LOCAL DUALITY FOR REPRESENTATIONS OF FINITE GROUP SCHEMES

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Abstract. A duality theorem for the stable module category of representations of a finite group scheme is proved. One of its consequences is an analogue of Serre duality, and the existence of Auslander-Reiten triangles, for the $\mathfrak{p}$-local and $\mathfrak{p}$-torsion subcategories of the stable category, for each homogeneous prime ideal $\mathfrak{p}$ in the cohomology ring of the group scheme.

1. Introduction

This work concerns the modular representation theory of finite groups and group schemes. A starting point for it is a duality theorem for finite groups due to Tate, that appears already in Cartan and Eilenberg [23]. For our purposes it is useful to recast this theorem in terms of stable module categories. The stable module category of a finite group scheme G over a field k is the category obtained from the (abelian) category of finite dimensional G-modules by annihilating morphisms that factor through a projective module; we denote it $\mathsf{stmod}\, G$, and write $\underline{\mathrm{Hom}}_G(-,-)$ for the morphisms in this category. The category $\mathsf{stmod}\, G$ is triangulated with suspension Ω^{-1}, and Tate duality translates to the statement that for all finite dimensional G-modules M and N there are natural isomorphisms

$$\mathrm{Hom}_k(\underline{\mathrm{Hom}}_G(M,N),k) \cong \underline{\mathrm{Hom}}_G(N, \Omega\delta_G \otimes_k M)\,.$$

Here δ_G is the modular character of G, described in Jantzen [34, §I.8.8]; it is isomorphic to the trivial representation k when G is a finite group. Tate duality can be deduced from a formula of Auslander and Reiten [1] that applies to general associative algebras; see Theorem 4.2.

In the language introduced by Bondal and Kapranov [20] the isomorphism above says that $\mathsf{stmod}\, G$ has Serre duality with Serre functor $M \mapsto \Omega\delta_G \otimes_k M$. One of the main results of our work is that such a duality also holds *locally.*

The precise statement involves a natural action of the cohomology ring $H^*(G,k)$ of G on the graded abelian group

$$\underline{\mathrm{Hom}}^*_G(M,N) = \bigoplus_{n\in\mathbb{Z}} \underline{\mathrm{Hom}}_G(M, \Omega^{-n}N)\,.$$

The ring $H^*(G,k)$ is graded commutative, and also finitely generated as a k-algebra, by a result of Friedlander and Suslin [29]. Fix a homogeneous prime ideal $\mathfrak{p}$ not containing $H^{\geqslant 1}(G,k)$ and consider the triangulated category $\gamma_{\mathfrak{p}}(\mathsf{stmod}\, G)$ that is obtained from $\mathsf{stmod}\, G$ by localising the graded morphisms at $\mathfrak{p}$ and then taking the full subcategory of objects such that the graded endomorphisms are $\mathfrak{p}$-torsion; see Section 7 for details. Our interest in the subcategories $\gamma_{\mathfrak{p}}(\mathsf{stmod}\, G)$ stems from the fact that they are the building blocks of $\mathsf{stmod}\, G$ and play a key role in the

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classification of its tensor ideal thick subcategories; see [15]. These subcategories may thus be viewed as analogues of the $K(n)$-local spectra in stable homotopy theory that give the chromatic filtration of a spectrum; see [42].

The following is our version of local Serre duality:

Theorem 1.1. *Let $\mathsf{C} := \gamma_{\mathfrak{p}}(\operatorname{stmod} G)$ and d the Krull dimension of $H^*(G,k)/\mathfrak{p}$. For all M,N in C there are natural isomorphisms*

$$\operatorname{Hom}_{H^*(G,k)}(\operatorname{Hom}^*_{\mathsf{C}}(M,N), I(\mathfrak{p})) \cong \operatorname{Hom}_{\mathsf{C}}(N, \Omega^d \delta_G \otimes_k M)$$

where $I(\mathfrak{p})$ is the injective hull of the graded $H^*(G,k)$-module $H^*(G,k)/\mathfrak{p}$.

One corollary is that $\gamma_{\mathfrak{p}}(\operatorname{stmod} G)$ has Auslander-Reiten triangles, so one can bring to bear the techniques of AR theory to the study of G-modules. These results are contained in Theorem 7.10.

We deduce Theorem 1.1 from a more general result concerning $\operatorname{StMod} G$, the stable category of all (including infinite dimensional) G-modules. Consider its subcategory $\varGamma_{\mathfrak{p}}(\operatorname{StMod} G)$ consisting of the $\mathfrak{p}$-local $\mathfrak{p}$-torsion modules; in other words, the G-modules whose support is contained in $\{\mathfrak{p}\}$. This is a compactly generated triangulated category and the full subcategory of compact objects is equivalent, up to direct summands, to $\gamma_{\mathfrak{p}}(\operatorname{stmod} G)$; this is explained in Remark 7.2. There is an idempotent functor $\varGamma_{\mathfrak{p}}\colon \operatorname{StMod} G \to \operatorname{StMod} G$ with image the $\mathfrak{p}$-local $\mathfrak{p}$-torsion modules; see Section 2 for details. The central result of this work is that $\varGamma_{\mathfrak{p}}(\delta_G)$ is a dualising object for $\varGamma_{\mathfrak{p}}(\operatorname{StMod} G)$, in the following sense.

Theorem 1.2. *For any G-module M and $i \in \mathbb{Z}$ there is a natural isomorphism*

$$\widehat{\operatorname{Ext}}^i_G(M, \varGamma_{\mathfrak{p}}(\delta_G)) \cong \operatorname{Hom}_{H^*(G,k)}(H^{*-d-i}(G,M), I(\mathfrak{p})).$$

This result is proved in Section 5. In the isomorphism, the vector space on the left is $\underline{\operatorname{Hom}}_G(M, \Omega^{-i}\varGamma_{\mathfrak{p}}(\delta_G))$. The statement is in terms of $\widehat{\operatorname{Ext}}$ to underscore its similarity to Serre duality on a non-singular projective variety X of dimension n:

$$\operatorname{Ext}^i_X(\mathcal{F}, \omega_X) \cong \operatorname{Hom}_k(H^{n-i}(X,\mathcal{F}), k),$$

for any coherent sheave $\mathcal{F}$ on X; see, for example, Hartshorne [33].

When G is a finite group $\varGamma_{\mathfrak{p}}(k)$ is the Rickard idempotent module κ_V, introduced by Benson, Carlson, and Rickard [8], that is associated to the irreducible subvariety V of $\operatorname{Proj} H^*(G,k)$ defined by $\mathfrak{p}$. In this context, Theorem 1.2 was proved by Benson and Greenlees [9]; see the paragraph following Theorem 5.1 below for a detailed comparison with their work and that of Benson [7].

Concerning $\varGamma_{\mathfrak{p}}(k)$, the following consequences of Theorem 1.2 have been anticipated in [6] when G is a finite group.

Theorem 1.3. *Assume $\delta_G \cong k$. The $H^*(G,k)$-module $\widehat{\operatorname{Ext}}^*_G(k, \varGamma_{\mathfrak{p}}(k))$ is injective and isomorphic to a twist of $I(\mathfrak{p})$. Also, there is an isomorphism of k-algebras*

$$\widehat{\operatorname{Ext}}^*_G(\varGamma_{\mathfrak{p}}(k), \varGamma_{\mathfrak{p}}(k)) \cong (H^*(G,k)_{\mathfrak{p}})^{\wedge}$$

where $(-)^{\wedge}$ denotes completion with respect to the $\mathfrak{p}$-adic topology, and the G-module $\varGamma_{\mathfrak{p}}(k)$ is pure injective.

Theorem 1.2 can be interpreted to mean that the category $\operatorname{StMod} G$ is Gorenstein, for it is analogous to Grothendieck's result that a commutative noetherian ring A is Gorenstein if, and only if, $\varGamma_{\mathfrak{p}}A$ is the injective hull of $A/\mathfrak{p}$, up to suspension, for each $\mathfrak{p}$ in $\operatorname{Spec} A$. In Section 6 we propose a general notion of a Gorenstein triangulated category, in an attempt to place these results in a common framework.

To prove Theorem 1.2 we use a technique from algebraic geometry in the tradition of Zariski and Weil; namely, the construction of generic points for algebraic varieties. Given a point $\mathfrak{p} \subseteq H^*(G,k)$, there is a purely transcendental extension K of k and a closed point $\mathfrak{m}$ of $\operatorname{Proj} H^*(G_K, K)$ lying above the point $\mathfrak{p}$ in $\operatorname{Proj} H^*(G,k)$. Here, G_K denotes the group scheme that is obtained from G by extending the field to K. The crux is that *one can choose* $\mathfrak{m}$ such that the following statement holds.

Theorem 1.4. *Restriction of scalars induces an exact functor*

$$\mathsf{stmod}\, G_K \supseteq \gamma_{\mathfrak{m}}(\mathsf{stmod}\, G_K) \longrightarrow \gamma_{\mathfrak{p}}(\mathsf{stmod}\, G)$$

that is surjective on objects, up to isomorphism.

This result is proved in Section 3, building on our work in [17]. It gives a remarkable description of the compact objects in $\varGamma_{\mathfrak{p}}(\mathsf{StMod}\, G)$: they are obtained from the finite dimensional objects in $\varGamma_{\mathfrak{m}}(\mathsf{StMod}\, G_K)$ by restriction of scalars. This allows one to reduce the proof of Theorem 1.2 to the case of a closed point, where it is essentially equivalent to classical Tate duality. The theorem above has other consequences; for example, it implies that the compact objects in $\varGamma_{\mathfrak{p}}(\mathsf{StMod}\, G)$ are endofinite G-modules in the sense of Crawley-Boevey [25]; see Section 3.

2. Cohomology and localisation

In this section we recall basic notions concerning certain localisation functors on triangulated categories with ring actions. The material is needed to state and prove the results in this work. The main triangulated category of interest is the stable module category of a finite group scheme, but the general framework is needed in Sections 6 and 7. Primary references for the material presented here are [11, 12]; see [17] for the special case of the stable module category.

Triangulated categories with central action. Let T be a triangulated category with suspension Σ. For objects X and Y in T set

$$\operatorname{Hom}^*_{\mathsf{T}}(X,Y) := \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathsf{T}}(X, \Sigma^i Y) \quad \text{and} \quad \operatorname{End}^*_{\mathsf{T}}(X) := \operatorname{Hom}^*_{\mathsf{T}}(X,X)\,.$$

Composition makes $\operatorname{End}^*_{\mathsf{T}}(X)$ a graded ring and $\operatorname{Hom}^*_{\mathsf{T}}(X,Y)$ a left-$\operatorname{End}^*_{\mathsf{T}}(Y)$ right-$\operatorname{End}^*_{\mathsf{T}}(X)$ module.

Let R be a graded commutative noetherian ring. In what follows we will only be concerned with homogeneous elements and ideals in R. In this spirit, 'localisation' will mean homogeneous localisation, and $\operatorname{Spec} R$ will denote the set of homogeneous prime ideals in R.

We say that a triangulated category T is *R-linear* if for each X in T there is a homomorphism of graded rings $\phi_X \colon R \to \operatorname{End}^*_{\mathsf{T}}(X)$ such that the induced left and right actions of R on $\operatorname{Hom}^*_{\mathsf{T}}(X,Y)$ are compatible in the following sense: For any $r\in R$ and $\alpha \in \operatorname{Hom}^*_{\mathsf{T}}(X,Y)$, one has

$$\phi_Y(r)\alpha = (-1)^{|r||\alpha|}\alpha\phi_X(r)\,.$$

An exact functor $F\colon \mathsf{T}\to\mathsf{U}$ between R-linear triangulated categories is *R-linear* if the induced map

$$\operatorname{Hom}^*_{\mathsf{T}}(X,Y) \longrightarrow \operatorname{Hom}^*_{\mathsf{U}}(FX,FY)$$

of graded abelian groups is R-linear for all objects X,Y in T.

In what follows, we fix a compactly generated R-linear triangulated category T and write T^{c} for its full subcategory of compact objects.

Localisation. Fix an ideal $\mathfrak{a}$ in R. An R-module M is $\mathfrak{a}$-*torsion* if $M_{\mathfrak{q}} = 0$ for all $\mathfrak{q}$ in $\operatorname{Spec} R$ with $\mathfrak{a} \not\subseteq \mathfrak{q}$. Analogously, an object X in T is $\mathfrak{a}$-*torsion* if the R-module $\operatorname{Hom}^*_{\mathsf{T}}(C, X)$ is $\mathfrak{a}$-torsion for all $C \in \mathsf{T}^{\mathsf{c}}$. The full subcategory of $\mathfrak{a}$-torsion objects

$$\mathit{\Gamma}_{\mathcal{V}(\mathfrak{a})}\mathsf{T} := \{X \in \mathsf{T} \mid X \text{ is } \mathfrak{a}\text{-torsion}\}$$

is localising and the inclusion $\mathit{\Gamma}_{\mathcal{V}(\mathfrak{a})}\mathsf{T} \subseteq \mathsf{T}$ admits a right adjoint, denoted $\mathit{\Gamma}_{\mathcal{V}(\mathfrak{a})}$.

Fix a $\mathfrak{p}$ in $\operatorname{Spec} R$. An R-module M is $\mathfrak{p}$-*local* if the localisation map $M \to M_{\mathfrak{p}}$ is invertible, and an object X in T is $\mathfrak{p}$-*local* if the R-module $\operatorname{Hom}^*_{\mathsf{T}}(C, X)$ is $\mathfrak{p}$-local for all $C \in \mathsf{T}^{\mathsf{c}}$. Consider the full subcategory of T of $\mathfrak{p}$-local objects

$$\mathsf{T}_{\mathfrak{p}} := \{X \in \mathsf{T} \mid X \text{ is } \mathfrak{p}\text{-local}\}$$

and the full subcategory of $\mathfrak{p}$-local and $\mathfrak{p}$-torsion objects

$$\mathit{\Gamma}_{\mathfrak{p}}\mathsf{T} := \{X \in \mathsf{T} \mid X \text{ is } \mathfrak{p}\text{-local and } \mathfrak{p}\text{-torsion}\}.$$

Note that $\mathit{\Gamma}_{\mathfrak{p}}\mathsf{T} \subseteq \mathsf{T}_{\mathfrak{p}} \subseteq \mathsf{T}$ are localising subcategories. The inclusion $\mathsf{T}_{\mathfrak{p}} \to \mathsf{T}$ admits a left adjoint $X \mapsto X_{\mathfrak{p}}$ while the inclusion $\mathit{\Gamma}_{\mathfrak{p}}\mathsf{T} \to \mathsf{T}_{\mathfrak{p}}$ admits a right adjoint. We denote by $\mathit{\Gamma}_{\mathfrak{p}} \colon \mathsf{T} \to \mathit{\Gamma}_{\mathfrak{p}}\mathsf{T}$ the composition of those adjoints; it is the *local cohomology functor* with respect to $\mathfrak{p}$; see [11, 12] for the construction of this functor.

The following observation is clear.

Lemma 2.1. *For any element r in $R \setminus \mathfrak{p}$, say of degree n, and $\mathfrak{p}$-local object X, the natural map $X \xrightarrow{r} \Sigma^n X$ is an isomorphism.* □

The functor $\mathit{\Gamma}_{\mathcal{V}(\mathfrak{a})}$ commutes with exact functors preserving coproducts.

