore, W is even-dimensional and can be written as $W = X \oplus Y$, where X and Y are maximal isotropic subspace with respect to the form B, and the pairing $X \times Y \to k$ given by $(x,y) \mapsto B(x,y)$ is nondegenerate. Moreover, we can find elements $x_1, \dots, x_{n-1}, y_1, \dots, y_{n_1}$ of $\mathfrak{g}$ such that $\varphi(x_1), \dots, \varphi(x_{n-1})$ is a basis of X, $\varphi(y_1), \dots, \varphi(y_{n-1})$ is a basis of Y, and the matrix for the bilinear form B with respect to these bases of X, Y has the standard form

$$B = \begin{pmatrix} 0 & I_{n-1} \\ -I_{n-1} & 0 \end{pmatrix}$$

with I_{n-1} being the identity matrix (see, for example, [Lam, ch.1]).

We have thus shown that $\dim \mathfrak{g} = \dim W + 1$ is odd, and that the basis $\{x_1, \dots, x_{n-1}, y_1, \dots, y_{n_1}, y_n\}$ satisfies the required conditions. □

We recall that a subspace L of a symplectic vector space W is said to be Lagrangian if L is an isotropic subspace (i.e., if the pairing of any two elements of L is 0) of maximal dimension. Consequently, if $\dim W = 2n$, then $\dim L = n$. We denote by $\mathrm{LG}(n, W)$ the *Lagrangian Grassmannian* of W, the homogeneous space parameterizing the Lagrangian subspaces of W. We note for future reference that we have an isomorphism of varieties

$$\mathrm{LG}(n, W) \simeq \mathrm{Sp}_{2n} / P_{\alpha_n}$$

where P_{α_n} is the unique standard cominuscule parabolic subgroup of Sp_{2n}.

Proposition 2.3. *Let $\mathfrak{g}$ be an extraspecial restricted Lie algebra of dimension $2n-1$ with trivial restriction map. Equip $W = \mathfrak{g}/\mathfrak{z}$ with the symplectic form as in the proof of Lemma 2.2.*

1. *The maximal dimension of an elementary subalgebra of $\mathfrak{g}$ is n.*
2. $\mathbb{E}(n, \mathfrak{g}) \simeq \mathrm{LG}(n-1, W)$.

Proof. We adopt the notation of the proof of Lemma 2.2 with $\varphi : \mathfrak{g} \to W = \mathfrak{g}/\mathfrak{z}$ being the projection map. Observe that if a subalgebra ϵ of $\mathfrak{g}$ is elementary then $\varphi(\epsilon)$ is an isotropic linear subspace of W. Since $\dim \varphi(\epsilon) + \dim \varphi(\epsilon)^{\perp} = \dim W$ (where $\varphi(\epsilon)^{\perp}$ denotes the orthogonal complement with respect to the sympectic form) and $\varphi(\epsilon) \subset \varphi(\epsilon)^{\perp}$ since $\varphi(\epsilon)$ is isotropic, we get that $\dim \varphi(\epsilon) \leq (\dim W)/2 = n-1$, and, consequently, $\dim \epsilon \leq n$. Moreover, the equality holds if and only if $\epsilon/\mathfrak{z}$ is a Lagrangian subspace of W. Hence, $\mathbb{E}(n, \mathfrak{g}) \simeq \mathrm{LG}(n-1, W)$. □

Let G be a semi-simple algebraic group. We fix a maximal torus T and a subset of simple roots $\Delta = \{\alpha_1, \dots, \alpha_n\}$ inside the root system Φ, determining the Borel subgroup B and its unipotent radical U. We follow the convention in [Bur, ch.6] in the numbering of simple roots.

Let $\mathfrak{g} = \operatorname{Lie} G$, $\mathfrak{h}$ be the Cartan subalgebra, $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ be the standard triangular decomposition. Denote by x_α the root vector corresponding to the root α.

For a simple root $\alpha \in \Delta$, we denote by P_α, $\mathfrak{p}_\alpha$ the corresponding standard maximal parabolic subgroup and its Lie algebra. It is easy to see that the following unipotent radicals of certain parabolics give examples of extraspecial Lie algebras with trivial restriction maps.

Example 2.4. (1) Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and assume that $p > 2$. Let $\mathfrak{p} \subset \mathfrak{g}$ be a standard parabolic subalgebra defined by the subset $I = \{\alpha_2, \ldots, \alpha_{n-1}\}$ of simple roots, that is, $\mathfrak{p} = \mathfrak{h} \oplus \bigoplus\limits_{\alpha \in \Phi_I^- \cup \Phi^+} kx_\alpha$, where Φ_I is the root subsystem of Φ generated by the subset of simple roots I. Then the unipotent radical $\mathfrak{u} = \bigoplus\limits_{\alpha \in \Phi^+ \backslash \Phi_I^+} kx_\alpha$ of $\mathfrak{p}$ is an extraspecial Lie algebra with trivial restriction of dimension $2n-1$. In matrix terms, this is the subalgebra of strictly upper triangular matrices with non-zero entries in the top row or the rightmost column.

(2) Let $\mathfrak{g} = \mathfrak{sp}_{2n}$. Let $\mathfrak{p} = \mathfrak{p}_{\alpha_1}$ be the maximal parabolic subalgebra corresponding to the simple root α_1. Let $\gamma_n = 2\alpha_1 + \ldots + 2\alpha_{n-1} + \alpha_n$ be the highest long root, and let further

$$\beta_i = \alpha_1 + \alpha_2 + \ldots + \alpha_i, \quad \gamma_{n-i} = \gamma_n - \beta_i. \tag{2.4.1}$$

Then $\mathfrak{u}_{\alpha_1}$, the nilpotent radical of $\mathfrak{p}_{\alpha_1}$ is an extraspecial Lie algebra with trivial restriction and the basis $\{x_{\beta_1}, \ldots, x_{\beta_{n-1}}, x_{\gamma_{n-1}}, \ldots, x_{\gamma_1}, x_{\gamma_n}\}$ satisfying the conditions of Lemma 2.2.

(3) *Type E_7.* Let $\mathfrak{p} = \mathfrak{p}_{\alpha_1}$. Then the nilpotent radical of $\mathfrak{p}$ is an extraspecial Lie algebra with trivial restriction.

Definition 2.5. For α a simple root, the (maximal) parabolic P_α is called *cominuscule* if α enters with coefficient at most 1 in any positive root.

The following is a complete list of cominuscule parabolics for simple groups (see, for example, [BL00] or [RRS92]):

(1) Type A_n. P_α for any $\alpha \in \{\alpha_1, \ldots, \alpha_n\}$.
(2) Type B_n. P_{α_1}.
(3) Type C_n. P_{α_n} (α_n is the unique long simple root).
(4) Type D_n. P_α for $\alpha \in \{\alpha_1, \alpha_{n-1}, \alpha_n\}$.
(5) Type E_6. P_α for $\alpha \in \{\alpha_1, \alpha_6\}$.
(6) Type E_7. P_{α_7}.

For types E_8, F_4, G_2 there are no cominuscule parabolics.

In the following lemma we remind the reader about the equivalent description of cominuscule parabolics which also underscores their relevance to our consideration of elementary subalgebras.

Lemma 2.6. [RRS92, Lemma 2.2] *Let G be a simple algebraic group and P be a proper standard parabolic subgroup. Assume $p \neq 2$ if $\Phi(G)$ has two different root lengths. Then the nilpotent radical of $\mathfrak{p} = \operatorname{Lie}(P)$ is abelian if and only if P is a cominuscule parabolic.*

Corollary 2.7. *Let P_α be a maximal parabolic subgroup of a simple algebraic group G, and assume $p \neq 2$ for types B, C. The nilpotent radical $\mathfrak{u}$ of $\mathfrak{p} = \mathrm{Lie}(P)$ is an elementary subalgebra if and only if P_α is cominuscule.*

Proof. If $\mathfrak{u}$ is elementary then, in particular, it is abelian and, hence, P_α is cominuscule by Lemma 2.6. Assume P_α is cominuscule of type A,B,C or D and consider the standard embedding on $\mathfrak{g}$ in $\mathfrak{gl}_n$ (such as in [Hum, Ch.1]). In each case, the nilpotent radical $\mathfrak{u}$ embeds as a block subalgebra with square zero, hence, it is elementary. The groups E_6, E_7 can be done by inspection. □

Proposition 2.8. *Let G be a simple algebraic group and P be a standard parabolic subgroup of G. Let $\mathfrak{p} = \mathrm{Lie}(P)$ and $\mathfrak{u}$ be the nilpotent radical of $\mathfrak{p}$. Assume that $p \neq 2$. Then*

(1) $[\mathfrak{u}, \mathfrak{p}] = \mathfrak{u}$;
(2) *If P is cominuscule then* $\mathfrak{p} = [\mathfrak{u}, \mathfrak{g}]$.

Proof. (1). Since $\mathfrak{u}$ is a Lie ideal in $\mathfrak{p}$, we have $[\mathfrak{u}, \mathfrak{p}] \subset \mathfrak{u}$. By the structure theory for classical Lie algebras, for any $\alpha \in \Phi^+$ there exists $h_\alpha \in \mathfrak{h}$ such that $[h_\alpha, x_\alpha] = 2x_\alpha$. Hence, $\mathfrak{u} = [\mathfrak{h}, \mathfrak{u}] \subset [\mathfrak{p}, \mathfrak{u}]$.

(2). Let $P = P_{\alpha_i}$, let $I = \Delta \backslash \{\alpha_i\}$ and let $\Phi_I \subset \Phi$ be the root system corresponding to the subset of simple roots I. We have $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{u}^-$ where $\mathfrak{u}^- = \sum\limits_{\beta \in \Phi^+ \backslash \Phi_I^+} k x_{-\beta}$. Note that $\Phi^+ \backslash \Phi_I^+$ consists of all positive roots into which α_i enters with coefficient 1. Let $\beta \in \Phi^+ \backslash \Phi_I^+$ and let γ be any root. If $\beta + \gamma$ is a root, then α_i enters into $\beta + \gamma$ with coefficient 0 or 1. Therefore, $x_{\beta+\gamma} \notin \mathfrak{u}^-$. Hence, $[x_\beta, x_\gamma] \in \mathfrak{p}$. Since x_β for $\beta \in \Phi^+ \backslash \Phi_I^+$ generate $\mathfrak{u}$, we conclude that $[\mathfrak{u}, \mathfrak{g}] \subset \mathfrak{p}$.

For the opposite inclusion, we first show that $\mathfrak{h} \subset [\mathfrak{u}, \mathfrak{g}]$. Let $S \subset \Phi^+ \backslash \Phi_I^+$ be the set of all positive roots of the form $a_1\alpha_1 + \ldots + a_n\alpha_n$ such that $a_i = 1$ and $a_j \in \{0, 1\}$ for all $j \neq i$. For any subset $J \subset \Delta$ of simple roots such that the subgraph of the Dynkin diagram corresponding to J is connected, we have that $\sum\limits_{\alpha_j \in J} \alpha_j$ is a root ([Bur, VI.1.6, Cor. 3 of Prop. 19]). This easily implies that for any simple root α_j, $j \neq i$, we can find $\beta_1, \beta_2 \in S$ such that $\beta_2 - \beta_1 = \alpha_j$. Hence, $\{\beta\}_{\beta \in S}$ generate the integer root lattice $\mathbb{Z}\Phi$. Consider the simply laced case first (A, D, E). Since the bijection $\alpha \to \alpha^\vee$ is linear in this case, we conclude that $\{\beta^\vee\}_{\beta \in S}$ generate the integer coroot lattice $\mathbb{Z}\Phi^\vee$. This, in turn, implies that $\{h_\beta\}_{\beta \in S}$ generate the integer form $\mathrm{Lie}(T_\mathbb{Z})$ of the Lie algebra $\mathrm{Lie}(T) = \mathfrak{h}$ over $\mathbb{Z}$, and, therefore, generate $\mathfrak{h} = \mathrm{Lie}(T_\mathbb{Z}) \otimes_\mathbb{Z} k$ over k (see [Jan, II.I.11]).

In the non-simply laced case (B or C), the relation $\beta_1 - \beta_2 = \alpha_j$ leads to $c_1\beta_1^\vee - c_2\beta_2^\vee = c_3\alpha_j^\vee$ where $c_1, c_2, c_3 \in \{1, 2\}$. Hence, in this case $\{\beta\}_{\beta \in S}$ generate the lattice $\mathbb{Z}[\frac{1}{2}]\Phi^\vee$. Since $p \neq 2$, this still implies that $\{h_\beta\}_{\beta \in S}$ generate $\mathfrak{h} = \mathrm{Lie}(T_\mathbb{Z}) \otimes_\mathbb{Z} k$ over k.

In either case, since $h_\beta = [x_\beta, x_{-\beta}] \in [\mathfrak{u}, \mathfrak{g}]$ for $\beta \in S$, we conclude that $\mathfrak{h} \subset [\mathfrak{u}, \mathfrak{g}]$.

The inclusion $\mathfrak{h} \subset [\mathfrak{u}, \mathfrak{g}]$ implies $[\mathfrak{p}, \mathfrak{h}] \subset [\mathfrak{p}, [\mathfrak{u}, \mathfrak{g}]]$. Hence, by the Jacobi identity, we have

$$[\mathfrak{p}, \mathfrak{h}] \subset [\mathfrak{p}, [\mathfrak{u}, \mathfrak{g}]] = [[\mathfrak{p}, \mathfrak{u}], \mathfrak{g}]] + [\mathfrak{u}, [\mathfrak{p}, \mathfrak{g}]] = [\mathfrak{u}, \mathfrak{g}] + [\mathfrak{u}, \mathfrak{p}] \subset [\mathfrak{u}, \mathfrak{g}].$$

Consequently, $\mathfrak{p} = [\mathfrak{p}, \mathfrak{h}] + \mathfrak{h} \subset [\mathfrak{u}, \mathfrak{g}]$. □

We consider the special linear Lie algebra in two parallel theorems, one for $\mathfrak{sl}_{2m}$ and one for $\mathfrak{sl}_{2m+1}$. For $\mathfrak{sl}_n$, we use the notation $P_{r,n-r}$, $\mathfrak{p}_{r,n-r}$, and $\mathfrak{u}_{r,n-r}$ to denote the maximal parabolic, its Lie algebra, and the nilpotent radical corresponding to the simple root α_r. We denote by $\mathfrak{u}_n$ the nilpotent radical of $\mathfrak{sl}_n$ itself.

The first parts of both theorems are well-known in the context of maximal elementary abelian subgroups in $\mathrm{GL}_n(\mathbb{F}_p)$ (see, for example, [G70] or [MP87]). We use the approach in [MP87] to compute conjugacy classes.

Theorem 2.9. *Assume $p > 2$. Let $n = 2m$.*

1. *The maximal dimension of an elementary abelian subalgebra of $\mathfrak{sl}_{2m}$ is m^2.*
2. *An elementary abelian subalgebra of dimension m^2 is conjugate to $\mathfrak{u}_{m,m}$, the nilpotent radical of the standard maximal parabolic $P_{m,m}$.*
3. *$\mathbb{E}(m^2, \mathfrak{sl}_{2m}) \simeq \mathrm{Grass}_{2m,m}$, the Grassmannian of m-planes in a $2m$-dimensional vector space.*

Proof. We prove the following statement by induction: any elementary subalgebra of $\mathfrak{sl}_{2m}$ has dimension at most m^2 and any subalgebra of such dimension inside the nilpotent radical $\mathfrak{n}$ must coincide with $\mathfrak{u}_{m,m}$. This will imply claims (1) and (2) of the theorem.

The statement is clear for $m = 1$. Assume it is proved for $m-1$. Let ϵ be an elementary subalgebra of $\mathfrak{sl}_{2m}$. Since ϵ is nilpotent, it can be conjugated into upper-triangular form. Let $J = \{\alpha_2, \ldots, \alpha_{2m-2}\}$ and let $\mathfrak{u}_J$ be the nilpotent radical of the standard parabolic P_J determined by J (as in Example 2.4(1)). Since $[\mathfrak{u}_{2m}, \mathfrak{u}_J] \subset \mathfrak{u}_J$, this is a Lie ideal in $\mathfrak{u}_{2m}$. We consider extension

$$0 \longrightarrow \mathfrak{u}_J \longrightarrow \mathfrak{u}_{2m} \longrightarrow \mathfrak{u}_{2m}/\mathfrak{u}_J \simeq \mathfrak{u}_{2m-2} \longrightarrow 0.$$

By induction, the dimension of the projection of ϵ onto $\mathfrak{u}_{2m-2}$ is at most $(m-1)^2$, and this dimension is attained if and only if the image of ϵ under the projection is the subalgebra of $\mathfrak{u}_{2m-2}$ of all block matrices of the form $\begin{pmatrix} 0 & \mathbf{A} \\ 0 & 0 \end{pmatrix}$, where $\mathbf{A}$ is a matrix in M_{m-1}. By Lemma 2.2, the maximal elementary subalgebra of $\mathfrak{u}_J$ has dimension $2m-1$. Hence, $\dim \epsilon \leq (m-1)^2 + 2m - 1 = m^2$. For this dimension to be attained we must have that for any $\mathbf{A} \in M_{m-1}$ there exists an element in ϵ of the form

$$\begin{pmatrix} 0 & \mathbf{v_2} & \mathbf{v_1} & * \\ 0 & 0 & \mathbf{A} & \mathbf{w_1} \\ 0 & 0 & 0 & \mathbf{w_2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.9.1}$$

where $\mathbf{v_i}, (\mathbf{w_i})^T \in k^{m-1}$.

Let $\begin{pmatrix} 0 & \mathbf{v_2'} & \mathbf{v_1'} & * \\ 0 & 0 & 0 & \mathbf{w_1'} \\ 0 & 0 & 0 & \mathbf{w_2'} \\ 0 & 0 & 0 & 0 \end{pmatrix}$ be an element in $\epsilon \cap \mathfrak{u}_J$. Taking a bracket of this element with a general element in ϵ of the form as in (2.9.1), we get

$$\begin{pmatrix} 0 & 0 & \mathbf{v_2'}\mathbf{A} & * \\ 0 & 0 & 0 & \mathbf{A}\mathbf{w_2'} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Since ϵ is abelian, we conclude that $\mathbf{v_2'}\mathbf{A} = 0$, $\mathbf{A}\mathbf{w_2'} = 0$ for any $\mathbf{A} \in M_{m-1}$. Hence, $\mathbf{v_2'} = 0$, $\mathbf{w_2'} = 0$ which implies that $\epsilon \cap \mathfrak{u}_J \subset \mathfrak{u}_{m,m}$. Moreover, for the dimension to be maximal, we need $\dim \epsilon \cap \mathfrak{u}_J = 2m-1$. Hence, for any $\mathbf{v_1}, (\mathbf{w_1})^T \in k^{m-1}$, the matrix $\begin{pmatrix} 0 & 0 & \mathbf{v_1} & 0 \\ 0 & 0 & 0 & \mathbf{w_1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ is in ϵ.

It remains to show that for an arbitrary element of ϵ, necessarily of the form (2.9.1), we must have $\mathbf{v_2} = 0, \mathbf{w_2} = 0$. We prove this by contradiction. Suppose $\begin{pmatrix} 0 & \mathbf{v_2} & \mathbf{v_1} & * \\ 0 & 0 & \mathbf{A} & \mathbf{w_1} \\ 0 & 0 & 0 & \mathbf{w_2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \epsilon$ with $\mathbf{v_2} \neq 0$. Subtracting a multiple of $E_{1,2m}$ which is necessarily in ϵ, we get that $M = \begin{pmatrix} 0 & \mathbf{v_2} & \mathbf{v_1} & 0 \\ 0 & 0 & \mathbf{A} & \mathbf{w_1} \\ 0 & 0 & 0 & \mathbf{w_2} \\ 0 & 0 & 0 & 0 \end{pmatrix}$ belongs to ϵ. As observed above, we also have $M' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\mathbf{v_2})^T \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ in ϵ. Therefore, $[M, M']$ has a non-trivial entry $\mathbf{v_2} \cdot (\mathbf{v_2})^T$ in the $(1, 2m)$ spot which contradicts commutativity of ϵ. Hence, $\mathbf{v_2} = \mathbf{0}$. Similarly, $\mathbf{w_2} = \mathbf{0}$. This finishes the proof of the claim.

To show (3), we recall that $P_{m,m} = \mathrm{Stab}_{\mathrm{SL}_{2m}}(\mathfrak{u}_{m,m})$ under the adjoint action of SL_{2m}. Indeed, for any parabolic P its unipotent radical U is a normal subgroup. Hence, the adjoint action of P stabilizes the Lie algebra $\mathfrak{u} = \mathrm{Lie}(U)$. We conclude that $P_{m,m} \subset \mathrm{Stab}_{\mathrm{SL}_{2m}}(\mathfrak{u}_{m,m})$. In particular, $\mathrm{Stab}_{\mathrm{SL}_{2m}}(\mathfrak{u}_{m,m})$ contains the Borel subgroup and, hence, is a standard parabolic subgroup of SL_{2m}. Since $P_{m,m}$ is maximal, we conclude that $P_{m,m} = \mathrm{Stab}_{\mathrm{SL}_{2m}}(\mathfrak{u}_{m,m})$.

By (2), $\mathbb{E}(m^2, \mathfrak{sl}_{2m})$ is the orbit of $\mathfrak{u}_{m,m}$ under the adjoint action of SL_{2m}. Hence,

$$\mathbb{E}(m^2, \mathfrak{sl}_{2m}) \simeq \mathrm{SL}_{2m} / P_{m,m} \ \simeq\ \mathrm{Grass}_{2m,m}\,.$$

$\square$

Theorem 2.10. *Let $n = 2m+1$ and assume $m > 1$, $p > 2$.*

(1) *The maximal dimension of an elementary abelian subalgebra of $\mathfrak{sl}_{2m+1}$ is $m(m+1)$.*
(2) *There are two distinct conjugacy classes of such elementary subalgebras, represented by $\mathfrak{u}_{m,m+1}$ and $\mathfrak{u}_{m+1,m}$.*
(3) *The variety $\mathbb{E}(m^2, \mathfrak{sl}_{2m+1})$ is a disjoint union of two connected components each isomorphic to* $\mathrm{Grass}_{2m+1,m}$.

Proof. One can check by a straightforward calculation that the following is a complete list of two-dimensional elementary subalgebras of $\mathfrak{u}_3$, the nilpotent radical of $\mathfrak{sl}_3$:

- $\mathfrak{u}_{1,2} = \{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid a, b \in k \}$,
- $\mathfrak{u}_{2,1} = \{ \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix} \mid a, b \in k \}$,

- a one-parameter family $\{\begin{pmatrix} 0 & a & b \\ 0 & 0 & xa \\ 0 & 0 & 0 \end{pmatrix} \mid a, b \in k\}$ for a fixed $x \in k^*$.

We prove the following statements by induction: For any $m > 1$, an elementary subalgebra of $\mathfrak{sl}_{2m+1}$ has dimension at most $m(m+1)$. Any subalgebra of such dimension inside $\mathfrak{u}_{2m+1}$ must coincide either with $\mathfrak{u}_{m,m+1}$ or $\mathfrak{u}_{m+1,m}$. This will imply claims (1) and (2) of the theorem.

Base case: $m = 2$. Any elementary subalgebra can be conjugated to the upper-triangular form. So it suffices to prove the statement for an elementary subalgebra ϵ of $\mathfrak{u}_5$, the nilpotent radical of $\mathfrak{sl}_5$. Just as in the proof of Theorem 2.9, we consider a short exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{u}_J \longrightarrow \mathfrak{u}_5 \xrightarrow{\mathrm{pr}} \mathfrak{u}_3 \longrightarrow 0$$

where $J = \{\alpha_2, \alpha_3\}$ (and, hence, $\mathfrak{u}_J \subset \mathfrak{u}_5$ is the subalgebra of upper triangular matrices with zeros everywhere except for the top row and the rightmost column). Since $\dim(\mathrm{pr}(\epsilon)) \leq 2$ by the remark above, and $\dim(\epsilon \cap \mathfrak{u}_J) \leq 4$ by Lemma 2.2, we get that $\dim \epsilon \leq 6$. For the equality to be attained, we need $\mathrm{pr}(\epsilon)$ to be one of the two-dimensional elementary subalgebras listed above. If $\mathrm{pr}(\epsilon) = \mathfrak{u}_{2,1}$ then arguing exactly as in the proof for the even-dimensional case, we conclude that $\epsilon = \mathfrak{u}_{3,2} \subset \mathfrak{u}_5$. Similarly, if $\mathrm{pr}(\epsilon) = \mathfrak{u}_{1,2}$, then $\epsilon = \mathfrak{u}_{2,3}$. We now assume that

$$\mathrm{pr}(\epsilon) = \{\begin{pmatrix} 0 & a & b \\ 0 & 0 & xa \\ 0 & 0 & 0 \end{pmatrix} \mid a, b \in k\}.$$

Let $A' = \begin{pmatrix} 0 & a_{12} & a_{13} & * & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & a_{35} \\ 0 & 0 & 0 & 0 & a_{45} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \epsilon \cap \mathfrak{u}_J$, and let $A = \begin{pmatrix} 0 & * & * & * & * \\ 0 & 0 & a & b & * \\ 0 & 0 & 0 & xa & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \epsilon$.

Then

$$[A', A] = \begin{pmatrix} 0 & 0 & aa_{12} & xaa_{13} + ba_{12} & * \\ 0 & 0 & 0 & 0 & -aa_{35} - ba_{45} \\ 0 & 0 & 0 & 0 & -xaa_{45} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since ϵ is abelian, and since the values of a, b run through all elements of k, we conclude that $a_{12} = a_{13} = a_{35} = a_{45} = 0$. Therefore, $\dim \epsilon \cap \mathfrak{u}_J \leq 3$ and $\dim \epsilon \leq 5$. Hence, the maximum is not attained in this case. This finishes the proof in the base case $m = 2$.

We omit the induction step since it is very similar to the even-dimensional case proved in Theorem 2.9. Hence, it remains to show (3).

Note that $\mathfrak{u}_{m,m+1}$ and $\mathfrak{u}_{m+1,m}$ are not conjugate under the adjoint action of SL_{2m+1} since their nullspaces in the standard representation of $\mathfrak{sl}_{2m+1}$ have different dimensions. Arguing as in the end of the proof of the even dimensional case, we conclude that the stabilizer of $\mathfrak{u}_{m+1,m}$ (resp. $\mathfrak{u}_{m,m+1}$) under the adjoint action is the standard parabolic $P_{m+1,m}$ (resp, $P_{m+1,m}$). Hence,

$$\mathbb{E}(m(m+1), \mathfrak{sl}_{2m+1}) \simeq \mathrm{SL}_{2m+1}/P_{m,m+1} \cup \mathrm{SL}_{2m+1}/P_{m+1,m} \simeq \mathrm{Grass}_{2m+1,m} \cup \mathrm{Grass}_{2m+1,m}.$$

$\square$

We make an immediate observation that the results of Theorems 2.9 and 2.10 apply equally well to $\mathfrak{gl}_n$.

