CLASSIFICATION OF CONJUGACY CLASSES OF MAXIMAL ELEMENTARY ABELIAN SUBGROUPS OF $GL(4, \mathbb{F}_p)$.

Strategy: we divide elementary abelian subgroups into several classes according to the maximal Jordan form occuring in the subgroup.

Any elementary abelian subgroup can be conjugated into $U(4, \mathbb{F}_p)$. Thus, any element has all eigenvalues equal to 1. Let $J_{\underline{\lambda}}$ be the matrix in the standard Jordan form corresponding to the partition $\underline{\lambda}$ with 0's on the main diagonal. Observe that if a subgroup E has an element of standard Jordan form $I + J_{\underline{\lambda}}$ then E can be conjugated into a subgroup of the centralizer of the element $J_{\underline{\lambda}} \in sl_4$.

We use notation $[i]$ for a single Jordan block of size $i \times i$ with zeros on the main diagonal. Let V be the standard representation of GL_4.

There are 5 possibilities for nilpotent Jordan forms in sl_4:

(I) $[4] \quad p > 3$
(II) $[3] + [1] \quad p > 2$
(III) $[2]^2$
(IV) $[2] + [1]^2$
(V) $[1]^4$

We can dismiss (V) right away since it corresponds to a trivial subgroup which is not elementary abelian and is certainly never maximal such.

Case **I**. $[4]$, $p > 3$. The unipotent part of the centralizer of the regular nilpotent element is the abelian group

$$E_1 = \left\{ \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & a & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad a, b, c \in \mathbb{F}_p \right\}$$

This group is elementary abelian, and, thus, maximal among such.

CASE **II**. $[3] + [1]$, $p > 2$.

$$J_{3,1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad \text{The centralizer is} \begin{pmatrix} r & a & b & c \\ 0 & r & a & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & d & s \end{pmatrix} \text{(see [1]).}$$

Using permutation matrix

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Date: December 2, 2005.

we conjugate $J_{3,1}$ to

$$u = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The unipotent part of the centralizer becomes

$$C_u = \left\{ \begin{pmatrix} 1 & a & c & b \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad a, b, c \in \mathbb{F}_p \right\}$$

Let $A = \begin{pmatrix} 1 & a & c & b \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $A' = \begin{pmatrix} 1 & a' & c' & b' \\ 0 & 1 & 0 & a' \\ 0 & 0 & 1 & d' \\ 0 & 0 & 0 & 1 \end{pmatrix}$ be two elements of C_u. Then

$$AA' - A'A = \begin{pmatrix} 0 & 0 & 0 & cd' - c'd \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, any two elements from an abelian subgroup of C_u must satisfy the relation

$$cd' = c'd$$

This leaves us with the following choices:

(1) c = 0. $E_2 = \begin{pmatrix} 1 & a & 0 & b \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(2) $d = 0$. $E_3 = \begin{pmatrix} 1 & a & c & b \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(3) $d = \eta c$, $\eta \in \mathbb{F}_p^*$. $E_{4,\eta} = \begin{pmatrix} 1 & a & c & b \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & \eta c \\ 0 & 0 & 0 & 1 \end{pmatrix}$

(i) None of the groups corresponding to [3]+[1] are conjugate to E_1 since they have the same rank but do not have elements of the standard Jordan form [4].

(ii) The groups from the class (c) - ($E_{4,\eta}$) - have $(p^3 - p)$ or $(p^3 - 3p + 2)$ elements of Jordan form [3]+[1] (see (iv)), whereas the groups E_2 and E_3 have $(p^3 - p^2)$ such elements each (when $a \neq 0$). Thus, $E_{4,\eta}$ are not conjugate to E_2, E_3.

(iii) We now show that E_2, E_3 give two different conjugacy classes. Let

$$B_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 & b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad b, d \in \mathbb{F}_p \right\} \subset E_2,$$

$$B_3 = \left\{ \begin{pmatrix} 1 & 0 & c & b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad b, c \in \mathbb{F}_p \right\} \subset E_3$$

Subgroups B_2, B_3 consist of all elements of the corresponding groups E_2, E_3 of standard Jordan type strictly less than the "generic" Jordan type for E_2, E_3. Namely, of elements of the types [2]+$[1]^2$ and $[1]^4$. Suppose there exists $g \in GL(V)$ such that $E_2^g = E_3$. Then the above observation implies that $B_2^g = B_3$. On the other hand, we have $\mathrm{Im}\,(I(kB_2)V) = \mathrm{Im}\,(\begin{pmatrix} 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d \\ 0 & 0 & 0 & 0 \end{pmatrix} V)$ is 2-dimensional whereas $\mathrm{Im}\,(I(kB_3)V) = \mathrm{Im}\,(\begin{pmatrix} 0 & 0 & c & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} V)$ is 1-dimensional. Here, I denotes the augmentation ideal of the corresponding group algebra. Hence, B_2, B_3 are not conjugate.