Lemma 2.2. *Let $F \colon \mathsf{T} \to \mathsf{U}$ be an exact functor between R-linear compactly generated triangulated categories such that F is R-linear and preserves coproducts. Suppose that the action of R on U factors through a homomorphism $f \colon R \to S$ of graded commutative rings. For any ideal $\mathfrak{a}$ of R there is a natural isomorphism*

$$F \circ \mathit{\Gamma}_{\mathcal{V}(\mathfrak{a})} \cong \mathit{\Gamma}_{\mathcal{V}(\mathfrak{a}S)} \circ F$$

of functors $\mathsf{T} \to \mathsf{U}$, where $\mathfrak{a}S$ denotes the ideal of S that is generated by $f(\mathfrak{a})$.

Proof. The statement follows from an explicit description of $\mathit{\Gamma}_{\mathcal{V}(\mathfrak{a})}$ in terms of homotopy colimits; see [12, Proposition 2.9]. □

Injective cohomology objects. Given an object C in T^{c} and an injective R-module I, Brown representability yields an object $T(C, I)$ in T such that

$$\operatorname{Hom}_R(\operatorname{Hom}^*_{\mathsf{T}}(C, -), I) \cong \operatorname{Hom}_{\mathsf{T}}(-, T(C, I))\,. \tag{2.1}$$

This yields a functor

$$T \colon \mathsf{T}^{\mathsf{c}} \times \operatorname{\mathsf{Inj}} R \longrightarrow \mathsf{T}.$$

For each $\mathfrak{p}$ in $\operatorname{Spec} R$, we write $I(\mathfrak{p})$ for the injective hull of $R/\mathfrak{p}$ and set

$$T_{\mathfrak{p}} := T(-, I(\mathfrak{p}))\,,$$

viewed as a functor $\mathsf{T}^{\mathsf{c}} \to \mathsf{T}$.

Tensor triangulated categories. Let $\mathsf{T} = (\mathsf{T}, \otimes, \mathbb{1})$ be a tensor triangulated category such that R acts on T via a homomorphism of graded rings $R \to \operatorname{End}^*_{\mathsf{T}}(\mathbb{1})$. Brown representability yields functions objects $\mathcal{H}om(X, Y)$ satisfying an adjunction isomorphism

$$\operatorname{Hom}_{\mathsf{T}}(X \otimes Y, Z) \cong \operatorname{Hom}_{\mathsf{T}}(X, \mathcal{H}om(Y, Z)) \quad \text{for all } X, Y, Z \text{ in } \mathsf{T}.$$

Set $X^\vee := \mathcal{H}om(X, \mathbb{1})$ for each X in T. It is part of our definition of a tensor triangulated category that the unit, $\mathbb{1}$, is compact, and that compact objects are *rigid*: For all C, X in T with C compact the natural map

$$C^\vee \otimes X \longrightarrow \mathcal{H}om(C, X)$$

is an isomorphism; see [11, §8] for details.

The functors $\varGamma_{\mathfrak p}$ and $T_{\mathfrak p}$ can be computed as follows:

$$\varGamma_{\mathfrak p} \cong \varGamma_{\mathfrak p}(\mathbb{1}) \otimes - \qquad \text{and} \qquad T_{\mathfrak p} \cong T_{\mathfrak p}(\mathbb{1}) \otimes - \tag{2.2}$$

Indeed, the first isomorphism is from [11, Corollary 8.3], while the second one holds because for each $X \in \mathsf{T}$ and compact object C there are isomorphisms

$$\begin{aligned}
\operatorname{Hom}_{\mathsf T}(X, T_{\mathfrak p}(\mathbb{1}) \otimes C) &\cong \operatorname{Hom}_{\mathsf T}(X, \mathcal{H}om(C^\vee, T_{\mathfrak p}(\mathbb{1}))) \\
&\cong \operatorname{Hom}_{\mathsf T}(X \otimes C^\vee, T_{\mathfrak p}(\mathbb{1})) \\
&\cong \operatorname{Hom}_R(\operatorname{Hom}^*_{\mathsf T}(\mathbb{1}, X \otimes C^\vee), I(\mathfrak p)) \\
&\cong \operatorname{Hom}_R(\operatorname{Hom}^*_{\mathsf T}(\mathbb{1}, \mathcal{H}om(C, X), I(\mathfrak p)) \\
&\cong \operatorname{Hom}_R(\operatorname{Hom}^*_{\mathsf T}(C, X), I(\mathfrak p))\,.
\end{aligned}$$

The first and the fourth isomorphisms above hold because C is rigid; the second and the last one are adjunction isomorphisms; the third one is by the defining isomorphism (2.1).

We turn now to modules over finite group schemes, following the notation and terminology from [17].

The stable module category. Let G be a finite group scheme over a field k of positive characteristic. The coordinate ring and the group algebra of G are denoted $k[G]$ and kG, respectively. These are finite dimensional Hopf algebras over k that are dual to each other. We write $\mathsf{Mod}\, G$ for the category of G-modules and $\mathsf{mod}\, G$ for its full subcategory consisting of finite dimensional G-modules. We often identity $\mathsf{Mod}\, G$ with the category of kG-modules, which is justified by [34, I.8.6].

We write $H^*(G,k)$ for the cohomology algebra, $\operatorname{Ext}^*_G(k,k)$, of G. This is a graded commutative k-algebra, because kG is a Hopf algebra, and acts on $\operatorname{Ext}^*_G(M,N)$, for any G-modules M,N. Moreover, the k-algebra $H^*(G,k)$ is finitely generated, and, when M,N are finite dimensional, $\operatorname{Ext}^*_G(M,N)$ is finitely generated over it; this is by a theorem of Friedlander and Suslin [29].

The stable module category $\mathsf{StMod}\, G$ is obtained from $\mathsf{Mod}\, G$ by identifying two morphisms between G-modules when they factor through a projective G-module. An isomorphism in $\mathsf{StMod}\, G$ will be called a *stable isomorphism*, to distinguish it from an isomorphism in $\mathsf{Mod}\, G$. In the same vein, G-modules M and N are said to be *stably isomorphic* if they are isomorphic in $\mathsf{StMod}\, G$; this is equivalent to the condition that M are N are isomorphic in $\mathsf{Mod}\, G$, up to projective summands.

The tensor product over k of G-modules passes to $\mathsf{StMod}\, G$ and yields a tensor triangulated category with unit k and suspension Ω^{-1}, the inverse of the syzygy functor. The category $\mathsf{StMod}\, G$ is compactly generated and the subcategory of compact objects identifies with $\mathsf{stmod}\, G$, the stable module category of finite dimensional G-modules. See Carlson [22, §5] and Happel [30, Chapter I] for details.

We use the notation $\underline{\operatorname{Hom}}_G(M,N)$ for the Hom-sets in $\mathsf{StMod}\, G$. The cohomology algebra $H^*(G,k)$ acts on $\mathsf{StMod}\, G$ via a homomorphism of k-algebras

$$- \otimes_k M \colon H^*(G,k) = \operatorname{Ext}^*_G(k,k) \longrightarrow \underline{\operatorname{Hom}}^*_G(M,M)\,.$$

Thus, the preceding discussion on localisation functors on triangulated categories applies to the $H^*(G,k)$-linear category $\mathsf{StMod}\, G$.

Koszul objects. Each b in $H^d(G,k)$ corresponds to a morphism $k \to \Omega^{-d}k$ in $\mathsf{StMod}\, G$; let $k/\!\!/b$ denote its mapping cone. This gives a morphism $k \to \Omega^d(k/\!\!/b)$. For a sequence of elements $\boldsymbol{b} := b_1, \dots, b_n$ in $H^*(G,k)$ and a G-module M, we set

$$k/\!\!/\boldsymbol{b} := (k/\!\!/b_1) \otimes_k \cdots \otimes_k (k/\!\!/b_n) \qquad \text{and} \qquad M/\!\!/\boldsymbol{b} := M \otimes_k k/\!\!/\boldsymbol{b}.$$

It is easy to check that for a G-module N and $s = \sum_i |b_i|$, there is an isomorphism

$$\underline{\mathrm{Hom}}_G(M, N/\!\!/\boldsymbol{b}) \cong \underline{\mathrm{Hom}}_G(\Omega^{n+s} M/\!\!/\boldsymbol{b}, N). \tag{2.3}$$

Let $\mathfrak{b} = (\boldsymbol{b})$ denote the ideal of $H^*(G,k)$ generated by $\boldsymbol{b}$. By abuse of notation we set $M/\!\!/\mathfrak{b} := M/\!\!/\boldsymbol{b}$. If $\boldsymbol{b}'$ is a finite set of elements in $H^*(G,k)$ such that $\sqrt{(\boldsymbol{b}')} = \sqrt{(\boldsymbol{b})}$, then, by [15, Proposition 3.10], for any M in $\mathsf{StMod}\, G$ there is an equality

$$\mathsf{Thick}(M/\!\!/\boldsymbol{b}) = \mathsf{Thick}(M/\!\!/\boldsymbol{b}')\,. \tag{2.4}$$

Fix $\mathfrak{p}$ in $\operatorname{Spec} H^*(G,k)$. We will repeatedly use the fact that $\varGamma_{\mathfrak{p}}(\mathsf{StMod}\, G)^{\mathsf{c}}$ is generated as a triangulated category by the family of objects $(M/\!\!/\mathfrak{p})_{\mathfrak{p}}$ with M in $\mathsf{stmod}\, G$; see [12, Proposition 3.9]. In fact, if S denotes the direct sum of a representative set of simple G-modules, then there is an equality

$$\varGamma_{\mathfrak{p}}(\mathsf{StMod}\, G)^{\mathsf{c}} = \mathsf{Thick}((S/\!\!/\mathfrak{p})_{\mathfrak{p}})\,. \tag{2.5}$$

It turns out that one has $\varGamma_{\mathfrak{m}}(\mathsf{StMod}\, G) = \{0\}$ where $\mathfrak{m}$ denotes $H^{\geqslant 1}(G,k)$, the ideal of elements of positive degree; see Lemma 2.5 below. For this reason, it is customary to focus on $\operatorname{Proj} H^*(G,k)$, the set of homogeneous prime ideals not containing $\mathfrak{m}$, when dealing with $\mathsf{StMod}\, G$.

Tate cohomology. By construction, the action of $H^*(G,k)$ on $\mathsf{StMod}\, G$ factors through $\underline{\mathrm{Hom}}^*_G(k,k)$, the graded ring of endomorphisms of the identity. The latter ring is not noetherian in general, which is one reason to work with $H^*(G,k)$. In any case, there is little difference, vis a vis their action on $\mathsf{StMod}\, G$, as the next remarks should make clear.

Remark 2.3. Let M and N be G-modules. The map $\mathrm{Hom}_G(M,N) \to \underline{\mathrm{Hom}}_G(M,N)$ induces a map $\mathrm{Ext}^*_G(M,N) \to \underline{\mathrm{Hom}}^*_G(M,N)$ of $H^*(G,k)$-modules. This map is surjective in degree zero, with kernel $\mathrm{PHom}_G(M,N)$, the maps from M to N that factor through a projective G-module. It is bijective in positive degrees and hence one gets an exact sequence of graded $H^*(G,k)$-modules

$$0 \longrightarrow \mathrm{PHom}_G(M,N) \longrightarrow \mathrm{Ext}^*_G(M,N) \longrightarrow \underline{\mathrm{Hom}}^*_G(M,N) \longrightarrow X \longrightarrow 0 \tag{2.6}$$

with $X^i = 0$ for $i \ge 0$. For degree reasons, the $H^*(G,k)$-modules $\mathrm{PHom}_G(M,N)$ and X are $\mathfrak{m}$-torsion. Consequently, for $\mathfrak{p}$ in $\operatorname{Proj} H^*(G,k)$ the induced localised map is an isomorphism:

$$\mathrm{Ext}^*_G(M,N)_{\mathfrak{p}} \xrightarrow{\cong} \underline{\mathrm{Hom}}^*_G(M,N)_{\mathfrak{p}}\,. \tag{2.7}$$

More generally, for each r in $\mathfrak{m}$ localisation induces an isomorphism

$$\mathrm{Ext}^*_G(M,N)_r \xrightarrow{\cong} \underline{\mathrm{Hom}}^*_G(M,N)_r$$

of $H^*(G,k)_r$-modules. This means that $\operatorname{Proj} H^*(G,k)$ has a finite cover by affine open sets on which ordinary cohomology and stable cohomology agree.

Given the finite generation result due to Friedlander and Suslin mentioned earlier, the next remark can be deduced from the exact sequence (2.6).

Remark 2.4. When M, N are finite dimensional G-modules, $\underline{\mathrm{Hom}}^{\geqslant s}_G(M,N)$ is a finitely generated $H^*(G,k)$-module for any $s \in \mathbb{Z}$. Moreover the $H^*(G,k)_{\mathfrak{p}}$-module

$$\underline{\mathrm{Hom}}^*_G(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \cong \underline{\mathrm{Hom}}^*_G(M,N)_{\mathfrak{p}}$$

is finitely generated for each $\mathfrak{p}$ in $\operatorname{Proj} H^*(G,k)$.

Lemma 2.5. *One has* $\varGamma_{\mathfrak{m}}(\mathsf{StMod}\, G) = \{0\}$*, where* $\mathfrak{m} = H^{\geqslant 1}(G,k)$.

Proof. Given (2.5) it suffices to check that $S/\!\!/\mathfrak{m} = 0$ in $\mathsf{StMod}\, G$, where S is the direct sum of representative set of simple G-modules. For any G-module M, the $H^*(G,k)$-module $\underline{\mathrm{Hom}}_G^*(M, S/\!\!/\mathfrak{m})$ is $\mathfrak{m}$-torsion; see [11, Lemma 5.11(1)]. Thus, when M is finite dimensional, the $H^*(G,k)$-module $\underline{\mathrm{Hom}}_G^{\geqslant 0}(M, S/\!\!/\mathfrak{m})$ is $\mathfrak{m}$-torsion and finitely generated, by Remark 2.4, so it follows that $\underline{\mathrm{Hom}}_G^i(M, S/\!\!/\mathfrak{m}) = 0$ for $i \gg 0$. This implies that $S/\!\!/\mathfrak{m}$ is projective, since kG is self-injective. □

To gain a better understanding of the discussion above, it helps to consider the homotopy category of $\mathsf{Inj}\, G$, the injective G-modules.

The homotopy category of injectives. Let $\mathsf{K}(\mathsf{Inj}\, G)$ and $\mathsf{D}(\mathsf{Mod}\, G)$ denote the homotopy category of $\mathsf{Inj}\, G$ and the derived category of $\mathsf{Mod}\, G$, respectively. These are also $H^*(G,k)$-linear compactly generated tensor triangulated categories, with the tensor product over k. The unit of the tensor product on $\mathsf{K}(\mathsf{Inj}\, G)$ is an injective resolution of the trivial G-module k, while that of $\mathsf{D}(\mathsf{Mod}\, G)$ is k. The canonical quotient functor $\mathsf{K}(\mathsf{Inj}\, G) \to \mathsf{D}(\mathsf{Mod}\, G)$ induces an equivalence of triangulated category $\mathsf{K}(\mathsf{Inj}\, G)^{\mathsf{c}} \xrightarrow{\sim} \mathsf{D}^b(\mathsf{mod}\, G)$, where the target is the bounded derived category of $\mathsf{mod}\, G$; see [37, Proposition 2.3].