Corollary 2.11. *Assume $p > 2$.*

1. *The maximal dimension of an elementary abelian subalgebra of $\mathfrak{gl}_n$ is $\lfloor \frac{n^2}{4} \rfloor$.*
2. $\mathbb{E}(m^2, \mathfrak{gl}_{2m}) \simeq \mathrm{Grass}_{2m,m}$ *for any $m \geq 1$.*
3. $\mathbb{E}(m(m+1), \mathfrak{gl}_{2m+1}) \simeq \mathrm{Grass}_{2m+1,m} \sqcup \mathrm{Grass}_{2m+1,m}$ *for any $m \geq 2$.*

Remark 2.12. In the case $n = 3$, excluded above, the variety $\mathbb{E}(2, \mathfrak{gl}_3)$ is connected and irreducible (see Example 3.20).

We now prove an analogous result in the symplectic case. Recall that α_n denotes the unique long simple root for type C_n, and, hence, P_{α_n} is the only standard cominuscule parabolic of Sp_{2n}.

Theorem 2.13. *Let $\mathfrak{g} = \mathfrak{sp}_{2n}$ be a simple Lie algebra of type C_n. Assume $p > 3$. Then*

1. *For any elementary subalgebra $\epsilon \subset \mathfrak{g}$, $\dim \epsilon \leq \frac{n(n+1)}{2}$*
2. *Any elementary subalgebra ϵ of maximal dimension is conjugate to $\mathfrak{u}_{\alpha_n}$*
3. $\mathbb{E}(\frac{n(n+1)}{2}, \mathfrak{sp}_{2n}) \simeq \mathrm{Sp}_{2n}/P_{\alpha_n}$, *the Lagrangian Grassmannian.*

Proof. The proof is by induction. For $n = 1$ we have $\mathfrak{sp}_2 = \mathfrak{sl}_2$ for which the statement is trivially true.

Induction step $n-1 \to n$. Let $\mathfrak{p}_{\alpha_1} = \mathfrak{l}_{\alpha_1} \oplus \mathfrak{u}_{\alpha_1}$ be the maximal parabolic subalgebra corresponding to the simple root α_1 with the Levi factor $\mathfrak{l}_{\alpha_1}$ and the nilpotent radical $\mathfrak{u}_{\alpha_1}$. Let $\mathfrak{u}_{\mathfrak{l}_{\alpha_1}}$ be the nilpotent radical of $\mathfrak{l}_{\alpha_1}$, and $\mathfrak{u}_{\mathfrak{sp}_{2n}}$ be the nilpotent radical of $\mathfrak{sp}_{2n}$. We have a short exact sequence

$$0 \longrightarrow \mathfrak{u}_{\alpha_1} \longrightarrow \mathfrak{u}_{\mathfrak{sp}_{2n}} \xrightarrow{\mathrm{pr}} \mathfrak{u}_{\mathfrak{l}_{\alpha_1}} \longrightarrow 0.$$

We can apply the induction hypothesis to $\mathfrak{l}_{\alpha_1}$ since it is a reductive Lie algebra of type C_{n-1}.

Let ϵ be an elementary subalgebra of $\mathfrak{g}$. Conjugating by an element in Sp_{2n}, we can assume that $\epsilon \subset \mathfrak{u}_{\mathfrak{sp}_{2n}}$. By our induction hypothesis, $\dim \mathrm{pr}(\epsilon) \leq \frac{n(n-1)}{2}$. Since $\mathfrak{u}_{\alpha_1}$ is an extraspecial Lie algebra of dimension $2n-1$ (see Example 2.4), Lemma 2.2 implies that $\dim \mathfrak{u}_{\alpha_1} \cap \epsilon \leq n$. Hence, $\dim \epsilon \leq n + \frac{n(n-1)}{2}$. This proves (1).

To prove (2), we observe that the induction hypothesis implies that for an elementary subalgebra ϵ to attain the maximal dimension, we must have that

$$\mathrm{pr} \downarrow_\epsilon \colon \epsilon \to \mathfrak{u}_{\mathfrak{l}_{\alpha_1}}$$

is surjective onto $\mathfrak{u}_{\mathfrak{l}_{\alpha_1}} \cap \mathfrak{u}_{\alpha_n}$, the nilpotent radical of the unique standard cominuscule parabolic of $\mathfrak{l}_{\alpha_1}$.

Let $\{x_{\beta_i}, x_{\gamma_i}\}$ be a basis of $\mathfrak{u}_{\alpha_1}$ as defined in (2.4.1). Let $x = \sum b_i x_{\beta_i} + \sum c_i x_{\gamma_i} \in \mathfrak{u}_{\alpha_1} \cap \epsilon$. We want to show that $x \in \mathfrak{u}_{\alpha_n}$ or, equivalently, that coefficients by x_{β_i} are zero. Assume, to the contrary, that $b_i \neq 0$ for some i, $1 \leq i \leq n-1$. Let $\mu = \gamma_{n-1} - \beta_i = \alpha_2 + \ldots + \alpha_i + 2\alpha_{i+1} + \ldots + 2\alpha_{n-1} + \alpha_n$. Then $x_\mu \in \mathfrak{u}_{\mathfrak{l}_{\alpha_1}} \cap \mathfrak{u}_{\alpha_n} \subset \mathrm{pr}(\epsilon)$. Therefore, there exists $y = x' + x_\mu \in \epsilon$ for some $x' \in \mathfrak{u}_{\alpha_1}$. Note that $[x, x'] \subset [\mathfrak{u}_{\alpha_1}, \mathfrak{u}_{\alpha_1}] = k x_{\gamma_n}$, and that $\mu + \gamma_i$ is never a root, and $\mu + \beta_j$ is not a root unless $i = j$. Hence,

$$[x, y] = [x, x'] + [x, x_\mu] = c x_{\gamma_n} + b_i [x_{\beta_i}, x_\mu] = c x_{\gamma_n} + b_i c_{\beta_i \mu} x_{\gamma_{n-1}} \neq 0.$$

Here, $c_{\beta_i\mu}$ is the structure constant from the equation $[x_{\beta_i}, x_\mu] = c_{\beta_i\mu}x_{\beta_i+\mu} = c_{\beta_i\mu}x_{\gamma_{n-1}}$ which is non-zero since $p > 3$ (see [Sel65, II.4.1]). We get a contradiction with the commutativity of ϵ. Hence, $b_i = 0$ for all i, $1 \le i \le n-1$, and, therefore, $\mathfrak{u}_{\alpha_1} \cap \epsilon \subset \mathfrak{u}_{\alpha_n}$. Moreover, since we assume that $\dim \epsilon$ is maximal, we must have $\dim \mathfrak{u}_{\alpha_1} \cap \epsilon = n$, and, therefore, $\mathfrak{u}_{\alpha_1} \cap \epsilon = \bigoplus\limits_{i=1}^{n} kx_{\gamma_i}$.

Now let $x + a$ be any element in ϵ where $x \in \mathfrak{u}_{\alpha_1}$ and $a \in \mathfrak{u}_{\mathfrak{l}_{\alpha_1}} \cap \mathfrak{u}_{\alpha_n}$. We need to show that $x \in \mathfrak{u}_{\alpha_n}$, that is, $x \in \bigoplus\limits_{i=1}^{n} kx_{\gamma_i}$. Let $x = \sum b_i x_{\beta_i} + \sum c_i x_{\gamma_i}$ and assume to the contrary that $b_i \neq 0$ for some i. Note that $[x_{\gamma_j}, \mathfrak{u}_{\mathfrak{l}_{\alpha_1}} \cap \mathfrak{u}_{\alpha_n}] = 0$ for any j, $1 \le j \le n$ since both x_{γ_j} and any $a \in \mathfrak{u}_{\mathfrak{l}_{\alpha_1}} \cap \mathfrak{u}_{\alpha_n}$ are linear combinations of root vectors for roots that have coefficient 1 by α_n. Hence, $[x + a, \gamma_{n-i}] = b_i[x_{\beta_i}, \gamma_{n-i}] \neq 0$. We get a contradiction again. Therefore, $\epsilon \subset \mathfrak{u}_{\alpha_n}$. This proves (2).

Finally, (c) follows immediately from the fact that P_{α_n} is the stabilizer of $\mathfrak{u}_{\alpha_n}$ under the adjoint action of Sp_{2n}. □

Proposition 2.14. *Assume that $p > 2$.*

1. *The maximal dimension of an elementary subalgebra of the standard parabolic subalgebra $\mathfrak{p}_{1,2m}$ of $\mathfrak{sl}_{2m+1}$ is $m(m+1)$.*
2. *For $m \ge 2$, $\mathbb{E}(m(m+1), \mathfrak{p}_{1,2m})$ is a disjoint union of two connected components isomorphic to $\mathrm{Grass}_{2m,m}$ and $\mathrm{Grass}_{2m,m-1}$:*

$$\mathbb{E}(m(m+1), \mathfrak{p}_{1,2m}) \;\simeq\; \mathrm{Grass}_{2m,m} \coprod \mathrm{Grass}_{2m,m-1}\,.$$

Proof. Let $\epsilon \subset \mathfrak{p}_{1,2m}$ be an elementary subalgebra. Since $\mathfrak{p}_{1,2m} \subset \mathfrak{sl}_{2m+1}$, Theorem 2.10 implies that $\dim \epsilon \le m(m+1)$. Since $\mathfrak{u}_{m,m+1}$ is a subalgebra of $\mathfrak{p}_{1,2m}$, we conclude that the maximal dimension is precisely $m(m+1)$. This proves (1).

To show (2), we first show that any elementary subalgebra ϵ of maximal dimension is conjugate to either $\mathfrak{u}_{m,m+1}$ or $\mathfrak{u}_{m+1,m}$ under the adjoint action of $P_{1,2m}$. By Theorem 2.10, ϵ is conjugate to $\mathfrak{u}_{m,m+1}$ or $\mathfrak{u}_{m+1,m}$ under the adjoint action of SL_{2m+1}. Assume that $\epsilon = g\mathfrak{u}_{m+1,m}g^{-1}$ for some $g \in \mathrm{SL}_{2m+1}$ (the case of $\mathfrak{u}_{m,m+1}$) is strictly analogous). We proceed to show that there exists $\tilde{g} \in P_{1,2m}$ such that $\epsilon = \tilde{g}\mathfrak{u}_{m+1,m}\tilde{g}^{-1}$.

Let $W(\mathrm{SL}_{2m+1}) \simeq N_{\mathrm{SL}_{2m+1}}(T)/Z_{\mathrm{SL}_{2m+1}}(T)$ be the Weyl group, U_{2m+1} be the unipotent radical, and B_{2m+1} the Borel subgroup of SL_{2m+1}. For an element $w \in W(\mathrm{SL}_{2m+1})$, we denote by $\widetilde{w}$ a fixed coset representative of w in $N_{\mathrm{SL}_{2m+1}}(T)$.

Using the Bruhat decomposition, we can write $g = g_1\widetilde{w}g_2$ where $g_1 \in U_{2m+1}$, $g_2 \in B_{2m+1}$, and $w \in W(\mathrm{SL}_{2m+1})$. Since both $\mathfrak{u}_{m+1,m}$ and $P_{1,2m}$ are stable under the conjugation by U_{2m+1} and B_{2m+1}, it suffices to prove the statement for $g = \widetilde{w}$, where w is a Weyl group element. We make the standard identifications $W(\mathrm{SL}_{2m+1}) \simeq S_{2m+1}$, $W(L_{1,2m}) \simeq S_{2m}$ and $W(L_{m+1,m}) \simeq S_{m+1} \times S_m$ where $L_{i,j}$ is the Levi factor of a standard parabolic $P_{i,j}$.

We further decompose

$$S_{2m+1} = W(\mathrm{SL}_{2m+1}) = \bigsqcup_{s \in S_{2m}\backslash S_{2m+1}/(S_{m+1}\times S_m)} S_{2m}s(S_{m+1} \times S_m)$$

into double cosets, where S_{2m} is the Weyl group of the Levi of $P_{1,2m}$ which is isomorphic to the subgroup of all permutations in S_{2m+1} which fix 1. We can choose coset representatives $\{t\}$ of $S_{2m+1}/S_{m+1} \times S_m$ in such a way that if $t^{-1}(1) = j \neq 1$ then $j > m+1$. Indeed, let t be any permutation and let $t^{-1}(1) = j \neq 1$.

Multiplying on the right by the transposition $(1j)$, we get a new permutation that fixes 1. If $j \leq m+1$, then $(1j) \in S_{m+1}$, and, hence, t and $t \cdot (1j)$ represent the same coset.

Let $w \in S_{2m+1}$, and assume that $\widetilde{w}\mathfrak{u}_{m+1,m}\widetilde{w}^{-1} \subset \mathfrak{p}_{1,2m}$. Write $w = w_1 s w_2$, where $w_1 \in S_{2m}, w_2 \in S_{m+1} \times S_m$ and s is a double coset representative. If $s^{-1}(1) = 1$, then $w_1 s \in S_{2m}$ and, hence, $\widetilde{w}_1\widetilde{s} \in P_{1,2m}$. Since $\widetilde{w}_2$ stabilizes $\mathfrak{u}_{m+1,m}$, the conjugates of $\mathfrak{u}_{m+1,m}$ under $\widetilde{w}$ and $\widetilde{w}_1\widetilde{s}$ coincide. But $\widetilde{w}_1\widetilde{s}$ is an element of $P_{1,2m}$ which finishes the proof in the case $s^{-1}(1) = 1$.

Now assume $s(1) = j \neq 1$. By the discussion above, we can assume that $S^{-1}(1) = j$, $j > m+1$. Since $w_1(1) = 1$, we get that $\widetilde{w}E_{ij}\widetilde{w}^{-1} = E_{w(i)w(j)} = E_{w(i)1} \notin \mathfrak{p}_{1,2m}$ if $w(i) \neq 1$. Since $E_{ij} \in \mathfrak{u}_{m+1,m}$ for all i, $1 \leq i \leq m$, we conclude that $\widetilde{w}\mathfrak{u}_{m+1,m}\widetilde{w}^{-1} \not\subset \mathfrak{p}_{1,2m}$. This leads to a contradiction. Therefore, $s(1) = 1$, and we can take $\widetilde{g} = g_1\widetilde{w}_1\widetilde{s} \in P_{1,2m}$.

The above discussion implies that $\mathbb{E}(r, \mathfrak{p}_{1,2m}) = P_{1,2m} \cdot \mathfrak{u}_{m+1,m} \cup P_{1,2m} \cdot \mathfrak{u}_{m,m+1}$. The $P_{1,2m}$-stabilizer of $\mathfrak{u}_{m+1,m}$ equals $P_{1,m,m} = P_{m+1,m} \cap P_{1,2m} \subset \mathrm{SL}_{2m+1}$. Thus, $P_{1,2m} \cdot \epsilon \simeq P_{1,2m}/P_{1,m,m} \simeq \mathrm{Grass}_{2m,m}$. Similarly, the $P_{1,2m}$-stabilizer of $\mathfrak{u}_{m,m+1}$ equals $P_{1,m-1,m+1} = P_{m,m+1} \cap P_{1,2m} \subset \mathrm{SL}_{2m+1}$. Hence, $P_{1,2m} \cdot \mathfrak{u}_{m,m+1} \simeq \mathrm{Grass}_{2m,m-1}$. Moreover, $\mathfrak{u}_{m+1,m}$ and $\mathfrak{u}_{m,m+1}$ are not conjugate since their nullspaces in the standard representation of $\mathfrak{sl}_{2m+1}$ have different dimensions. Therefore, $\mathbb{E}(m(m+1), \mathfrak{p}_{1,2m}) \simeq \mathrm{Grass}_{2m,m} \coprod \mathrm{Grass}_{2m,m-1}$.

$\square$

Proposition 2.14 has the following immediate corollary.

Corollary 2.15. *Let $\mathfrak{g}_{1,2m} \subset \mathfrak{gl}_{2m+1}$ be as defined in Example 1.11(1). The maximal dimension of an elementary subalgebra of $\mathfrak{g}_{1,2m}$ is $m(m+1)$. For $m \geq 2$,*

$$\mathbb{E}(m(m+1), \mathfrak{g}_{1,2m}) \simeq \mathrm{Grass}_{2m,m} \coprod \mathrm{Grass}_{2m,m-1}.$$

3. Radicals, socles, and geometric invariants for $\mathfrak{u}(\mathfrak{g})$-modules

We now proceed to consider invariants for a (finite dimensional) $\mathfrak{u}(\mathfrak{g})$-module M defined in terms of restrictions of M to elementary subalgebras $\epsilon \subset \mathfrak{g}$. If $\epsilon \subset \mathfrak{g}$ is an elementary subalgebra and M a $\mathfrak{u}(\mathfrak{g})$-module, then we shall denote by $\epsilon^* M$ the restriction of M to $\mathfrak{u}(\epsilon) \subset \mathfrak{u}(\mathfrak{g})$.

The following is a natural extension of the usual support variety in the case $r = 1$ (see [FP86]) and of the variety $\mathrm{Grass}(r, V)_M$ of [CFP12, 1.4] for $\mathfrak{g} = \mathfrak{g}_a^{\oplus n}$.

Definition 3.1. For any $\mathfrak{u}(\mathfrak{g})$-module M and any positive integer r, we define

$$\mathbb{E}(r, \mathfrak{g})_M = \{\epsilon \in \mathbb{E}(r, \mathfrak{g}); \epsilon^* M_K \text{ is not projective}\}.$$

Remark 3.2. Let $\underline{\mathfrak{g}}$ denote the height 1 infinitesimal group scheme associated to $\mathfrak{g}$, that is, $k[\underline{\mathfrak{g}}] = u(\mathfrak{g})^*$. We denote by $k\underline{\mathfrak{g}}$ the dual Hopf algebra to the coordinate algebra $k[\underline{\mathfrak{g}}]$ and call it the *group algebra* of $\underline{\mathfrak{g}}$. We identify $\mathbb{E}(1, \mathfrak{g})$ with the projectivization of the conical affine variety of 1-parameter subgroups of $\underline{\mathfrak{g}}$. This conical variety is isomorphic to the p-nilpotent cone $\mathcal{N}_p(\mathfrak{g})$ as in Example 1.5. Then for any finite dimensional $\mathfrak{u}(\mathfrak{g})$-module M,

$$\mathbb{E}(1, \mathfrak{g})_M = \mathrm{Proj}\, k[V(\underline{\mathfrak{g}})_M],$$

where the rank variety $V(\underline{\mathfrak{g}})_M \subset V(\underline{\mathfrak{g}})$ is defined in [SFB2] as the affine subvariety of those 1-parameter subgroups $\mathbb{G}_{a(1)} \to \underline{\mathfrak{g}}$ restricted to which M is not projective. In particular, $\mathbb{E}(1, \mathfrak{g})_M \subset \mathbb{E}(1, \mathfrak{g})$ is a closed subvariety.

The following proposition tells us that the geometric invariant $M \mapsto \mathbb{E}(r,\mathfrak{g})_M$ can be computed in terms of the more familiar (projectivized) support variety $\mathbb{E}(1,\mathfrak{g})_M \ = \ \mathrm{Proj}(V(\underline{\mathfrak{g}})_M)$.

Proposition 3.3. *For any $\mathfrak{u}(\mathfrak{g})$-module M, and positive integer r, and any $\epsilon \in \mathbb{E}(r,\mathfrak{g})$,*

$$\mathbb{E}(r,\mathfrak{g})_M \ = \ \{\epsilon \in \mathbb{E}(r,\mathfrak{g});\ \epsilon \cap V(\underline{\mathfrak{g}})_M \neq 0\} \tag{3.3.1}$$

where the intersection $\epsilon \cap V(\underline{\mathfrak{g}})_M$ is as subvarieties of $\mathfrak{g}$.

Proof. By definition, $\epsilon \in \mathbb{E}(r,\mathfrak{g})_M$ if and only if $\epsilon^* M$ is not free which is the case if and only if $V(\underline{\epsilon})_{\epsilon^* M} \neq 0$. Since $\epsilon \subset \mathfrak{g}$ induces an isomorphism

$$V(\underline{\epsilon})_{\epsilon^*(M)} \xrightarrow{\sim} V(\underline{\epsilon}) \cap V(\underline{\mathfrak{g}})_M$$

(see [FP86]), this is equivalent to $\epsilon \cap V(\underline{g})_M \neq 0$. □

Proposition 3.4. *For any $\mathfrak{u}(\mathfrak{g})$-module M and for any $r \geq 1$,*

$$\mathbb{E}(r,\mathfrak{g})_M \ \subset \ \mathbb{E}(r,\mathfrak{g})$$

is a closed subvariety.

Moreover, if G is an algebraic group with $\mathfrak{g} = \mathrm{Lie}(G)$ and M is a rational G-module, then $\mathbb{E}(r,\mathfrak{g})_M \ \subset \ \mathbb{E}(r,\mathfrak{g})$ is G-stable.

Proof. Let $\mathrm{Proj}\,\epsilon \subset \mathbb{E}(1,\mathfrak{g})$ be the projectivization of the linear subvariety $\epsilon \subset \mathfrak{g}$. Let $X_M = \{\epsilon \in \mathrm{Grass}(r,\mathfrak{g}) \mid \mathrm{Proj}\,\epsilon \cap \mathbb{E}(1,\mathfrak{g})_M \neq \emptyset\}$. Then $X_M \subset \mathrm{Grass}(r,\mathfrak{g})$ is a closed subvariety (see [Harr, ex. 6.14]). Since $\mathbb{E}(r,\mathfrak{g})_M = \mathbb{E}(r,\mathfrak{g}) \cap X_M$ by Prop. 3.3, we conclude that $\mathbb{E}(r,\mathfrak{g})_M$ is a closed subvariety of $\mathbb{E}(r,\mathfrak{g})$.

If $\mathfrak{g} = \mathrm{Lie}(G)$ and M is a rational G-module, then $M \simeq M^x$ as $\mathfrak{u}(\mathfrak{g})$-modules and the pull-back of M along the isomorphism $x^{-1} : \mathfrak{u}(\epsilon^x) \xrightarrow{\sim} \mathfrak{u}(\epsilon)$ equals $(\epsilon^x)^*(M^x)$ for any $x \in G(k)$. Thus, $\mathbb{E}(r,\mathfrak{g})_M$ is G-stable. □

Proposition 3.3 implies the following result concerning the realization of subsets of $\mathbb{E}(r,\mathfrak{g})$ as subsets of the form $X = \mathbb{E}(r,\mathfrak{g})_M$. We remind the reader of the definition of the module L_ζ associated to a cohomology class $\zeta \in \mathrm{H}^n(\mathfrak{u}(\mathfrak{g}),k)$: L_ζ is the kernel of the map $\zeta : \Omega^n(k) \to k$ determined by ζ, where $\Omega^n(k)$ is the n^{th} Heller shift of the trivial module k (see [Ben] or Example 4.6).

Corollary 3.5. *A subset $X \subset \mathbb{E}(r,\mathfrak{g})$ has the form $X = \mathbb{E}(r,\mathfrak{g})_M$ for some $\mathfrak{u}(\mathfrak{g})$-module M if and only if there exists a closed subset $Z \subset \mathbb{E}(1,\mathfrak{g})$ such that*

$$X \ = \ \{\epsilon \in \mathbb{E}(r,\mathfrak{g});\ \mathrm{Proj}\,\epsilon \cap Z \neq \emptyset\}. \tag{3.5.1}$$

Moreover, such an M can be chosen to be a tensor product of modules L_ζ with each ζ of even cohomological degree.

Proof. We recall that any closed, conical subvariety of $V(\underline{\mathfrak{g}})$ (i.e., any closed subvariety of $\mathbb{E}(1,\mathfrak{g})$) can be realized as the (affine) support of a tensor product of modules L_ζ (see [FP86]) and that the support of any finite dimensional $k\underline{\mathfrak{g}}$-module is a closed, conical subvariety of $V(\underline{\mathfrak{g}})$. Thus, the proposition follows immediately from Proposition 3.3. □

Example 3.6. As one specific example of Proposition 3.5, we take some even degree cohomology class $0 \neq \zeta \in \mathrm{H}^{2m}(\mathfrak{u}(\mathfrak{g}), k)$ and $M = L_\zeta$. We identify $V(\underline{\mathfrak{g}})$ with the spectrum of $\mathrm{H}^{\mathrm{ev}}(\mathfrak{u}(\mathfrak{g}), k)$ (for $p > 2$), so that ζ is a (homogeneous) algebraic function on $V(\underline{\mathfrak{g}})$. Thus $V(\underline{\mathfrak{g}})_{L_\zeta} = Z(\zeta) \subset V(\underline{\mathfrak{g}})$, the zero locus of the function ζ. Then,

$$\mathbb{E}(r, \mathfrak{g})_{L_\zeta} \;=\; \{\epsilon \in \mathbb{E}(r, \mathfrak{g});\ \epsilon \cap Z(\zeta) \neq \{0\}\}.$$

On the other hand, if $\zeta \in \mathrm{H}^{2m+1}(\mathfrak{u}(\mathfrak{g}), k)$ has odd degree and $p > 2$, then $V(\underline{\mathfrak{g}})_{L_\zeta} = V(\underline{\mathfrak{g}})$, so that $\mathbb{E}(r, \mathfrak{g})_{L_\zeta} \;=\; \mathbb{E}(r, \mathfrak{g})$.

Remark 3.7. As pointed out in [CFP12, 1.10] in the special case $\mathfrak{g} = \mathfrak{g}_a^{\oplus 3}$ and $r = 2$, not every closed subset $X \subset \mathbb{E}(r, \mathfrak{g})$ has the form (3.5.1).

Example 3.8. We consider another computation of $\mathbb{E}(r, \mathfrak{g})_M$. Let G be a reductive group and assume that p is good for G. Let λ be a dominant weight and consider the induced module $M = \mathrm{H}^0(\lambda) = \mathrm{Ind}_B^G \lambda$. By a result of Nakano, Parshall, and Vella [NPV02, 6.2.1], $V(\underline{\mathfrak{g}})_{\mathrm{H}^0(\lambda)} = G \cdot \mathfrak{u}_J$, where $\mathfrak{u}_J$ is the nilpotent radical of a suitably chosen parabolic subgroup $P_J \subset G$. Then,

$$\mathbb{E}(r, \mathfrak{g})_{\mathrm{H}^0(\lambda)} \;=\; G \cdot \{\epsilon \in \mathbb{E}(r, \mathfrak{g});\ \epsilon \cap \mathfrak{u}_J \neq \{0\}\}.$$

We now proceed to consider invariants of $\mathfrak{u}(\mathfrak{g})$-modules associated to $\mathbb{E}(r, \mathfrak{g})$ which for $r > 1$ are not determined by the case $r = 1$. As before, for a given M and a given $r \geq 1$, we consider the restrictions $\epsilon^*(M)$ for $\epsilon \in \mathbb{E}(r, \mathfrak{g})$.