(iv) Let's analyze the type (c). Let

$$A = \begin{pmatrix} 1 & a & c & b \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & \eta c \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Assume either c or a is non-zero. Then $rk\,(A - I) = 2$. We also have

$$(A - I)^2 = \begin{pmatrix} 0 & 0 & 0 & \eta c^2 + a^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, the Jordan type of A is [3]+[1] if $\eta \neq -(a^2/c^2)$ in $\mathbb{F}_p$ and [2]+[2] otherwise. More precisely, we get the following:

(a) If $\eta \neq -N^2$, then there are $p^3 - p$ elements of Jordan type [3]+[1], $p - 1$ elements of Jordan type [2]+[1]+[1], and a trivial element.

(b) If $\eta = -N^2$, then there are $p^3 - p - 2(p-1)$ elements of Jordan form [3]+[1], $2(p-1)$ elements of Jordan type [2]+[2] (one for each pair (a, c) such that $\eta = -a^2/c^2$), $p - 1$ elements of the type [2]+[1]+[1], and a trivial element.

The following lemma guarantees that types (a) and (b) above provide exactly one new conjugacy class each.

Lemma. *Let $\eta = x^2\zeta$. Then $E_{4,\eta}$ is conjugate to $E_{4,\zeta}$.*

Proof. Let

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then

$$g^{-1}\cdot\begin{pmatrix}1&a&c&b\\0&1&0&a\\0&0&1&\eta c\\0&0&0&1\end{pmatrix}\cdot g=\begin{pmatrix}1&a&xc&b\\0&1&0&a\\0&0&1&x^{-1}\eta c\\0&0&0&1\end{pmatrix}=\begin{pmatrix}1&a&(xc)&b\\0&1&0&a\\0&0&1&\zeta(xc)\\0&0&0&1\end{pmatrix}$$

□

CONCLUSION: There are **4** conjugacy classes in this case.

CASE **III**. $[2]^2$

$J_{2,2}=\begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix}$. The centralizer is $\begin{pmatrix}r&a&e&b\\0&r&0&e\\f&c&s&d\\0&f&0&s\end{pmatrix}$ (see [1]).

Using the same permutation matrix S as before we conjugate $J_{2,2}$ into the element

$$u=\begin{pmatrix}0&0&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

The centralizer becomes

$$C_u=\left\{\begin{pmatrix}r&e&a&b\\f&s&c&d\\0&0&r&e\\0&0&f&s\end{pmatrix}\quad a,b,c,d,e,f\in\mathbb{F}_p, rs-ef=1\right\}$$

Let E be an abelian subgroup of C_u containing u (any conjugacy class of elementary abelians arising in this case will have such representative). Write a general element of E as a block matrix

$$\begin{pmatrix}T&R\\0&T\end{pmatrix}$$

The matrices T must form an elementary abelian subgroup of $GL(2,p)$. Any such subgroup can be conjugated into $U(2,p)$. Thus, we can find a 2×2 matrix C such that $\begin{pmatrix}C&0\\0&C\end{pmatrix}\cdot E\cdot\begin{pmatrix}C^{-1}&0\\0&C^{-1}\end{pmatrix}$ is upper triangular. Moreover, we preserved the property that the entries $(1,2)$ and $(3,4)$ are the same and that element $I+u$ is in the subgroup. Thus, we may assume that E is a subgroup of

$$\left\{\begin{pmatrix}1&e&a&b\\0&1&c&d\\0&0&1&e\\0&0&0&1\end{pmatrix}\quad a,b,c,d,e,\in\mathbb{F}_p\right\}$$

containing

$$I+u=\begin{pmatrix}1&0&1&0\\0&1&0&1\\0&0&1&0\\0&0&0&1\end{pmatrix}.$$

For further analysis we need to separate out the case $p = 2$. Assume $\mathbf{p} > \mathbf{2}$. Let $A = \begin{pmatrix} 1 & e & a & b \\ 0 & 1 & c & d \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{pmatrix} \in E$. Since $I+u \in E$, we have $A' = (I+u)A = \begin{pmatrix} 1 & e & 1+a & b+e \\ 0 & 1 & c & 1+d \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{pmatrix} \in E$.

We must have

$$(A-I)^2 = \begin{pmatrix} 0 & 0 & 0 & e(a+d) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0$$

since the maximal allowed Jordan type here is $[2]^2$. Similarly,

$$(A'-I)^2 = \begin{pmatrix} 0 & 0 & 0 & e(a+d+2) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0$$

Since $p \neq 2$, at least one of $a+d$ and $a+d+2$ is non-zero. Thus, we obtain $e = 0$.
We therefore get the subgroup

$$E_6 = \left\{ \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad a, b, c, d \in \mathbb{F}_p \right\},$$

the only subgroup arising in this class. Since it has rank 4, it is clearly new.