Taking Tate resolutions identifies $\mathsf{StMod}\, G$ with $\mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, G)$, the full subcategory of acyclic complexes in $\mathsf{K}(\mathsf{Inj}\, G)$. In detail, let $\mathsf{p}k$ and $\mathsf{i}k$ be a projective and an injective resolution of the trivial G-module k, respectively, and let $\mathsf{t}k$ be the mapping cone of the composed morphism $\mathsf{p}k \to k \to \mathsf{i}k$; this is a Tate resolution of k. Since projective and injective G-modules coincide, one gets the exact triangle

$$\mathsf{p}k \longrightarrow \mathsf{i}k \longrightarrow \mathsf{t}k \longrightarrow \tag{2.8}$$

in $\mathsf{K}(\mathsf{Inj}\, G)$. For a G-module M, the complex $M \otimes_k \mathsf{t}k$ is a Tate resolution of M and the assignment $M \mapsto M \otimes_k \mathsf{t}k$ induces an equivalence of categories

$$\mathsf{StMod}\, G \xrightarrow{\sim} \mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, G),$$

with quasi-inverse $X \mapsto Z^0(X)$, the submodule of cycles in degree 0. Assigning X in $\mathsf{K}(\mathsf{Inj}\, G)$ to $X \otimes_k \mathsf{t}k$ is a left adjoint of the inclusion $\mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, G) \to \mathsf{K}(\mathsf{Inj}\, G)$. These results are contained in [37, Theorem 8.2]. Consider the composed functor

$$\pi\colon \mathsf{K}(\mathsf{Inj}\, G) \xrightarrow{-\otimes_k \mathsf{t}k} \mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, G) \xrightarrow{\ \sim\ } \mathsf{StMod}\, G\,.$$

A straightforward verification yields that these functors are $H^*(G,k)$-linear. The result below is the categorical underpinning of Remark 2.3 and Lemma 2.5.

Lemma 2.6. *There is a natural isomorphism $\varGamma_{\mathfrak{m}} X \cong X \otimes_k \mathsf{p}k$ for $X \in \mathsf{K}(\mathsf{Inj}\, G)$. For each $\mathfrak{p}$ in $\operatorname{Proj} H^*(G,k)$, the functor π induces triangle equivalences*

$$\mathsf{K}(\mathsf{Inj}\, G)_{\mathfrak{p}} \xrightarrow{\sim} (\mathsf{StMod}\, G)_{\mathfrak{p}} \qquad \text{and} \qquad \varGamma_{\mathfrak{p}}(\mathsf{K}(\mathsf{Inj}\, G)) \xrightarrow{\sim} \varGamma_{\mathfrak{p}}(\mathsf{StMod}\, G).$$

Proof. We identify $\mathsf{StMod}\, G$ with $\mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, G)$. This entails $\varGamma_{\mathfrak{m}}(\mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, G)) = \{0\}$, by Lemma 2.5. It is easy to check that kG is $\mathfrak{m}$-torsion, and hence so is $\mathsf{p}k$, for it is in the localising subcategory generated by kG, and the class of $\mathfrak{m}$-torsion objects in $\mathsf{K}(\mathsf{Inj}\, G)$ is a tensor ideal localising subcategory; see, for instance, [11, Section 8]. Thus, applying $\varGamma_{\mathfrak{m}}(-)$ to the exact triangle (2.8) yields $\mathsf{p}k \cong \varGamma_{\mathfrak{m}}(\mathsf{i}k)$. It then follows from (2.2) that $X \otimes_k \mathsf{p}k \cong \varGamma_{\mathfrak{m}} X$ for any X in $\mathsf{K}(\mathsf{Inj}\, G)$.

From the construction of π and (2.8), the kernel of π is the subcategory

$$\{X \in \mathsf{K}(\mathsf{Inj}\, G) \mid X \otimes_k \mathsf{p}k \cong X\}.$$

These are precisely the $\mathfrak{m}$-torsion objects in $\mathsf{K}(\mathsf{Inj}\, G)$, by the already established part of the result. Said otherwise, $X \in \mathsf{K}(\mathsf{Inj}\, G)$ is acyclic if and only if $\varGamma_{\mathfrak{m}} X = 0$. It follows that $\mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, G)$ contains the subcategory $\mathsf{K}(\mathsf{Inj}\, G)_{\mathfrak{p}}$ of $\mathfrak{p}$-local objects, for each $\mathfrak{p}$ in $\operatorname{Proj} H^*(G,k)$. On the other hand, the inclusion $\mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, G) \subseteq \mathsf{K}(\mathsf{Inj}\, G)$ preserves coproducts, so its left adjoint π preserves compactness of objects and all compacts of $\mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, G)$ are in the image of π. Given this a simple calculation shows that $\mathsf{K}(\mathsf{Inj}\, G)_{\mathfrak{p}}$ contains $\mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, G)_{\mathfrak{p}}$. Thus $\mathsf{K}_{\mathrm{ac}}(\mathsf{Inj}\, G)_{\mathfrak{p}} = \mathsf{K}(\mathsf{Inj}\, G)_{\mathfrak{p}}$. □

3. Passage to closed points

Let G be a finite group scheme over a field k of positive characteristic. In this section we describe a technique that relates the $\mathfrak{p}$-local $\mathfrak{p}$-torsion objects in $\mathsf{StMod}\, G$, for a point $\mathfrak{p}$ in $\operatorname{Proj} H^*(G,k)$, to the corresponding modules at a closed point defined over a field extension of k. Recall that a point $\mathfrak{m}$ is *closed* when it is maximal with respect to inclusion: $\mathfrak{m} \subseteq \mathfrak{q}$ implies $\mathfrak{m} = \mathfrak{q}$ for all $\mathfrak{q}$ in $\operatorname{Proj} H^*(G,k)$. In what follows, $k(\mathfrak{p})$ denotes the graded residue field of $H^*(G,k)$ at $\mathfrak{p}$.

For a field extension K/k extension of scalars and restriction give exact functors

$$K \otimes_k (-)\colon \mathsf{StMod}\, G \longrightarrow \mathsf{StMod}\, G_K \quad \text{and} \quad (-){\downarrow_G}\colon \mathsf{StMod}\, G_K \longrightarrow \mathsf{StMod}\, G\,.$$

The functors form a adjoint pair, with the left adjoint $K \otimes_k (-)$ mapping k to K and respecting tensor products, so one has a well-known projection formula:

$$M \otimes_k N{\downarrow_G} \cong (M_K \otimes_K N){\downarrow_G} \tag{3.1}$$

for a G-module M and G_K-module N; see [3, (2.16)] or [16, Lemma 2.2].

The functor $K \otimes_k (-)$ yields a homomorphism $H^*(G,k) \to H^*(G_K,K)$ of rings. There is a natural isomorphism $H^*(G_K,K) \xrightarrow{\sim} K \otimes_k H^*(G,k)$ of K-algebras, so the preceding map above is just extension of scalars. There is an induced map

$$\operatorname{Proj} H^*(G_K,K) \longrightarrow \operatorname{Proj} H^*(G,k)\,,$$

with $\mathfrak{q}$ mapping to $\mathfrak{p} := \mathfrak{q} \cap H^*(G,k)$. We say that $\mathfrak{q}$ *lies over* $\mathfrak{p}$ to indicate this.

The main objective of this section is the proof of the following result. We say a functor is *dense* if it is surjective on objects, up to isomorphism.

Theorem 3.1. *Fix $\mathfrak{p}$ in $\operatorname{Proj} H^*(G,k)$ and K/k a purely transcendental extension of degree $\dim(H^*(G,k)/\mathfrak{p}) - 1$. There exists a closed point $\mathfrak{m}$ in $\operatorname{Proj} H^*(G_K,K)$ lying over $\mathfrak{p}$ with $k(\mathfrak{m}) \cong k(\mathfrak{p})$ such that the functor $(-){\downarrow_G}$ restricts to functors*

$$\varGamma_{\mathfrak{m}}(\mathsf{StMod}\, G_K) \to \varGamma_{\mathfrak{p}}(\mathsf{StMod}\, G) \quad \textit{and} \quad \varGamma_{\mathfrak{m}}(\mathsf{StMod}\, G_K)^{\mathsf{c}} \to \varGamma_{\mathfrak{p}}(\mathsf{StMod}\, G)^{\mathsf{c}}$$

that are dense.

The proof of the theorem yields more: There is a subcategory of $\varGamma_{\mathfrak{m}}(\mathsf{StMod}\, G_K)$ on which $(-){\downarrow_G}$ is full and dense; ditto for the category of compact objects. However, the functor need not be full on all of $\varGamma_{\mathfrak{m}}(\mathsf{StMod}\, G_K)^{\mathsf{c}}$; see Example 3.7.

Here is one consequence of Theorem 3.1.

Corollary 3.2. *The compact objects in $\varGamma_{\mathfrak{p}}(\mathsf{StMod}\, G)$ are, up to isomorphism, the restrictions of finite dimensional G_K-modules in $\varGamma_{\mathfrak{m}}(\mathsf{StMod}\, G_K)$.*

Proof. By [11, Theorem 6.4], for any ideal $\mathfrak{a}$ in $H^*(G,k)$, we have

$$\varGamma_{\mathcal{V}(\mathfrak{a})}(\mathsf{StMod}\, G)^{\mathsf{c}} = \varGamma_{\mathcal{V}(\mathfrak{a})}(\mathsf{StMod}\, G) \cap \mathsf{stmod}\, G\,.$$

Applying this observation to the ideal $\mathfrak{m}$ of $H^*(G_K,K)$ and noting that $\varGamma_{\mathfrak{m}} = \varGamma_{\mathcal{V}(\mathfrak{m})}$, since $\mathfrak{m}$ is a closed point, the desired result follows from Theorem 3.1. □

The closed point in Theorem 3.1 depends on the choice of a Noether normalisation of $H^*(G,k)/\mathfrak{p}$ as is explained in the construction below, from [17, §7].

Construction 3.3. Fix $\mathfrak{p}$ in $\operatorname{Proj} H^*(G,k)$; the following construction is relevant only when $\mathfrak{p}$ is not a closed point. Choose elements $\boldsymbol{a} := a_0, \dots, a_{d-1}$ in $H^*(G,k)$ of the same degree such that their image in $H^*(G,k)/\mathfrak{p}$ is algebraically independent over k and $H^*(G,k)/\mathfrak{p}$ is finitely generated as a module over the subalgebra $k[\boldsymbol{a}]$.

Thus the Krull dimension of $H^*(G,k)/\mathfrak{p}$ is d. Set $K := k(t_1,\dots,t_{d-1})$, the field of rational functions in indeterminates $t_1,\dots,t_{d-1}$ and

$$b_i := a_i - a_0 t_i \quad \text{for } i = 1,\dots,d-1$$

viewed as elements in $H^*(G_K,K)$. Let $\mathfrak{p}'$ be the extension of $\mathfrak{p}$ to $H^*(G_K,K)$ and

$$\mathfrak{q} := \mathfrak{p}' + (\boldsymbol{b}) \qquad \text{and} \qquad \mathfrak{m} := \sqrt{\mathfrak{q}}\,.$$

It is proved as part of [17, Theorem 7.7] that the ideal $\mathfrak{m}$ is a closed point in $\operatorname{Proj} H^*(G_K,K)$ with the property that $\mathfrak{m}\cap H^*(G,k) = \mathfrak{p}$. What is more, it follows from the construction (see in particular [17, Lemma 7.6, and (7.2)]) that the induced extension of fields is an isomorphism

$$k(\mathfrak{p}) \xrightarrow{\cong} k(\mathfrak{m})\,.$$

The sequence of elements $\boldsymbol{b}$ in $H^*(G_K,K)$ yields a morphism $K \to \Omega^s(K/\!\!/\boldsymbol{b})$, where $s = \sum_i |b_i|$, and composing its restriction to G with the canonical morphism $k \to K{\downarrow_G}$ gives in $\mathsf{StMod}\, G$ a morphism

$$f\colon k \longrightarrow \Omega^s(K/\!\!/\boldsymbol{b}){\downarrow_G}\,.$$

Since the a_i are not in $\mathfrak{p}$, Lemma 2.1 yields a natural stable isomorphism

$$\Omega^s M \cong M \tag{3.2}$$

for any $\mathfrak{p}$-local G-module M. This remark will be used often in the sequel.

The result below extends [17, Theorem 8.8]; the latter is the case $M = k/\!\!/\mathfrak{p}$.

Theorem 3.4. *For any G-module M, the morphism $M\otimes_k f$ induces a natural stable isomorphism of G-modules*

$$\varGamma_{\mathfrak{p}} M \cong M\otimes_k \varGamma_{\mathfrak{m}}(K/\!\!/\boldsymbol{b}){\downarrow_G} \cong (M_K \otimes_K \varGamma_{\mathfrak{m}}(K/\!\!/\boldsymbol{b})){\downarrow_G}\,.$$

When M is $\mathfrak{p}$-torsion, these induce natural stable isomorphisms

$$\varGamma_{\mathfrak{p}} M \cong M_{\mathfrak{p}} \cong M\otimes_k (K/\!\!/\boldsymbol{b}){\downarrow_G} \cong (M_K \otimes_K K/\!\!/\boldsymbol{b}){\downarrow_G}\,.$$

Proof. We begin by verifying the second set of isomorphisms. As M is $\mathfrak{p}$-torsion so is $M_{\mathfrak{p}}$ and then it is clear that the natural map $\varGamma_{\mathfrak{p}} M = \varGamma_{\mathcal{V}(\mathfrak{p})} M_{\mathfrak{p}} \to M_{\mathfrak{p}}$ is an isomorphism. The third of the desired isomorphisms follows from (3.1). It thus remains to check that $M\otimes_k f$ induces an isomorphism

$$M_{\mathfrak{p}} \cong M\otimes_k (K/\!\!/\boldsymbol{b}){\downarrow_G}\,.$$

It is easy to verify that the modules M having this property form a tensor ideal localising subcategory of $\mathsf{StMod}\, G$. Keeping in mind (3.2), from [17, Theorem 8.8] one obtains that this subcategory contains $k/\!\!/\mathfrak{p}$. The desired assertion follows since the $\mathfrak{p}$-torsion modules form a tensor ideal localising subcategory of $\mathsf{StMod}\, G$ that is generated by $k/\!\!/\mathfrak{p}$; see [12, Proposition 2.7].

Now we turn to the first set of isomorphisms. There the second one is by (3.1), so we focus on the first. Let M be an arbitrary G-module, and let $\mathfrak{p}'$ be as in Construction 3.3. Since $\varGamma_{\mathcal{V}(\mathfrak{p})} M$ is $\mathfrak{p}$-torsion, the already established isomorphism yields the second one below.

$$\begin{aligned}
\varGamma_{\mathfrak{p}} M &\cong (\varGamma_{\mathcal{V}(\mathfrak{p})} M)_{\mathfrak{p}}\\
&\cong ((\varGamma_{\mathcal{V}(\mathfrak{p})} M)_K \otimes_K K/\!\!/\boldsymbol{b}){\downarrow_G}\\
&\cong (\varGamma_{\mathcal{V}(\mathfrak{p}')}(M_K) \otimes_K K/\!\!/\boldsymbol{b}){\downarrow_G}\\
&\cong (M_K \otimes_K \varGamma_{\mathcal{V}(\mathfrak{p}')}(K/\!\!/\boldsymbol{b})){\downarrow_G}\\
&\cong (M_K \otimes_K \varGamma_{\mathcal{V}(\mathfrak{p}'+(\boldsymbol{b}))}(K/\!\!/\boldsymbol{b})){\downarrow_G}\\
&\cong (M_K \otimes_K \varGamma_{\mathfrak{m}}(K/\!\!/\boldsymbol{b})){\downarrow_G}
\end{aligned}$$

The third one is by Lemma 2.2, applied to the functor $K\otimes_k(-)$ from $\mathsf{StMod}\,G$ to $\mathsf{StMod}\,G_K$. The next one is standard, the penultimate one holds as $K/\!\!/\boldsymbol{b}$ is $(\boldsymbol{b})$-torsion, and the last follows from $\sqrt{\mathfrak{p}'+(\boldsymbol{b})}=\mathfrak{m}$. This completes the proof. $\square$

In the next remark we recast part of Theorem 3.4.