Definition 3.9. Let $\mathfrak{g}$ be a finite dimensional p-restricted Lie algebra and M a finite dimensional $\mathfrak{u}(\mathfrak{g})$-module. For any $r \geq 1$, any $\epsilon \in \mathbb{E}(r, \mathfrak{g})$, and any $j, 1 \leq j \leq (p-1)r$, we consider

$$\mathrm{Rad}^j(\epsilon^*(M)) \;=\; \sum_{j_1+\cdots+j_r=j} \mathrm{Im}\{u_1^{j_1} \cdots u_r^{j_r} : M \to M\}$$

and

$$\mathrm{Soc}^j(\epsilon^*(M)) \;=\; \bigcap_{j_1+\cdots+j_r=j} \mathrm{Ker}\{u_1^{j_1} \cdots u_r^{j_r} : M \to M\},$$

where $\{u_1, \ldots, u_r\}$ is a basis for ϵ.

For each $r \geq 1$ and each $j, 1 \leq j \leq (p-1)r$, we define the local (r, j)-radical rank of M and the local (r, j)-socle rank of M to be the (non-negative) integer valued functions

$$\epsilon \in \mathbb{E}(r, \mathfrak{g}) \;\mapsto\; \dim \mathrm{Rad}^j(\epsilon^*(M))$$

and

$$\epsilon \in \mathbb{E}(r, \mathfrak{g}) \;\mapsto\; \dim \mathrm{Soc}^j(\epsilon^*(M))$$

respectively.

Remark 3.10. If M is a $\mathfrak{u}(\mathfrak{g})$-module, we denote by $M^\# = \mathrm{Hom}_k(M, k)$ the dual of M whose $\mathfrak{u}(\mathfrak{g})$-module structure arises from that on M using the antipode of $\mathfrak{u}(\mathfrak{g})$. Thus, if $X \in \mathfrak{g}$ and $f \in M^\#$, then $(X \circ f)(m) = -f(X \circ m)$. If $i : L \subset M$ is a $\mathfrak{u}(\mathfrak{g})$-submodule, then we denote by $L^\perp \subset M^\#$ the submodule defined as the kernel of $i^\# : M^\# \to L^\#$. We remind the reader that

$$\mathrm{Soc}^j(\epsilon^*(M^\#)) \;\simeq\; (\mathrm{Rad}^j(\epsilon^* M))^\perp \tag{3.10.1}$$

(as shown in [CFP12, 2.2]).

The following elementary observation will enable us to conclude that constructions of §4 determine vector bundles on G-orbits of $\mathbb{E}(r, \operatorname{Lie} G)$.

Proposition 3.11. *If $\mathfrak{g} = \operatorname{Lie}(G)$ and M is a rational G-module, then the local (r,j)-radical rank of M and the local (r,j)-socle rank of M are constant on G-orbits of $\mathbb{E}(r,\mathfrak{g})$.*

Proof. Let $g \in G$, and let $\epsilon \in \mathbb{E}(r,\mathfrak{g})$. We denote by $\epsilon^g \in \mathbb{E}(r,\mathfrak{g})$ the image of ϵ under the action of G on $\mathbb{E}(r,\mathfrak{g})$, and let $g\cdot(-) : M \to M$ be the action of G on M. Observe that

$$g : M \xrightarrow{m\mapsto gm} M^g$$

defines an isomorphism of rational G-modules, where the action of $x \in G$ on $m \in M^g$ is given by the action of gxg^{-1} on m (with respect to the G-module structure on M). Thus, the proposition follows from the observation that the pull-back of $\epsilon^{g*}(M^g)$ equals $\epsilon^*(M)$ under the isomorphism $g : \mathfrak{u}(\epsilon) \xrightarrow{\sim} \mathfrak{u}(\epsilon^g)$. □

The following discussion leads to Proposition 3.14 which establishes the lower and upper semi-continuity of local (r,j)-radical rank and local (r,j)-socle rank respectively.

Notation 3.12. We fix a basis $\{x_1,\dots,x_n\}$ of $\mathfrak{g}$ and use it to identify $M_{n,r} \simeq \mathfrak{g}^{\oplus r}$ as in the beginning of §1. Let $\Sigma \subset \{1,\dots,n\}$ be an r-subset. Recall the section $s_\Sigma : U_\Sigma \to \mathbb{M}^o_{n,r}$ of (1.1.1) that sends an r-plane $\epsilon \in U_\Sigma$ to the $n\times r$ matrix $A^\Sigma(\epsilon)$ with the $r \times r$ submatrix corresponding to Σ being the identity and the columns generating the plane ϵ. Extend the map s_Σ to $s_\Sigma : U_\Sigma \to \mathbb{M}_{n,r}$ and consider the induced map on coordinate algebras:

$$k[\mathbb{M}_{n,r}] = k[T_{i,s}] \xrightarrow{s^*_\Sigma} k[U_\Sigma\ ] \tag{3.12.1}$$

We define

$$T^\Sigma_{i,s} \equiv s^*_\Sigma(T_{i,s})$$

It follows from the definition that $T^\Sigma_{i,s} = \delta_{\alpha^{-1}(i),s}$ for $i \in \Sigma$, where $\alpha : \{1,\dots,r\} \to \Sigma$ is the function with $\alpha(1) < \dots < \alpha(r)$, and that $T^\Sigma_{i,s}$ for $i \notin \Sigma$ are algebraically independent generators of $k[U_\Sigma]$.

Let $V_\Sigma \equiv \mathbb{E}(r,\mathfrak{g}) \cap U_\Sigma$. We define the set $\{Y^\Sigma_{i,s}\}$ of algebraic generators of $k[V_\Sigma]$ as images of $\{T^\Sigma_{i,s}\}$ under the map of coordinate algebras induced by the closed immersion $V_\Sigma \subset U_\Sigma$:

$$k[U_\Sigma] \twoheadrightarrow k[V_\Sigma]\ , \quad T^\Sigma_{i,s} \mapsto Y^\Sigma_{i,s}$$

It again follows that $Y^\Sigma_{i,s} = \delta_{\alpha^{-1}(i),s}$, for $i \in \Sigma$ and α as above. For each $\epsilon \in V_\Sigma \subset U_\Sigma$ (implicitly assumed to be a k-rational point), we have

$$Y^\Sigma_{i,s}(\epsilon) = T^\Sigma_{i,s}(\epsilon) = s^*_\Sigma(T^\Sigma_{i,s})(\epsilon) = T_{i,s}(s_\Sigma(\epsilon)).$$

Hence,

$$A^\Sigma(\epsilon) = [Y^\Sigma_{i,s}(\epsilon)]. \tag{3.12.2}$$

Definition 3.13. For a $\mathfrak{u}(\mathfrak{g})$-module M, and for a given $s, 1 \leq s \leq r$, we define the endomorphism of $k[V_\Sigma]$–modules

$$\Theta_s^\Sigma \equiv \sum_{i=1}^n x_i \otimes Y_{i,s}^\Sigma : M \otimes k[V_\Sigma] \to M \otimes k[V_\Sigma], \tag{3.13.1}$$

via

$$m \otimes 1 \mapsto \sum_i x_i m \otimes Y_{i,s}^\Sigma.$$

We refer the reader to [Hart, III.12] for the definition of an upper/lower semi-continuous function on a topological space.

Proposition 3.14. *Let M be a $\mathfrak{u}(\mathfrak{g})$-module, r a positive integer, and j an integer satisfying $1 \leq j \leq (p-1)r$. Then the local (r,j)-radical rank of M is a lower semi-continuous function and the local (r,j)-socle rank of M is an upper semicontinuous function on $\mathbb{E}(r, \mathfrak{g})$.*

Proof. It suffices to show that the local (r,j)-radical rank of M is lower semi-continuous when restricted along each of the open immersions $V_\Sigma \subset \mathbb{E}(r, \mathfrak{g})$. For $\epsilon \in V_\Sigma$ with residue field K, the specialization of Θ_s^Σ at ϵ defines a linear operator $\Theta_s^\Sigma(\epsilon) = \sum_{i=1}^n Y_{i,s}^\Sigma(\epsilon) x_i$ on M_K:

$$m \mapsto \Theta_s^\Sigma(\epsilon) \cdot m = \sum_{i=1}^n Y_{i,s}^\Sigma(\epsilon) x_i m.$$

Since the columns of $[Y_{i,s}^\Sigma(\epsilon)]$ generate ϵ by (3.12.2), we get that

$$\mathrm{Rad}(\epsilon^* M) \ = \ \sum_{s=1}^r \mathrm{Im}\{\Theta_s^\Sigma(\epsilon) : M_K \to M_K\} \tag{3.14.1}$$

and

$$\mathrm{Rad}^j(\epsilon^* M) \ = \sum_{j_1 + \cdots + j_r = j} \mathrm{Im}\{\Theta_1^\Sigma(\epsilon)^{j_1} \dots \Theta_r^\Sigma(\epsilon)^{j_r} : M_K \to M_K\} = \tag{3.14.2}$$

$$\mathrm{Im}\{\bigoplus_{j_1 + \cdots + j_r = j} \Theta_1^\Sigma(\epsilon)^{j_1} \dots \Theta_r^\Sigma(\epsilon)^{j_r} : M_K^{\oplus r(j)} \to M_K\}$$

where $r(j)$ is the number of ways to write j as the sum of non-negative integers $j_1 + \cdots + j_r$. Hence, the usual argument for lower semicontinuity of the dimension of images of a homomorphism of finitely generated free modules applied to the $k[V_\Sigma]$-linear map

$$\bigoplus_{j_1 + \cdots + j_r = j} (\Theta_1^\Sigma)^{j_1} \dots (\Theta_r^\Sigma)^{j_r} : (M \otimes k[V_\Sigma])^{\oplus r(j)} \to M \otimes k[V_\Sigma].$$

enables us to conclude that the function

$$\epsilon \in \mathbb{E}(r, \mathfrak{g}) \ \mapsto \ \dim \mathrm{Rad}^j(\epsilon^* M) \quad \text{ is lower semi-continuous.} \tag{3.14.3}$$

The upper semi-continuity of socle ranks now follows by Remark 3.10. □

Remark 3.15. To get some understanding of the operators $\Theta_s^\Sigma(\epsilon)$ occurring in the proof of Proposition 3.14, we work out the very special case in which $\mathfrak{g} = \mathfrak{g}_a \oplus \mathfrak{g}_a$, $r = 1$ (so that $\mathbb{E}(r, \mathfrak{g}) = \mathbb{P}^1$), and $j = 1$. We fix a basis $\{x_1, x_2\}$ for $\mathfrak{g}$ which induces the identification $\mathfrak{g} \simeq \mathbb{A}^2$. The two possibilities for $\Sigma \subset \{1, 2\}$ are $\{1\}, \{2\}$. Let $k[T_1, T_2]$ be the coordinate ring for $\mathbb{A}^2$ (corresponding to the fixed basis $\{x_1, x_2\}$.

Let $\Sigma = \{1\}$. We have $V_{\{1\}} = U_{\{1\}} = \{[a:b]\,|\,a \neq 0\} \simeq \mathbb{A}^1$ and the section $s_{\{1\}} : V_{\{1\}} \to \mathbb{A}^2$ given explicitly as $[a:b] \mapsto (1, b/a)$. The corresponding map of coordinate algebras as in (3.12.1) is given by

$$k[\mathbb{A}^2] = k[T_1, T_2] \to k[V_{\{1\}}] \simeq k[\mathbb{A}^1]$$

$$T_1 \mapsto 1, T_2 \mapsto s^*_{\{1\}}(T_2)$$

Then for a $\mathfrak{u}(\mathfrak{g})$-module M, $\epsilon = \langle a, b\rangle \in \mathbb{P}^1$ with $a \neq 0$, and $m \in M$, we have

$$\Theta^{\{1\}} \;=\; x_1 \otimes 1 + x_2 \otimes s^*_{\{1\}}(T_2) : M \otimes k[V_{\{1\}}] \to M \otimes k[V_{\{1\}}]; \tag{3.15.1}$$

$$\Theta^{\{1\}}(\epsilon) \;=\; x_1 + \frac{b}{a}x_2, \quad m \mapsto x_1(m) + \frac{b}{a}x_2(m).$$

We extend the formulation of "generalized support varieties" introduced in [FP10] for $r = 1$ and in [CFP12] for elementary abelian p-groups (or, equivalently, for $\mathfrak{g} = \mathfrak{g}_a^{\oplus r}$) to any r and an arbitrary restricted Lie algebra $\mathfrak{g}$.

Definition 3.16. For any finite dimensional $\mathfrak{u}(\mathfrak{g})$-module M, any positive integer r, and any j, $1 \leq j \leq (p-1)r$, we define

$$\mathbb{R}\mathrm{ad}^{\mathrm{j}}(\mathrm{r}, \mathfrak{g})_{\mathrm{M}} \;\equiv\; \{\epsilon \in \mathbb{E}(\mathrm{r}, \mathfrak{g}) : \dim(\mathrm{Rad}^{\mathrm{j}}(\epsilon^*\mathrm{M})) < \max_{\epsilon' \in \mathbb{E}(\mathrm{r},\mathfrak{g})} \dim \mathrm{Rad}^{\mathrm{j}}(\epsilon'^*\mathrm{M})\}$$

$$\mathbb{S}\mathrm{oc}^{\mathrm{j}}(\mathrm{r}, \mathfrak{g})_{\mathrm{M}} \;\equiv\; \{\epsilon \in \mathbb{E}(\mathrm{r}, \mathfrak{g}) : \dim(\mathrm{Soc}^{\mathrm{j}}(\epsilon^*\mathrm{M})) > \min_{\epsilon' \in \mathbb{E}(\mathrm{r},\mathfrak{g})} \dim \mathrm{Soc}^{\mathrm{j}}(\epsilon'^*\mathrm{M})\}$$

It follows from Prop. 3.14 that $\mathbb{R}\mathrm{ad}^{\mathrm{j}}(\mathrm{r}, \mathfrak{g})_{\mathrm{M}}$, $\mathbb{S}\mathrm{oc}^{\mathrm{j}}(\mathrm{r}, \mathfrak{g})_{\mathrm{M}}$ are closed subvarieties in $\mathbb{E}(\mathfrak{g})$.

Proposition 3.17. *Let M be a finite-dimensional $\mathfrak{g}$-module, and let r, j be positive integers such that $1 \leq j \leq (p-1)r$. Then $\mathbb{R}\mathrm{ad}^{\mathrm{j}}(\mathrm{r}, \mathfrak{g})_{\mathrm{M}}$, $\mathbb{S}\mathrm{oc}^{\mathrm{j}}(\mathrm{r}, \mathfrak{g})_{\mathrm{M}}$ are proper closed subvarieties in $\mathbb{E}(r, \mathfrak{g})$.*

To give our first application, we need the following elementary fact.

Lemma 3.18. *Let $k[x_1, \ldots, x_n]$ be a polynomial ring, let $x_1^{i_1} \ldots x_n^{i_n}$ be a monomial of degree i and assume that $p = char\, k > i$. There exist linear polynomials without constant term $\lambda_0, \ldots, \lambda_m$ on the variables $x_1, \ldots, x_n$, and scalars $a_0, \ldots, a_m \in k$ such that*

$$x_1^{i_1} \ldots x_n^{i_n} = a_0\lambda_0^i + \ldots + a_m\lambda_m^i.$$

Proof. It suffices to prove the statement for $n = 2$, thanks to an easy induction argument (with respect to n). Hence, we assume that we have only two variables, x and y.

Let $\lambda_j = jx + y$ for $j = 0, \ldots, i$, so that we have $i + 1$ equalities:

$$\begin{array}{rcl} y^i & = & \lambda_0^i \\ (x+y)^i & = & \lambda_1^i \\ (2x+y)^i & = & \lambda_2^i \\ \vdots & & \vdots \\ (ix+y)^i & = & \lambda_i^i \end{array}$$

Treating monomials on x, y as variables, we interpret this as a system of $i+1$ equations on $i+1$ variables with the matrix

$$\begin{pmatrix} 0 & 0 & \dots & 0 & \dots & 0 & 1 \\ 1 & i & \dots & \binom{i}{j} & \dots & i & 1 \\ 2^i & 2^{i-1}i & \dots & 2^{i-j}\binom{i}{j} & \dots & 2i & 1 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ i^i & i^{i-1}i & \dots & i^{i-j}\binom{i}{j} & \dots & i^2 & 1 \end{pmatrix}$$

By canceling the coefficient $\binom{i}{j}$ in the $j+1$st column (which is non-trivial since $p > i$) we reduce the determinant of this matrix to a non-trivial Vandermonde determinant. Hence, the matrix is invertible. We conclude the monomials $x^j y^{i-j}$ can be expressed as linear combinations of the free terms $\lambda_0^i, \dots, \lambda_i^i$. $\square$

Determination of the closed subvarieties $\mathbb{R}\mathrm{ad}^{\mathrm{j}}(\mathrm{r}, \mathfrak{g})_{\mathrm{M}}$, $\mathbb{S}\mathrm{oc}^{\mathrm{j}}(\mathrm{r}, \mathfrak{g})_{\mathrm{M}}$ of $\mathbb{E}(r, \mathfrak{g})$ appears to be highly non-trivial. The reader will find a few computer-aided calculations in [CFP12] for $\mathfrak{g} = \mathfrak{g}_a^{\oplus n}$. The following proposition presents some information of $\mathbb{E}(n-1, \mathfrak{gl}_n)$.

Proposition 3.19. *Assume that $p \geq n$. Let $X \in \mathfrak{gl}_n$ be a regular nilpotent element, and let $\epsilon \in \mathbb{E}(n-1, \mathfrak{gl}_n)$ be an $n-1$-plane with basis $\{X, X^2, \dots, X^{n-1}\}$. Then $\mathrm{GL}_n \cdot \epsilon$ is an open GL_n-orbit for $\mathbb{E}(n-1, \mathfrak{gl}_n)$.*

Proof. Let V be the standard n-dimensional representation of $\mathfrak{gl}_n$. Let ϵ' be any elementary Lie subalgebra of $\mathfrak{gl}_n$ of dimension $n-1$. If ϵ' contains a regular nilpotent element Y, then ϵ' has basis $\{Y, Y^2, \dots, Y^{n-1}\}$, since the centralizer of a regular nilpotent element in $\mathfrak{gl}_n$ is generated as a linear space by the powers of that nilpotent element. Hence, in this case ϵ' is conjugate to the fixed plane ϵ. Moreover, $\mathrm{Rad}^{n-1}(\epsilon'^*V) = \mathrm{Im}\{Y^{n-1} : V \to V\}$, and, hence, $\dim \mathrm{Rad}^{n-1}(\epsilon'^*V) = 1$.

Suppose ϵ' does not contain a regular nilpotent element. Then for any matrix $Y \in \epsilon'$, we have $Y^{n-1} = 0$. Lemma 3.18 implies that any monomial of degree $n-1$ on elements of ϵ' is trivial. Therefore, $\mathrm{Rad}^{n-1}(\epsilon'^*V) = 0$. We conclude that $\mathrm{GL}_n \cdot \epsilon$ is the complement to $\mathbb{R}\mathrm{ad}^{\mathrm{n-1}}(\mathrm{n}-1, \mathfrak{gl}_{\mathrm{n}})_{\mathrm{V}}$ in $\mathbb{E}(n-1, \mathfrak{gl}_n)$. Proposition 3.14 now implies that $\mathrm{GL}_n \cdot \epsilon$ is open. $\square$

Example 3.20. In this example we describe the geometry of $\mathbb{E}(2, \mathfrak{gl}_3)$ making an extensive use of the GL_3-action. Further calculations involving more geometry will appear elsewhere.

Assume $p > 3$. Fix a regular nilpotent element $X \in \mathfrak{gl}_3$. Let $\epsilon_1 = \langle X, X^2 \rangle$ be the 2-plane in $\mathfrak{gl}_3$ with the basis X, X^2, and let

$$C_1 = \mathrm{GL}_3 \cdot \epsilon_1 \subset \mathbb{E}(2, \mathfrak{gl}_3)$$

be the orbit of ϵ_1 in $\mathbb{E}(2, \mathfrak{gl}_3)$. By Proposition 3.19, this is an open subset of $\mathbb{E}(2, \mathfrak{gl}_3)$. Since $\mathbb{E}(2, \mathfrak{gl}_3)$ is irreducible (see Example 1.6), C_1 is dense. We have $\dim C_1 = \dim \overline{C_1} = \dim \mathbb{E}(2, \mathfrak{gl}_3) = 4$.

The closure of C_1 contains two more (closed) GL_3 stable subvarieties, each one of dimension 2. They are the GL_3 saturations in $\mathbb{E}(2, \mathfrak{gl}_3)$ of the elementary subalgebras $\mathfrak{u}_{1,2}$ (spanned by $E_{1,2}$ and $E_{1,3}$), and $\mathfrak{u}_{2,1}$ (spanned by $E_{1,3}$ and $E_{2,3}$). Since the stabilizer of $\mathfrak{u}_{1,2}$ (resp. $\mathfrak{u}_{2,1}$) is the standard parabolic $P_{1,2}$ (resp. $P_{2,1}$), the corresponding orbit is readily identified with $\mathrm{GL}_3 / P_{1,2} \simeq \mathrm{Grass}_{2,3} = \mathbb{P}^2$ (resp., $\mathrm{GL}_3 / P_{2,1} \simeq \mathbb{P}^2$).

Proposition 3.21. *Let $\mathfrak{u}$ be a nilpotent p-restricted Lie algebra such that $x^{[p]} = 0$ for any $x \in \mathfrak{u}$. Then the locus of elementary subalgebras $\epsilon \in \mathbb{E}(r, \mathfrak{u})$ such that ϵ is maximal (that is, not properly contained in any other elementary subalgebra of $\mathfrak{u}$) is an open subset of $\mathbb{E}(r, \mathfrak{u})$.*

Proof. Regard $\mathfrak{u}$ as acting on itself via adjoint representation. Note that we necessarily have $\epsilon \subset \mathrm{Soc}(\epsilon^*(\mathfrak{u}_{\mathrm{ad}}))$. Moreover, our hypothesis that $x^{[p]} = 0$ for any $x \in \mathfrak{u}$ implies that this inclusion is an equality if and only if ϵ is a maximal elementary subalgebra. Hence,

$$\dim \mathrm{Soc}(\epsilon^*(\mathfrak{u}_{\mathrm{ad}})) \geq \dim \epsilon = r$$

with equality if and only if ϵ is maximal. We conclude that the locus of elementary subalgebras $\epsilon \in \mathbb{E}(r, \mathfrak{u})$ such that ϵ is nonmaximal equals the nonminimal socle variety $\mathbb{S}\mathrm{oc}(\mathrm{r}, \mathfrak{u})_{\mathfrak{u}_{\mathrm{ad}}}$. The statement now follows from Proposition 3.17. □

4. Modules of constant (r, j)-radical rank and/or and constant (r, j)-socle rank

In previous work with coauthors, we have considered the interesting class of modules of constant Jordan type (see, for example [CFP08]). In the terminology of this paper, these are $\mathfrak{u}(\mathfrak{g})$-modules M with the property that the isomorphism type of $\epsilon^* M$ is independent of $\epsilon \in \mathbb{E}(1, \mathfrak{g})$. In the special case $\mathfrak{g} = \mathfrak{g}_a^{\oplus n}$, further classes of special modules were considered by replacing this condition on the isomorphism type of $\epsilon^* M$ for $\epsilon \in \mathbb{E}(1, \mathfrak{g}_a^{\oplus n})$ by the "radical" or "socle" type of $\epsilon^* M$ for $\epsilon \in \mathbb{E}(r, \mathfrak{g}_a^{\oplus n})$.

In this section, we consider $\mathfrak{u}(\mathfrak{g})$-modules of constant (r, j)-radical rank and constant r-radical type (and similarly for socles). As already seen in [CFP12] in the special case $\mathfrak{g} = \mathfrak{g}_a^{\oplus n}$, the variation of radical and socle behavior for $r > 1$ can be quite different. Moreover, having constant r radical type does not imply the constant behavior for a different r.

As we shall see in the next section, a $\mathfrak{u}(\mathfrak{g})$-module of constant (r, j)-radical rank or constant (r, j)-socle rank determines a vector bundle on $\mathbb{E}(r, \mathfrak{g})$, thereby providing good motivation for studying such modules.

Definition 4.1. Fix integers $r > 0$ and $j, 1 \leq j < (p-1)r$. A $\mathfrak{u}(\mathfrak{g})$-module M is said to have constant (r, j)-radical rank (respectively, (r, j)-socle rank) if the dimension of $\mathrm{Rad}^j(\epsilon^* M)$ (resp., $\mathrm{Soc}^j(\epsilon^* M)$) is independent of $\epsilon \in \mathbb{E}(r, \mathfrak{g})$.

We say that M has constant r-radical type (respectively, r-socle type) if M has constant (r, j)-radical rank (resp., (r, j)-socle rank) for all $j, 1 \leq j \leq (p-1)r$.

Remark 4.2. For $r > 1$, the condition that the r-radical type of M is constant does not imply that the isomorphism type of $\epsilon^* M$ is independent of $\epsilon \in \mathbb{E}(r, \mathfrak{g})$. The condition that $\dim \mathrm{Rad}^j(\epsilon^*(M)) = \dim \mathrm{Rad}^j(\epsilon'^* M)$ for all j is much weaker than the condition that $\epsilon^* M \simeq \epsilon'^* M$. Indeed, examples are given in [CFP12] (with $\mathfrak{g} = \mathfrak{g}_a^{\oplus n}$) of modules M whose r-radical type is constant but whose r-socle type is not constant, thereby implying that the isomorphism type of $\epsilon^* M$ varies with $\epsilon \in \mathbb{E}(r, \mathfrak{g})$.

Proposition 4.3. *A $\mathfrak{u}(\mathfrak{g})$-module M has constant (r, j)-radical rank (respectively, (r, j)-socle rank) if and only if $\mathbb{R}\mathrm{ad}^j(r, \mathfrak{g})_M = \emptyset$ (resp., $\mathbb{S}\mathrm{oc}^j(r, \mathfrak{g})_M = \emptyset$.)*

Proof. This follows from the fact that there is a non-maximal radical rank if and only if the radical rank is not constant, a non-minimal socle rank if and only if the socle rank is not constant. □

Proposition 4.4. *Let G be an affine algebraic group, and let $\mathfrak{g} = \mathrm{Lie}(G)$. If $\mathbb{E}(r, \mathfrak{g})$ consists of a single G-orbit, then any finite dimensional rational G-module has constant r-radical type and constant r-socle type.*

Proof. Follows immediately from Proposition 3.11. □

Example 4.5. If P is a finite dimensional projective $\mathfrak{u}(\mathfrak{g})$-module, then $\epsilon^* P$ is a projective (and thus free) $\mathfrak{u}(\epsilon)$-module for any elementary subalgebra $\epsilon \subset \mathfrak{g}$. Thus, the r-radical type and r-socle type of P are constant.

Example 4.6. Let $\mathfrak{g}$ be a finite dimensional p-restricted Lie algebra. Recall that $\Omega^s(k)$ for $s > 0$ is the kernel of $P_{s-1} \stackrel{d}{\to} P_{s-2}$, where d is the differential in the minimal projective resolution $P_* \to k$ of k as a $\mathfrak{u}(\mathfrak{g})$-module; if $s < 0$, then $\Omega^s(k)$ is the cokernel of $I^{-s-2} \stackrel{d}{\to} I^{-s-1}$, where d is the differential in the minimal injective resolution $k = I^{-1} \to I^*$ of k as a $\mathfrak{u}(\mathfrak{g})$-module. Then for any $s \in \mathbb{Z}$, the s-th Heller shift $\Omega^s(k)$ has constant r-radical type and constant r-socle type for each $r > 0$.