Now let $\mathbf{p} = \mathbf{2}$, and let $A = \begin{pmatrix} 1 & e & a & b \\ 0 & 1 & c & d \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{pmatrix}$. We consider two cases:

- $e = 0$. Then we get E_6 as in the case $p > 2$.
- $e \neq 0$. Then $c = 0$ or otherwise A would have an element of order greater than 2. Since $(A-I)^2 = \begin{pmatrix} 0 & 0 & 0 & e(a+d) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = 0$, we get $a + d = 0$. Hence, $a = d$. We are therefore reduced to a subgroup

$$E^{sp} = \left\{ \begin{pmatrix} 1 & e & a & b \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad a, b, e \in \mathbb{F}_p \right\}$$

Proceeding as in II.(iii), we observe that $\mathrm{Im}\,(I(kE^{sp})V) = \mathrm{Im}\,(\begin{pmatrix} 0 & e & a & b \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & e \\ 0 & 0 & 0 & 0 \end{pmatrix} V)$ is 3-dimensional whereas $\mathrm{Im}\,(I(kE_6)V) = \mathrm{Im}\,(\begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} V)$ is 2-dimensional. Hence, E^{sp} gives a new conjugacy class in the special case $p = 2$.

We conclude that CASE III yields one new conjugacy class - the one of rank 4 represented by E_6 - when $p > 2$. When $p = 2$ we get yet another one represented by E^{sp}.

Remark. *The subgroup E^{sp} does, in fact, show up for other p as well, but in that case the corresponding "generic" Jordan type is [3]+[1] and E^{sp} belongs to the conjugacy class of $E_{4,-1}$.*

CASE **IV**. $[2] + [1]^2$. Since any p-subgroup can be conjugated into $U(4, \mathbb{F}_p)$, we may assume that an elementary subgroup E from this class is upper-triangular. If every matrix $A \in E$ satisfies $A_{12} = A_{34} = 0$ then E is a subgroup of E_6 and, hence, is not maximal. Thus, we can find a matrix $A \in E$ such that either $A_{12} \neq 0$ or $A_{34} \neq 0$. Assume E contains an element A such that $A_{12} \neq 0$. For any such A we must have that the 2nd and 3rd rows of $A - I$ are all zero since otherwise $A - I$ will have rank at least 2. Thus, $A = \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ with $a \neq 0$. Let $A' = \begin{pmatrix} 1 & 0 & b' & c' \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix}$ be any element in E such that $A'_{12} = 0$. We have $AA' = \begin{pmatrix} 1 & a & * & * \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix}$. One immediately observes that if any of e, d or f is non-zero, then $\mathrm{rk}\, AA' - I > 1$. Thus, all elements of E only have non-zero elements in the first row. We get a subgroup

$$E_7 = \left\{ \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad a, b, c \in \mathbb{F}_p \right\}$$

Arguing similarly in the case when $A_{34} \neq 0$, we get

$$E_8 = \left\{ \begin{pmatrix} 1 & 0 & 0 & c \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad a, b, c \in \mathbb{F}_p \right\}$$

Finally, we check that these two subgroups are not conjugate and new:
- E_7 and E_8 are not conjugate since $\dim I(kE_7)V = 1$ and $\dim I(kE_8)V = 3$.
- E_7 and E_8 each have $p^3 - 1$ elements of Jordan type [2]+[1]+[1]. This is more than any of the previous groups have. Thus, these two groups are new.

And now we are done! The result is **8** for $\mathbf{p > 3}$ and here is the complete list:

$$\begin{pmatrix} 1 & a & b & c \\ 0 & 1 & a & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & a & 0 & b \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & a & c & b \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & a & c & b \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & \eta c \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{where } (-\eta) \text{ is a quadratic residue modulo } p,$$

$$\begin{pmatrix} 1 & a & c & b \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & \eta c \\ 0 & 0 & 0 & 1 \end{pmatrix}, \text{where } (-\eta) \text{ is a non-quadratic residue modulo } p,$$

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & c \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For $\mathbf{p} = \mathbf{3}$, the first group from the above list is missing, the count is **7**.

For $\mathbf{p} = \mathbf{2}$, we get **4**: the last 3 subgroups on the list plus the subgroup

$$E^{sp} = \left\{ \begin{pmatrix} 1 & e & a & b \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad a, b, e \in \mathbb{F}_p \right\}$$

References

[1] J. Jantzen, *Nilpotent orbits in representation theory*, Lie theory, Progr. Math., **228**, 1-211, Birkhuser Boston, Boston, MA, 2004.