Remark 3.5. Fix a point $\mathfrak{p}$ in $\operatorname{Proj} H^*(G,k)$, and let K, $\boldsymbol{b}$ and $\mathfrak{m}$ be as in Construction 3.3. Consider the following adjoint pair of functors.

$$\begin{aligned} \lambda\colon \mathsf{StMod}\,G &\longrightarrow \mathsf{StMod}\,G_K \quad\text{and}\quad & \rho\colon \mathsf{StMod}\,G_K &\longrightarrow \mathsf{StMod}\,G\\ \lambda(M)&=M_K\otimes_K K/\!\!/\boldsymbol{b} & \rho(N)&=\operatorname{Hom}_K(K/\!\!/\boldsymbol{b},N){\downarrow_G}\end{aligned}$$

It is easy to check that this induces an adjoint pair

$$\varGamma_{\mathfrak{p}}(\mathsf{StMod}\,G)\ \underset{\rho}{\overset{\lambda}{\rightleftarrows}}\ \varGamma_{\mathfrak{m}}(\mathsf{StMod}\,G_K)\,.$$

Theorem 3.4 implies that $(\lambda M){\downarrow_G}\cong M$ for any M in $\varGamma_{\mathfrak{p}}(\mathsf{StMod}\,G)$.

Proof of Theorem 3.1. Let $\mathfrak{m}$, $\mathfrak{q}$, and $\boldsymbol{b}$ be as in Construction 3.3. As noted there, $\mathfrak{m}$ is a closed point in $\operatorname{Proj} H^*(G_K,K)$ lying over $\mathfrak{p}$ and $k(\mathfrak{m})\cong k(\mathfrak{p})$. The modules in $\varGamma_{\mathfrak{p}}(\mathsf{StMod}\,G)$ are precisely those with support contained in $\{\mathfrak{p}\}$. It then follows from [17, Proposition 6.2] that $(-){\downarrow_G}$ restricts to a functor

$$\varGamma_{\mathfrak{m}}(\mathsf{StMod}\,G_K)\longrightarrow \varGamma_{\mathfrak{p}}(\mathsf{StMod}\,G)\,.$$

This functor is dense because for any G-module M that is $\mathfrak{p}$-local and $\mathfrak{p}$-torsion one has $M\cong(\lambda M){\downarrow_G}$ where λ is the functor from Remark 3.5.

Consider the restriction of $(-){\downarrow_G}$ to compact objects in $\varGamma_{\mathfrak{m}}(\mathsf{StMod}\,G_K)$. First we verify that its image is contained in the compact objects of $\varGamma_{\mathfrak{p}}(\mathsf{StMod}\,G)$. To this end, it suffices to check that there exists a generator of $\varGamma_{\mathfrak{m}}(\mathsf{StMod}\,G_K)^{\mathsf{c}}$, as a thick subcategory, whose restriction is in $\varGamma_{\mathfrak{p}}(\mathsf{StMod}\,G)^{\mathsf{c}}$.

Let S be the direct sum of a representative set of simple G-modules. Each simple G_K-module is (isomorphic to) a direct summand of S_K, so from (2.5) one gets the first equality below:

$$\varGamma_{\mathfrak{m}}(\mathsf{StMod}\,G_K)^{\mathsf{c}}=\mathsf{Thick}(S_K/\!\!/\mathfrak{m})=\mathsf{Thick}(S_K/\!\!/\mathfrak{q})\,.$$

The second one is by (2.4), since $\sqrt{\mathfrak{q}}=\mathfrak{m}$. From Theorem 3.4 one gets isomorphisms of G-modules

$$(S_K/\!\!/\mathfrak{q}){\downarrow_G}\cong((S/\!\!/\mathfrak{p})_K\otimes_K K/\!\!/\boldsymbol{b}){\downarrow_G}\cong(S/\!\!/\mathfrak{p})_{\mathfrak{p}}\,.$$

It remains to note that $(S/\!\!/\mathfrak{p})_{\mathfrak{p}}$ is in $\varGamma_{\mathfrak{p}}(\mathsf{StMod}\,G)^{\mathsf{c}}$, again by (2.5).

The last item to verify is that restriction is dense also on compacts. Since $K/\!\!/\boldsymbol{b}$ is compact, the functor ρ from Remark 3.5 preserves coproducts, and hence its left adjoint λ preserves compactness. Thus Theorem 3.4 gives the desired result. $\square$

Theorem 3.4 yields that $f\cong(f_K){\downarrow_G}$ for any morphism f in $\varGamma_{\mathfrak{p}}(\mathsf{StMod}\,G)$; in particular, the restriction functor is full and dense on the subcategory of $\varGamma_{\mathfrak{m}}(\mathsf{StMod}\,G_K)$ consisting of objects of the form λM, where M is a $\mathfrak{p}$-local $\mathfrak{p}$-torsion G-module. It need not be full on the entire category, or even on its subcategory of compact objects; see Example 3.7, modeled on the following one from commutative algebra.

Example 3.6. Let k be a field and $k[a]$ the polynomial ring in an indeterminate a. Let $\mathsf{D}(k[a])$ denote its derived category; it is $k[a]$-linear in an obvious way. For the prime $\mathfrak{p}:=(0)$ of $k[a]$ the $\mathfrak{p}$-local $\mathfrak{p}$-torsion subcategory $\varGamma_{\mathfrak{p}}(\mathsf{D}(k[a]))$ is naturally identified with the derived category of $k(a)$, the field of rational functions in a.

With $k(t)$ denoting the field of rational functions in an indeterminate t, the maximal ideal $\mathfrak{m}:=(a-t)$ of $k(t)[a]$ lies over the prime ideal $\mathfrak{p}$ of $k[a]$. The inclusion $k[a]\subset k(t)[a]$ induces an isomorphism $k(a)\cong k(t)[a]/\mathfrak{m}\cong k(t)$. The analogue of

Theorem 3.4 is that restriction of scalars along the inclusion $k[a] \subset k(t)[a]$ induces a dense functor

$$\mathit{\Gamma}_{\mathfrak{m}}(\mathsf{D}(k(t)[a])) \longrightarrow \mathit{\Gamma}_{\mathfrak{p}}(\mathsf{D}(k[a])) \simeq \mathsf{D}(k(a))\,.$$

This property can be checked directly: The $\mathfrak{m}$-torsion module $k(t)[a]/(a-t)$ restricts to $k(a)$, and each object in $\mathsf{D}(k(a))$ is a direct sum of shifts of $k(a)$. This functor is however not full: For $n \geq 1$, the $k(t)[a]$-module $L := k(t)[a]/(a-t)^n$ is $\mathfrak{m}$-torsion, and satisfies

$$\operatorname{rank}_{k(a)} \operatorname{End}_{\mathsf{D}}(L) = n \quad \text{and} \quad \operatorname{rank}_{k(a)} \operatorname{End}_{\mathsf{D}}(L{\downarrow_{k[a]}}) = n^2$$

where D stands for the appropriate derived category. In particular, if $n \geq 2$, the canonical map $\operatorname{End}_{\mathsf{D}}(L) \to \operatorname{End}_{\mathsf{D}}(L{\downarrow_{k[a]}})$ is not surjective.

Indeed, the module of endomorphisms of L in $\mathsf{D}(k(t)[a])$ is

$$\operatorname{End}_{\mathsf{D}}(L) = \operatorname{Hom}_{k(t)[a]}(L, L) \cong L\,.$$

In particular, it has rank n as an $k(a)$-vector space. On the other hand, restricted to $k[a]$, the $k(t)[a]$-module $k(t)/(a-t)$ is isomorphic to $k(a)$. It then follows from the exact sequences

$$0 \longrightarrow \frac{k(t)[a]}{(a-t)} \xrightarrow{1 \mapsto (a-t)^i} \frac{k(t)[a]}{(a-t)^{i+1}} \longrightarrow \frac{k(t)[a]}{(a-t)^i} \longrightarrow 0$$

of $k(t)[a]$-modules that L restricts to a direct sum of n copies of $k(a)$, so that

$$\operatorname{End}_{\mathsf{D}}(L{\downarrow_{k[a]}}) = \operatorname{Hom}_{k(a)}(k(a)^n, k(a)^n) \cong k(a)^{n^2}\,.$$

In particular, this has rank n^2 as a $k(a)$-vector space.

Example 3.7. Let $V = \mathbb{Z}/2 \times \mathbb{Z}/2$ and k a field of characteristic two. As k-algebras, one has $H^*(V, k) \cong k[a, b]$, where a and b are indeterminates of degree one. For the prime ideal $\mathfrak{p} = (0)$ of $k[a, b]$, Construction 3.3 leads to the field extension $K := k(t)$ of k, and the closed point $\mathfrak{m} = (b - at)$ of $\operatorname{Proj} H^*(V_K, K)$.

Set $F := \underline{\operatorname{End}}_V(k_{\mathfrak{p}})$; this is the component in degree 0 of the graded field $k[a, b]_{\mathfrak{p}}$ and can be identified with K; see Construction 3.3.

Fix an integer $n \geq 1$ and set $N := K /\!\!/ (b - at)^n$. This is a finite-dimensional $\mathfrak{m}$-torsion V_K-module and hence compact in $\mathit{\Gamma}_{\mathfrak{m}}(\mathsf{StMod}\, V_K)$. We claim that

$$\operatorname{rank}_F \underline{\operatorname{End}}_{V_K}(N) = 2n \quad \text{and} \quad \operatorname{rank}_F \underline{\operatorname{End}}_V(N{\downarrow_V}) = n^2\,,$$

and hence that the map $\underline{\operatorname{End}}_{V_K}(N) \to \underline{\operatorname{End}}_V(N{\downarrow_V})$ is not surjective when $n \geq 3$.

The claim can be checked as follows: Set $S := \underline{\operatorname{End}}^*_{V_K}(K)_{\mathfrak{m}} \cong K[a, b]_{\mathfrak{m}}$. Since $(b - at)^n$ is not a zerodivisor on S, applying $\underline{\operatorname{Hom}}_{V_K}(K, -)$ to the exact triangle

$$K \xrightarrow{(b-at)^n} \Omega^{-n} K \longrightarrow N \longrightarrow$$

one gets that $\underline{\operatorname{Hom}}^*_{V_K}(K, N)$ is isomorphic to $S/(b - at)^n$, as an S-module; in particular $(b - at)^n$ annihilates it. Given this, applying $\underline{\operatorname{Hom}}_{V_K}(-, N)$ to the exact triangle above yields that the rank of $\underline{\operatorname{End}}_{V_K}(N)$, as an F-vector space, is $2n$.

As to the claim about $N{\downarrow_V}$: the category $\mathit{\Gamma}_{\mathfrak{p}}(\mathsf{StMod}\, V)$ is semisimple for its generator $k_{\mathfrak{p}}$ has the property that $\underline{\operatorname{End}}^*_V(k_{\mathfrak{p}})$ is a graded field. It thus suffices to verify that $N{\downarrow_V} \cong k_{\mathfrak{p}}^n$; equivalently, that $\operatorname{rank}_F \underline{\operatorname{Hom}}_V(k_{\mathfrak{p}}, N{\downarrow_V}) = n$. This follows from the isomorphisms

$$\underline{\operatorname{Hom}}_V(k_{\mathfrak{p}}, N{\downarrow_V}) \cong \underline{\operatorname{Hom}}_V(k, N{\downarrow_V}) \cong \underline{\operatorname{Hom}}_{V_K}(K, N) \cong F^n\,.$$

The first one holds because $N{\downarrow_V}$ is $\mathfrak{p}$-local, the second one is by adjunction.

There is a close connection between this example and Example 3.6. Namely, the Bernstein-Gelfand-Gelfand correspondence sets up an equivalence between $\mathsf{StMod}\, V$ and the derived category of dg modules over $R := k[a, b]$, viewed as a dg algebra with zero differential, modulo the subcategory of (a, b)-torsion dg modules; see [19] and also [14, §5.2.2]. The BGG correspondence induces the equivalences in the following commutative diagram of categories.

$$\begin{tikzcd} \mathit{\Gamma}_{\mathfrak{m}}(\mathsf{StMod}\, V_K) \arrow[d, "(-)\downarrow_V"'] & \mathit{\Gamma}_{\mathfrak{m}}(\mathsf{D}(S)) \arrow[l, "\simeq"'] \arrow[d] \\ \mathit{\Gamma}_{\mathfrak{p}}(\mathsf{StMod}\, V) & \mathsf{D}(R_{\mathfrak{p}}) \arrow[l, "\simeq"'] \end{tikzcd}$$

where $\mathsf{D}(-)$ denotes the derived category of dg modules. The functor on the right is restriction of scalars along the homomorphism of rings $R_{\mathfrak{p}} \to S$, which is induced by the inclusion $R = k[a,b] \subset K[a,b]$. Under the BGG equivalence, the V_K-module N corresponds to $S/(b-at)^n$, viewed as dg S-module with zero differential. Since $R_{\mathfrak{p}}$ is a graded field, isomorphic to $K[a^{\pm 1}]$, each dg $R_{\mathfrak{p}}$-module is isomorphic to a direct sum of copies of $R_{\mathfrak{p}}$. Arguing as in Example 3.6 one can verify that the dg S-module $S/(b-at)^n$ restricts to a direct sum of n copies of $R_{\mathfrak{p}}$. This is another way to compute the endomorphism rings in question.

The remainder of this section is devoted to a further discussion of the compact objects in $\mathit{\Gamma}_{\mathfrak{p}}(\mathsf{StMod}\, G)$. This is not needed in the sequel.

Endofiniteness. Following Crawley-Boevey [24, 25], a module X over an associative ring A is *endofinite* if X has finite length as a module over $\operatorname{End}_A(X)$.

An object X of a compactly generated triangulated category T is *endofinite* if the $\operatorname{End}_{\mathsf{T}}(X)$-module $\operatorname{Hom}_{\mathsf{T}}(C,X)$ has finite length for all $C \in \mathsf{T}^{\mathsf{c}}$; see [40].

Let A be a self-injective algebra, finite dimensional over some field. Then an A-module is endofinite if and only if it is endofinite as an object of $\mathsf{StMod}\, A$. This follows from the fact X is an endofinite A-module if and only if the $\operatorname{End}_A(X)$-module $\operatorname{Hom}_A(C,X)$ has finite length for every finite dimensional A-module C.