Namely, for any $\epsilon \in \mathbb{E}(r, \mathfrak{g})$, $\epsilon^*(\Omega^s(k))$ is the direct sum of the s-th Heller shift of the trivial module k and a free $\mathfrak{u}(\epsilon)$-module (whose rank is independent of the choice of $\epsilon \in \mathbb{E}(r, \mathfrak{g})$).

The following example is one of many we can realize using Proposition 4.3.

Example 4.7. Let $\mathfrak{g} = \mathfrak{gl}_{2n}$ and $r = n^2$. If M is any finite dimensional rational GL_{2n}-module, then it has constant r-radical type and constant r-socle type by Corollary 2.11.

In Example 4.7, the dimension r of elementary subalgebras $\epsilon \subset \mathfrak{g}$ is maximal. We next consider an example of non-maximal elementary subalgebras.

Example 4.8. Choose $r > 0$ such that no elementary subalgebra of dimension r in $\mathfrak{g}$ is maximal. Let $\zeta \in \widehat{\mathrm{H}}^n(\mathfrak{u}(\mathfrak{g}), k)$ for $n < 0$ be an element in negative Tate cohomology. Consider the associated short exact sequence

$$0 \longrightarrow k \longrightarrow E \longrightarrow \Omega^{n-1}(k) \longrightarrow 0. \tag{4.8.1}$$

Then E has constant r-radical rank and constant r-socle rank for every j, $1 \le j \le (p-1)r$.

Namely, we observe that the restriction of the exact sequence (4.8.1) to ϵ^* splits for every $\epsilon \in \mathbb{E}(r, \mathfrak{g})$. This splitting is a consequence of [CFP12, 3.8] (stated for an elementary abelian p-group and equally applicable to any elementary subalgebra $\mathfrak{f} \subset \mathfrak{g}$ which strictly contains ϵ). The assertion is now proved with an appeal to Example 4.6.

We next proceed to consider modules L_ζ, adapting to the context of p-restricted Lie algebras the results of [CFP12, §5].

Proposition 4.9. *(see* [CFP12, 5.5]*) Suppose that we have a non-zero cohomology class $\zeta \in \mathrm{H}^m(\mathfrak{u}(\mathfrak{g}), k)$ satisfying the condition that*

$$Z(\zeta) \ \subset \ \mathcal{N}_p(\mathfrak{g}) \ \subset \ \mathfrak{g}$$

does not contain a linear subspace of dimension r for some $r \geq 1$. Then the $\mathfrak{u}(\mathfrak{g})$-module L_ζ has constant r-radical type.

Proof. Consider $\epsilon \in \mathbb{E}(r, \mathfrak{g})$. We identify $\epsilon_* : \mathcal{N}_p(\epsilon) \to \mathcal{N}_p(\mathfrak{g})$ with the composition $\epsilon \to \mathcal{N}_p(\mathfrak{g}) \subset \mathfrak{g}$. Thus, our hypothesis implies that ϵ is not contained in $Z(\zeta)$. Hence, $\zeta \downarrow_\epsilon \in \mathrm{H}^m(\mathfrak{u}(\epsilon), k)$ is not nilpotent, and, therefore, is not a zero-divisor. Proposition 5.3 of [CFP12] applied to ϵ implies that

$$\mathrm{Rad}(L_{\zeta\downarrow_\epsilon}) = \mathrm{Rad}(\Omega^n(k\downarrow_\epsilon)), \tag{4.9.1}$$

where $\Omega^n(k\downarrow_\epsilon)$ is the n-th Heller shift of the trivial $\mathfrak{u}(\epsilon)$-module. We note that the statement and proof of [CFP12, Lemma 5.4] generalizes immediately to the map $\mathfrak{u}(\epsilon) \to \mathfrak{u}(\mathfrak{g})$ yielding the statement that $\dim \mathrm{Rad}(\epsilon^*(L_\zeta)) - \dim \mathrm{Rad}(L_{\zeta\downarrow_\epsilon}) = \dim \mathrm{Rad}(\epsilon^*(\Omega^n(k))) - \dim \mathrm{Rad}(\Omega^n(k\downarrow_\epsilon))$ is independent of ϵ whenever $\zeta\downarrow_\epsilon \neq 0$. Combined with (4.9.1), this allows us to conclude that

$$\dim \mathrm{Rad}(\epsilon^*(L_\zeta)) = \dim \mathrm{Rad}(\epsilon^*(\Omega^n(k))).$$

Since $\epsilon^*(L_\zeta)$ is a submodule of $\epsilon^*(\Omega^n(k))$ this further implies that equality of radicals

$$\mathrm{Rad}^j(\epsilon^*(L_\zeta)) = \mathrm{Rad}^j(\epsilon^*(\Omega^n(k)))$$

for all $j > 0$. Since $\Omega^n(k)$ has constant r-radical type by Example 4.6, we conclude that the same holds for L_ζ. □

Utilizing another result of [CFP12], we obtain a large class of $\mathfrak{u}(\mathfrak{g})$-modules of constant radical type.

Proposition 4.10. *For any $d > 0$, there exists some $0 \neq \zeta \in \mathrm{H}^{2d}(\mathfrak{u}(\mathfrak{g}), k)$ such that L_ζ has constant r-radical type.*

Proof. The embedding $V(\underline{\mathfrak{g}}) \simeq \operatorname{Spec} \mathrm{H}^{\mathrm{ev}}(\mathfrak{u}(\mathfrak{g}), k) \to \mathfrak{g}$ (for $p > 2$) is given by the natural map $S^*(\mathfrak{g}^\#[2]) \to \mathrm{H}^*(\mathfrak{u}(\mathfrak{g}), k)$ determined by the Hochschild construction $\mathfrak{g}^\# \to \mathrm{H}^2(\mathfrak{u}(\mathfrak{g}), k)$ (see, for example, [FP83]). (Here, $\mathfrak{g}^\#[2]$ is the vector space dual to the underlying vector space of $\mathfrak{g}$, placed in cohomological degree 2.) As computed in [CFP12, 5.7], the set of all homogeneous polynomials F of degree d in $S^*(\mathfrak{g}^\#[2])$ such that the zero locus $Z(F) \subset \mathrm{Proj}(\mathfrak{g})$ does not contain a linear hyperplane isomorphic to $\mathbb{P}^{r-1}$ is dense in the space of all polynomials of degree d. Let ζ be the restriction to $\operatorname{Proj} k[V(\underline{\mathfrak{g}})]$ of such an $F \in S^*(\mathfrak{g}^\#[2])$; since such an F can be chosen from a dense subset of homogeneous polynomials of degree d, we may find such an F whose associated restriction ζ is non-zero. Now, we may apply Proposition 4.9 to conclude that L_ζ has constant r-radical type. □

The following closure property for modules of constant radical and socle types is an extension of a similar property for modules of constant Jordan type.

Proposition 4.11. *Let M be a $\mathfrak{u}(\mathfrak{g})$-module of constant (r, j)-radical rank (respectively, constant (r, j)-socle rank) for some r, j. Then any $\mathfrak{u}(\mathfrak{g})$-summand M' of M also has constant (r, j)-radical rank (resp., constant (r, j)-socle rank).*

Proof. Write $M = M' \oplus M''$, and set m equal to the (r, j)-radical rank of M. Since the local (r, j)-radical types of M', M'' are both lower semicontinuous by Proposition 3.14 and since the sum of these local radical types is the constant function m, we conclude that both M', M'' have constant (r, j)-radical rank.

The argument for (r, j)-socle rank is essentially the same. □

5. Coherent sheaves on locally closed subvarieties of $\mathbb{E}(r, \mathfrak{g})$

Extending the construction of vector bundles given in [CFP12], we construct image coherent sheaves $\mathcal{I}m^j(M)$ and kernel coherent sheaves $\mathcal{K}er^j(M)$ on $\mathbb{E}(r, \mathfrak{g})$ associated to a $\mathfrak{u}(\mathfrak{g})$-module M. Our construction in Theorem 5.4 involves the patching of images (respectively, kernels) of explicit linear maps on affine opens of $\mathbb{E}(r, \mathfrak{g})$. The reader should keep in mind that these image and kernel sheaves are not images and kernels of the action of $\mathfrak{u}(\mathfrak{g})$, but rather globalizations of images and kernels of local actions on $M \otimes k[V_\Sigma]$ whose fibers above a generic point $\epsilon \in V_\Sigma$ are given by the images and kernels of the action of $\mathfrak{u}(\epsilon)$. In Theorem 5.18, we show that these kernel and image sheaves are the *same* as subsheaves of the free coherent sheaf $M \otimes \mathcal{O}_{\mathbb{E}(r,\mathfrak{g})}$ as the coherent sheaves we obtain by equivariant descent with respect to the GL_r-torsor $\mathcal{N}_p^r(\mathfrak{g})^o \to \mathbb{E}(r, \mathfrak{g})$ and globally defined operators on free coherent sheaves of $\mathcal{O}_{\mathcal{N}_p^r(\mathfrak{g})^o}$-modules

Throughout this section, we adopt most of the terminology introduced in Notation 3.12 and Definition 3.13. In particular, we fix a basis $\{x_1, \ldots, x_n\}$ for $\mathfrak{g}$; as shown in Corollary **??**, the image and kernel sheaves we introduce are do not depend upon this choice of basis, though it is used in their formulation.

Throughout this section, we adopt most of the terminology introduced in Notation 3.12 and Definition 3.13. We make a minor generalization in the definition of the local coordinates $Y_{i,s}^\Sigma$ extending our considerations from $\mathbb{E}(r, \mathfrak{g})$ to an arbitrary closed subset $W \subset \mathrm{Grass}(r, \mathfrak{g})$. Indeed, we could consider locally closed subsets $X \subset \mathrm{Grass}(r, \mathfrak{g})$ at the cost of working with coherent sheaves on X rather than the simpler situation of modules for $k[W]$.

Notation 5.1. Let $W \subset \mathrm{Grass}(r, \mathfrak{g})$ be a closed subset. Let $\Sigma \subset \{1, \ldots, n\}$ be a subset of cardinality r. Recall the closed embedding $s_\Sigma : U_\Sigma \hookrightarrow \mathbb{M}_{n,r}$ induced by the section $s_\Sigma : U_\Sigma \hookrightarrow \mathbb{M}_{n,r}^o$ and the local variables $T_{i,s}^\Sigma = s_\Sigma^*(T_{i,s})$ generating $k[U_\Sigma]$.

Let $W_\Sigma = W \cap U_\Sigma$. Define $Y_{i,s}^{W,\Sigma}$ to be the image of $T_{i,s}^\Sigma$ under the projection

$$k[U_\Sigma] \twoheadrightarrow k[W_\Sigma]\ , \quad T_{i,s}^\Sigma \mapsto Y_{i,s}^{W,\Sigma}$$

We reserve the notation $Y_{i,s}^\Sigma$ for the special case $W = \mathbb{E}(r, \mathfrak{g})$ which is of most interest to us. For a $\mathfrak{u}(\mathfrak{g})$-module M we define the following $k[W_\Sigma]$-endomorphism generalizing Definition 3.13 to an arbitrary closed subset W:

$$\Theta_s^{W,\Sigma} \equiv \sum_{i=1}^n x_i \otimes Y_{i,s}^{W,\Sigma} : M \otimes k[W_\Sigma] \to M \otimes k[W_\Sigma], \tag{5.1.1}$$

via

$$m \otimes 1 \mapsto \sum_i x_i(m) \otimes Y_{i,s}^{W,\Sigma}.$$

We again reserve the notation Θ_s^Σ for $W = \mathbb{E}(r, \mathfrak{g})$.

Definition 5.2. (cf. [CFP12, 6.1]) Let M be a $\mathfrak{u}(\mathfrak{g})$-module, let $W \subset \mathrm{Grass}_r(\mathfrak{g})$ be a closed subset, and let $\Sigma \subset \{1, \ldots, n\}$ be a subset of cardinality r. We define two $k[W_\Sigma]$-submodules of the free module $M \otimes k[W_\Sigma]$:

$$\mathcal{I}m(M)_{W_\Sigma} \equiv \sum_{s=1}^r \mathrm{Im}\, \Theta_s^{W,\Sigma} = \mathrm{Im}\{\Theta_1^{W,\Sigma} + \ldots + \Theta_r^{W,\Sigma} : (M \otimes k[W_\Sigma])^{\oplus r} \to M \otimes k[W_\Sigma]\}$$

$$\mathcal{K}er(M)_{W_\Sigma} \equiv \bigcap_{s=1}^{r} \operatorname{Ker} \Theta_s^{W,\Sigma} = \operatorname{Ker}\{\bigoplus_{s=1}^{r} \Theta_s^{W,\Sigma} : M \otimes k[W_\Sigma] \to (M \otimes k[W_\Sigma])^{\oplus r}\}.$$

We identify these $k[W_\Sigma]$-submodules of $M \otimes k[W_\Sigma]$ with coherent subsheaves of the free $\mathcal{O}_W$-module $M \otimes \mathcal{O}_W$ restricted to the affine open subvariety $W_\Sigma \subset W$.

We remind the reader of the following elementary lemma.

Lemma 5.3. *Let X be a topological space with an open covering $\{U_i, i \in I\}$, and let G be a sheaf on X. Suppose one is given subsheaves $F_i \subset G_{|U_i}$ for all i such that $(F_i)_{|U_{i,j}} = (F_j)_{|U_{i,j}}$ for all pairs i, j, where $U_{i,j} = U_i \cap U_j$. Then the F_i patch together to determine a uniquely defined subsheaf $F \subset G$ satisfying $F_{|U_i} = F_i$ (for all $i \in I$).*

We now are in a position to construct image and kernel sheaves.

Theorem 5.4. *Retain the notation of Definition 5.2. The coherent subsheaves*

$$\mathcal{I}m(M)_{W_\Sigma} \subset M \otimes (\mathcal{O}_W)_{|W_\Sigma} = M \otimes k[W_\Sigma]$$

on W_Σ patch together to determine a (unique) coherent subsheaf

$$\mathcal{I}m^W(M) \subset M \otimes \mathcal{O}_W \tag{5.4.1}$$

on W. Similarly, the coherent subsheaves

$$\mathcal{K}er(M)_{W_\Sigma} \subset M \otimes (\mathcal{O}_W)_{|W_\Sigma}$$

patch together to determine a (unique) coherent subsheaf

$$\mathcal{K}er^W(M) \subset M \otimes \mathcal{O}_W. \tag{5.4.2}$$

Proof. By Lemma 5.3, it suffices to prove

$$\mathcal{K}er(M)_{W_\Sigma} \otimes_{k[W_\Sigma]} k[W_{\Sigma,\Sigma'}] = \mathcal{K}er(M)_{W_{\Sigma'}} \otimes_{k[W_{\Sigma'}]} k[W_{\Sigma,\Sigma'}]$$

$$\mathcal{I}m(M)_{W_\Sigma} \otimes_{k[W_\Sigma]} k[W_{\Sigma,\Sigma'}] = \mathcal{I}m(M)_{W_{\Sigma'}} \otimes_{k[W_{\Sigma'}]} k[W_{\Sigma,\Sigma'}]$$

for any pair of subsets $\Sigma, \Sigma' \subset \{1, \ldots, n\}$, where $W_{\Sigma,\Sigma'} = W_\Sigma \cap W_{\Sigma'}$.

Since localization is exact, the equality of kernels (respectively, images) restricted to $W_{\Sigma,\Sigma'}$ is equivalent to the equality of kernels (resp., images) of the localized maps:

$$\bigoplus_{s=1}^{r} \Theta_s^{W,\Sigma} : M \otimes_{k[W_\Sigma]} k[W_{\Sigma,\Sigma'}] \to (M \otimes_{k[W_\Sigma]} k[W_{\Sigma,\Sigma'}])^{\oplus r} \tag{5.4.3}$$

and

$$\bigoplus_{s=1}^{r} \Theta_s^{W,\Sigma'} : M \otimes_{k[W_{\Sigma'}]} k[W_{\Sigma,\Sigma'}] \to (M \otimes_{k[W_{\Sigma'}]} k[W_{\Sigma,\Sigma'}])^{\oplus r}$$

We express our operators in matrix terms:

$$\bigoplus_{s=1}^{r} \Theta_s^{W,\Sigma} = [x_1, \ldots, x_n] \otimes [Y_{i,s}^{W,\Sigma}], \quad \bigoplus_{s=1}^{r} \Theta_s^{W,\Sigma'} = [x_1, \ldots, x_n] \otimes [Y_{i,s}^{W,\Sigma'}] \tag{5.4.4}$$

There is an invertible matrix $A_{\Sigma,\Sigma'}^W \in \mathrm{GL}_n(k[W_{\Sigma,\Sigma'}])$ which is the "change of generators" matrix from $Y_{i,s}^{W,\Sigma}$ variables to $Y_{i,s}^{W,\Sigma'}$ variables. We have

$$\boxed{[Y_{i,s}^{W,\Sigma'}] = A_{\Sigma,\Sigma'}^W [Y_{i,s}^{W,\Sigma}].} \tag{5.4.5}$$

Hence,

$$\operatorname{Ker} \bigoplus_{s=1}^{r} \Theta_s^{W,\Sigma'} = \operatorname{Ker}[x_1, \ldots, x_n] \otimes [Y_{i,s}^{W,\Sigma'}] = \operatorname{Ker}[x_1, \ldots, x_n] \otimes A_{\Sigma,\Sigma'}^{W}[Y_{i,s}^{W,\Sigma}] =$$

$$\operatorname{Ker}([1] \otimes A_{\Sigma,\Sigma'}^{W})([x_1, \ldots, x_n] \otimes [Y_{i,s}^{W,\Sigma}]) = \operatorname{Ker}[x_1, \ldots, x_n] \otimes [Y_{i,s}^{W,\Sigma}] = \operatorname{Ker} \bigoplus_{s=1}^{r} \Theta_s^{W,\Sigma}$$

in $M \otimes k[W_{\Sigma,\Sigma'}]$. The proof for images is strictly analogous.

We now explain the construction of the change of generators matrix $A^W_{\Sigma,\Sigma'}$ in more detail. Let $\mathfrak{p}_\Sigma(T) \in k[T_{i,s}] = k[\mathbb{M}_{n,r}]$ be the Σ-minor of the $n \times r$ matrix $[T_{i,s}]$. Let

$$\alpha_\Sigma : \{1, \ldots, r\} \to \Sigma$$

be the bijection with $\alpha_\Sigma(1) < \alpha_\Sigma(2) < \cdots < \alpha_\Sigma(r)$, and let A^Σ be an $n \times r$ matrix given by the rule:

$$A^\Sigma_{i,j} = \begin{cases} T_{i,j} \text{ for } i \notin \Sigma \\ \delta_{\alpha_\Sigma^{-1}(i),j} \text{ for } i \in \Sigma. \end{cases}$$

Define $A_{\Sigma,\Sigma'} \in M_n(k[T_{i,j}, p_{\Sigma'}^{-1}])$ to be the transition matrix from A^Σ to $A^{\Sigma'}$:

$$A_{\Sigma,\Sigma'} A^\Sigma = A^{\Sigma'}. \tag{5.4.6}$$

To construct this transition matrix, we first multiply A^Σ by an appropriate matrix in $M_n(k[T_{i,j}])$ to get $[T_{i,j}]$, and then by another matrix to make the submatrix corresponding to Σ' identity. The second matrix requires inverting the minor $\mathfrak{p}_{\Sigma'}(T)$. The transition matrix $A_{\Sigma,\Sigma'}$ is invertible in $M_n(k[T_{i,j}, \mathfrak{p}_\Sigma^{-1}(T), \mathfrak{p}_{\Sigma'}^{-1}(T)])$ since the analogously constructed matrix $A_{\Sigma',\Sigma} \in M_n(k[T_{i,j}, \mathfrak{p}_\Sigma^{-1}(T)])$ is the left inverse.

Since s_Σ is a section of the GL_r-torsor $p : \mathbb{M}^o_{n,r} \to \operatorname{Grass}_r(\mathfrak{g})$ on U_Σ, we have the following relations

$$p^*(T^\Sigma_{i,s}) = p^*(s^*_\Sigma(T_{i,s})) = \begin{cases} T_{i,s} \text{ for } i \notin \Sigma \\ \delta_{\alpha^{-1}(i),s} \text{ for } i \in \Sigma, \end{cases}$$

Hence, we can rewrite (5.4.6) as

$$A_{\Sigma,\Sigma'}[p^*(T^\Sigma_{i,s})] = [p^*(T^{\Sigma'}_{i,s})]. \tag{5.4.7}$$

Applying $s^*_{\Sigma'}$ entry-wise to both sides, we get

$$A^{\mathrm{Gr}}_{\Sigma,\Sigma'}[T^\Sigma_{i,s}] \equiv (s^*_{\Sigma'} \circ A_{\Sigma,\Sigma'} \circ p^*)[T^\Sigma_{i,s}] = [s^*_{\Sigma'} p^*(T^{\Sigma'}_{i,s})] = [T^{\Sigma'}_{i,s}]. \tag{5.4.8}$$

Finally, we set $A^W_{\Sigma,\Sigma'} \equiv i^*(A^{\mathrm{Gr}}_{\Sigma,\Sigma'})$ where $i^* : k[U_\Sigma \cap U_{\Sigma'}] \to k[W_\Sigma \cap W_{\Sigma'}]$ is the map on algebras induced by the embedding $i : W \subset \operatorname{Grass}_r(\mathfrak{g})$. The equation (5.4.5) follows by applying i^* to (5.4.8). $\square$

Definition 5.5. Let M be a $\mathfrak{u}(\mathfrak{g})$-module and let $X \subset \operatorname{Grass}(r, V)$ be a locally closed subvariety. We define the image sheaf

$$\mathcal{I}m^X(M) \subset M \otimes \mathcal{O}_X$$

to be the coherent sheaf obtained by restricting the coherent sheaf $\mathcal{I}m^W(M)$ of Theorem 5.4 to X, where W is the closure of X in $\operatorname{Grass}(r, \mathbb{V})$.

Similarly, we define the kernel sheaf

$$\mathcal{K}er^X(M) \subset M \otimes \mathcal{O}_X$$

to be the coherent sheaf obtained by restricting the coherent sheaf $\mathcal{K}er^W(M)$ of Theorem 5.4 to X.

We now assume that $W \subset \mathbb{E}(r,\mathfrak{g})$. Under this assumption, the operators $\Theta_s^{\Sigma,W}$ commute which allows us to extend Definitions 5.2 as follows:

Definition 5.6. Let M be a $\mathfrak{u}(\mathfrak{g})$-module, let $W \subset \mathbb{E}(r,\mathfrak{g})$ be a closed subset, let $\Sigma \subset \{1,\dots,n\}$ be a subset of cardinality r, and let j be a positive integer $\leq (p-1)r$. We define the following $k[W_\Sigma]$-submodules of the free module $M \otimes k[W_\Sigma]$:

$$\mathcal{K}er^j(M)_{W_\Sigma} \equiv \operatorname{Ker}\{\bigoplus_{j_1+\cdots+j_r=j} (\Theta_1^{W,\Sigma})^{j_1}\dots(\Theta_r^{W,\Sigma})^{j_r} : M\otimes k[W_\Sigma] \to (M\otimes k[W_\Sigma])^{\oplus r(j)}\}$$

$$\mathcal{I}m^j(M)_{W_\Sigma} \equiv \operatorname{Im}\{\sum_{j_1+\cdots+j_r=j} (\Theta_1^{W,\Sigma})^{j_1}\dots(\Theta_r^{W,\Sigma})^{j_r} : (M\otimes k[W_\Sigma])^{\oplus r(j)} \to M\otimes k[W_\Sigma]\}$$

where $r(j)$ is the number of ways j can be written as a sum of r non-negative integers, $j = j_1 + \cdots + j_r$.

Notation 5.7. In the special case $j=1$, we continue to use $\mathcal{K}er^W(M)$, $\mathcal{I}m^W(M)$ rather than $\mathcal{K}er^{j,W}(M)$, $\mathcal{I}m^{j,W}(M)$. In the special case in which W equals $\mathbb{E}(r,\mathfrak{g})$, we drop the superscript W and simply write $\mathcal{K}er^j(M)$, $\mathcal{I}m^j(M)$.

Theorem 5.8. *The coherent subsheaves*

$$\mathcal{I}m^j(M)_{W_\Sigma} \subset M \otimes (\mathcal{O}_W)_{|W_\Sigma} = M \otimes k[W_\Sigma]$$

of Definition 5.6 patch together to determine a (unique) coherent subsheaf

$$\mathcal{I}m^{j,W}(M) \subset M \otimes \mathcal{O}_W. \tag{5.8.1}$$

Similarly, the coherent subsheaves

$$\mathcal{K}er^j(M)_{W_\Sigma} \subset M \otimes (\mathcal{O}_W)_{|W_\Sigma})$$

of Definition 5.6 patch together to determine a (unique) coherent subsheaf

$$\mathcal{K}er^{j,W}(M) \subset M \otimes \mathcal{O}_W. \tag{5.8.2}$$

Proof. We need to prove an analogue of Theorem 5.4 for $j>1$. The proof proceeds exactly as in the $j=1$ case with a minor change that we describe. Let $A^W_{\Sigma,\Sigma}$ be the change of variables matrix as in (5.4.5). We have

$$\operatorname{Ker} \bigoplus_{\sum j_\ell = j} (\Theta_1^{W,\Sigma'})^{j_1}\dots(\Theta_r^{W,\Sigma'})^{j_r} = \operatorname{Ker} \bigoplus_{\sum j_\ell = j} ([x_i]\otimes[Y_{i,1}^{W,\Sigma'}])^{j_1}\dots([x_i]\otimes[Y_{i,r}^{W,\Sigma'}])^{j_r} =$$

$$\operatorname{Ker} \bigoplus_{j_1+\cdots+j_r=j} ([x_i]\otimes A^W_{\Sigma,\Sigma'}[Y_{i,1}^{W,\Sigma}])^{j_1}\dots([x_i]\otimes A^W_{\Sigma,\Sigma'}[Y_{i,r}^{W,\Sigma}])^{j_r}) =$$

$$\operatorname{Ker}\ ([1]\otimes A^W_{\Sigma,\Sigma'})^j(\bigoplus_{j_1+\cdots+j_r=j} ([x_i]\otimes[Y_{i,1}^{W,\Sigma}])^{j_1}\dots([x_i]\otimes[Y_{i,r}^{W,\Sigma}])^{j_r}) =$$

$$\operatorname{Ker} \bigoplus_{j_1+\cdots+j_r=j} ([x_i]\otimes[Y_{i,1}^{W,\Sigma}])^{j_1}\dots([x_i]\otimes[Y_{i,r}^{W,\Sigma}])^{j_r} = \operatorname{Ker} \bigoplus_{j_1+\cdots+j_r=j} (\Theta_1^{W,\Sigma})^{j_1}\dots(\Theta_r^{W,\Sigma})^{j_r}.$$

The proof for images is similar. $\square$

Definition 5.9. Let M be a $\mathfrak{u}(\mathfrak{g})$-module and let $X \subset \mathbb{E}(r,\mathfrak{g})$ be a locally closed subvariety. We define the j-image sheaf

$$\mathcal{I}m^{j,X}(M) \ \subset \ M \otimes \mathcal{O}_X$$

to be the coherent sheaf obtained by restricting the coherent sheaf $\mathcal{I}m^{j,W}(M)$ of Theorem 5.8 to X, where W is the closure of X in $\mathrm{Grass}(r,\mathbb{V})$.

Similarly, we define the j-kernel sheaf

$$\mathcal{K}er^{j,X}(M) \ \subset \ M \otimes \mathcal{O}_X$$

to be the coherent sheaf obtained by restricting the coherent sheaf $\mathcal{K}er^{j,W}(M)$ of Theorem 5.4 to X.

We also define the sheaf $\mathcal{C}oker^{j,W}(M)$ to be the cokernel of the embedding $\mathcal{I}m^{j,X}(M) \to M \otimes \mathcal{O}_X$

The following elementary proposition identifies the "generic" fibers of the image and kernel sheaves of Definition 5.9. This is particularly useful when the locally closed subset $X \subset \mathbb{E}(r,\mathfrak{g})$ is an orbit closure.

Proposition 5.10. *Let M be a $\mathfrak{u}(\mathfrak{g})$-module, $X \subset \mathrm{Grass}_r(\mathfrak{g})$ be a locally closed subset, $W = \overline{X}$ be the closure of X, and r, j be positive integers with $j \leq (p-1)r$. If $j > 1$, we further assume that $X \subset \mathbb{E}(r,\mathfrak{g})$. For any $\Sigma \subset \{1,\ldots,n\}$ of cardinality r there exists an open dense subset $U \subset X \cap W_\Sigma$ such that for any point $\epsilon \in U$ with residue field K there are natural identifications*

$$\mathcal{I}m^{j,X}(M)_\epsilon = \mathcal{I}m^j(M)_{W_\Sigma} \otimes_{k[W_\Sigma]} K \ = \ \mathrm{Rad}^j(\epsilon^*(M_K)),$$

$$\mathcal{K}er^{j,X}(M)_\epsilon = \mathcal{K}er^j(M)_{W_\Sigma} \otimes_{k[W_\Sigma]} K \ = \ \mathrm{Soc}^j(\epsilon^*(M_K)).$$

Proof. Since X is open dense in W, we may assume that $W = X$. For $\epsilon \in W_\Sigma$ a generic point, the given identifications are immediate consequences of the exactness of localization and (3.14.1) (together with its analogue for kernels/socles). The fact that these identifications apply to an open subset now follows from the generic flatness of the $k[W_\Sigma]$-modules $\mathcal{I}m^j(M)_{W_\Sigma}$, $\mathcal{K}er^j(M)_{W_\Sigma}$. $\square$

Remark 5.11. To see why the isomorphism $\mathcal{I}m^j(M)_\epsilon \ \simeq \ \mathrm{Rad}^j(\epsilon^*M)$ is not valid for a general $\mathfrak{u}(\mathfrak{g})$-module M and an arbitrary point $\epsilon \in \mathbb{E}(r,\mathfrak{g})$, we consider the short exact sequence of bundles on $\mathbb{E}(r,\mathfrak{g})$,

$$0 \to \mathcal{I}m^j(M) \to M \otimes \mathcal{O}_{\mathbb{E}(r,\mathfrak{g})} \to \mathcal{C}oker^j(M) \to 0. \tag{5.11.1}$$

and specialize at some point $\epsilon \in \mathbb{E}(r,\mathfrak{g})$. The equality $\mathcal{I}m^j(M)_\epsilon = \mathrm{Rad}^j(\epsilon^*M)$ is equivalent to (left) exactness of specializations at the point ϵ. In particular, if $R = \mathcal{O}_{\mathbb{E}(r,\mathfrak{g}),\epsilon}$, the stalk of the structure sheaf at ϵ, a sufficient (but apparently not necessary) condition for $\mathcal{I}m^j(M_X)_\epsilon \ \simeq \ \mathrm{Rad}^j(\epsilon^*M)$ is vanishing of $Tor^1_R(k, \mathcal{C}oker^j(M)_R)$.

For an elementary example of the failure of the isomorphism $\mathcal{K}er^j(M)_\epsilon \ \simeq \ \mathrm{Soc}^j(\epsilon^*M)$ outside of an open subset of W_Σ, we consider $\mathfrak{g} = \mathfrak{g}_a \oplus \mathfrak{g}_a$, take $r = 1$ and $j = 1$ as in Remark 3.15. Let $\{x_1, x_2\}$ be a fixed basis of $\mathfrak{g}$, and let M be the four dimensional module with basis $\{m_1,\ldots,m_4\}$, such that $x_1m_1 = m_4$, $x_1m_2 = x_1m_3 = x_1m_4 = 0$ and $x_2m_1 = m_3$, $x_2m_2 = m_4$, $x_2m_3 = x_2m_4 = 0$. We can picture M as follows:

$$\begin{tikzcd}[column sep=small]
m_2 \arrow[dr, "x_2"] & & m_1 \arrow[dl, "x_1"] \arrow[dr, "x_2"] & \\
& m_4 & & m_3.
\end{tikzcd}$$

The kernel of

$$x_1 \otimes 1 + x_2 \otimes T_2^{\{1\}} : M \otimes k[T_2^{\{1\}}] \to M \otimes k[T_2^{\{1\}}]$$

(as in (3.15.1)) is a free $k[T_2^{\{1\}}]$-module of rank 2, generated by $m_3 \otimes 1$ and $m_4 \otimes 1$. The specialization of this module at the point $\epsilon = kx_1$ (letting $T_2 \to 0$) is vector space of dimension 2. This is a proper subspace of $\mathrm{Soc}(\epsilon^*(M))$ which is spanned by m_2, m_3, m_4.

Assume now that $\mathfrak{g} = \mathrm{Lie}(G)$ for an algebraic group G (over k) and that the $\mathfrak{u}(\mathfrak{g})$-module M comes from a rational G-module structure on M. Then the action map $\mathfrak{u}(\mathfrak{g}) \otimes M \to M$ is G-equivariant; in other words, for $g \in G$, $x \in \mathfrak{g}$, and $m \in M$,

$$(x \circ m)^g \;=\; x^g \circ m^g, \tag{5.11.2}$$

where the action $x \mapsto x^g$ is the adjoint action of g on x and the action $m \mapsto m^g$ is the given rational action of G on M.

For $\epsilon \in \mathbb{E}(r, \mathfrak{g})$ denote by ϵ^g the result of the (adjoint) action of G of $\mathbb{E}(r, \mathfrak{g})$. This action induces an action of G on $\mathcal{O}_{\mathbb{E}(r,\mathfrak{g})}$ in the usual way. For $U \subset \mathbb{E}(r, \mathfrak{g})$, and $f \in \mathcal{O}_{\mathbb{E}(r,\mathfrak{g})}(U)$, we have $f^g \in \mathcal{O}_{\mathbb{E}(r,\mathfrak{g})}(U^{g^{-1}})$.

Let Θ_s^Σ be as in (3.13.1) (or (3.15.1) for $W = \mathbb{E}(r, \mathfrak{g})$), and let $\epsilon \in V_\Sigma$. Applying (5.11.2) to $X = \Theta_s^\Sigma(\epsilon)$ we get the following equality:
(5.11.3)

$$(\Theta_s^\Sigma(\epsilon) m)^g \;=\; (\sum Y_{i,s}^\Sigma(\epsilon) x_i)(m))^g \;= \sum Y_{i,s}^\Sigma(\epsilon) x_i^g m^g = (\Theta_s^\Sigma)^g(\epsilon^{g^{-1}})(m^g),$$

where

$$(\Theta_s^\Sigma)^g \equiv \sum_{i=1}^n x_i^g \otimes (Y_{i,s}^\Sigma)^g : M \otimes k[V_\Sigma^{g^{-1}}] \to M \otimes k[V_\Sigma^{g^{-1}}].$$

Hence, specialization of the following diagram at each point $\epsilon \in V_\Sigma$ is commutative:

$$\begin{tikzcd}
M \otimes k[V_\Sigma] \arrow[r, "g"] \arrow[d, "(\Theta_1^\Sigma)^{j_1} \dots (\Theta_r^\Sigma)^{j_r}"'] & M \otimes k[V_\Sigma^{g^{-1}}] \arrow[d, "((\Theta_1^\Sigma)^{j_1} \dots (\Theta_r^\Sigma)^{j_r})^g"] \\
M \otimes k[V_\Sigma] \arrow[r, "g"] & M \otimes k[V_\Sigma^{g^{-1}}].
\end{tikzcd} \tag{5.11.4}$$

This implies that the diagram is commutative. Indeed, suppose $f : M \otimes k[V_\Sigma] \to M \otimes k[V_\Sigma^{g^{-1}}]$ is a map of modules compatible with the isomorphism $k[V_\Sigma] \xrightarrow{g} k[V_\Sigma^{g^{-1}}]$ and such that the specialization of f at each point $\epsilon \in V_\Sigma$ is zero. Since specialization is right exact, this implies that the surjection $M \otimes k[V_\Sigma^{g^{-1}}] \to \mathrm{Coker}\, f$ is an isomorphism when specialized to any point of V_Σ. This, in turn, implies that $M \otimes k[V_\Sigma^{g^{-1}}] \simeq \mathrm{Coker}\, f$ (see, for example, [BP12, 3.1]). Therefore, $f = 0$. To conclude commutativity of the diagram (5.11.4) we apply this argument to $((\Theta_1^\Sigma)^{j_1} \dots (\Theta_r^\Sigma)^{j_r})^g \circ g - g \circ (\Theta_1^\Sigma)^{j_1} \dots (\Theta_r^\Sigma)^{j_r} : M \otimes k[V_\Sigma] \to M \otimes k[V_\Sigma^{g^{-1}}]$.

Let G be an affine algebraic group and X an algebraic variety on which G acts. A quasi-coherent sheaf $\mathcal{F}$ on X is said to be G-equivariant if one has an algebraic (i.e., functorial with respect to base change from k to any finitely generated commutative k-algebra R) action of G on $\mathcal{F}$ compatible with the action of G on X: for all open subset $U \subset X$ and every $h, g \in G(R)$, an $\mathcal{O}_X(U_R)$-isomorphism $(-)^g : \mathcal{F}(U_R) \to \mathcal{F}(U_R^{g^{-1}})$ such that $(-)^h \circ (-)^g = (-)^{hg}$. This is equivalent to the following data: an isomorphism $\theta : \mu^*\mathcal{F} \overset{\sim}{\to} p^*\mathcal{F}$ (where $\mu,\ p : G \times X \to X$ are the action and projection maps) together with a cocycle condition on the pull-backs of θ to $G \times G \times X$ insuring that $(-)^h \circ (-)^g = (-)^{hg}$.

If X is a point, then a G-equivariant sheaf on X is simply a rational G-module. If G acts of X and if M is a rational G-module, then the trivial vector bundle $M \otimes \mathcal{O}_X$ is a G–equivariant vector bundle (that is, a vector bundle which is also a G-equivariant sheaf). If X is a G-orbit (that is, G acts transitively on X), then any G-equivariant sheaf on X is a G-equivariant vector bundle.

The following proposition shows that the $\mathcal{I}m^j(M)$, $\mathcal{K}er^j(M)$, $\mathcal{C}oker^j(M)$ are G-equivariant sheaves whenever $\mathfrak{g} = \mathrm{Lie}(G)$ and M is a rational G-module.

Proposition 5.12. *Let G be an affine algebraic group, $\mathfrak{g} = \mathrm{Lie}(G)$, and M a rational G-module. Fix some $r \geq 1$ and j with $1 \leq j \leq (p-1)r$. Then $\mathcal{I}m^j(M)$ (respectively, $\mathcal{C}oker^j(M)$; resp., $\mathcal{K}er^j(M)$) is a G-equivariant sheaf on $\mathbb{E}(r, \mathfrak{g})$.*

More generally, if $X \subset \mathrm{Grass}(\mathfrak{g})$ is a G-stable, locally closed subset with $X \subset \mathbb{E}(r, \mathfrak{g})$ for $j > 1$, then $\mathcal{I}m^{j,X}(M)$ (resp. $\mathcal{C}oker^{j,X}(M)$; resp., $\mathcal{K}er^{j,X}(M)$) is a G-equivariant sheaf on W.

Proof. Let $\mathcal{O}_{\mathbb{E}}$ denote the structure sheaf of the projective variety $\mathbb{E}(r, \mathfrak{g})$. To prove that $\mathcal{I}m^j(M)$ is a G-equivariant sheaf on $\mathbb{E}(r, \mathfrak{g})$, it suffices to prove that $\mathcal{I}m^j(M) \subset M \otimes \mathcal{O}_{\mathbb{E}}$ is a G-stable subsheaf. For this, it suffices to show that the action of each $g \in G$ sends the stalk $\mathcal{I}m^j(M)_{(\epsilon)}$ at ϵ to the stalk $\mathcal{I}m^j(M)_{(\epsilon^{g^{-1}})}$.

Localizing (5.11.4) we obtain the commutative square

(5.12.1)
$$\begin{tikzcd} M \otimes k[V_\Sigma]_{(\epsilon)} \arrow[r, "g"] \arrow[d, "(\Theta_1^\Sigma)^{j_1} \cdots (\Theta_r^\Sigma)^{j_r}"'] & M \otimes k[V_\Sigma^{g^{-1}}]_{(\epsilon^{g^{-1}})} \arrow[d, "((\Theta_1^\Sigma)^{j_1} \cdots (\Theta_r^\Sigma)^{j_r})^g"] \\ M \otimes k[V_\Sigma]_{(\epsilon)} \arrow[r, "g"] & M \otimes k[V_\Sigma^{g^{-1}}]_{(\epsilon^{g^{-1}})}. \end{tikzcd}$$

Since kernels and images commute with taking the stalk, we conclude that $\mathcal{I}m^j(M)$, $\mathcal{K}er^j(M)$, $\mathcal{C}oker^j(M)$ are G-equivariant sheaves.

The proof of the second assertion for $X \subset \mathrm{Grass}_r(\mathfrak{g})$ a G-stable, locally closed subset can be obtained from the above proof by making minor notational changes. □

As in [CFP12, §6.2], we give an alternative construction of image and kernel sheaves $\mathcal{I}m^j(M)$, $\mathcal{K}er^j(M)$ on $\mathbb{E}(r, \mathfrak{g})$ which does not rely on a patching argument. Rather than identify these sheaves on $\mathcal{N}_p^r(\mathfrak{g})$ local charts $V_\Sigma \subset \mathbb{E}(r, \mathfrak{g})$, this alternative construction exploits the technique of equivariant descent to obtain the sheaves from a global construction on $\mathcal{N}_p^r(\mathfrak{g})^o$, a GL_r-torsor over $\mathbb{E}(r, \mathfrak{g})$ obtained from the classical Stiefel fibration $\mathbb{M}_{n,r}^o \to \mathrm{Grass}_{n,r}$. This is a natural extension to $r > 1$ of the construction of the global nilpotent operator given in [FP11].

As we shall see, one advantage of this approach is that it easily leads to the verification that the kernel and image sheaves do not depend upon our choice $\{x_1, \ldots, x_n\}$ of basis for $\mathfrak{g}$.

The natural action of GL_r on $\mathfrak{g}^{\oplus r}$ induces an action of GL_r on $\mathfrak{g}^{\times r}$, the affine variety associated to $\mathfrak{g}^{\oplus r}$ (isomorphic to the affine space $\mathbb{A}^{nr}$); if $\mathfrak{g} = \mathrm{Lie}(G)$, then G also acts on $\mathfrak{g}^{\times r}$ by the diagonal adjoint action and this action commutes with that of GL_r. We set $(\mathfrak{g}^{\times r})^o \subset \mathfrak{g}^{\times r}$ to be the open subvariety of those r-tuples of elements of $\mathfrak{g}$ which are linearly independent. We consider the following diagram of quasi-projective varieties over k with Cartesian (i.e., pull-back) squares

$$
\begin{tikzcd}
\mathcal{N}_p^r(\mathfrak{g}) \arrow[r] & \mathfrak{g}^{\times r} \\
\mathcal{N}_p^r(\mathfrak{g})^o \arrow[r] \arrow[u] \arrow[d] & (\mathfrak{g}^{\times r})^o \arrow[u] \arrow[d] \\
\mathbb{E}(r,\mathfrak{g}) \arrow[r] & \mathrm{Grass}_r(\mathfrak{g})
\end{tikzcd}
\tag{5.12.2}
$$

whose upper vertical maps are open immersions, lower vertical maps are quotient maps by the GL_r actions, and horizontal maps are closed immersions.

Our choice of basis $\{x_1, \ldots, x_n\}$ of $\mathfrak{g}$ as in Notation 3.12 determines an identification of $\mathfrak{g}^{\otimes r}$ with $\mathbb{M}_{n,r}$. Under this identification, the matrix function $T_{i,s}$ is the linear dual to the element $(\ldots, 0, x_i, 0, \ldots) \in \mathfrak{g}^{\oplus r}$ with x_i in the sth spot. Furthermore, $Y_{i,s} \in k[\mathcal{N}_p^r(\mathfrak{g})]$ is defined to be the image of the matrix function $T_{i,s}$ under the surjective map

$$k[\mathbb{M}_{n,r}] \simeq k[\mathfrak{g}^{\times r}] \twoheadrightarrow k[\mathcal{N}_p^r(\mathfrak{g})], \quad T_{i,s} \mapsto Y_{i,s}.$$

For any $s,\ 1 \le s \le r$, we define

$$\Theta_s \equiv \sum_{i=1}^n x_i \otimes Y_{i,s} \in \mathfrak{g} \otimes k[\mathcal{N}_p^r(\mathfrak{g})] \tag{5.12.3}$$

and use the same notation to denote the operator

$$\Theta_s : M \otimes k[\mathcal{N}_p^r(\mathfrak{g})] \to M \otimes k[\mathcal{N}_p^r(\mathfrak{g})], \quad \Theta_s(m \otimes f) = \sum_{i=1}^n x_i m \otimes Y_{i,s} f$$

for any finite dimensional $\mathfrak{u}(\mathfrak{g})$-module M.

Proposition 5.13. *The operator Θ_s of (5.12.3) does not depend upon the choice of basis of $\mathfrak{g}$.*

Proof. Let $\{y_1, \ldots, y_n\}$ be another choice of basis of $\mathfrak{g}$, and set $Z_{i,s}$ equal to the image of $T_{i,s}$ under the surjective map $k[\mathbb{M}_{n,r}] \to k[\mathcal{N}_p^r(\mathfrak{g})]$ determined by this choice. Let $(a_{i,j}) \in \mathrm{GL}_n(k)$ be the change of basis matrix, so that $y_j = \sum_i a_{i,j} x_i$. Since $Y_{i,s}$'s are the images of the linear duals to x_i's under the projection $k[\mathbb{M}_{n,r}] \to k[\mathcal{N}_p^r(\mathfrak{g})]$ (and similarly for $Z_{i,s}$), we conclude that $Z_{j,s} = \sum_i b_{j,i} Y_{i,s}$ where $(b_{i,j}) = (a_{i,j})^{-1}$. To prove the proposition, it suffices to observe that

$$\sum_j y_j \otimes Z_{j,s} \equiv \sum_j (\sum_i a_{i,j} x_i) \otimes (\sum_i b_{j,i} Y_{i,s}) = \sum_i x_i \otimes Y_{i,s}.$$

This follows directly from the fact that $(a_{i,j}) \cdot (b_{i,j})$ equals the identity matrix. $\square$

Let $j : X \subset \mathbb{E}(r,\mathfrak{g})$ be a locally closed embedding, and denote by $\widetilde{X} \to X$ the restriction of the GL_r-torsor $\mathcal{N}_p^r(\mathfrak{g})^o \to \mathbb{E}(r,\mathfrak{g})$ to X so that there is a Cartesian square

$$\begin{tikzcd} \widetilde{X} \arrow[r, hook, "\widetilde{j}"] \arrow[d] & \mathcal{N}_p^r(\mathfrak{g})^o \arrow[d] \\ X \arrow[r, hook, "j"] & \mathbb{E}(r,\mathfrak{g}). \end{tikzcd} \tag{5.13.1}$$

We extend (5.12.3) by defining

$$\Theta_s^{\widetilde{X}} : M \otimes \mathcal{O}_{\widetilde{X}} \ \to\ M \otimes \mathcal{O}_{\widetilde{X}}, \quad \Theta_s(m \otimes f) = \sum_{i=1}^n x_i m \otimes \widetilde{j}^*(Y_{i,s}) f. \tag{5.13.2}$$

Definition 5.14. For any finite-dimensional $\mathfrak{u}(\mathfrak{g})$-module M, and any j, $1 \leq j \leq p(r-1)$, we define the following submodules of $M \otimes k[\mathcal{N}_p^r(\mathfrak{g})]$:

$$\mathrm{Im}\{\Theta^j, M\} = \mathrm{Im}\{\sum_{\sum j_\ell = j} \Theta_1^{j_1} \cdots \Theta_r^{j_r} : (M \otimes k[\mathcal{N}_p^r(\mathfrak{g})])^{\oplus r(j)} \to M \otimes k[\mathcal{N}_p^r(\mathfrak{g})]\},$$

$$\mathrm{Ker}\{\Theta^j, M\} = \mathrm{Ker}\{[\Theta_1^{j_1} \cdots \Theta_r^{j_r}]_{\sum j_\ell = j} : M \otimes k[\mathcal{N}_p^r(\mathfrak{g})] \to (M \otimes k[\mathcal{N}_p^r(\mathfrak{g})])^{\oplus r(j)}\},$$

where $r(j)$ is the number of ways to write j as a sum of non-negative integers.

Moreover, for any locally closed subset $X \subset \mathbb{E}(r,\mathfrak{g})$, we define the following coherent sheaves on $\widetilde{X}$:

$$\mathrm{Im}\{\Theta^{j,\widetilde{X}}, M\} = \mathrm{Im}\{\sum_{\Sigma j_\ell = j} (\Theta_1^{\widetilde{X}})^{j_1} \cdots (\Theta_r^{\widetilde{X}})^{j_r} : (M \otimes \mathcal{O}_{\widetilde{X}})^{\oplus r(j)} \to M \otimes \mathcal{O}_{\widetilde{X}}\},$$

$$\mathrm{Ker}\{\Theta^{j,\widetilde{X}}, M\} = \mathrm{Ker}\{[(\Theta_1^{\widetilde{X}})^{j_1} \cdots (\Theta_r^{\widetilde{X}})^{j_r}]_{\Sigma j_\ell = j} : M \otimes \mathcal{O}_{\widetilde{X}} \to (M \otimes \mathcal{O}_{\widetilde{X}}^{\oplus r(j)}\}.$$

Remark 5.15. By Proposition 5.13, $\mathrm{Im}\{\Theta^{j,\widetilde{X}}, M\}$, $\mathrm{Ker}\{\Theta^{j,\widetilde{X}}, M\}$ do not depend upon our choice of basis for $\mathfrak{g}$.

The argument of [CFP12, Lemma 6.7] now applies to show the following:

Lemma 5.16. *Let M be a $\mathfrak{u}(\mathfrak{g})$-module. For any locally closed subset $X \subset \mathbb{E}(r,\mathfrak{g})$, $\mathrm{Ker}\{\Theta^{j,\widetilde{X}}, M\}$, $\mathrm{Im}\{\Theta^{j,\widetilde{X}}, M\}$ are GL_r-invariant $\mathcal{O}_{\widetilde{X}}$-submodules of $M \otimes \mathcal{O}_{\widetilde{X}}$.*

The relevance of the previous proposition to our consideration of coherent sheaves on $\mathbb{E}(r,\mathfrak{g})$ becomes evident in view of the following categorical equivalence.

Proposition 5.17. *Since $\mathcal{N}_p^r(\mathfrak{g})^o \to \mathbb{E}(r,\mathfrak{g})$ is a GL_r-torsor, there is a natural equivalence of categories*

$$\eta : \mathrm{Coh}^{\mathrm{GL}_r}(\mathcal{N}_p^r(\mathfrak{g})^o) \xrightarrow{\ \sim\ } \mathrm{Coh}(\mathbb{E}(r,\mathfrak{g})) \tag{5.17.1}$$

between the GL_r-equivariant coherent sheaves on $\mathcal{N}_p^r(\mathfrak{g})^o$ and coherent sheaves on $\mathbb{E}(r,\mathfrak{g})$.

Moreover, (5.17.1) restricts to an equivalence of categories

$$\eta_X : \mathrm{Coh}^{\mathrm{GL}_r}(\widetilde{X}) \xrightarrow{\ \sim\ } \mathrm{Coh}(X) \tag{5.17.2}$$

for any locally closed subset $X \in \mathbb{E}(r,\mathfrak{g})$ and $\widetilde{X} \to X$ as in (5.13.1).

Proof. See, for example, [CFP12, 6.5]. □

We now identify the "patching" construction of sheaves in Theorem 5.8 and the construction obtained via equivariant descent. Even though the proof of the following theorem is very similar to that of [CFP12, 6.8], we provide it here for completeness.

Theorem 5.18. *Let M be a $\mathfrak{u}(\mathfrak{g})$-module, let r, j be positive integers with $j \leq (p-1)r$, let $X \subset \mathbb{E}(r,\mathfrak{g})$ be a locally closed subset, and let $\widetilde{X} \to X$ be GL_r-torsor as in* (5.13.1)*. Then*

$$\mathcal{I}m^{j,X}(M) \;=\; \eta_X(\mathrm{Im}\{\Theta^{j,\widetilde{X}}, M\})$$

$$\mathcal{K}er^{j,X}(M) \;=\; \eta_X(\mathrm{Ker}\{\Theta^{j,\widetilde{X}}, M\})$$

as subsheaves of the free coherent sheaf $M \otimes \mathcal{O}_X$.

Proof. We give the proof in the special case $X = \mathbb{E}(r,\mathfrak{g})$; the general proof proceeds exactly as that given, only with cumbersome additional notation involving base change from $\mathbb{E}(r,\mathfrak{g})$ to X. Since $\mathcal{N}_p^r(\mathfrak{g})^o$ is open dense in $\mathcal{N}_p^r(\mathfrak{g})$, the exactness of localization implies that for $X = \mathbb{E}(r,\mathfrak{g})$, $\widetilde{X} = \mathcal{N}_p^r(\mathfrak{g})^o$,

$$\mathrm{Im}\{\Theta^{j,\widetilde{X}}, M\} = \mathrm{Im}\{\Theta^j, M\}_{|\widetilde{X}}, \quad \mathrm{Ker}\{\Theta^{j,\widetilde{X}}, M\} = \mathrm{Ker}\{\Theta^j, M\}_{\widetilde{X}}.$$

Hence, in the case $X = \mathbb{E}(r,\mathfrak{g})$, the statement reduces to the following equalities:

(5.18.1) $\mathcal{I}m^j(M) \;=\; \eta(\mathrm{Im}\{\Theta^j, M\}_{|\mathcal{N}_p^r(\mathfrak{g})^o}), \quad \mathcal{K}er^j(M) \;=\; \eta(\mathrm{Ker}\{\Theta^j, M\}_{|\mathcal{N}_p^r(\mathfrak{g})^o}).$

It suffices to show that the asserted equalities of sheaves are valid when restricted to each open chart $V_\Sigma \subset \mathbb{E}(r,\mathfrak{g})$ as Σ runs through subsets of cardinality r in $\{1, 2, \ldots, n\}$.