Lemma 3.8. *Let $F\colon \mathsf{T} \to \mathsf{U}$ be a functor between compactly generated triangulated categories that preserves products and coproducts. Let X be an object in T. If X is endofinite, then so is FX and the converse holds when F is fully faithful.*

Proof. By Brown representability, F has a left adjoint, say F'. It preserves compactness, as F preserves coproducts. For $X \in \mathsf{T}$ and $C \in \mathsf{U}^{\mathsf{c}}$, there is an isomorphism

$$\operatorname{Hom}_{\mathsf{U}}(C, FX) \cong \operatorname{Hom}_{\mathsf{T}}(F'C, X)$$

of $\operatorname{End}_{\mathsf{T}}(X)$-modules. Thus if X is endofinite, then $\operatorname{Hom}_{\mathsf{U}}(C, FX)$ is a module of finite length over $\operatorname{End}_{\mathsf{T}}(X)$, and therefore also over $\operatorname{End}_{\mathsf{U}}(FX)$. For the converse, observe that each compact object in T is isomorphic to a direct summand of an object of the form $F'C$ for some $C \in \mathsf{U}^{\mathsf{c}}$. □

Proposition 3.9. *Let $\mathfrak{p}$ be a point in $\operatorname{Proj} H^*(G,k)$ and M a G-module that is compact in $\mathit{\Gamma}_{\mathfrak{p}}(\mathsf{StMod}\, G)$. Then M is endofinite both in $\mathsf{StMod}\, G$ and in $\mathit{\Gamma}_{\mathfrak{p}}(\mathsf{StMod}\, G)$.*

Proof. By Corollary 3.2, the module M is of the form $N{\downarrow_G}$ for a finite dimensional G_K-module N. Clearly, N is endofinite in $\mathsf{StMod}\, G_K$ and $(-){\downarrow_G}$ preserves products and coproducts, so it follows by Lemma 3.8 that M is endofinite in $\mathsf{StMod}\, G$. By the same token, as the inclusion $(\mathsf{StMod}\, G)_{\mathfrak{p}} \to \mathsf{StMod}\, G$ preserves products and coproducts, M is endofinite in $(\mathsf{StMod}\, G)_{\mathfrak{p}}$ as well. Finally, the functor $\mathit{\Gamma}_{\mathcal{V}(\mathfrak{p})}$ is a right adjoint to the inclusion $\mathit{\Gamma}_{\mathfrak{p}}(\mathsf{StMod}\, G) \to (\mathsf{StMod}\, G)_{\mathfrak{p}}$. It preserves products, being a right adjoint, and also coproducts. Thus M is endofinite in $\mathit{\Gamma}_{\mathfrak{p}}(\mathsf{StMod}\, G)$, again by Lemma 3.8. □

4. G-modules and Tate duality

Now we turn to various dualities for modules over finite group schemes. We begin by recalling the construction of the transpose and the dual of a module over a finite group scheme, and certain functors associated with them. Our basic reference for this material is Skowroński and Yamagata [44, Chapter III].

Throughout G will be a finite group scheme over k. We write $(-)^\vee = \operatorname{Hom}_k(-, k)$.

Transpose and dual. Let G^{op} be the opposite group scheme of G; it can be realised as the group scheme associated to the cocommutative Hopf algebra $(kG)^{\mathrm{op}}$. Since kG is a G-bimodule, the assignment $M \mapsto \operatorname{Hom}_G(M, kG)$ defines a functor

$$(-)^t \colon \operatorname{\mathsf{Mod}} G \longrightarrow \operatorname{\mathsf{Mod}} G^{\mathrm{op}}\,.$$

Let M be a finite dimensional G-module and $P_1 \xrightarrow{f} P_0 \to M$ a minimal projective presentation. The *transpose* of M is the G^{op}-module $\operatorname{Tr} M := \operatorname{Coker}(f^t)$. By construction, there is an exact sequence of G^{op}-modules:

$$0 \longrightarrow M^t \longrightarrow P_0^t \xrightarrow{f^t} P_1^t \longrightarrow \operatorname{Tr} M \longrightarrow 0\,.$$

The P_i^t are projective G^{op}-modules, so this yields an isomorphism of G^{op}-modules

$$M^t \cong \Omega^2 \operatorname{Tr} M\,.$$

Given a G^{op}-module N, the k-vector space $\operatorname{Hom}_k(N, k)$ has a natural structure of a G-module, and the assignment $N \mapsto \operatorname{Hom}_k(N, k)$ yields a functor

$$D := \operatorname{Hom}_k(-, k) \colon \operatorname{\mathsf{stmod}} G^{\mathrm{op}} \longrightarrow \operatorname{\mathsf{stmod}} G\,.$$

The Auslander-Reiten translate. In what follows we write τ for the Auslander-Reiten translate of G:

$$\tau := D \circ \operatorname{Tr} \colon \operatorname{\mathsf{stmod}} G \to \operatorname{\mathsf{stmod}} G$$

Given an extension of fields K/k, for any finite dimensional G-module M there is a stable isomorphism of G_K-modules

$$(\tau M)_K \cong \tau(M_K)\,.$$

Nakayama functor. The Nakayama functor

$$\nu \colon \operatorname{\mathsf{Mod}} G \xrightarrow{\sim} \operatorname{\mathsf{Mod}} G$$

is given by the assignment

$$M \mapsto D(kG) \otimes_{kG} M \cong \delta_G \otimes_k M\,.$$

where $\delta_G = \nu(k)$ is the *modular character* of G; see [34, I.8.8]. Since the group of characters of G is finite, by [45, §2.1 & §2.2], there exists a positive integer d such that $\delta_G^{\otimes d} \cong k$ and hence as functors on $\operatorname{\mathsf{Mod}} G$ there is an equality

$$\nu^d = \mathrm{id}\,. \tag{4.1}$$

When M is a finite dimensional G-module, there are natural stable isomorphisms

$$\nu M \cong D(M^t) \cong \Omega^{-2} \tau M\,.$$

When in addition M is projective, one has

$$\operatorname{Hom}_G(M, -)^\vee \cong (M^t \otimes_{kG} -)^\vee \cong \operatorname{Hom}_G(-, \nu M).$$

Let K/k be an extension of fields. For any G-module M there is a natural isomorphism of G_K-modules

$$(\nu M)_K \cong \nu(M_K)\,. \tag{4.2}$$

This is clear for $M = kG$ since

$$K \otimes_k \operatorname{Hom}_k(kG, k) \cong \operatorname{Hom}_k(kG, K) \cong \operatorname{Hom}_K(K \otimes_k kG, K)\,,$$

and the general case follows by taking a free presentation of M.

Remark 4.1. A finite group scheme is *unimodular* if the character δ_G is trivial; equivalently, when kG is symmetric. Examples include finite groups, unipotent groups schemes, and Frobenius kernels of reductive groups; see [34, I.8.9, II.3.4(a)]. Group schemes that are not unimodular also abound.

Frobenius kernels of Borel subgroups of reductive groups are not unimodular for $p \geq 3$; see [34, II.3.4(c)]. The finite group scheme associated to a Lie algebra is unimodular if and only if $\operatorname{tr}(\operatorname{ad} x) = 0$ for any x in the Lie algebra; see [34, I.9.7]. This condition fails for upper triangular matrices inside sl_2 for $p \geq 3$.

Tate duality. For finite groups, the duality theorem below is classical and due to Tate [23, Chapter XII, Theorem 6.4]. A proof of the extension to finite group schemes was sketched in [16, §2], and is reproduced here for readers convenience.

Theorem 4.2. *Let G be a finite group scheme over a field k. For any G-modules M, N with M finite dimensional, there are natural isomorphisms*

$$\underline{\operatorname{Hom}}_G(M, N)^\vee \cong \underline{\operatorname{Hom}}_G(N, \Omega^{-1}\tau M) \cong \underline{\operatorname{Hom}}_G(N, \Omega\nu M)\,.$$

Proof. A formula of Auslander and Reiten [1, Proposition I.3.4], see also [36, Corollary p. 269], yields the first isomorphism below

$$\underline{\operatorname{Hom}}_G(M, N)^\vee \cong \operatorname{Ext}^1_G(N, \tau M) \cong \underline{\operatorname{Hom}}_G(N, \Omega^{-1}\tau M)$$

The second isomorphism is standard. It remains to recall that $\tau M \cong \Omega^2\nu M$. □

Restricted to finite dimensional G-modules, Tate duality is the statement that the k-linear category $\mathsf{stmod}\, G$ has Serre duality, with Serre functor $\Omega\nu$. A refinement of this Serre duality will be proved in Section 7.

5. Local cohomology versus injective cohomology

Let k be a field and G a finite group scheme over k. In this section we establish the main result of this work; it identifies for a prime ideal $\mathfrak{p}$ in $H^*(G, k)$, up to some twist and some suspension, the local cohomology object $\varGamma_{\mathfrak{p}}(k)$ with the injective cohomology object $T_{\mathfrak{p}}(k)$.

Theorem 5.1. *Fix a point $\mathfrak{p}$ in $\operatorname{Proj} H^*(G, k)$ and let d be the Krull dimension of $H^*(G, k)/\mathfrak{p}$. There is a stable isomorphism of G-modules*

$$\varGamma_{\mathfrak{p}}(\delta_G) \cong \Omega^{-d} T_{\mathfrak{p}}(k)\,;$$

equivalently, for any G-module M there is a natural isomorphism

$$\underline{\operatorname{Hom}}_G(M, \Omega^d \varGamma_{\mathfrak{p}}(\delta_G)) \cong \operatorname{Hom}_{H^*(G,k)}(H^*(G, M), I(\mathfrak{p}))\,.$$

When G is the group scheme arising from a finite group the modular character δ_G is trivial, and the result above was proved by Benson and Greenlees [9, Theorem 2.4] using Gorenstein duality for cochains on the classifying space of G. Benson [7, Theorem 2] gave a different proof by embedding G into a general linear group and exploiting the fact that its cohomology ring is a polynomial ring, as was proved by Quillen. These results have been extended to compact Lie groups; see [10, Theorem 6.10], and work of Barthel, Heard, and Valenzuela [4, Proposition 4.33].

Theorem 5.1 is established using (by necessity) completely different arguments, thereby giving yet another proof in the case of finite groups that is, in a sense, more elementary than the other ones for it is based on classical Tate duality.

A caveat: In [7, 9] it is asserted that $\mathit{\Gamma}_{\mathfrak{p}}(k) \cong \Omega^d T_{\mathfrak{p}}(k)$. However, this is incorrect and the correct shift is the one in the preceding theorem. We illustrate this by computing these modules directly for the quaternions.

Example 5.2. Let $G := Q_8$, the quaternions, viewed as a group scheme over a field k of characteristic 2. In this case $\delta_G = k$, the trivial character. The cohomology algebra of G is given by

$$H^*(G,k) = k[z] \otimes_k B \quad \text{where } B = k[x,y]/(x^2+xy+y^2, x^2y+xy^2)\,,$$

with $|x| = 1 = |y|$ and $|z| = 4$; see, for instance, [5, p. 186]. Thus $\operatorname{Proj} H^*(G,k)$ consists of a single point, namely $\mathfrak{m} := (x,y)$. In particular, $\mathit{\Gamma}_{\mathfrak{m}} k = k$, in $\mathsf{StMod}\, G$.

Next we compute $I(\mathfrak{m})$ as a module over $H^*(G,k)_{\mathfrak{m}} \cong k[z^{\pm 1}] \otimes_k B$, using Lemma A.3. The extension $k[z^{\pm 1}] \subseteq k[z^{\pm 1}] \otimes_k B$ is evidently finite (and hence also residually finite). Since $\mathfrak{m} \cap k[z^{\pm 1}] = (0)$ and $k[z^{\pm 1}]$ is a graded field, from Lemma A.3 one gets an isomorphism of $H^*(G,k)_{\mathfrak{m}}$-modules.

$$\begin{aligned} I(\mathfrak{m}) &\cong \operatorname{Hom}_{k[z^{\pm 1}]}(k[z^{\pm 1}] \otimes_k B, k[z^{\pm 1}]) \\ &\cong k[z^{\pm 1}] \otimes_k \operatorname{Hom}_k(B,k) \\ &\cong k[z^{\pm 1}] \otimes_k \Sigma^3 B \\ &\cong \Sigma^3 H^*(G,k)_{\mathfrak{m}} \end{aligned}$$

This yields the first isomorphism below of G-modules

$$T_{\mathfrak{m}}(k) \cong \Omega^{-3} k \cong \Omega^1 k\,,$$

and the second one holds because $\Omega^4 k \cong k$ in $\mathsf{StMod}\, G$.

In the proof of Theorem 5.1 the following simple observation will be used repeatedly; it is a direct consequence of Yoneda's lemma.

Lemma 5.3. *Let X and Y be G-modules that are $\mathfrak{p}$-local and $\mathfrak{p}$-torsion. There is an isomorphism $X \cong Y$ in $\mathsf{StMod}\, G$ if and only if there is a natural isomorphism*

$$\underline{\operatorname{Hom}}_G(M,X) \cong \underline{\operatorname{Hom}}_G(M,Y)$$

for all $\mathfrak{p}$-local and $\mathfrak{p}$-torsion (equivalently, for all) G-modules M. □

Proof of Theorem 5.1. The G-module $T_{\mathfrak{p}}(k)$ is $\mathfrak{p}$-local and $\mathfrak{p}$-torsion; see, for example, [13, Lemma 11.10]. This fact will be used in the sequel, without comment.

The first isomorphism of the theorem is equivalent to the second: By (2.7) for any $\mathfrak{p}$-local $H^*(G,k)$-module I there is an isomorphism

$$\operatorname{Hom}_{H^*(G,k)}(H^*(G,M), I) \cong \operatorname{Hom}_{H^*(G,k)}(\underline{\operatorname{Hom}}^*_G(k,M), I)\,.$$

Consequently, one can rephrase the defining isomorphism (2.1) for the object $T_{\mathfrak{p}}(k)$ as a natural isomorphism

$$\underline{\operatorname{Hom}}_G(M, T_{\mathfrak{p}}(k)) \cong \operatorname{Hom}_{H^*(G,k)}(H^*(G,M), I(\mathfrak{p}))\,.$$

The desired equivalence is then a consequence of Lemma 5.3.

The main task is to prove that there is a stable isomorphism:

$$\mathit{\Gamma}_{\mathfrak{p}}(\nu k) \cong \Omega^{-d} T_{\mathfrak{p}}(k)\,.$$

Recall that $\nu k = \delta_G$.

We first verify the isomorphism above for closed points of $\operatorname{Proj} H^*(G,k)$ and then use a reduction to closed points.

Claim. The desired isomorphism holds when $\mathfrak{m}$ is a closed point of $\operatorname{Proj} H^*(G,k)$.

Set $A := H^*(G,k)$ and $R := A_{\mathfrak{m}}$. The injective hull, $I(\mathfrak{m})$, of the A-module $A/\mathfrak{m}$ is the same as that of the R-module $k(\mathfrak{m})$, viewed as an A-module via restriction of scalars along the localisation map $A \to R$. Thus $I(\mathfrak{m})$ is the module I described in Lemma A.2; this is where the fact that $\mathfrak{m}$ is closed is used.

Let M be a G-module that is $\mathfrak{m}$-local and $\mathfrak{m}$-torsion. Given Lemma 5.3, the claim is a consequence of the following computation:

$$\begin{aligned}\underline{\operatorname{Hom}}_G(M, \Omega\mathit{\Gamma}_{\mathfrak{m}}(\nu k)) &\cong \underline{\operatorname{Hom}}_G(M, \Omega\nu k)\\ &\cong \underline{\operatorname{Hom}}_G(k, M)^{\vee}\\ &\cong \operatorname{Hom}_R(\underline{\operatorname{Hom}}^*_G(k, M), I(\mathfrak{m}))\\ &\cong \operatorname{Hom}_A(\underline{\operatorname{Hom}}^*_G(k, M), I(\mathfrak{m}))\\ &\cong \underline{\operatorname{Hom}}_G(M, T_{\mathfrak{m}}(k))\end{aligned}$$

The first isomorphism holds because M is $\mathfrak{m}$-torsion; the second is Tate duality, Theorem 4.2, and the next one is by Lemma A.2, which applies because $\underline{\operatorname{Hom}}^*_G(k, M)$ is $\mathfrak{m}$-local and $\mathfrak{m}$-torsion as an A-module. The penultimate one holds because the A-module $\underline{\operatorname{Hom}}^*_G(k, M)$ is $\mathfrak{m}$-local, and the last one is by definition (2.1).