The operator Θ_s of (5.12.3) is given as a product (written symbolically) $\Theta_s = [x_i] \otimes [Y_{i,s}]$, where $\{x_1, \ldots, x_n\}$ is our chosen basis for $\mathfrak{g}$.

Let $\Sigma = \{i_1, \ldots, i_r\}$, let $V_\Sigma = U_\Sigma \cap \mathbb{E}(r,\mathfrak{g})$, and let $\widetilde{V_\Sigma} \to V_\Sigma$ be the GL_r-torsor obtained by pulling-back the GL_r-torsor $\mathcal{N}_p^r(\mathfrak{g})^o \to \mathbb{E}(r,\mathfrak{g})$ along the open immersion $V_\Sigma \subset \mathbb{E}(r,\mathfrak{g})$. The GL_r-torsor $\widetilde{V_\Sigma} \to V_\Sigma$ is trivial (see, for example, [CFP12, Rem. 6.6]). The section s_Σ of Remark 3.12 gives a splitting $\widetilde{V_\Sigma} = V_\Sigma \times \mathrm{GL}_r$ given explicitly as follows. Let $\epsilon \in V_\Sigma$, let $s(\epsilon) = A^\Sigma(\epsilon)$ as defined in Prop. 1.1 or Notation 3.12 and let $g \in \mathrm{GL}_r$. Then the isomorphism $V_\Sigma \times \mathrm{GL}_r \xrightarrow{\sim} \widetilde{V_\Sigma} \subset \mathbb{M}_{n,r}$ is given by

$$\epsilon \times g \mapsto A^\Sigma(\epsilon) \cdot g^{-1}$$

where we use matrix multiplication to multiply $A^\Sigma(\epsilon)$ and g^{-1}. We use g^{-1} here because the action of GL_r on $\mathbb{M}_{n,r}^o$ in the GL_r-torsor $\mathbb{M}_{n,r}^o \to \mathrm{Grass}_r(\mathfrak{g})$ is via multiplication by the inverse on the right. Let $Z_{i_t,j}$, $1 \leq t, j \leq r$ be the standard polynomial generators of $k[\mathrm{GL}_r]$. The effect of the associated isomorphism on coordinate algebras $k[\widetilde{V_\Sigma}] \xrightarrow{\sim} k[V_\Sigma] \otimes k[\mathrm{GL}_r] = k[Y_{i,s}^\Sigma] \otimes k[Z_{i_t,s}, \det^{-1}]$ can be written symbolically as follows:

$$\begin{pmatrix} Y_{1,1} & \dots & Y_{1,r} \\ \vdots & \ddots & \vdots \\ & & \\ \vdots & \ddots & \vdots \\ Y_{n,1} & \dots & Y_{n,r} \end{pmatrix} \xrightarrow{\sim} \begin{pmatrix} Y^{\Sigma}_{1,1} & \dots & Y^{\Sigma}_{1,r} \\ \vdots & \ddots & \vdots \\ & & \\ \vdots & \ddots & \vdots \\ Y^{\Sigma}_{n,1} & \dots & Y^{\Sigma}_{n,r} \end{pmatrix} \otimes \begin{pmatrix} Z_{i_1,1} & \dots & Z_{i_1,r} \\ \vdots & \ddots & \vdots \\ Z_{i_r,1} & \dots & Z_{i_r,r} \end{pmatrix}^{-1}$$

Hence, we obtain the following decomposition (independent of s) of the operator $\Theta_s \downarrow_{\widetilde{V}_\Sigma}$ on $M \otimes k[\widetilde{V}_\Sigma] \simeq M \otimes k[V_\Sigma] \otimes k[\mathrm{GL}_r]$:

$$\Theta_s \downarrow_{\widetilde{V}_\Sigma} = \Theta_s^\Sigma \otimes [Z_{i_t,j}]^{-1} \in \ (M \otimes k[Y^\Sigma_{i,s}]) \otimes k[\mathrm{GL}_r].$$

Because this decomposition is independent of s, this gives decompositions

$$\Theta_1^{j_1} \cdots \Theta_r^{j_r} \ = \ (\Theta_1^\Sigma)^{j_1} \cdots (\Theta_r^\Sigma)^{j_r} \otimes [Z_{i_t,j}]^{-j} \tag{5.18.2}$$

where $j = j_1 + \cdots + j_r$. The exactness of localization enables us to conclude that

$$\mathrm{Im}\{\Theta^j, M\} \downarrow_{\widetilde{V}_\Sigma} \ = \ \mathcal{I}m^j(M)_{V_\Sigma} \otimes k[\mathrm{GL}_r] \ = \ (\eta \downarrow_{V_\Sigma})^{-1}(\mathrm{Im}^j(M)_{V_\Sigma}),$$

$$\mathrm{Ker}\{\Theta^j, M\} \downarrow_{\widetilde{V}_\Sigma} \ = \ \mathcal{K}er^j(M)_{V_\Sigma} \otimes k[\mathrm{GL}_r] \ = \ (\eta \downarrow_{V_\Sigma})^{-1}(\mathrm{Ker}^j(M)_{V_\Sigma}),$$

where the right hand equalities hold by the triviality of the GL_r-torsor $\widetilde{V}_\Sigma \to V_\Sigma$ In other words, we conclude

$$\eta_{V_\Sigma}(\mathrm{Im}\{\Theta^j, M\} \downarrow_{\widetilde{V}_\Sigma}) \ = \ \mathrm{Im}^j(M)_{V_\Sigma},$$

$$\eta_{V_\Sigma}(\mathrm{Ker}\{\Theta^j, M\} \downarrow_{\widetilde{V}_\Sigma}) \ = \ \mathrm{Ker}^j(M)_{V_\Sigma},$$

which finishes the proof. □

Theorem 5.18 combined with Proposition 5.13 (see also Remark 5.15) immediately imply the following corollary.

Corollary 5.19. *Let M be a finite-dimensional $\mathfrak{u}(\mathfrak{g})$-module, and let $X \subset \mathbb{E}(r, \mathfrak{g})$ be a locally closed subset. Then the sheaves $\mathcal{I}m^{j,X}(M)$, $\mathcal{K}er^{j,X}(M)$ are independent of the choice of basis of $\mathfrak{g}$.*

For the remainder of this section and in Section 6, we analyze the situation in which the sheaves $\mathcal{K}er^j(M), \mathcal{I}m^j(M)$ are algebraic vector bundles. This generalizes the work of the last two authors [FP11] for $r = 1$ where this connection was first observed and also the work [CFP12] for the special case of elementary abelian groups.

The following proposition is a significant extension of [CFP12, 6.2], whose proof applies in this more general context. For notation, see 3.16.

Proposition 5.20. *Let M be an $\mathfrak{u}(\mathfrak{g})$-module, and let $Z = \mathbb{R}\mathrm{ad}^{\mathrm{j}}(\mathrm{r}, \mathfrak{g})_{\mathrm{M}}$ (resp, $Z = \mathbb{S}\mathrm{oc}^{\mathrm{j}}(\mathrm{r}, \mathfrak{g})_{\mathrm{M}}$). Let $X = \mathbb{E}(r, \mathfrak{g}) \setminus Z$. Then $\mathcal{I}m^{j,X}(M) = \mathcal{I}m^j(M)_{|X}$ (resp., $\mathcal{K}er^{j,X}(M)$) is an algebraic vector bundle on X.*

Moreover, the fiber of $\mathcal{I}m^j(M)$ (reps., $\mathcal{K}er^j(M)$) at $\epsilon \in X$ is naturally identified with $\mathrm{Rad}^j(\epsilon^* M)$ (resp. $\mathrm{Soc}^j(\epsilon^* M)$).

Proof. It suffices to restrict to an arbitrary $\Sigma \subset \{1,\ldots,n\}$ of cardinality r and prove that the $\mathcal{O}_{V_\Sigma\cap X}$-modules $\mathcal{I}m^j(M)_{|V_\Sigma\cap X}$ (resp., $(\mathcal{K}er^j(M)_{|V_\Sigma\cap X})$ are locally free.

Recall that $\Theta_s^\Sigma(\epsilon)$ for $\epsilon\in\mathbb{E}(r,\mathfrak{g})$ is the specialization of Θ_s^Σ at the point ϵ, which is the result of tensoring $(-)\otimes_{k[V_\Sigma]} k$ along evaluation at ϵ. Since specialization is right exact,

$$\mathrm{Coker}\{\sum_{\sum j_i=j}(\Theta_1^\Sigma)^{j_1}\cdots(\Theta_r^\Sigma)^{j_r}\}\otimes_{k[V_\Sigma]} k \;=\; \mathrm{Coker}\{\sum_{\sum j_i=j}\Theta_1^\Sigma(\epsilon)^{j_1}\cdots\Theta_r^\Sigma(\epsilon)^{j_r}\}.$$

Exactly as in the proof of [CFP12, 6.2], the hypothesis that $\dim \mathrm{Rad}^j(\epsilon^*M)$ is the same for any $\epsilon\in X$ implies that $\mathrm{Coker}\{\sum\limits_{\sum j_i=j}(\Theta_1^\Sigma)^{j_1}\cdots(\Theta_r^\Sigma)^{j_r}\}_{|V_\Sigma\cap X}$ is a locally free $\mathcal{O}_{V_\Sigma\cap X}$-module. The short exact sequence

$$0\longrightarrow \mathcal{I}m^j(M)_{V_\Sigma}\longrightarrow (M\otimes k[V_\Sigma])^{\oplus r(j)}\longrightarrow \mathrm{Coker}\{\sum\limits_{\sum j_i=j}(\Theta_1^\Sigma)^{j_1}\cdots(\Theta_r^\Sigma)^{j_r}\}\longrightarrow 0$$

localized at $V_\Sigma\cap X$ now implies that $\mathcal{I}m(M)^j_{|V_\Sigma\cap X}$ is locally free, as well as enables the identification of the fiber above $\epsilon\in V_\Sigma\cap X$.

The proof for $\mathcal{K}er^j(M)$ is a minor adaptation of above; see also the proof of Theorem 6.2 of [CFP12]. □

As an immediate corollary we verify that $\mathcal{I}m^j(M)$ (respectively, $\mathcal{K}er^j(M)$) is an algebraic vector bundle on $\mathbb{E}(r,\mathfrak{g})$ provided that M has constant (r,j)-radical rank (respectively, constant (r,j)-socle rank).

Corollary 5.21. *Let M be an $\mathfrak{u}(\mathfrak{g})$-module which has constant (r,j)-radical rank (respectively, (r,j)-socle rank). Then the coherent sheaf $\mathcal{I}m^j(M)$ (respectively, $\mathcal{K}er^j(M)$) is an algebraic vector bundle on $\mathbb{E}(r,\mathfrak{g})$.*

Moreover, the fiber of $\mathcal{I}m^j(M)$ (resp., $\mathcal{K}er^j(M)$) at ϵ is naturally identified with $\mathrm{Rad}^j(\epsilon^*M)$ *(resp.* $\mathrm{Soc}^j(\epsilon^*M)$*).*

Proof. The condition of constant (r,j)-radical rank (respectively, (r,j)-socle rank) implies that $\mathbb{R}\mathrm{ad}^{\mathrm{j}}(\mathrm{r},\mathfrak{g})_{\mathrm{M}}=\emptyset$ (resp, $\mathbb{S}\mathrm{oc}^{\mathrm{j}}(\mathrm{r},\mathfrak{g})_{\mathrm{M}}=\emptyset$). Hence, the corollary is a special case of Proposition 5.20 for $X=\mathbb{E}(r,\mathfrak{g})$. □

Example 5.22. Let $\mathfrak{u}$ be a nilpotent restricted Lie algebra such that $x^{[p]}=0$ for any $x\in\mathfrak{u}$, and let $\mathfrak{u}_{\mathrm{ad}}$ denote the adjoint representation of $\mathfrak{u}$ on itself. Let $X\subset\mathbb{E}(r,\mathfrak{u})$ denote the open subvariety of maximal elementary subalgebras of dimension r as in Prop. 3.21. Then $\mathcal{K}er(\mathfrak{u}_{\mathrm{ad}})_{|X}\subset\mathfrak{u}_{\mathrm{ad}}\otimes\mathcal{O}_X$ is isomorphic to the restriction along $X\subset\mathbb{E}(r,\mathfrak{u})\subset\mathrm{Grass}_r(\mathfrak{g})$ of the canonical rank r subbundle $\gamma_r\subset\mathfrak{u}\otimes\mathcal{O}_{\mathrm{Grass}_r(\mathfrak{u})}$.

Indeed, as noted in the proof of Prop. 3.21, X is an open subset of $\mathbb{E}(r,\mathfrak{u})$ equal to the complement of $\mathbb{S}\mathrm{oc}(\mathrm{r},\mathfrak{u})_{\mathfrak{u}_{\mathrm{ad}}}$. Hence, by Proposition 5.20, $\mathcal{K}er(\mathfrak{u}_{\mathrm{ad}})_{|X}$ is a vector bundle with the fiber $\mathcal{K}er(\mathfrak{u}_{\mathrm{ad}})_\epsilon=\mathrm{Soc}(\epsilon^*(\mathfrak{u}_{\mathrm{ad}}))$ for any $\epsilon\in X$. Since ϵ is maximal, $\mathrm{Soc}(\epsilon^*(\mathfrak{u}_{\mathrm{ad}}))=\epsilon$ which finishes the proof.

In the next example we specialize to $\mathfrak{u}=\mathfrak{u}_3$ as in Example 1.7

Example 5.23. Let $\mathfrak{g}=\mathfrak{u}_3\subset\mathfrak{gl}_3$ so that $\mathbb{E}(2,\mathfrak{u}_3)\simeq\mathbb{P}^1$. Consider $\mathfrak{u}_3$ as a module over itself via the adjoint action. Then

$$\mathcal{K}er(\mathfrak{u}_3)\;\simeq\;\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}\;\subset\;\mathfrak{u}_3\otimes\mathcal{O}_{\mathbb{P}^1}.$$

The following proposition refines the analysis given in [FP11] of projective modules on $\mathfrak{sl}_2^{\oplus r}$. We implicitly use the isomorphism $\mathbb{E}(r, \mathfrak{sl}_2^{\oplus r}) \simeq (\mathbb{P}^1)^{\times r}$ of Corollary 1.13.

Proposition 5.24. *Let $\mathfrak{g} = \mathfrak{sl}_2^{\oplus r}$ and let $\pi_s : \mathfrak{g} \to \mathfrak{sl}_2$ be the s-th projection, $1 \leq s \leq r$. For each λ, $0 \leq \lambda \leq p-1$, let P_λ be the indecomposable projective $\mathfrak{u}(\mathfrak{sl}_2)$-module of highest weight λ. Then for each $(\lambda, s) \neq (\lambda', s')$, there exists some j such that the vector bundle $\mathcal{K}er^j(\pi_s^*(P_\lambda))$ on $\mathbb{E}(r, \mathfrak{g})$ is not isomorphic to $\mathcal{K}er^j(\pi_{s'}^*(P_{\lambda'}))$.*

Proof. Observe that $\operatorname{Soc}^j(\epsilon^*(\pi_s^* M)) = \operatorname{Soc}^j(\epsilon_s^* M)$ for any $\mathfrak{u}(\mathfrak{sl}_2)$-module M and any $j, 1 \leq j < r$, where $\epsilon = (\epsilon_1, \ldots, \epsilon_r) \in \mathbb{E}(r, \mathfrak{sl}_2^{\oplus r})$; in particular, the action of ϵ on $\epsilon^*(\pi_s^* M)$ factors through ϵ_s. This implies that $\mathcal{K}er^j(\pi_s^*(P_\lambda)) \simeq \pi_s^*(\mathcal{K}er^j(P_\lambda))$. The proposition now follows from the computation given in [FP11, 6.3]. □

6. Vector bundles on G-orbits of $\mathbb{E}(r, \mathfrak{g})$

Our explicit examples of algebraic vector bundles involve the restrictions of image, cokernel, and kernel sheaves to G-orbits, where G is an algebraic group, $\mathfrak{g}$ is the Lie algebra of G, and M is a rational G-module. Proposition 6.1 verifies that the image and kernel sheaves determine algebraic vector bundles on G-orbits inside $\mathbb{E}(r, \mathfrak{g})$, which are interpreted in Theorem 6.5 in terms of the well-known induction functor from rational H-modules to vector bundles on G/H. .

We begin this section with a general discussion of such bundles and conclude with explicit examples.

Let $\epsilon \subset \mathbb{E}(r, \mathfrak{g})$ be an elementary subalgebra, and let $X = G \cdot \epsilon$ be the G-orbit of ϵ in $\mathbb{E}(r, \mathfrak{g})$. Then X is open in its closure $i \circ j : X \subset \overline{X} \subset \mathbb{E}(r, \mathfrak{g})$, and, hence, to any finite-dimensional rational G-representation M and any j, $1 \leq j \leq (p-1)r$, we can associate coherent sheaves $\mathcal{I}m^{j,X}(M)$, $\mathcal{K}er^{j,X}(M)$ as in Definition 5.9.

The following proposition can be viewed as a generalization of Proposition 4.4.

Proposition 6.1. *Let G be an affine algebraic group, $\mathfrak{g} = \operatorname{Lie}(G)$, and M a rational G-module. Let $\epsilon \in \mathbb{E}(r, \mathfrak{g})$ be an elementary subalgebra of rank r, and let $X = G \cdot \epsilon \subset \mathbb{E}(r, \mathfrak{g})$ be the orbit of ϵ under the adjoint action of G.*

Then

$$\mathcal{I}m^{j,X}(M), \quad \mathcal{K}er^{j,X}(M), \quad \mathcal{C}oker^{j,X}(M)$$

are algebraic vector bundles on X.

Moreover, for any $x = \epsilon^g \in X$, we have natural identifications

$$\mathcal{I}m^{j,X}(M)_x \simeq g \operatorname{Rad}^j(\epsilon^* M), \quad \mathcal{K}er^{j,X}(M)_x \simeq g \operatorname{Soc}^j(\epsilon^* M)$$

of fibers at the point x.

Proof. The sheaves $\mathcal{I}m^{j,X}(M)$, $\mathcal{C}oker^{j,X}(M)$, $\mathcal{K}er^{j,X}(M)$ are G-equivariant by Proposition 5.12 since X is a G-stable locally closed subset of $\mathbb{E}(r, \mathfrak{g})$. If $x = \epsilon^g$ for some $g \in G$, then the action of g on one of these sheaves sends the fiber at ϵ isomorphically to the fiber at x. Since X is Noetherian, we conclude that the sheaves are locally free (see, for example, [FP11, 4.11] or [Hart, 5. ex. 5.8]).

The identification of fibers follows from Proposition 5.10 and the fact that for a G-rational module M we have equalities $g \operatorname{Rad}^j(\epsilon^* M) = \operatorname{Rad}^j((\epsilon^g)^* M)$ (resp., $g \operatorname{Soc}^j(\epsilon^* M) = \operatorname{Soc}^j((\epsilon^g)^* M)$). □

To identify the bundles on homogeneous varieties, we recall the following well known induction functor ([Jan]).

Proposition 6.2. [Jan, II.6.1] *Let G be a affine algebraic group, $H \subset G$ a closed subgroup. For each (finite dimensional) rational H-module W, consider the sheaf of $\mathcal{O}_{G/H}$-modules $\mathcal{L}_{G/H}(W)$ which sends an open subset $U \subset G/H$ to the sections of $G \times^H W \to G/H$ above U:*

$$\mathcal{L}_{G/H}(W)(U) \;=\; \{\textit{sections of } G \times^H W \to G/H \textit{ above } U\}. \tag{6.2.1}$$

1. *So defined, $W \mapsto \mathcal{L}_{G/H}(W)$ is a functor from finite dimensional rational representations W of H to G-equivariant locally free coherent sheaves on G/H.*
2. *If W is the restriction of a rational G-module, then $\mathcal{L}_{G/H}(W)$ is isomorphic as a coherent sheaf to $W \otimes \mathcal{O}_{G/H}$.*
3. *$\mathcal{L}_{G/H}(-)$ is exact and commutes with tensor powers $(-)^{(\otimes^i)}$, duals $(-)^{\#}$, symmetric powers $S^i(-)$, divided powers $\Gamma^i(-)$, exterior powers $\Lambda^i(-)$, and Frobenius twists $(-)^{(i)}$.*

In the next proposition we remind the reader that the functor $\mathcal{L}$ of (6.2.1) determines an equivalence of categories between rational H-modules and G-equivariant vector bundles on G/H. The statement of the proposition follows from the observation that $p : G \to G/H$ is an H-torsor in the fppf topology (cf. [Jan, I.5.6]) and that the sheaf of sections of such an H-torsor is locally trivial in the Zariski topology by [DG, III.§4,2.4]. We thank Burt Totaro for this last reference.

Proposition 6.3. *Let G be an affine algebraic group, and $H \subset G$ be a closed subgroup. For any G-equivariant vector bundle $\mathcal{E}$ on G/H we have an isomorphism of G-equivariant algebraic vector bundles on G/H*

$$\mathcal{E} \simeq \mathcal{L}_{G/H}(W),$$

where W is the fiber of $\mathcal{E}$ over the coset $eH \in G/H$ with H-module structure obtained by restricting the action of G to H (which stabilizes this fiber).

Moreover, for any $x = gH \in G/H$, there is a natural identification of H-modules

$$\mathcal{E}_x \simeq gW$$

where $\mathcal{E}_x$ is the fiber of $\mathcal{E}$ at the point x.

Example 6.4. We identify some standard bundles using the functor $\mathcal{L}$. Let $G = \mathrm{GL}_n$, $P = P_{r,n-r}$ be a maximal parabolic with the Levi factor $L \simeq \mathrm{GL}_r \times \mathrm{GL}_{n-r}$. Set $X = \mathrm{Grass}_{n,r} = G/P$ and let V be the defining representation for G. Denote by W the representation of P given by composition of the projection $P \to L \to \mathrm{GL}_r$ followed by the defining representation for GL_r. We set

$$\gamma_r \;\simeq\; \mathcal{L}_X(W), \quad \delta_{n-r} \;\simeq\; \mathcal{L}_X((V/W)^{\#}). \tag{6.4.1}$$

Thus, γ_r is the canonical rank r subbundle on $\mathrm{Grass}_{n,r}$. Observe that we have a short exact sequence of algebraic vector bundles on X:

$$0 \longrightarrow \gamma_r \longrightarrow V \otimes \mathcal{O}_X \longrightarrow \delta_{n-r}^{\vee} \longrightarrow 0\,, \tag{6.4.2}$$

where we denote by $\mathcal{E}^{\vee}$ the dual sheaf to $\mathcal{E}$.

Let $F(-)$ be one of the functors of Proposition 6.2.3. Then Proposition 6.2.3 implies that

$$F(\gamma_r) \;\simeq\; \mathcal{L}_X(F(W)).$$

Combining Propositions 6.1 and 6.2, we conclude the following "identifications" of the vector bundles on a G-orbit in $\mathbb{E}(r,\mathfrak{g})$ associated to a rational G-module. The proof follows immediately from these propositions.

Theorem 6.5. *Let G be an algebraic group and M be a rational G-module. Set $\mathfrak{g} = \mathrm{Lie}(G)$, and let r be a positive integer. Let $H \subset G$ denote the stabilizer of some $\epsilon \in \mathbb{E}(r,\mathfrak{g})$, set $X \simeq G/H$, and consider $\mathcal{L}_X : H\text{-mod} \to G/H\text{-bundles}$ as in* (6.2.1).

For any j, $1 \le j \le (p-1)r$, we have the following isomorphisms of G-equivariant vector bundles

$$\mathcal{I}m^{j,X}(M) \;\simeq\; \mathcal{L}_X(\mathrm{Rad}^j(\epsilon^* M)), \quad \mathcal{K}er^{j,X}(M) \;\simeq\; \mathcal{L}_X(\mathrm{Soc}^j(\epsilon^* M))$$

as subbundles of the trivial bundle $\mathcal{L}_X(M) = M \otimes \mathcal{O}_X$, where $\mathrm{Rad}^j(\epsilon^* M))$, $\mathrm{Soc}^j(\epsilon^* M))$ *are endowed with the action of H induced by the action of G on M.*

The following proposition, a generalization of [CFP12, 7.9], enables us to identify kernel bundles provided we know corresponding image bundles and vice versa.

Proposition 6.6. *Retain the notation and hypotheses of Theorem 6.5. Then there is a natural short exact sequence of vector bundles on $G/H \simeq X \subset \mathbb{E}(r,\mathfrak{g})$*

$$0 \longrightarrow \mathcal{K}er^{j,X}(M^\#) \longrightarrow (M^\#) \otimes \mathcal{O}_X \longrightarrow (\mathcal{I}m^{j,X}(M))^\vee \longrightarrow 0. \tag{6.6.1}$$

Proof. The proof is a repetition of that of [CFP12, 7.19]. By Remark 3.10, the sequence

$$0 \longrightarrow \mathrm{Soc}^j(\epsilon^*(M^\#)) \longrightarrow M^\# \longrightarrow (\mathrm{Rad}^j(\epsilon^* M))^\# \longrightarrow 0. \tag{6.6.2}$$

is an exact sequence of H-modules. Applying the functor $\mathcal{L}$ to (6.6.2) (which preserves exactness by Proposition 6.2) and appealing to Theorem 6.5, we conclude the exactness of (6.6.1). □

We next work out specific examples of vector bundles on $\mathbb{E}(m,\mathfrak{gl}_n)$.

Proposition 6.7. *Let $G = \mathrm{GL}_n$, and let V be the n-dimensional defining representation. Let $\epsilon = \mathfrak{u}_{r,n-r} \in \mathbb{E}(r(n-r),\mathfrak{gl}_n)$ for some $r < n$. Consider the (closed) GL_n-orbit*

$$X \equiv \mathrm{GL}_n \cdot \epsilon \simeq \mathrm{Grass}_{n,r}$$

of $\mathbb{E}(r(n-r),\mathfrak{gl}_n)$. We have the following isomorphisms of algebraic vector bundles on X:

(1) $\mathcal{I}m^X(V) \;\simeq\; \mathcal{K}er^X(V) \;\simeq\; \gamma_r$,
$\mathcal{I}m^{j,X}(V) = 0$ *for* $j > 1$.

(2) $\mathcal{C}oker^X(V) \simeq \delta^\vee_{n-r}$,
$\mathcal{C}oker^{j,X}(V) = 0$ *for* $j > 1$.

(3) $\mathcal{K}er^X(\Lambda^{n-1}(V)) \simeq \mathcal{I}m^X(\Lambda^{n-1}(V)) \simeq \delta^\vee_{n-r}$,
$\mathcal{I}m^{j,X}(\Lambda^{n-1}(V)) = 0$ *for* $j > 1$.