Let $\mathfrak{p}$ be a point in $\operatorname{Proj} H^*(G, k)$ that is not closed, and let K, $\boldsymbol{b}$, and $\mathfrak{m}$ be as in Construction 3.3. Recall that $\mathfrak{m}$ is a closed point in $H^*(G_K, K)$ lying over $\mathfrak{p}$.

Claim. There is a stable isomorphism of G-modules

$$(T_{\mathfrak{m}}(K)/\!\!/\boldsymbol{b}){\downarrow_G} \cong \Omega^{-d+1} T_{\mathfrak{p}}(k) \tag{5.1}$$

where d is the Krull dimension of $H^*(G, k)/\mathfrak{p}$.

Let M be a G-module that is $\mathfrak{p}$-local and $\mathfrak{p}$-torsion. Then Theorem 3.4 applies and yields isomorphisms of G-modules.

$$(M_K/\!\!/\boldsymbol{b}){\downarrow_G} \cong (M_K \otimes_K K/\!\!/\boldsymbol{b}){\downarrow_G} \cong M\,.$$

This gives the sixth isomorphism below.

$$\begin{aligned}\underline{\operatorname{Hom}}_G(M, \Omega^{d-1}(T_{\mathfrak{m}}(K)/\!\!/\boldsymbol{b}){\downarrow_G}) &\cong \underline{\operatorname{Hom}}_{G_K}(M_K, \Omega^{d-1}(T_{\mathfrak{m}}(K)/\!\!/\boldsymbol{b}))\\ &\cong \underline{\operatorname{Hom}}_{G_K}(M_K/\!\!/\boldsymbol{b}, T_{\mathfrak{m}}(K))\\ &\cong \operatorname{Hom}_{H^*(G_K,K)}(\underline{\operatorname{Hom}}^*_{G_K}(K, M_K/\!\!/\boldsymbol{b}), I(\mathfrak{m}))\\ &\cong \operatorname{Hom}_{H^*(G,k)}(\underline{\operatorname{Hom}}^*_{G_K}(K, M_K/\!\!/\boldsymbol{b}), I(\mathfrak{p}))\\ &\cong \operatorname{Hom}_{H^*(G,k)}(\underline{\operatorname{Hom}}^*_{G}(k, (M_K/\!\!/\boldsymbol{b}){\downarrow_G}), I(\mathfrak{p}))\\ &\cong \operatorname{Hom}_{H^*(G,k)}(\underline{\operatorname{Hom}}^*_{G}(k, M), I(\mathfrak{p}))\\ &\cong \underline{\operatorname{Hom}}_G(M, T_{\mathfrak{p}}(k))\end{aligned}$$

The first and the fifth isomorphisms are by adjunction. The second isomorphism is a direct computation using (2.3) and (3.2). The next one is by definition and the fourth isomorphism is by Lemma A.3 applied to the canonical homomorphism $H^*(G,k) \to H^*(G_K, K)$; note that the $H^*(G_K, K)$-module $\underline{\operatorname{Hom}}^*_{G_K}(K, M_K/\!\!/\boldsymbol{b})$ is $\mathfrak{m}$-torsion. The desired isomorphism (5.1) holds by Lemma 5.3, because both modules in question are $\mathfrak{p}$-torsion; for the one on the left, see [17, Proposition 6.2].

We are now ready to wrap up the proof of the theorem. Since $\mathfrak{m}$ is a closed point in $\operatorname{Proj} H^*(G_K, K)$, the first claim yields that the G_K-modules $\mathit{\Gamma}_{\mathfrak{m}}(\nu K)$ and $\Omega^{-1} T_{\mathfrak{m}}(K)$ are isomorphic. This then gives an isomorphism of G_K-modules.

$$\nu K \otimes_K \mathit{\Gamma}_{\mathfrak{m}}(K/\!\!/\boldsymbol{b}) \cong \mathit{\Gamma}_{\mathfrak{m}}(\nu K)/\!\!/\boldsymbol{b} \cong \Omega^{-1} T_{\mathfrak{m}}(K)/\!\!/\boldsymbol{b}$$

Restricting to G and applying (5.1) gives the last of the following isomorphisms of G-modules.

$$\begin{aligned}\varGamma_{\mathfrak{p}}(\nu k) &\cong ((\nu k)_K \otimes_K \varGamma_{\mathfrak{m}}(K/\!\!/\boldsymbol{b}){\downarrow_G}\\ &\cong (\nu K \otimes_K \varGamma_{\mathfrak{m}}(K/\!\!/\boldsymbol{b})){\downarrow_G}\\ &\cong \Omega^{-d} T_{\mathfrak{p}}(k)\end{aligned}$$

The first one is by Theorem 3.4 and the second by (4.2). □

The following consequence of Theorem 5.1 was anticipated in [6, pp. 203]. It concerns the *Tate cohomology* of a G-module M, namely the $H^*(G,k)$-module

$$\widehat{H}^*(G,M) := \widehat{\operatorname{Ext}}^*_G(k,M) \cong \underline{\operatorname{Hom}}^*_G(k,M)\,.$$

For a graded module $N = \bigoplus_{p\in\mathbb{Z}} N^p$ and $i \in \mathbb{Z}$ the twist $N(i)$ is the graded module with $N(i)^p = N^{p+i}$.

Corollary 5.4. *Fix $\mathfrak{p}$ in $\operatorname{Proj} H^*(G,k)$. With d the Krull dimension of $H^*(G,k)/\mathfrak{p}$ there are isomorphisms of $H^*(G,k)$-modules*

$$\widehat{H}^*(G,\varGamma_{\mathfrak{p}}(k)) \cong \operatorname{Hom}^*_{H^*(G,k)}(H^*(G,\delta_G), I(\mathfrak{p}))(d)$$

and

$$\widehat{H}^*(G,\operatorname{End}_k(\varGamma_{\mathfrak{p}}(k))) \cong (H^*(G,k)_{\mathfrak{p}})^{\wedge}$$

where $(-)^{\wedge}$ denotes completion with respect to the $\mathfrak{p}$-adic topology.

Proof. Set $R = H^*(G,k)$. The first of the stated isomorphisms is a composition of the following isomorphisms of R-modules.

$$\begin{aligned}\underline{\operatorname{Hom}}^*_G(k,\varGamma_{\mathfrak{p}}(k)) &\cong \underline{\operatorname{Hom}}^*_G(\delta_G,\varGamma_{\mathfrak{p}}(\delta_G))\\ &\cong \underline{\operatorname{Hom}}^*_G(\delta_G,\Omega^{-d}T_{\mathfrak{p}}(k))\\ &\cong \underline{\operatorname{Hom}}^*_G(\delta_G,T_{\mathfrak{p}}(k))(d)\\ &\cong \operatorname{Hom}^*_R(H^*(G,\delta_G),I(\mathfrak{p}))(d)\end{aligned}$$

The second isomorphism is by Theorem 5.1, the one after is standard, while the last one is by the definition of $T_{\mathfrak{p}}(k)$.

In the same vein, one has the following chain of isomorphisms.

$$\begin{aligned}\underline{\operatorname{Hom}}^*_G(k,\operatorname{End}_k(\varGamma_{\mathfrak{p}}(k))) &\cong \underline{\operatorname{Hom}}^*_G(\varGamma_{\mathfrak{p}}(k),\varGamma_{\mathfrak{p}}(k))\\ &\cong \underline{\operatorname{Hom}}^*_G(T_{\mathfrak{p}}(k),T_{\mathfrak{p}}(k))\\ &\cong \operatorname{Hom}^*_R(\underline{\operatorname{Hom}}^*_G(k,T_{\mathfrak{p}}(k)),I(\mathfrak{p}))\\ &\cong \operatorname{Hom}^*_R(I(\mathfrak{p}),I(\mathfrak{p}))\\ &\cong (R_{\mathfrak{p}})^{\wedge}\end{aligned}$$

The second isomorphism is by Theorem 5.1 and the rest are standard. □

Remark 5.5. Another consequence of Theorem 5.1 is that $\varGamma_{\mathfrak{p}}(\delta_G)$, and hence also $\varGamma_{\mathfrak{p}}k$, is an indecomposable pure injective object in $\mathsf{StMod}\,G$; see [18, Theorem 5.1].

6. The Gorenstein property

In this section we introduce a notion of a Gorenstein triangulated category and reinterpret Theorem 5.1 to mean that $\mathsf{StMod}\,G$ has this property. The definition is justified by its consequences for modular representations; it yields duality results which will be discussed in the subsequent Section 7.

Let R be a graded commutative noetherian ring and T a compactly generated R-linear triangulated category. The *support* of T is by definition the set

$$\operatorname{supp}_R(\mathsf{T}) = \{\mathfrak{p} \in \operatorname{Spec} R \mid \varGamma_{\mathfrak{p}} \neq 0\}\,.$$

Definition 6.1. We say that T is *R-Gorenstein* (or simply *Gorenstein* when the action of R is clear) if there is an R-linear triangle equivalence

$$F\colon \mathsf{T}^{\mathsf{c}} \xrightarrow{\sim} \mathsf{T}^{\mathsf{c}}$$

and for every $\mathfrak{p}$ in $\operatorname{supp}_R(\mathsf{T})$ there is an integer $d(\mathfrak{p})$ and a natural isomorphism

$$\varGamma_{\mathfrak{p}} \circ F \cong \Sigma^{d(\mathfrak{p})} \circ T_{\mathfrak{p}}$$

of functors $\mathsf{T}^{\mathsf{c}} \to \mathsf{T}$.

In this context we call F a *global Serre functor*, because in Proposition 7.3 we show that localising at $\mathfrak{p}$ induces a functor $\Sigma^{-d(\mathfrak{p})}F_{\mathfrak{p}}$ which is an analogue of a Serre functor in the sense of Bondal and Kapranov [20].

We have not as yet explored fully the dependence of the Gorenstein property of T on the ring R; a beginning has been made in the work of Yuliawan [46].

Let $\mathsf{T} = (\mathsf{T}, \otimes, \mathbb{1})$ be a tensor triangulated category such that R acts on T via a homomorphism of graded rings $R \to \operatorname{End}^*_{\mathsf{T}}(\mathbb{1})$. We assume that $\mathbb{1}$ is compact and that each compact object is rigid; see Section 2 for details. The Gorenstein property is implied by the existence of a *dualising* object in T, as explained below.

Lemma 6.2. *Let T be a tensor triangulated category with an R-action. Suppose that there exists a compact object W with the following properties:*

(1) *There is a compact object W^{-1} such that $W \otimes W^{-1} \cong \mathbb{1}$;*
(2) *For each $\mathfrak{p}$ in $\operatorname{supp}_R(\mathsf{T})$ there exists an integer $d(\mathfrak{p})$ and an isomorphism*

$$\varGamma_{\mathfrak{p}} W \cong \Sigma^{d(\mathfrak{p})} T_{\mathfrak{p}}(\mathbb{1})\,.$$

Then T is R-Gorenstein, with global Serre functor $F := W \otimes -$.

Proof. Since W is compact, so is $W \otimes C$ for any compact object C of T. Thus F induces a functor on compact objects. It is an equivalence of categories with quasi-inverse $W^{-1} \otimes -$, by condition (1). Moreover for any compact object C and $\mathfrak{p} \in \operatorname{supp}_R(\mathsf{T})$ one has isomorphisms

$$\Sigma^{d(\mathfrak{p})} T_{\mathfrak{p}}(C) \cong \Sigma^{d(\mathfrak{p})} T_{\mathfrak{p}}(\mathbb{1}) \otimes C \cong \varGamma_{\mathfrak{p}} W \otimes C \cong \varGamma_{\mathfrak{p}}(W \otimes C)$$

where the first and the last one are consequences of (2.2), and the middle one is by condition (2). ☐

The example below justifies the language of Gorenstein triangulated categories.

Example 6.3. Let A be a commutative noetherian ring and D the derived category of A-modules. This is an A-linear compactly generated tensor triangulated category, with compact objects the perfect complexes of A-modules, that is to say, complexes quasi-isomorphic to bounded complexes of finitely generated projective A-modules.

Recall that the ring A is *Gorenstein* if for each $\mathfrak{p} \in \operatorname{Spec} A$ the injective dimension of $A_{\mathfrak{p}}$, as a module over itself, is finite; see [21, 3.1]. By Grothendieck's local duality theorem [21, §3.5], this is equivalent to an isomorphism of $A_{\mathfrak{p}}$-modules

$$\varGamma_{\mathfrak{p}} A \cong \Sigma^{-\dim A_{\mathfrak{p}}} I(\mathfrak{p})\,.$$

Thus D is Gorenstein with dualising object A and $d(\mathfrak{p}) = -\dim A_{\mathfrak{p}}$; see Lemma 6.2. Conversely, it is not difficult to check that D is Gorenstein only if A is Gorenstein.

For a finite group scheme G over a field k, the Gorenstein property for $\mathsf{StMod}\,G$ is basically a reformulation of Theorem 5.1.

Corollary 6.4. *As an $H^*(G,k)$-linear triangulated category, $\mathsf{StMod}\,G$ is Gorenstein, with $F = \delta_G \otimes_k -$ the Nakayama functor and $d(\mathfrak{p}) = \dim H^*(G,k)/\mathfrak{p}$ for each $\mathfrak{p}$ in $\operatorname{Proj} H^*(G,k)$.* ☐

Next we discuss the Gorenstein property for $\mathsf{K}(\operatorname{Inj} G)$ for a finite group scheme G. To this end observe that the assignment $X \mapsto \nu X$ induces triangle equivalences

$$\mathsf{K}(\operatorname{Inj} G) \xrightarrow{\sim} \mathsf{K}(\operatorname{Inj} G) \qquad \text{and} \qquad \mathsf{D}^b(\operatorname{mod} G) \xrightarrow{\sim} \mathsf{D}^b(\operatorname{mod} G).$$

We are ready to establish the Gorenstein property for $\mathsf{K}(\operatorname{Inj} G)$.

Theorem 6.5. *Let G be a finite group scheme over a field k. Then $\mathsf{K}(\operatorname{Inj} G)$ is Gorenstein as an $H^*(G,k)$-linear triangulated, with F induced by the Nakayama functor and $d(\mathfrak{p}) = \dim H^*(G,k)/\mathfrak{p}$ for each $\mathfrak{p} \in \operatorname{Spec} H^*(G,k)$.*

Proof. Set $R := H^*(G,k)$ and $\mathfrak{m} := R^{\geqslant 1}$, the maximal ideal of R. It is easy to verify that $I(\mathfrak{m}) := \operatorname{Hom}_k(R,k)$ is the injective hull of $R/\mathfrak{m}$ as an R-module and hence that $\operatorname{Hom}_R(-, I(\mathfrak{m})) = \operatorname{Hom}_k(-,k)$ on the category of graded R-modules; see [21, Proposition 3.6.16]. This observation is used in the first isomorphism below.