Proof. Let $e_1,\ldots,e_n$ be the standard basis of V, so that both $\mathrm{Rad}(\epsilon^* V)$ and $\mathrm{Soc}(\epsilon^* M)$ are spanned by $e_1,\ldots,e_r$. That is, $\mathrm{Rad}(\epsilon^* V) = \mathrm{Soc}(\epsilon^* M) = W$ as $P_{r,n-r}$-modules in the notation of Example 6.4. Hence, Theorem 6.5 implies that

$$\mathcal{I}m^X(V) \simeq \mathcal{L}_X(W) = \gamma_r \quad \text{and} \quad \mathcal{K}er^X(V) \simeq \mathcal{L}_X(W) = \gamma_r.$$

This proves the first part of (1). The vanishing $\mathcal{I}m^{j,X}(V) = 0$ follows immediately from the fact that $\mathrm{Rad}^j(\epsilon^*(V)) = 0$ for $j \geq 2$.

Part (2) follows from the exactness of (6.4.1). The last assertion follows from the elementary identification of $\Lambda^{n-1}V$ with $V^{\#}$ as GL_n-modules and Proposition 6.6. $\square$

Proposition 6.8. *We retain the hypotheses and notation of Proposition 6.7. For any positive integer $m \leq n - r$,*

(1) $\mathcal{I}m^{m,X}(V^{\otimes m}) \;=\; \gamma_r^{\otimes m}$,
(2) $\mathcal{I}m^{m,X}(S^m(V)) \;=\; S^m(\gamma_r)$,
(3) $\mathcal{I}m^{m,X}(\Lambda^m(V)) \;=\; \Lambda^m(\gamma_r)$.

Proof. Write $\mathfrak{u}(\epsilon) = k[t_{i,j}]/(t_{i,j}^p), 1 \leq i \leq r, r+1 \leq j \leq n$. The action of $t_{i,j}$ on V is given by the rule $t_{i,j}e_j = e_i$ and $t_{i,j}e_\ell = 0$ for $\ell \neq j$. Let $W = \mathrm{Rad}(\epsilon^* V)$ as in the proof of Prop. 6.7. On a tensor product $M \otimes N$ of modules the action is given by $t_{i,j}(v \otimes w) = t_{i,j}v \otimes w + v \otimes t_{i,j}w$; thus $\mathrm{Rad}^m(\epsilon^*(V^{\otimes m}))$ is contained in the subspace of $V^{\otimes m}$ spanned by all elements $e_{i_1} \otimes \cdots \otimes e_{i_m}$, where $1 \leq i_1, \ldots, i_m \leq r$, which is $W^{\otimes m}$. On the other hand, for any sequence $i_1, \ldots, i_m$, with $1 \leq i_1, \ldots, i_m \leq r$, we have that

$$e_{i_1} \otimes \cdots \otimes e_{i_m} \;=\; (t_{i_1,r+1} \ldots t_{i_m,r+m})(e_{r+1} \otimes \cdots \otimes e_{r+m}) \tag{6.8.1}$$

since $r + m \leq n$. Hence, $\mathrm{Rad}^m(\epsilon^*(V^{\otimes m})) = W^{\otimes m}$. Therefore, the equality $\mathcal{I}m^{m,X}(V^{\otimes m}) = \mathcal{L}_X(W^{\otimes m}) \;=\; \gamma_r^{\otimes m}$ follows from Proposition 6.2.3, Theorem 6.5, and Example 6.4.

To show (2), note that the action of $\mathfrak{u}(\epsilon)$ on $S^m(V)$ is induced by the action on $V^{\otimes m}$ via the projection $V^{\otimes m} \twoheadrightarrow S^m(V)$. Hence, the formula (6.8.1) is still valid in $S^m(V)$, and implies the inclusion $S^m(W) \subset \mathrm{Rad}^m(\epsilon^*(S^m(V))$. The reverse inclusion is immediate just as in the tensor powers case. Therefore, $\mathrm{Rad}^m(\epsilon^*(S^m(V))) = S^m(W)$, and we conclude the equality $\mathcal{I}m^{m,X}(S^m(V)) \;=\; S^m(\gamma_r)$ appealing to Theorem 6.5.

The proof for exterior powers is completely analogous. $\square$

We provide similar computations for the symplectic group Sp_{2n}.

Proposition 6.9. *Consider $G = \mathrm{Sp}_{2n}$ and its defining representation V (of dimension $2n$); assume $p > 3$. Let $P \subset \mathrm{Sp}_{2n}$ be the unique cominuscule parabolic subgroup (as described in Definition 2.5), and let $\mathfrak{p} = \mathrm{Lie}(P)$. Let ϵ be the nilpotent radical of $\mathfrak{p}$, an elementary subalgebra of $\mathfrak{sp}_{2n}$ of dimension $m = \binom{n+1}{2}$. Applying Theorem 2.13, we consider*

$$Y \;=\; \mathbb{E}(m, \mathfrak{sp}_{2n}) \;\simeq \mathrm{LG}(n, V).$$

Let $\gamma_n \;\subset\; \mathcal{O}_Y^{\oplus 2n}$ be the canonical subbundle of rank n. We have the following natural identifications of algebraic vector bundles on Y:

(1) $\mathcal{I}m(V) \;\simeq\; \gamma_n, \quad \mathcal{I}m^j(V) = 0$ *for* $j > 1$.

(2) $\mathcal{I}m(\Lambda^{2n-1}(V)) \simeq \gamma_n^\vee, \quad \mathcal{I}m^j(\Lambda^{2n-1}(V)) = 0$ *for* $j > 1$.

(3) *For $m \leq n$,*
 (a) $\mathcal{I}m^m(V^{\otimes m}) = (\gamma_n)^{\otimes m}$,
 (b) $\mathcal{I}m^m(S^m(V)) = S^m(\gamma_n)$,

(c) $\mathcal{I}m^m(\Lambda^m(V)) = \Lambda^m(\gamma_n)$.

Proof. We view Sp_{2n} as the stabilizer of the form

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

so that $\mathfrak{sp}_{2n}$ is the set of matrices of the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $D = -A^T$ and B and C are $n \times n$ symmetric matrices. Then $\mathfrak{p} \subset \mathfrak{sp}_{2n}$ is defined by $C = 0$ (this can be easily verified from the explicit description of roots and roots spaces as in, for example, [EW06, 12.5]). We view V as the space of column vectors on which these matrices act from the left, and give V the standard basis $e_1, \ldots, e_{2n}$.

The restricted enveloping algebra of ϵ has the form $k[t_{i,j}]/(t_{i,j}^p)$ where $1 \leq i \leq n$ and $n+i \leq j \leq 2n$. The generator $t_{i,j}$ acts on V by the matrix $E_{i,j}$ if $j = n+i$ and by $E_{i,j} + E_{j-n,i+n}$ otherwise. Here, $E_{i,j}$ is the matrix with 1 in the (i,j) position and 0 elsewhere. Thus we have that

$$t_{i,j}e_j = e_i, \quad t_{i,j}e_{i+n} = e_{j-n}, \quad \text{and} \quad t_{i,j}e_\ell = 0 \tag{6.9.1}$$

whenever $\ell \neq j, i+n$. These relations immediately imply that $\mathrm{Rad}(\epsilon^* V) = \mathrm{Soc}(\epsilon^*(V)) = W$ where $W \subset V$ is the P-stable subspace generated by $e_1, \ldots, e_n$. Moreover, we also have that $\mathrm{Rad}^j(\epsilon^* V) = \mathrm{Soc}^j(\epsilon^* V) = 0$ for any $j > 1$. Applying Theorem 6.5, we get

$$\mathcal{I}m(V) = \mathcal{K}er(V) \simeq \mathcal{L}_Y(W) = \gamma_n, \quad \mathcal{I}m^j(V) = \mathcal{K}er^j(V) = 0 \text{ for } j > 1.$$

Part (2) follows from (1) and the fact that $\Lambda^{2n-1}(V)$ is the dual of $\mathfrak{g}$-module V.

We proceed to show that $\mathcal{I}m^m(V^{\otimes m}) \simeq (\gamma_n)^{\otimes m}$ for $m \leq n$. We note that as in the proof of Proposition 6.7 it is only necessary to show that $(\mathrm{Rad}(\epsilon^* V))^{\otimes m} \subseteq \mathrm{Rad}^m(\epsilon^*(V^{\otimes m}))$, since the reverse inclusion is obvious.

Since $\mathrm{Rad}^2(\epsilon^* V) = 0$, the action of $\mathrm{Rad}^m(\mathfrak{u}(\epsilon))$ on $V^{\otimes m}$ is given by the formula

$$(t_{i_1,n+j_1} \cdots t_{i_m,n+j_m})(e_{s_1} \otimes \cdots \otimes e_{s_m}) = \tag{6.9.2}$$

$$\sum_{\pi \in \Sigma_m} t_{i_{\pi(1)},n+j_{\pi(1)}} e_{s_1} \otimes \cdots \otimes t_{i_{\pi(1)},n+j_{\pi(m)}} e_{s_m}.$$

To prove the inclusion $(\mathrm{Rad}(\epsilon^* V))^{\otimes m} \subseteq \mathrm{Rad}^m(\epsilon^*(V^{\otimes m}))$, we need to show that for any m-tuple of indices $(i_1, \ldots, i_m)$, $1 \leq i_j \leq n$, we have $e_{i_1} \otimes \cdots \otimes e_{i_m} \in \mathrm{Rad}^m(\epsilon^*(V^{\otimes m}))$. We first show the following

Claim. For any simple tensor $e_{i_1} \otimes \cdots \otimes e_{i_m}$ in $(\mathrm{Rad}(\epsilon^* V))^{\otimes m}$ there exists a permutation $w \in \Sigma_m$ such that $e_{w(i_1)} \otimes \cdots \otimes e_{w(i_m)} \in \mathrm{Rad}^m(\epsilon^*(V^{\otimes m}))$.

We proceed to prove the claim. Let $e_{i_1} \otimes \cdots \otimes e_{i_m}$ be any simple tensor in $(\mathrm{Rad}(\epsilon^* V))^{\otimes m}$. Applying a suitable permutation $\pi \in \Sigma_m$ to $(1, \ldots, m)$, we may assume that $(i_1, \ldots, i_m)$ has the form $(i_1^{a_1}, i_2^{a_2}, \ldots, i_\ell^{a_\ell})$ where $i_1 > i_2 > \cdots > i_\ell$ and $a_1 + \ldots + a_\ell = m$. Applying yet another permutation, we may assume that the string of indices $(i_1, \ldots, i_m)$ has the form

$$(i_1, i_2, \ldots, i_\ell, i_1^{a_1-1}, \ldots, i_\ell^{a_\ell-1}),$$

with $i_1 > i_2 > \cdots > i_\ell$. To this string of indices we associate the string of indices $j_1, \ldots, j_m$ by the following rule:

$$j_1 = i_1, j_2 = i_2, \ldots, j_\ell = i_\ell$$

and $(j_{\ell+1}, \ldots, j_m)$ is a subset of $m-\ell$ distinct numbers from $\{1, \ldots, n\}\backslash\{i_1, i_2, \ldots, i_\ell\}$. We claim that

$$(6.9.3) \qquad (t_{i_1,n+j_1} \cdots t_{i_m,n+j_m})(e_{n+j_1} \otimes \cdots \otimes e_{n+j_m}) = e_{i_1} \otimes \cdots \otimes e_{i_m}.$$

Indeed, relations (6.9.1) imply that $t_{i_1,n+j_1}e_{n+j_1} \otimes \cdots \otimes t_{i_m,n+j_m}e_{n+j_m} = e_{i_1} \otimes \cdots \otimes e_{i_m}$. We need to show that all the other terms in (6.9.2) are zero. To have $t_{i_s,n+j_s}e_{n+j_r} \neq 0$, we must have either $j_s = j_r$ or $i_s = j_r$. By the choice of $(j_1, \ldots, j_m)$, the second condition $i_s = j_r$ implies that $s = r$ and, hence, $j_s = j_r$. Therefore, $t_{i_s,n+j_s}e_{n+j_r} \neq 0$ if and only if $j_s = j_r$. Since by construction all $(j_1, \ldots, j_m)$ are distinct, we conclude that $t_{i_{\pi(1)},n+j_{\pi(1)}}e_{n+j_1} \otimes \cdots \otimes t_{i_{\pi(1)},n+j_{\pi(m)}}e_{n+j_m} \neq 0$ if and only if π is the identity permutation which proves (6.9.3). This finishes the proof of the claim.

Now let $e_{i_1} \otimes \cdots \otimes e_{i_m}$ be an arbitrary tensor with $1 \leq i_j \leq n$. As we just proved, there exist $w \in \Sigma_m$ and indices $j_1, \ldots, j_m$ such that

$$(6.9.4) \quad (t_{w(i_1),n+j_1} \cdots t_{w(i_m),n+j_m})(e_{n+j_1} \otimes \cdots \otimes e_{n+j_m}) = e_{w(i_1)} \otimes \cdots \otimes e_{w(i_m)}.$$

The formula(6.9.2) implies that if we apply w^{-1} to (6.9.4) we get the desired result, that is

$$(t_{i_1,n+w^{-1}(j_1)} \cdots t_{i_m,n+w^{-1}(j_m)})(e_{n+w^{-1}(j_1)} \otimes \cdots \otimes e_{n+w^{-1}(j_m)}) = e_{i_1} \otimes \cdots \otimes e_{i_m}.$$

Therefore, $e_{i_1} \otimes \cdots \otimes e_{i_m} \in \operatorname{Rad}^m(\epsilon^*(V^{\otimes m}))$. The statement for symmetric and exterior powers follows just as in Proposition 6.8. □

In the next proposition we remind the reader of some standard constructions of bundles using the operator $\mathcal{L}$ in addition to γ and δ mentioned in Example 6.4. As this must be well-known, we only provide either references or short sketches of the proofs.

Proposition 6.10. *Let G be a reductive algebraic group and let P be a standard parabolic subgroup. Set $\mathfrak{g} = \operatorname{Lie} G$, $\mathfrak{p} = \operatorname{Lie} P$, and let $\mathfrak{u}$ be the nilpotent radical of $\mathfrak{p}$.*

(1) *$\mathbb{T}_{G/P} \simeq \mathcal{L}_{G/P}(\mathfrak{g}/\mathfrak{p})$, where $\mathbb{T}_{G/P}$ is the tangent bundle.*
(2) *Assume that $\mathfrak{g}$ has a nondegenerate G-invariant symmetric bilinear form (such as the Killing form). Then*
$$\Omega_{G/P} = \mathbb{T}^\vee_{G/P} \simeq \mathcal{L}_{G/P}(\mathfrak{u}),$$
where $\mathfrak{u}$ is viewed as P-module via the restriction of the adjoint action of P on $\mathfrak{p}$.
(3) *For $G = \mathrm{SL}_n, P = P_{r,n-r}$, $X = G/P = \mathrm{Grass}_{n,r}$, we have*
$$\mathbb{T}_X \simeq \gamma_r^\vee \otimes \delta_{n-r}^\vee, \quad \Omega_X \simeq \gamma_r \otimes \delta_{n-r}.$$
(4) *Let $G = \mathrm{Sp}_{2n}, P = P_{\alpha_n}$, the cominuscule parabolic. Let $Y = G/P$. Then*
$$\mathbb{T}_Y \simeq \mathcal{L}_{G/P}(\mathfrak{g}/\mathfrak{p}) \simeq S^2(\gamma_n^\vee).$$
Moreover, if $p > 2$ and does not divide $n+1$, then
$$\Omega_Y \simeq S^2(\gamma_n).$$

Proof. (1). See [Jan, II.6.1].

(2). This follows from (1) together with the isomorphism of P-modules $(\mathfrak{g}/\mathfrak{p})^\# \simeq \mathfrak{u}$, guaranteed by the existence of a nondegenerate form.

(3). We have $\mathfrak{g}_{\mathrm{ad}} = \mathrm{End}(V)$. Let $e_1, \dots, e_n$ be a basis of V, and choose a linear splitting of the sequence $0 \longrightarrow W \longrightarrow V \longrightarrow V/W \longrightarrow 0$ sending V/W to the subspace generated by $e_{r+1}, \dots, e_n$ (see notation introduced in Ex 6.4). We have

$$\mathrm{End}(V) = \mathrm{Hom}(W, V/W) \oplus \mathrm{Hom}(V/W, W) \oplus \mathrm{Hom}(W,W) \oplus \mathrm{Hom}(V/W, V/W),$$

where the sum of the last three summands is a P-stable subspace isomorphic to $\mathfrak{p}$. Hence, we have an isomorphism of P-modules: $\mathfrak{g}/\mathfrak{p} \simeq \mathrm{Hom}(W, V/W) \simeq W^\# \otimes V/W$. Therefore,

$$\mathbb{T}_X \simeq \mathcal{L}_{G/P}(\mathfrak{g}/\mathfrak{p}) \simeq \mathcal{L}(W^\#) \otimes \mathcal{L}(V/W) = \gamma_r^\vee \otimes \delta_{n-r}^\vee.$$

Consequently,

$$\Omega_X \simeq \gamma_r \otimes \delta_{n-r},$$

since the form $(x, y) \mapsto Tr(xy)$ is non-degenrate on $\mathfrak{sl}_n$.

(4). In this case W is an isotropic subspace of V, and $W \simeq (V/W)^\#$. Then $\mathfrak{g}/\mathfrak{p} \simeq \mathrm{Hom}_{Sym}(W, V/W) \simeq S^2(W^\#)$. Hence,

$$\mathbb{T}_Y \simeq \mathcal{L}_{G/P}(\mathfrak{g}/\mathfrak{p}) \simeq S^2(\gamma_n^\vee).$$

The condition on p assures that the Killing form is nondegenerate (see [Sel65]). Hence, we can dualize to obtain the last asserted isomorphism. □

We next show how to realize the tangent bundle of G/P for a cominuscule parabolic P of a simple algebraic group G as a cokernel bundle.

Proposition 6.11. *Let G be a simple algebraic group, and let P be a cominuscule parabolic subgroup of G. Set $\mathfrak{g} = \mathrm{Lie}\, G$, $\mathfrak{p} = \mathrm{Lie}\, P$, and let $\mathfrak{u}$ be the nilpotent radical of $\mathfrak{p}$. Let $G \cdot \mathfrak{u} \simeq G/P$ be the (closed) orbit of $\mathfrak{u} \in \mathbb{E}(\dim \mathfrak{u}, \mathfrak{g})$. Consider $\mathfrak{g}$ as the adjoint representation of G. We have isomorphisms of vector bundles on G/P:*

$$\mathcal{I}m^{G/P}(\mathfrak{g}) \;\simeq\; \mathcal{L}_{G/P}(\mathfrak{p})$$

and

$$\mathcal{C}oker^{G/P}(\mathfrak{g}) \;\simeq\; \mathbb{T}_{G/P}.$$

Proof. Let $X = G/P$, and let $\epsilon = \mathfrak{u}$. Then $\mathrm{Rad}(\epsilon^* \mathfrak{g}) = [u, \mathfrak{g}] = \mathfrak{p}$ by Prop. 2.8. Propositions 6.1 and 6.2 give an isomorphism

$$\mathrm{Im}^X(\mathfrak{g}) \;\simeq\; \mathcal{L}_X(\mathfrak{p})$$

as bundles on X. Applying Proposition 6.2 again, we conclude that the short exact sequence of rational P-modules

$$0 \to \mathfrak{p} \to \mathfrak{g} \to \mathfrak{g}/\mathfrak{p} \to 0$$

determines a short exact sequence of bundles on X:

$$0 \to \mathcal{L}_X(\mathfrak{p}) \;\to\; \mathfrak{g} \otimes \mathcal{O}_X \;\to\; \mathcal{L}_X(\mathfrak{g}/\mathfrak{p}) \to 0.$$

We conclude that

$$\mathcal{C}oker^X(\mathfrak{g}) \;\simeq\; (\mathfrak{g} \otimes \mathcal{O}_X)/\,\mathrm{Im}^X(\mathfrak{g}) \simeq (\mathfrak{g} \otimes \mathcal{O}_X)/\mathcal{L}_X(\mathfrak{p}) \simeq \mathcal{L}_X(\mathfrak{g}/\mathfrak{p}) \simeq \mathbb{T}_{G/P}.$$

□

We offer some other interesting bundles coming from the adjoint representation of $\mathfrak{g}$.

Proposition 6.12. *Under the assumptions of Proposition 6.11, we have*

$$\mathcal{I}m^{2,G/P}(\mathfrak{g}) \simeq \mathcal{L}_{G/P}(\mathfrak{u}),$$

where $\mathfrak{u}$ is viewed as a submodule of $\mathfrak{p}$ via the adjoint action of P.

Proof. Let $\epsilon = \mathfrak{u}$. By Proposition 2.8, $\mathrm{Rad}^2(\epsilon^*\mathfrak{g}) = [\mathfrak{u}, [\mathfrak{u}, \mathfrak{g}]] = \mathfrak{u}$. Hence, $\mathcal{I}m^{2,G/P}(\mathfrak{g}) \simeq \mathcal{L}_{G/P}(\mathfrak{u})$ by Theorem 6.5. □

In the next three examples we specialize Proposition 6.12 to the simple groups of types A_n, B_n, and C_n.

Example 6.13. Let $G = \mathrm{SL}_n$, $P = P_{r,n-r}$, $X = G/P \simeq \mathrm{Grass}_{n,r}$. We have an isomorphism of vector bundles on X

$$\mathcal{I}m^{2,X}(\mathfrak{g}) \simeq \Omega_X \simeq \gamma_r \otimes \delta_{n-r}.$$

Indeed, this follows immediately from Propositions 6.12 and 6.10(3).

Example 6.14. Let $G = \mathrm{SO}_{2n+1}$ be a simple algebraic group of type B_n so that $\mathfrak{g} = \mathfrak{so}_{2n+1}$, and let $P = P_{\alpha_1}$ be the standard cominuscule parabolic subgroup of G (we choose the symmetric form, the Cartan matrix, and the simple roots as in [EW06, 12.3]). Let $\mathfrak{u}$ be the nilpotent radical of $\mathfrak{p} = \mathrm{Lie}(P)$, and let $X \simeq G/P \simeq \mathbb{P}^{2n-1}$ be the G orbit of $\mathfrak{u}$ in $\mathbb{E}(2n-1, \mathfrak{g})$. Then

$$\mathcal{I}m^{2,X}(\mathfrak{g}) = \mathcal{L}_{G/P}(\mathfrak{u}) \simeq \mathcal{L}_{\mathbb{P}^{2n-1}}(V_{2n-1}),$$

and, if $p > 2$ and does not divide $2n-1$,

$$\mathcal{I}m^{2,X}(\mathfrak{g}) \simeq \Omega_{\mathbb{P}^{2n-1}}.$$

In the first formula above, V_{2n-1} is the natural module for the block of the Levy factor of P which has type B_{n-1}. More precisely, we have $P = LU$ where L is the Levi factor and U is the unipotent radical. The Levi factor L is a block matrix group with blocks of size 2 and $2n-1$. Factoring out the subgroup concentrated in the block of size 2, we get a simple algebraic group isomorphic to SO_{2n-1}. We take V_{2n-1} to be the standard module for this group inflated to the parabolic P.

To justify the claims, we note that the isomorphism $\mathcal{I}m^{2,X}(\mathfrak{g}) = \mathcal{L}_X(\mathfrak{u})$ is the content of Proposition 6.12, whereas the isomorphism $\mathcal{I}m^{2,X}(\mathfrak{g}) = \mathcal{L}_X(V_{2n-1})$ follows from an isomorphism of P-modules $\mathfrak{u} \simeq V_{2n-1}$ which can be checked by direct inspection. Finally, the last asserted isomorphism follows from Proposition 6.10, since the condition on p guarantees that the Killing form on $\mathfrak{g} = \mathrm{Lie}\, G$ is nondegenerate (see [Sel65]).

Example 6.15. Let $G = \mathrm{Sp}_{2n}$, $P = P_{\alpha_n}$, and assume that $p > 3$ does not divide $n+1$. We have an isomorphism of vector bundles on $\mathbb{E}(\binom{n+1}{2}, \mathfrak{g}) \simeq \mathrm{LG}_{2n,n}$:

$$\mathcal{I}m^2(\mathfrak{g}) \simeq S^2(\gamma_n).$$

Just as in the previous examples, this follows immediately from Theorem 2.13 which identifies $\mathbb{E}(\binom{n+1}{2}, \mathfrak{g})$ with $\mathrm{LG}_{2n,n}$, and Propositions 6.12 and 6.10(4). Proposition 6.10 is applicable here since the condition on p guarantees that the Killing form is nondegenerate (see [Sel65]).

Proposition 6.16. *Let G be a simple algebraic group and $P \subset G$ be a cominuscule parabolic. Let $\mathfrak{g} = \mathrm{Lie}\, G$, $\mathfrak{p} = \mathrm{Lie}\, P$, and let $\mathfrak{u}$ be the nilpotent radical of $\mathfrak{p}$. Let $X = G \cdot \mathfrak{u} \subset \mathbb{E}(r, \mathfrak{g})$ where $r = \dim \mathfrak{u}$. Consider the adjoint action of G on $\mathfrak{g}$. We have an isomorphism of bundles on $X \simeq G/P$:*

$$\mathcal{K}er^X(\mathfrak{g}) \;\simeq\; \mathcal{L}_{G/P}(\mathfrak{u}) \oplus \mathcal{O}_X.$$

If, in addition, $\mathfrak{g}$ has a nondegenerate G-invariant bilinear form, then

$$\mathcal{K}er^X(\mathfrak{g}) \;\simeq\; \Omega_X \oplus \mathcal{O}_X.$$

Proof. Let $\epsilon = \mathfrak{u}$ which is an elementary subalgebra by Lemma 2.6. We have $\mathrm{Soc}(\epsilon^*(\mathfrak{g})) = C_{\mathfrak{g}}(\mathfrak{u})$, the centralizer of $\mathfrak{u}$ in $\mathfrak{g}$. Since $\mathfrak{p}$ is the normalizer of $\mathfrak{u}$, we have $C_{\mathfrak{g}}(\mathfrak{u}) \subset \mathfrak{p}$. Moreover, since $\mathfrak{u} \subset \mathfrak{p}$ is a Lie ideal, so is $C_{\mathfrak{g}}(\mathfrak{u})$. Since $\mathfrak{p}/\mathfrak{u}$ is reductive, we conclude that $C_{\mathfrak{g}}(\mathfrak{u})/\mathfrak{u}$ is the center of the Levi subalgebra of $\mathfrak{p}$ which coincides with the center of $\mathfrak{p}$ itself. Denote the center of $\mathfrak{p}$ by $C(\mathfrak{p})$. We have $C_{\mathfrak{g}}(\mathfrak{u}) = \mathfrak{u} + C(\mathfrak{p})$; moreover, $C(\mathfrak{p})$ is a trivial one-dimensional P-module since P is a maximal parabolic. Therefore, $\mathcal{L}_{G/P}(C(\mathfrak{g})) \simeq \mathcal{O}_{G/P}$, a trivial line bundle. Combining this with Theorem 6.5 we get the following isomorphisms

$$\mathrm{Ker}^{G/P}(\mathfrak{g}) \simeq \mathcal{L}_{G/P}(C_{\mathfrak{g}}(\mathfrak{u})) \simeq \mathcal{L}_{G/P}(\mathfrak{u}) \oplus \mathcal{L}_{G/P}(C(\mathfrak{g})) \simeq \mathcal{L}_{G/P}(\mathfrak{u}) \oplus \mathcal{O}_{G/P}.$$

If $\mathfrak{g}$ has a nondegenerate G-invariant bilinear form then $\mathcal{L}_{G/P}(\mathfrak{u})$ is isomorphic to the cotangent bundle $\Omega_{G/P}$ by Proposition 6.10. Hence, in this case we have

$$\mathcal{K}er^{G/P}(\mathfrak{g}) \simeq \Omega_{G/P} \oplus \mathcal{O}_{G/P}.$$

□

We restate in the following example the special cases of Proposition 6.16 for which we have an identification $\mathbb{E}(r, \mathfrak{g}) = G \cdot \mathfrak{u} \simeq G/P$ as shown in Section 2.