Let $\mathsf{p}k$ be a projective resolution of the trivial representation. For any X in $\mathsf{K}(\operatorname{Inj} G)$ the complex $\mathsf{p}X := \mathsf{p}k \otimes_k X$ is a projective resolution of X.

Suppose X is compact in $\mathsf{K}(\operatorname{Inj} G)$; one may assume $X^n = 0$ for $n \ll 0$ and that $H^n(X)$ is finitely generated for all n and equal to 0 for $n \gg 0$; see [37, Proposition 2.3(2)]. Then, for each Y in $\mathsf{K}(\operatorname{Inj} G)$ from [39, Theorem 3.4] one gets the first of the following natural isomorphisms:

$$\begin{aligned}\operatorname{Hom}_R(\operatorname{Hom}^*_{\mathsf{K}}(X,Y), I(\mathfrak{m})) &\cong \operatorname{Hom}_{\mathsf{K}}(Y, D(kG) \otimes_{kG} \mathsf{p}X)\\ &\cong \operatorname{Hom}_{\mathsf{K}}(Y, \delta_G \otimes_k \mathsf{p}X)\\ &\cong \operatorname{Hom}_{\mathsf{K}}(Y, \mathsf{p}(\delta_G \otimes_k X))\,.\end{aligned}$$

The second one is by the definition of the modular character, and the last one is immediate from the definition of p and the commutativity of tensor products. Since $\mathsf{p}(\delta_G \otimes_k X) = \varGamma_{\mathfrak{m}}(\delta_G \otimes_k X)$, by Lemma 2.6, it follows that $T_{\mathfrak{m}} \cong \varGamma_{\mathfrak{m}} \circ F$.

For a prime ideal $\mathfrak{p} \neq \mathfrak{m}$, the assertion follows from Theorem 5.1, for localisation at $\mathfrak{p}$ yields a triangle equivalence $\varGamma_{\mathfrak{p}}(\mathsf{K}(\operatorname{Inj} G)) \xrightarrow{\sim} \varGamma_{\mathfrak{p}}(\operatorname{StMod} G)$ by Lemma 2.6. □

Remark 6.6. There are extensions of the Gorenstein property to differential graded algebras—see [28] and [26], for instance—and a natural question is how these relate to the Gorenstein property, in the sense of Definition 6.1, of the associated derived categories. We defer exploring these connections to another occasion.

Balmer, Dell'Ambrogio, and Sanders [3] have introduced a categorical framework extending the duality theory for schemes due to Grothendieck and Neeman. The relationship to our work might be explained thus: A commutative noetherian ring R is Gorenstein precisely when it has an invertible dualising complex. The framework in [3] captures the relative version (dealing with a morphism of rings, or schemes) of the Gorenstein property and its characterisation in terms of the relative dualising complex. We are interested in the characterisation of the Gorenstein property in terms of local cohomology, and in the fact that when R is Gorenstein, so is $R_{\mathfrak{p}}$ for each prime $\mathfrak{p}$ in $\operatorname{Spec} R$. Theorem 5.1 may be seen as an analogue of these results for modular representations.

7. Local Serre duality

In this section we introduce a notion of local Serre duality for an essentially small R-linear triangulated category and link it to the Gorenstein property from Section 6. We use the concept of a Serre functor for a triangulated category which is due to Bondal and Kapranov [20]; this provides a conceptual way to formulate classical Serre duality and Grothendieck's local duality in a triangulated setting.

In the second part of this section we discuss the existence of Auslander-Reiten triangles. These were introduced by Happel for derived categories of finite dimensional algebras [30], and in [31] he established the connection with the Gorenstein property, while Reiten and Van den Bergh [43] discovered the connection between Auslander-Reiten triangles and the existence of a Serre functor.

Small triangulated categories with central action. Let C be an essentially small R-linear triangulated category. Fix $\mathfrak{p} \in \operatorname{Spec} R$ and let $\mathsf{C}_\mathfrak{p}$ denote the triangulated category obtained from C by keeping the objects of C and setting

$$\operatorname{Hom}^*_{\mathsf{C}_\mathfrak{p}}(X,Y) := \operatorname{Hom}^*_C(X,Y)_\mathfrak{p}\,.$$

Then $\mathsf{C}_\mathfrak{p}$ is an $R_\mathfrak{p}$-linear triangulated category and localising the morphisms induces an exact functor $\mathsf{C} \to \mathsf{C}_\mathfrak{p}$; see [2, Theorem 3.6] or [15, Lemma 3.5].

Let $\gamma_\mathfrak{p}\mathsf{C}$ be the full subcategory of $\mathfrak{p}$-torsion objects in $\mathsf{C}_\mathfrak{p}$, namely

$$\gamma_\mathfrak{p}\mathsf{C} := \{X \in \mathsf{C}_\mathfrak{p} \mid \operatorname{End}^*_{\mathsf{C}_\mathfrak{p}}(X) \text{ is } \mathfrak{p}\text{-torsion}\}.$$

This is a thick subcategory of $\mathsf{C}_\mathfrak{p}$; see [15, p. 458f.]. In [15] this category is denoted $\varGamma_\mathfrak{p}\mathsf{C}$. The notation has been changed to avoid confusion.

Remark 7.1. Let $F\colon \mathsf{C} \to \mathsf{C}$ be an R-linear equivalence. It is straightforward to check that this induces triangle equivalences $F_\mathfrak{p}\colon \mathsf{C}_\mathfrak{p} \xrightarrow{\sim} \mathsf{C}_\mathfrak{p}$ and $\gamma_\mathfrak{p}\mathsf{C} \xrightarrow{\sim} \gamma_\mathfrak{p}\mathsf{C}$ making the following diagram commutative.

$$\begin{tikzcd}
\mathsf{C} \arrow[r, two heads] \arrow[d, "F"] & \mathsf{C}_\mathfrak{p} \arrow[d, "F_\mathfrak{p}"] & \gamma_\mathfrak{p}\mathsf{C} \arrow[l, tail] \arrow[d, "F_\mathfrak{p}"] \\
\mathsf{C} \arrow[r, two heads] & \mathsf{C}_\mathfrak{p} & \gamma_\mathfrak{p}\mathsf{C} \arrow[l, tail]
\end{tikzcd}$$

Remark 7.2. Let T be a compactly generated R-linear triangulated category. Set $\mathsf{C} := \mathsf{T}^\mathsf{c}$ and fix $\mathfrak{p} \in \operatorname{Spec} R$. The triangulated categories $\mathsf{T}_\mathfrak{p}$ and $\varGamma_\mathfrak{p}\mathsf{T}$ are compactly generated. The left adjoint of the inclusion $\mathsf{T}_\mathfrak{p} \to \mathsf{T}$ induces (up to direct summands) a triangle equivalence $\mathsf{C}_\mathfrak{p} \xrightarrow{\sim} (\mathsf{T}_\mathfrak{p})^\mathsf{c}$ and restricts to a triangle equivalence

$$\gamma_\mathfrak{p}\mathsf{C} \xrightarrow{\sim} (\varGamma_\mathfrak{p}\mathsf{T})^\mathsf{c}\,.$$

This follows from the fact that the localisation functor $\mathsf{T} \to \mathsf{T}_\mathfrak{p}$ taking X to $X_\mathfrak{p}$ preserves compactness and that for compact objects X, Y in T

$$\operatorname{Hom}^*_\mathsf{T}(X,Y)_\mathfrak{p} \xrightarrow{\sim} \operatorname{Hom}^*_{\mathsf{T}_\mathfrak{p}}(X_\mathfrak{p}, Y_\mathfrak{p})\,.$$

For details we refer to [11].

Local Serre duality. Let R be a graded commutative ring that is *local*; thus there is a unique homogeneous maximal ideal, say $\mathfrak{m}$. Extrapolating from Bondal and Kapranov [20, §3], we call an R-linear triangle equivalence $F\colon \mathsf{C} \xrightarrow{\sim} \mathsf{C}$ a *Serre functor* if for all objects X, Y in C there is a natural isomorphism

$$\operatorname{Hom}_R(\operatorname{Hom}^*_\mathsf{C}(X,Y), I(\mathfrak{m})) \xrightarrow{\sim} \operatorname{Hom}_\mathsf{C}(Y, FX)\,. \tag{7.1}$$

The situation when R is a field was the one considered in [20]. For a general ring R, the appearance of $\operatorname{Hom}^*_R(-, I(\mathfrak{m}))$, which is the Matlis duality functor, is natural for it is an extension of vector-space duality; see also Lemma A.2. The definition proposed above is not the only possible extension to the general context, but it is well-suited for our purposes.

For an arbitrary graded commutative ring R, we say that an R-linear triangulated category C satisfies *local Serre duality* if there exists an R-linear triangle equivalence $F\colon \mathsf{C} \xrightarrow{\sim} \mathsf{C}$ such that for every $\mathfrak{p} \in \operatorname{Spec} R$ and some integer $d(\mathfrak{p})$ the induced functor $\Sigma^{-d(\mathfrak{p})}F_\mathfrak{p}\colon \gamma_\mathfrak{p}\mathsf{C} \xrightarrow{\sim} \gamma_\mathfrak{p}\mathsf{C}$ is a Serre functor for the $R_\mathfrak{p}$-linear category $\gamma_\mathfrak{p}\mathsf{C}$. Thus for all objects X, Y in $\gamma_\mathfrak{p}\mathsf{C}$ there is a natural isomorphism

$$\operatorname{Hom}_R(\operatorname{Hom}^*_{\mathsf{C}_\mathfrak{p}}(X,Y), I(\mathfrak{p})) \xrightarrow{\sim} \operatorname{Hom}_{\mathsf{C}_\mathfrak{p}}(Y, \Sigma^{-d(\mathfrak{p})}F_\mathfrak{p}X)\,.$$

For a compactly generated triangulated category, the Gorenstein property 6.1 is linked to local Serre duality for the subcategory of compact objects.

Proposition 7.3. *Let R be a graded commutative noetherian ring and T a compactly generated R-linear triangulated category. Suppose that T is Gorenstein, with global Serre functor F and shifts $\{d(\mathfrak{p})\}$. Then for each $\mathfrak{p} \in \operatorname{supp}_R(\mathsf{T})$, object $X \in (\mathit{\Gamma}_{\mathfrak{p}}\mathsf{T})^{\mathsf{c}}$ and $Y \in \mathsf{T}_{\mathfrak{p}}$ there is a natural isomorphism*

$$\operatorname{Hom}_R(\operatorname{Hom}^*_{\mathsf{T}}(X,Y), I(\mathfrak{p})) \cong \operatorname{Hom}_{\mathsf{T}}(Y, \Sigma^{-d(\mathfrak{p})} F_{\mathfrak{p}}(X))\,.$$

Proof. Given Remark 7.2 we can assume $X = C_{\mathfrak{p}}$ for a $\mathfrak{p}$-torsion compact object C in T. The desired isomorphism is a concatenation of the following natural ones:

$$\begin{aligned}
\operatorname{Hom}_R(\operatorname{Hom}^*_{\mathsf{T}}(C_{\mathfrak{p}},Y), I(\mathfrak{p})) &\cong \operatorname{Hom}_R(\operatorname{Hom}^*_{\mathsf{T}}(C,Y), I(\mathfrak{p})) \\
&\cong \operatorname{Hom}_{\mathsf{T}}(Y, T_{\mathfrak{p}}(C)) \\
&\cong \operatorname{Hom}_{\mathsf{T}}(Y, \Sigma^{-d(\mathfrak{p})} \mathit{\Gamma}_{\mathfrak{p}} F(C)) \\
&\cong \operatorname{Hom}_{\mathsf{T}}(Y, \Sigma^{-d(\mathfrak{p})} \mathit{\Gamma}_{\mathcal{V}(\mathfrak{p})} F(C)_{\mathfrak{p}}) \\
&\cong \operatorname{Hom}_{\mathsf{T}}(Y, \Sigma^{-d(\mathfrak{p})} \mathit{\Gamma}_{\mathcal{V}(\mathfrak{p})} F_{\mathfrak{p}}(C_{\mathfrak{p}})) \\
&\cong \operatorname{Hom}_{\mathsf{T}}(Y, \Sigma^{-d(\mathfrak{p})} F_{\mathfrak{p}}(C_{\mathfrak{p}}))
\end{aligned}$$

In this chain, the first map is induced by the localisation $C \mapsto C_{\mathfrak{p}}$ and is an isomorphism because Y is $\mathfrak{p}$-local. The second one is by the definition of $T_{\mathfrak{p}}(C)$; the third is by the Gorenstein property of T; the fourth is by the definition of $\mathit{\Gamma}_{\mathfrak{p}}$; the last two are explained by Remark 7.2, where for the last one uses also the fact that $C_{\mathfrak{p}}$, and hence also $F_{\mathfrak{p}}(C_{\mathfrak{p}})$, is $\mathfrak{p}$-torsion. □

Corollary 7.4. *Let R be a graded commutative noetherian ring and T a compactly generated R-linear triangulated category. If T is Gorenstein, then T^{c} satisfies local Serre duality.*

Proof. Given Remark 7.2, the assertion follows from Proposition 7.3. □

Example 7.5. In the notation of Example 6.3, when A is a (commutative noetherian) Gorenstein ring, local Serre duality reads: For each $\mathfrak{p} \in \operatorname{Spec} A$ and integer n there are natural isomorphisms

$$\operatorname{Hom}_{A_{\mathfrak{p}}}(\operatorname{Ext}^n_{A_{\mathfrak{p}}}(X,Y), I(\mathfrak{p})) \cong \operatorname{Ext}^{n+\dim A_{\mathfrak{p}}}_{A_{\mathfrak{p}}}(Y,X)$$

where X is a perfect complexes of $A_{\mathfrak{p}}$-modules with finite length cohomology, and Y is a complex of $A_{\mathfrak{p}}$-modules.

Auslander-Reiten triangles. Let C be an essentially small triangulated category. Following Happel [30], an exact triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ in C is an *Auslander-Reiten triangle* if

(1) any morphism $X \to X'$ that is not a split monomorphism factors through α;
(2) any morphism $Z' \to Z$ that is not a split epimorphism factors through β;
(3) $\gamma \neq 0$.

In this case, the endomorphism rings of X and Z are local, and in particular the objects are indecomposable. Moreover, each of X and Z determines the AR-triangle up to isomorphism. Assuming conditions (2) and (3), the condition (1) is equivalent to the following:

(1′) The endomorphism ring of X is local.

See [35, §2] for details.

Let C be a *Krull-Schmidt category*, that is, each object decomposes into a finite direct sum of objects with local endomorphism rings. We say that C *has AR-triangles* if for every indecomposable object X there are AR-triangles

$$V \to W \to X \to \Sigma V \quad \text{and} \quad X \to Y \to Z \to \Sigma X\,.$$

The next proposition establishes the existence of AR-triangles; it is the analogue of a result of Reiten and Van den Bergh [43, I.2] for triangulated categories that are Hom-finite over a field.

Proposition 7.6. *Let R be a graded commutative ring that is local, and let C be an essentially small R-linear triangulated category that is Krull-Schmidt. If C has a Serre functor, then it has AR-triangles.*

Proof. Let F be a Serre functor for C and X an indecomposable object in C. The ring $\operatorname{End}_{\mathsf{C}}(X)$ is thus local; let J be its maximal ideal and I the right ideal of $\operatorname{End}^*_{\mathsf{C}}(X)$ that it generates. One has $I^0 = J$, by Remark 7.7, so $\operatorname{End}^*_{\mathsf{C}}(X)/I$ is nonzero. Choose a nonzero morphism $\operatorname{End}^*_{\mathsf{C}}(X)/I \to I(\mathfrak{m})$ and let $\gamma\colon X \to FX$ be the corresponding morphism in C provided by Serre duality (7.1). We claim that the induced exact triangle

$$\Sigma^{-1}FX \to W \to X \xrightarrow{\gamma} FX$$

is an AR-triangle. Indeed, by construction, if $X' \to X$ is not a split epimorphism, then the composition $\operatorname{Hom}^*_{\mathsf{C}}(X,X') \to \operatorname{End}^*_{\mathsf{C}}(X)/I \to I(\mathfrak{m})$ is zero, and therefore the naturality of (7.1) yields that the composition $X' \to X \xrightarrow{\gamma} FX$ is zero. Moreover, since X is indecomposable so is FX.

Applying this construction to $F^{-1}\Sigma X$ yields an AR-triangle starting at X. $\square$

The following observations about graded rings has been, and will again be, used.

Remark 7.7. Let E be a graded ring. For any J a right ideal in E^0, the right ideal JE of E it generates satisfies $JE \cap E^0 = J$; this can be verified directly, or by noting that E^0 is a direct summand of E, as right E^0-modules.

It follows that if E is artinian than so is E^0: any descending chain of ideals in E^0 stabilises, because the chain of ideals in E that they generate stabilises; confer the proof of [21, Theorem 1.5.5].

Let R be a graded commutative ring. An R-linear triangulated category C is *noetherian* if the R-module $\operatorname{Hom}^*_{\mathsf{C}}(X,Y)$ is noetherian for all X, Y in C.

Lemma 7.8. *Let R be a graded commutative ring and C an essentially small, noetherian, R-linear triangulated category. For each $\mathfrak{p} \in \operatorname{Spec} R$, the idempotent completion of $\gamma_{\mathfrak{p}}\mathsf{C}$ is a Krull-Schmidt category.*

Proof. The noetherian property implies that for any object X in $\gamma_{\mathfrak{p}}\mathsf{C}$ the $R_{\mathfrak{p}}$-module $E := \operatorname{End}^*_{\gamma_{\mathfrak{p}}\mathsf{C}}(X)$ is of finite length. The graded ring E is thus artinian, and then so is the ring E^0, by Remark 7.7. Aritinian rings are semi-perfect, so the idempotent completion of $\gamma_{\mathfrak{p}}\mathsf{C}$ is a Krull-Schmidt category; see [38, Corollary 4.4]. $\square$

Corollary 7.9. *Let T be a compactly generated R-linear triangulated category with T^{c} noetherian. If T is Gorenstein, then $(\varGamma_{\mathfrak{p}}\mathsf{T})^{\mathsf{c}}$ has AR-triangles for $\mathfrak{p} \in \operatorname{Spec} R$.*

Proof. Set $\mathsf{S} := (\varGamma_{\mathfrak{p}}\mathsf{T})^{\mathsf{c}}$; this is an idempotent complete, essentially small, $R_{\mathfrak{p}}$-linear triangulated category. Since T^{c} is noetherian, it follows from Remark 7.2 and Lemma 7.8, that S is a Krull-Schmidt category. The Gorenstein hypothesis implies that the $R_{\mathfrak{p}}$-linear category S has a Serre functor, by Proposition 7.3, and then Proposition 7.6 yields the existence of AR-triangles. $\square$

Next we consider local Serre duality for $\mathsf{D}^b(\mathsf{mod}\, G)$ and $\mathsf{stmod}\, G$. Recall from Lemma 2.6 that localisation at $\mathfrak{p} \in \operatorname{Proj} H^*(G,k)$ induces a triangle equivalence $\Gamma_{\mathfrak{p}}(\mathsf{K}(\mathsf{Inj}\, G)) \xrightarrow{\sim} \Gamma_{\mathfrak{p}}(\mathsf{StMod}\, G)$. Using Remark 7.2, this yields (up to direct summands) triangle equivalences

$$\gamma_{\mathfrak{p}}(\mathsf{D}^b(\mathsf{mod}\, G)) \xrightarrow{\sim} \Gamma_{\mathfrak{p}}(\mathsf{K}(\mathsf{Inj}\, G))^{\mathsf{c}} \xrightarrow{\sim} \Gamma_{\mathfrak{p}}(\mathsf{StMod}\, G)^{\mathsf{c}} \xrightarrow{\sim} \gamma_{\mathfrak{p}}(\mathsf{stmod}\, G).$$

The result below contains Theorem 1.1 in the introduction.

Theorem 7.10. *Let G be a finite group scheme over a field k. Then the $H^*(G,k)$-linear triangulated category $\mathsf{D} := \mathsf{D}^b(\mathsf{mod}\, G)$ satisfies local Serre duality. Said otherwise, given $\mathfrak{p}$ in $\operatorname{Spec} R$ and with d the Krull dimension of $H^*(G,k)/\mathfrak{p}$, for each M in $\gamma_{\mathfrak{p}}\mathsf{D}$ and N in $\mathsf{D}_{\mathfrak{p}}$, there are natural isomorphisms*

$$\operatorname{Hom}_{H^*(G,k)}(\operatorname{Hom}^*_{\mathsf{D}_{\mathfrak{p}}}(M,N), I(\mathfrak{p})) \cong \operatorname{Hom}_{\mathsf{D}_{\mathfrak{p}}}(N, \Omega^d \delta_G \otimes_k M)\,.$$

In particular, $\gamma_{\mathfrak{p}}\mathsf{D}$ has AR-triangles.

Proof. The first assertion follows from Theorem 6.5 and Proposition 7.3. The existence of AR-triangles then follows from Corollary 7.9, as D is noetherian. □

AR-components and periodicity. The existence of AR-triangles for a triangulated category C gives rise to an *AR-quiver*; see for example [32, 41]. The vertices are given by the isomorphism classes of indecomposable objects in C and an arrow $[X] \to [Y]$ exists if there is an irreducible morphism $X \to Y$.

In the context of $\mathsf{stmod}\, G$, one can describe part of the structure of the AR-quiver of the $\mathfrak{p}$-local $\mathfrak{p}$-torsion objects as the Serre functor is periodic.

Proposition 7.11. *Let G be a finite group scheme over a field k. Fix a point $\mathfrak{p}$ in $\operatorname{Proj} H^*(G,k)$ and set $d = \dim H^*(G,k)/\mathfrak{p}$. Then the Serre functor*

$$\Omega^d\nu \colon \gamma_{\mathfrak{p}}(\mathsf{stmod}\, G) \xrightarrow{\sim} \gamma_{\mathfrak{p}}(\mathsf{stmod}\, G)$$

is periodic, that is, $(\Omega^d\nu)^r = \mathrm{id}$ for some positive integer r.

Proof. Lemma 2.1 and (4.1) provide an integer $r \geq 0$ such that $\nu^r M \cong M$ and $\Omega^r M \cong M$ for M in $\gamma_{\mathfrak{p}}(\mathsf{stmod}\, G)$. Thus $(\Omega^d\nu)^r = \mathrm{id}$, since ν and Ω commute. □

This has the following consequence.

Corollary 7.12. *Every connected component of the AR-quiver of $\gamma_{\mathfrak{p}}(\mathsf{stmod}\, G)$ is a stable tube in case it is infinite; and otherwise, it is of the form $\mathbb{Z}\Delta/U$, where Δ is a quiver of Dynkin type and U is a group of automorphisms of $\mathbb{Z}\Delta$.*

Proof. Since the Serre functor on $\gamma_{\mathfrak{p}}(\mathsf{stmod}\, G)$ is periodic, the desired result follows from [41, Theorem 5.5]; see also [32]. □

The preceding result may be seen as a first step in the direction of extending the results of Farnsteiner's [27, §3] concerning $\mathsf{stmod}\, G$ to $\gamma_{\mathfrak{p}}(\mathsf{stmod}\, G)$ for a general (meaning, not necessarily closed) point $\mathfrak{p}$ of $\operatorname{Proj} H^*(G,k)$.

Appendix A. Injective modules at closed points

In this section we collect some remarks concerning injective hulls over graded rings, for use in Section 5. Throughout k will be a field and $A := \bigoplus_{i \geqslant 0} A^i$ will be a finitely generated graded commutative k-algebra with $A^0 = k$; we have in mind $H^*(G,k)$, for a finite group scheme G over k.

As usual $\operatorname{Proj} A$ denotes the homogeneous prime ideals in A that do not contain the ideal $A^{\geqslant 1}$. Given a point $\mathfrak{p}$ in $\operatorname{Proj} A$, we write $k(\mathfrak{p})$ for the graded residue field at $\mathfrak{p}$; this is the homogeneous field of fractions of the graded domain $A/\mathfrak{p}$. Observe that $k(\mathfrak{p})^0$ is a field extension of k and $k(\mathfrak{p})$ is of the form $k(\mathfrak{p})^0[t^{\pm 1}]$ for some indeterminate t over $k(\mathfrak{p})^0$; see, for example, [21, Lemma 1.5.7].

Lemma A.1. *The degree of* $k(\mathfrak{m})^0/k$ *is finite for any closed point* $\mathfrak{m}$ *in* $\operatorname{Proj} A$.

Proof. One way to verify this is as follows: The Krull dimension of $A/\mathfrak{m}$ is one so, by Noetherian normalisation, there exists a subalgebra $k[t]$ of $A/\mathfrak{m}$ where t is an indeterminate over k and the $A/\mathfrak{m}$ is finitely generated $k[t]$-module. Thus, inverting t, one gets that $(A/\mathfrak{m})_t$ is a finitely generated module over the graded field $k[t^{\pm 1}]$, and hence isomorphic to $k(\mathfrak{m})$. The finiteness of the extension $k(\mathfrak{m})/k[t^{\pm 1}]$ implies that the extension $k(\mathfrak{m})^0/k$ of fields is finite. □

The result below is familiar; confer [21, Proposition 3.6.16].

Lemma A.2. *Let* A *be as above, let* $\mathfrak{m}$ *be a closed point in* $\operatorname{Proj} A$ *and set* $R := A_{\mathfrak{m}}$. *The* R*-submodule* $I := \bigcup_{i\geqslant 0} \operatorname{Hom}^*_k(R/\mathfrak{m}^i, k)$ *of* $\operatorname{Hom}^*_k(R,k)$ *is the injective hull of* $k(\mathfrak{m})$*, and for any* $\mathfrak{m}$*-torsion* R*-module* N*, there is a natural isomorphism*

$$\operatorname{Hom}_R(N, I) \cong \operatorname{Hom}_k(N, k)\,.$$

Proof. Set $K = k(\mathfrak{m})^0$ and recall that $k(\mathfrak{m}) = K[t^{\pm 1}]$, for some indeterminate t over K. Thus, one has isomorphisms of graded $k(\mathfrak{m})$-modules

$$\begin{aligned}\operatorname{Hom}^*_k(k(\mathfrak{m}), k) &\cong \operatorname{Hom}^*_K(k(\mathfrak{m}), \operatorname{Hom}_k(K,k)) \\ &\cong \operatorname{Hom}^*_K(k(\mathfrak{m}), K) \\ &\cong k(\mathfrak{m})\,.\end{aligned} \tag{A.1}$$

The first isomorphism is adjunction, the second holds because $\operatorname{rank}_k K$ is finite, by Lemma A.1, and the last one is a direct verification.

The R-module $\operatorname{Hom}^*_k(R,k)$ is injective and hence so is its $\mathfrak{m}$-torsion submodule

$$\bigcup_{i\geqslant 0} \operatorname{Hom}^*_R(R/\mathfrak{m}^i, \operatorname{Hom}^*_k(R,k))\,.$$

This is precisely the R-module I, by standard adjunction. Thus I must be a direct sum of shifts of injective hulls of $k(\mathfrak{m})$. It remains to verify that I is in fact just the injective hull of $k(\mathfrak{m})$. To this end, note that for any $\mathfrak{m}$-torsion R-module N, one has isomorphisms of graded $k(\mathfrak{m})$-modules

$$\begin{aligned}\operatorname{Hom}^*_R(N, I) &\cong \operatorname{Hom}^*_R(N, \operatorname{Hom}^*_k(R,k)) \\ &\cong \operatorname{Hom}^*_k(N,k)\,.\end{aligned}$$

This settles the last assertion in the desired result and also yields the first isomorphism below of graded $k(\mathfrak{m})$-modules.

$$\operatorname{Hom}^*_R(k(\mathfrak{m}), I) \cong \operatorname{Hom}^*_k(k(\mathfrak{m}), k) \cong k(\mathfrak{m})$$

The second one is by (A.1). It follows that I is the injective hull of $k(\mathfrak{m})$. □

The next result, whose proof is rather similar to the one above, gives yet another way to get to the injective hull at a closed point of Proj.

Recall that $I(\mathfrak{p})$ denotes the injective hull of $A/\mathfrak{p}$ for any $\mathfrak{p}$ in $\operatorname{Spec} A$.

Lemma A.3. *Let* $A \to B$ *be a homomorphism of graded commutative algebras, let* $\mathfrak{m}$ *be a closed point in* $\operatorname{Proj} B$*, and set* $\mathfrak{p} := \mathfrak{m} \cap A$. *If the extension of residue fields* $k(\mathfrak{p}) \subseteq k(\mathfrak{m})$ *is finite, then the* B*-module* $\varGamma_{\mathcal{V}(\mathfrak{m})} \operatorname{Hom}^*_A(B, I(\mathfrak{p}))$ *is the injective hull of* $B/\mathfrak{m}$*, and for any* $\mathfrak{m}$*-torsion* B*-module* N*, adjunction induces an isomorphism*

$$\operatorname{Hom}_B(N, I(\mathfrak{m})) \cong \operatorname{Hom}_A(N, I(\mathfrak{p}))\,.$$

Proof. The B-module $I := \varGamma_{\mathcal{V}(\mathfrak{m})} \operatorname{Hom}^*_A(B, I(\mathfrak{p}))$ is injective, for it is the $\mathfrak{m}$-torsion submodule of the injective B-module $\operatorname{Hom}^*_A(B, I(\mathfrak{p}))$. As $\mathfrak{m}$ is a closed point, I is a direct sum of shifts of copies of $I(\mathfrak{m})$. It remains to make the computation below:

$$\begin{aligned}
\operatorname{Hom}^*_B(k(\mathfrak{m}), I) &\cong \operatorname{Hom}^*_B(k(\mathfrak{m}), \operatorname{Hom}^*_A(B, I(\mathfrak{p}))) \\
&\cong \operatorname{Hom}^*_A(k(\mathfrak{m}), I(\mathfrak{p})) \\
&\cong \operatorname{Hom}^*_{k(\mathfrak{p})}(k(\mathfrak{m}), \operatorname{Hom}^*_A(k(\mathfrak{p}), I(\mathfrak{p})) \\
&\cong \operatorname{Hom}^*_{k(\mathfrak{p})}(k(\mathfrak{m}), k(\mathfrak{p})) \\
&\cong k(\mathfrak{m})
\end{aligned}$$

These are all isomorphisms of $k(\mathfrak{m})$-modules. The last one is where the hypothesis that $k(\mathfrak{m})/k(\mathfrak{p})$ is finite is required. This implies that $I \cong I(\mathfrak{m})$. Given this, the last isomorphism follows by standard adjunction. □

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