Example 6.17. (1). For $G = \mathrm{SL}_{2n}$, $(x, y) = Tr(xy)$ defines a nondegenerate G-invariant bilinear form on $\mathfrak{g} = \mathfrak{sl}_{2n}$. Hence, specializing Proposition 6.16 to $G = \mathrm{SL}_{2n}$, $P = P_{n,n}$, $r = n^2$, and $\mathbb{E}(r, \mathfrak{sl}_{2n}) \simeq \mathrm{Grass}_{2n,n}$, we get an isomorphism of bundles on the Grassmannian:

$$\mathcal{K}er(\mathfrak{sl}_{2n}) \simeq \Omega_{\mathrm{Grass}_{2n,n}} \oplus \mathcal{O}_{\mathrm{Grass}_{2n,n}}.$$

(2). Assume that $p > 2$ and does not divide $n+1$. This guarantees that the Killing form on $\mathfrak{sp}_{2n}$ is nondegenerate (see [Sel65]). For $G = \mathrm{Sp}_{2n}$ and $r = \dim \mathfrak{u} = \binom{n+1}{2}$, we have $\mathbb{E}(r, \mathfrak{sp}_{2n}) = \mathrm{LG}(n, n)$ (the Lagrangian Grassmannian), by Theorem 2.13. Hence, we have an isomorphism of bundles on $\mathrm{LG}_{n,n}$:

$$\mathcal{K}er(\mathfrak{sp}_{2n}) \;\simeq\; \Omega_{\mathrm{LG}_{n,n}} \oplus \mathcal{O}_{\mathrm{LG}_{n,n}}.$$

7. Vector bundles associated to semi-direct products

In this section, we provide a reinterpretation of "GL-equivariant kE-modules" considered in [CFP12] as modules for the subgroup scheme $G_{(1),n} = \mathbb{V}_{(1)} \rtimes \mathrm{GL}_n$ of the algebraic group $G_{1,n} = \mathbb{V} \rtimes GL_n$ of Example 1.11. This leads to consideration of rational representations for semi-direct product group schemes $\mathbb{W}_{(1)} \rtimes H$ where H is any algebraic group H and W is any faithful rational H-representation.

Notation 7.1. Throughout this section, V is an n-dimensional vector space with chosen basis, so that we may identify $\mathrm{GL}(V)$ with GL_n and V with the defining representation of GL_n. Let $\mathbb{V} = \mathrm{Spec}(S^*(V^\#)) \simeq \mathbb{G}_a^{\oplus n}$ be the vector group associated to V, and let $\mathbb{V}_{(1)} \simeq (\mathbb{G}_{a(1)})^{\oplus n}$ be the first Frobenius kernel of $\mathbb{V}$. The standard action of GL_n on V induces an action on the vector group $\mathbb{V}$. Moreover, it is straightforward that this action stabilizes the subgroup scheme $\mathbb{V}_{(1)} \subset \mathbb{V}$. Hence, we can form the following semi-direct products:

$$G_{1,n} \stackrel{\text{def}}{=\!=} \mathbb{V} \rtimes \mathrm{GL}_n \qquad G_{(1),n} \stackrel{\text{def}}{=\!=} \mathbb{V}_{(1)} \rtimes \mathrm{GL}_n \,. \tag{7.1.1}$$

Let

$$\mathfrak{g}_{1,n} \stackrel{\text{def}}{=\!=} \mathrm{Lie}(G_{(1),n}) = \mathrm{Lie}(G_{1,n}) \,. \tag{7.1.2}$$

We view $V \simeq \mathrm{Lie}(\mathbb{V}_{(1)}) \subset g_{1,n}$ as an elementary subalgebra of $\mathfrak{g}_{1,n}$ which is also a Lie ideal stable under the adjoint action of $G_{1,n}$. For any r-dimensional subspace $\epsilon \subset V \subset \mathfrak{g}_{1,n}$, we consider the adjoint action of $G_{1,n}$ on ϵ. Since V is stable under the adjoint action, and the action of $\mathbb{V}$ on V is trivial, we get that the adjoint orbit $G_{1,n} \cdot \epsilon \subset \mathbb{E}(r, g_{1,n})$ can be identified with $\mathrm{GL}_n \cdot \epsilon \subset \mathrm{Grass}(r, \mathbb{V}) \simeq \mathbb{E}(r, \mathrm{Lie}(\mathbb{V}_{(1)})) \subset \mathbb{E}(r, g_{1,n})$.

We recall the notion of a GL-equivariant kE-module considered in [CFP12].

Definition 7.2. Let E be an elementary abelian p-group of rank n and choose some linear map $V \to \mathrm{Rad}(kE)$ such that the composition $V \to \mathrm{Rad}(kE) \to \mathrm{Rad}(kE)/\mathrm{Rad}^2(kE)$ is an isomorphism. This determines an identification $kE \simeq S^*(V)/\langle v^p, v \in V\rangle$. Then M is said to be a GL-*equivariant* kE-*module* (in the terminology of [CFP12, 3.5]) if M is provided with two pairings

$$S^*(V)/\langle v^p, v \in V\rangle \otimes M \;\to\; M, \quad \mathrm{GL}(V) \times M \to M \tag{7.2.1}$$

such that the first pairing is $\mathrm{GL}(V)$-equivariant with respect to the diagonal action of $\mathrm{GL}(V)$ on $S^*(V)/\langle v^p, v \in V\rangle \otimes M$.

As the next proposition explains, the consideration of GL-equivariant kE-modules has a natural interpretation as $G_{(1),n}$-representations for $G_{(1),n} = \mathbb{V}_{(1)} \rtimes \mathrm{GL}_n$.

Proposition 7.3. *There is a natural equivalence of categories between the category of rational modules for the group scheme $G_{(1),n}$ and the category of "*GL*-equivariant kE-modules".*

Proof. Assume that we are given a functorial action of the semi-direct product

$$(G_{(1),n})(A) = \mathbb{V}_{(1)}(A) \rtimes \mathrm{GL}_n(A) \quad \text{on } M \otimes A$$

as A runs over commutative k-algebras. We view this as a group action of pairs $(v, g) = (v, 1) \cdot (0, g)$ on M. Since $(0, g) \cdot (v, 1) = (v^g, g) = (v^g, 1) \cdot (0, g)$ in the semi-direct product, we conclude for any $m \in M$ that the action of $(0, g)$ on $(v, 1) \circ m$ equals the action of $(v^g, 1)$ on $(0, g) \circ m$. This is precisely the condition that the action of $\mathbb{V}_{(1)} \times M \to M$ is GL_n-equivariant for the diagonal action of GL_n on $\mathbb{V}_{(1)} \times M$. Consequently, once the identification $kE \simeq k\mathbb{V}_{(1)} = \mathfrak{u}(\mathrm{Lie}(\mathbb{V}))$ is chosen, to give a GL_n-equivariant action $kE \times M \to M$ is to give actions of $\mathbb{V}_{(1)}$ and GL_n on M which satisfy the condition that this pair of actions determines an action of the semi-direct product.

Conversely, given a GL_n-equivariant kE-module N, it is straightforward to check that the actions of GL_n and $kE \simeq k\mathbb{V}_{(1)}$ determine an action of $G_{(1),n}$ on the underlying vector space of N. $\square$

Note that we have GL_n acting on $\mathfrak{g}_{1,n}$ by restricting the adjoint action of $G_{1,n}$ on its Lie algebra. This, in turn, makes $\mathbb{E}(r, \mathfrak{g}_{1,n})$ into a GL_n-variety. We next observe that rational $G_{(1),n}$-representations (even those which are not restrictions of $G_{1,n}$-representations) lead to GL_n-equivariant sheaves on $\mathrm{Grass}(r, V)$.

Proposition 7.4. *Let $W \subset \mathbb{E}(r, \mathfrak{g}_{1,n})$ be a closed GL_n-invariant subset for some r, $1 \le r < n$. Let M be a finite dimensional rational representation of $G_{(1),n}$. Then for any j, $1 \le j \le (p-1)r$, the image and kernel sheaves $\mathcal{K}er^{j,W}(M)$, $\mathcal{I}m^{j,W}(M)$ on W are GL_n-equivariant.*

Consequently, $\mathcal{K}er^{j,X}(M)$ and $\mathcal{I}m^{j,X}(M)$ are GL_n-equivariant algebraic vector bundles on $X = \mathrm{GL}_n \cdot \epsilon \simeq \mathrm{Grass}_{n,r}$, where $\epsilon \subset V$ is an r-dimensional subspace.

Proof. As seen in the proof of Proposition 7.3, the action $\mathfrak{u}(\mathfrak{g}_{1,n}) \otimes M \to M$ restricts to an action of $\mathfrak{u}(\mathrm{Lie}\,\mathbb{V}) \otimes M \to M$ which is GL_n-equivariant. In other words,

$$(v \circ m)^g \;=\; v^g \circ m^g \tag{7.4.1}$$

for $v \in V = \mathrm{Lie}\,\mathbb{V},\ g \in \mathrm{GL}_n,\ m \in M$.

We consider the following analogue of (5.11.4):

$$\begin{tikzcd} M \otimes k[W_\Sigma] \arrow[r, "g"] \arrow[d, "(\Theta_1^{W,\Sigma})^{j_1} \dots (\Theta_r^{W,\Sigma})^{j_r}"'] & M \otimes k[W_\Sigma^{g^{-1}}] \arrow[d, "((\Theta_1^{W,\Sigma})^{j_1} \dots (\Theta_r^{W,\Sigma})^{j_r})^g"] \\ M \otimes k[W_\Sigma] \arrow[r, "g"] & M \otimes k[W_\Sigma^{g^{-1}}], \end{tikzcd} \tag{7.4.2}$$

where $g \in \mathrm{GL}_n$ and W_Σ is a principal open subset of W (the restriction of a principal open subset of $\mathrm{Grass}(r, \mathfrak{g}_{1,n})$), and $\epsilon \in W_\Sigma$. The operators Θ_s in (7.4.3) are defined as in (5.1.1) with $\mathfrak{u}(\mathfrak{g})$ replaced by $\mathfrak{u}(\mathrm{Lie}(\mathbb{V}))$. The commutativity of (7.4.2) follows from (7.4.1), just as the commutativity of (5.11.4) follows from (5.11.2).

To complete the proof, we argue as in the proof of Proposition 5.12 (replacing the commutative square (5.11.4) by the above commutative square (7.4.2)) to obtain the following commutative square analogous to (5.12.1):

$$\begin{tikzcd} M \otimes k[W_\Sigma]_{(\epsilon)} \arrow[r, "g"] \arrow[d, "(\Theta_1^{W,\Sigma})^{j_1} \dots (\Theta_r^{W,\Sigma})^{j_r}"'] & M \otimes k[W_\Sigma^{g^{-1}}]_{(\epsilon^{g^{-1}})} \arrow[d, "((\Theta_1^{W,\Sigma})^{j_1} \dots (\Theta_r^{W,\Sigma})^{j_r})^g"] \\ M \otimes k[W_\Sigma]_{(\epsilon)} \arrow[r, "g"] & M \otimes k[W_\Sigma^{g^{-1}}]_{(\epsilon^{g^{-1}})}. \end{tikzcd} \tag{7.4.3}$$

The proof of the proposition is completed by arguing exactly as at the end of the proof of Proposition 5.12: since kernels and images commute with taking stalks, $\mathcal{I}m^{W,j}(M)$, $\mathcal{K}er^{W,j}(M)$ are GL_n-equivariant sheaves.

The second statement of the proposition follows just as in the proof of Proposition 6.1. $\square$

Using the GL_n equivariance of image and kernel sheaves, we obtain the following comparison supplementing Proposition 7.4.

Proposition 7.5. *Let $M_{|E}$ denote the kE-module associated to the rational $G_{(1),n}$-module M. Choose some $r, 1 \leq r < n$, and some j with $1 \leq j \leq (p-1)r$. Let $\epsilon \subset V$ be an r-dimensional subspace. Then there are natural identifications of* GL_n*-equivariant vector bundles on* $X \simeq \mathrm{Grass}(r, V)$,

$$\mathcal{I}m^{j,X}(M) \simeq \mathcal{I}m^j(M_{|E}), \quad \mathcal{K}er^{j,X}(M) \simeq \mathcal{K}er^j(M_{|E}),$$

where the vector bundles $\mathcal{I}m^j(M_{|E})$, $\mathcal{K}er^j(M_{|E})$ on $\mathrm{Grass}(r, V)$ *are those constructed in* [CFP12].

Proof. The vector bundle $\mathcal{I}m^{j,X}(M)$ on $X \subset \mathbb{E}(r, \mathfrak{g}_{1,n})$ is GL_n-equivariant by Proposition 7.4 with the fiber at the point $\epsilon \in X$ isomorphic to $\mathrm{Rad}^j(\epsilon^* M)$ by Proposition 6.1. As proved in [CFP12, 7.5], the vector bundle $\mathcal{I}m^j(M_{|E})$ on $\mathrm{Grass}(r, V)$ is also GL_n-equivariant with fiber over $\epsilon \in \mathrm{Grass}(r, V)$ also isomorphic to $\mathrm{Rad}^j(\epsilon^* M)$. Hence, $\mathcal{I}m^{j,X}(M) \simeq \mathcal{I}m^j(M_{|E})$ by Proposition 6.3.

The argument for the kernels is strictly analogous. □

As an immediate corollary of Proposition 7.3, we conclude the following interpretation of the computations of [CFP12]. The representations N, M, R of the following proposition are rational $G_{(1),n}$-modules which do not extend to rational $G_{1,n}$-modules. The $G_{(1),n}$ action on $N = S^*(V)/S^{*\geq j+1}(V)$, for example, is given by $\mathbb{V}_{(1)}$ acting by "multiplying by V" which increases degree of this graded module, whereas the GL_n structure is a direct sum of actions on each symmetric power $S^i(V)$. See [CFP12, 3.6] for details of the $G_{(1),n}$-structures on N, M, R.

Proposition 7.6. [CFP12, 7.12,7.11,7.14] *Let $\epsilon \subset V$ be an r-plane for some integer r, $1 \leq r \leq n$, and let $X = \mathrm{GL}_n \cdot \epsilon \simeq \mathrm{Grass}_{n,r}$ be the orbit of ϵ in $\mathbb{E}(r, g_{1,n})$ as in Notation 7.1. There are isomorphisms of* GL_n*-equivariant vector bundles on* $\mathrm{Grass}_{n,r}$*:*

(1) For the rational $G_{(1),n}$-module $N = S^*(V)/S^{*\geq j+1}(V)$ and for any j, $1 \leq j \leq p-1$,

$$\mathcal{I}m^{j,X}(N) \ \simeq \ S^j(\gamma_r),$$

where γ_r is the canonical rank r subbundle of the trivial rank n bundle on $\mathrm{Grass}_{n,r}$.

(2) For the rational $G_{(1),n}$-module $M = \mathrm{Rad}^r(\Lambda^*(V))/\mathrm{Rad}^{r+2}(\Lambda^*(V))$,

$$\mathcal{K}er^X(M) \ \simeq \ \mathcal{O}_X(-1) \oplus \mathcal{O}_X^{\binom{n}{r+1}}.$$

(3) For the rational $G_{(1),n}$ module $R = S^{r(p-1)}(V)/\langle S^{r(p-1)+2}(V); v^p, v \in \mathbb{V}\rangle$,

$$\mathcal{K}er^X(R) \ \simeq \ \mathcal{O}_X(1-p) \oplus (\mathrm{Rad}(R) \otimes \mathcal{O}_X).$$

We point out that specializing Proposition 7.6(1) to the case $j = 1$ gives a realization of the canonical subbundle γ_r on the Grassmannian as an image bundle of the $G_{(1),n}$-module $S^*(V)/S^{\geq 2}(V)$ different from the realization of γ_r given in Proposition 6.7(1).

Our new examples of vector bundles arise by considering subgroup schemes of $G_{(1),n}$ which we now introduce.

Notation 7.7. Let H be an algebraic group and W a faithful, finite dimensional rational representation of H of dimension n; let $\mathbb{W}$ be the associated vector group ($\simeq \mathbb{G}_a^{\oplus n}$) equipped with the action of H. Let

$$G_{W,H} \ \equiv \ \mathbb{W}_{(1)} \rtimes H \subset \mathbb{W} \rtimes H,$$

and let

$$g_{W,H} = \mathrm{Lie}(G_{W,H}).$$

For any subspace $\epsilon \subset W$ of dimension r, we identify the $W \rtimes H$-orbit of $\epsilon \in \mathbb{E}(r, g_{W,H})$ with $Y = H \cdot \epsilon \subset \mathrm{Grass}(r, W) \subset \mathbb{E}(r, g_{W,H})$.

If $\rho : H \to GL_n$ defines the representation of H on W, then ρ induces closed embeddings

$$\mathbb{W} \rtimes H \ \subset\ G_{1,n}, \quad G_{W,H} \ \subset\ G_{(1),n}.$$

Our next proposition affirms that Proposition 7.4 extends to rational $G_{W,H}$-representations.

Proposition 7.8. *Let M be a finite dimensional rational $G_{W,H}$-module and choose some r, $1 \le r < n$. For any j, $1 \le j \le (p-1)r$, the image and kernel sheaves*

$$\mathcal{I}m^{j,Y}(M), \ \mathcal{K}er^{j,Y}(M)$$

on Y are H-equivariant algebraic vector bundles.

Proof. The proof is a repetition of the proof of Proposition 7.4 with slight notational changes. We leave it as an exercise for the interested reader. □

We easily extend the computations of Proposition 7.6 by considering the rational $G_{(1),n}$-modules N, M, R upon restriction to $G_{W,H} \subset G_{(1),n}$

Theorem 7.9. *In the Notation 7.7, we have the following isomorphisms of H-equivariant vector bundles on Y:*

(1) For the rational $G_{(1),n}$-modules $N = S^*(V)/S^{*\ge j+1}(V)$ and any j, $1 \le j \le p-1$,

$$\mathcal{I}m^{j,Y}(N_{|G_{W,H}}) \ \simeq\ S^j(\gamma_r)_{|Y},$$

where γ_r denotes the canonical rank r subbundle on $\mathrm{Grass}(r, W)$.

(2) For the rational $G_{(1),n}$-modules $M = \mathrm{Rad}^r(\Lambda^*(V)/\mathrm{Rad}^{r+2}(\Lambda(V))$,

$$\mathcal{K}er^Y(M_{|G_{W,H}}) \ \simeq\ \mathcal{O}_Y(-1) \oplus \mathcal{O}_Y^{\binom{n}{r+1}}.$$

(3) For the rational $G_{(1),n}$-modules $R = S^{r(p-1)}(V)/\langle S^{r(p-1)+2}(V); v^p, v \in V\rangle$,

$$\mathcal{K}er^Y(R_{|G_{W,H}}) \ \simeq\ \mathcal{O}_Y(1-p) \oplus (\mathrm{Rad}(R) \otimes \mathcal{O}_Y).$$

Proof. Let L be a rational representation of $G_{(1),n}$. Theorem 6.5 implies that the fibers above $\epsilon \in Y$ of $\mathcal{I}m^{j,Y}(L_{|G_{W,H}})$, $\mathcal{I}m^{j,X}(L)_{|Y}$ are both isomorphic to $\mathrm{Rad}^j(\epsilon^* L)$ as modules for $\mathrm{Stab}_H(\epsilon) \subset H$. Since both $\mathcal{I}m^{j,Y}(L_{|G_{W,H}})$, $\mathcal{I}m^{j,X}(L)_{|Y}$ are H-equivariant coherent sheaves on the H-orbit Y, we conclude that they are isomorphic by Theorem 6.5.

The proposition now follows immediately from Proposition 7.6 and the above observation applied to N, M, or R. □

We restate as a corollary the following special case of Theorem 7.9.1.

Corollary 7.10. *Let H be an algebraic group, and let W be a finite dimensional representation of H. Choose $\epsilon \subset W$ an r-dimensional subspace, let $S \subset H$ be the stabilizer of ϵ, and let $Y = H \cdot \epsilon \simeq H/S$. Then there exists a rational $G_{W,H}$-representation M such that*

$$\mathcal{L}_{H/S}(\epsilon) = (\gamma_r)_{|Y} \ \simeq\ \mathcal{I}m^Y(M)$$

as H-equivariant algebraic vector bundles on $Y \subset$ $\mathrm{Grass}(r, W) \subset \mathbb{E}(r, g_{W,H})$.

Proof. The isomorphism $(\gamma_r)_{|Y} \simeq \mathcal{I}m^Y(M)$ is a special case of Theorem 7.9(1) for $j = 1$.

Note that the given action of H on W induces an action on $\mathrm{Grass}(r, W)$ and also makes the canonical subbundle γ_r on $\mathrm{Grass}(r, W)$ H-equivariant. The action of S on the fiber ϵ of γ_r at the point ϵ is the restriction of the action of H on W. Similarly, the action of S on the fiber ϵ of $\mathcal{L}_{H/S}(\epsilon)$ is the restriction of the action of H on W. Hence, $\mathcal{L}_{H/S}(\epsilon) \simeq (\gamma_r)_{|Y}$ by Prop. 6.3. □

Proposition 7.11. *Let H be a simple algebraic group with a nondegenerate H-invariant bilinear form on $\mathfrak{h} = \mathrm{Lie}\, H$. Let P be a standard cominuscule parabolic of H, and let $\epsilon = \mathfrak{u}$ be the nilradical of $\mathrm{Lie}\, P$. Then for any j, $1 \leq j \leq p-1$, there exists a rational $G_{\mathfrak{h},H}$-representation N such that*

$$\mathcal{I}m^{j,Y}(N) \simeq S^j(\Omega_Y),$$

where

$$Y = H \cdot \epsilon \simeq G/P$$

is considered as a subvariety in $\mathbb{E}(r, g_{\mathfrak{h},H})$ for $r = \dim \epsilon$.

Proof. By Theorem 7.9.1, we can find a rational $G_{\mathfrak{h},H}$-representation N such that $\mathcal{I}m^{j,Y}(N) \simeq S^j(\gamma_r)_{|Y} = S^j((\gamma_r)_{|Y})$ where γ_r is the canonical rank r subbundle on $\mathrm{Grass}(r, \mathfrak{h})$. As shown in Corollary 7.10, $(\gamma_r)_{|Y} \simeq \mathcal{L}_Y(\epsilon)$. Since $\mathcal{L}_Y(\epsilon) \simeq \Omega_Y$ by Proposition 6.10, the statement follows. □

Applying Proposition 7.11 in combination with Proposition 6.10 to $H = \mathrm{Sp}_{2n}$, we get the following realization results for bundles on the Lagrangian Grassmannian. An interested reader can compare them to Proposition 6.9(3b).

Example 7.12. Take $H = \mathrm{Sp}_{2n}$, and let $\epsilon \subset \mathfrak{sp}_{2n}$ be the nilpotent radical of the Lie algebra of the standard cominuscule parabolic subgroup of H; let $r = \dim \epsilon = \binom{n+1}{2}$. Assume that $p > 2$ and does not divide $n+1$ (so that the Killing form is nondegenerate). Consider

$$Y = H \cdot \epsilon \simeq \mathrm{LG}_{n,n} \simeq \mathbb{E}(r, \mathfrak{sp}_{2n})$$

where $\mathrm{LG}_{n,n}$ is the Lagrangian Grassmannian (see Theorem 2.13 for details). Then we have

$$\mathcal{I}m^Y(N) \simeq \Omega_Y \simeq S^2(\gamma_n)$$

for a certain $G_{\mathfrak{sp}_{2n},\mathrm{Sp}_{2n}}$-representation N. Moreover, for any j, $1 \leq j \leq p-1$, we can find a rational $G_{\mathfrak{sp}_{2n},\mathrm{Sp}_{2n}}$-representation N such that

$$\mathcal{I}m^{j,Y}(N_{|G_{W,H}}) \simeq S^j(\Omega_Y) \simeq S^{2j}(\gamma_n).$$

Here, γ_n is the canonical rank n subbundle on $\mathrm{LG}_{n,n}$.

We finish with the following consequence of Corollary 7.10 – any image or kernel bundle on an H-orbit is the pull-back of an “image-1” bundle for some rational $G_{W,H}$-module.

Proposition 7.13. *Let H be an affine algebraic group, let $\epsilon \subset \mathrm{Lie}\, H$ be an elementary subalgebra of dimension r, and let $X = H \cdot \epsilon \subset \mathbb{E}(r, \mathrm{Lie}\, H)$. Let W be a rational H-module, and choose an integer j, $1 \leq j \leq (p-1)r$.*

(1) *Let* $\mu = \mathrm{Rad}^j(\epsilon^* W)$*, let* $Y = H\cdot\mu \simeq H/\operatorname{Stab}_H(\mu)$*, and let* $f : X \to Y$ *be the evident projection map. There exists a rational* $G_{W,H}$*-module* N *such that*

$$\mathcal{I}m^{j,X}(W) \ \simeq\ f^*(\mathcal{I}m^Y(N))$$

as H*-equivariant vector bundles on* X*.*

(2) *Similarly, let* $\nu = \mathrm{Soc}^j(\epsilon^* W)$*, let* $Z = H\cdot\nu \simeq H/\operatorname{Stab}_H(\nu)$*, and let* $\phi : X \to Y$ *be the projection map. There exists a rational* $G_{W,H}$*-module* L *such that*

$$\mathcal{K}er^{j,X}(W) \ \simeq\ f^*(\mathcal{I}m^Z(L)).$$

Proof. (1). Since the stabilizer of $\epsilon \in \mathbb{E}(r, \mathrm{Lie}(H))$ is contained in the stabilizer of μ, we, indeed, have an H-equivariant quotient map

$$f : X = H/\operatorname{Stab}_H(\epsilon) \ \to\ Y = H/\operatorname{Stab}_H(\mu).$$

By Corollary 7.10, $\mathcal{L}_{H/\operatorname{Stab}(\mu)}(\mu)$ is of the form $\mathcal{I}m^Y(N)$ for some choice of rational $G_{W,H}$-module N. The fibers of $f^*(\mathcal{I}m^Y(N))$ and $\mathcal{I}m^{j,X}(W)$ above $\epsilon \in X$ can both be identified with $\mu = \mathrm{Rad}^j(\epsilon^* W)$ as $\operatorname{Stab}_H(\epsilon)$-modules. The statement now follows from Proposition 6.3.

The proof for (2) is strictly analogous. □

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