LECTURES ON π-POINTS AND APPLICATIONS TO COHOMOLOGY AND REPRESENTATION THEORY LECTURE 3

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ABSTRACT. Applications: Modules of constant Jordan type and non-maximal rank varieties.

To finish up what's left from lecture 2.

Definition 2.1. Let α, β be two π-points of G.
$\alpha \sim \beta \iff$ for any finite-dimensional G–module M, $\alpha^* M$ is free if and only if $\beta^* M$ is free.

Remark 2.2. What is behind the equivalence relation? Let's bisect it in the case of elementary abelian p-group. Let $kE = k[x_1, \dots, x_n]/(x_i^p)$. Cyclic shifted subgroups and Rank varieties were defined in terms of these generators x_i. What if we change generators? Let $kE = k[x_1', \dots, x_n']/((x_i')^p)$. We need to compare restrictions of a module M to $\langle x_{\underline{a}}+1\rangle$ and $\langle x'_{\underline{a}}+1\rangle$ where $x_{\underline{a}} = a_1x_1 + \ldots + a_nx_n + 1$, $x'_{\underline{a}} = a_1x'_1 + \ldots + a_nx'_n + 1$.

Exercise. $x_{\underline{a}} - x'_{\underline{a}} \in I^2 = (x_1, \dots, x_n)^2$.

The equivalence relation in this case says the following:

$$M\downarrow_{\langle x_{\underline{a}}+1\rangle} \text{ is projective if and only if } M\downarrow_{\langle cx_{\underline{a}}+p(x_1,\dots,x_n)+1\rangle} \text{ is projective}$$

where $p(x_1, \dots, x_n)$ is any polynomial without constant or linear term, and $c \in k^*$.

Definition 2.3. Support space of a finite group scheme G:

$$\Pi(G) =< \pi\text{-points} > / \sim$$

Support space of a G-module M:

$$\Pi(G)_M =< [\alpha] : k[t]/t^p \to kG : \alpha^* M \text{ is not free} >$$

Topology: closed sets are $\Pi(G)_M$ for finite dimensional G-modules M.

This specializes to

- $\operatorname{Proj} V_E$ and $\operatorname{Proj} V_E(M)$ for $G = E$, an elementary abelian p-group.
- $\operatorname{Proj} \mathcal{N}^{[p]}$ and $V_{\mathfrak{g}}(M)$ for a restricted Lie algebra $\mathfrak{g}$
- SFB theory of varieties of one-parameter subgroups for Frobenius kernels

$\boxed{\texttt{main}}$ **Theorem 2.4** (Friedlander-P.)**.**

$$\Pi(G) \simeq \operatorname{Proj} |G|$$

$$\underbrace{\Pi(G)_M}_{\textit{local prop}} \simeq \underbrace{\operatorname{Proj} |G|_M}_{\textit{cohomology}}$$

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for any finite dimensional G-module M

$\Pi(G)$ has an intrinsic topology and a scheme structure. It's isomorphic to $\operatorname{Proj}|G|$ with respect to both of these structures.

Theorem 2.5 (Detection of projectivity $\sim$ Dade's lemma)**.** *M is projective $\Leftrightarrow$ $\Pi(G)_M = \emptyset$ $\Leftrightarrow$ M is free when restricted to any subalgebra $k[t]/t^p \to kG$.*

Credit: Dade, Chouinard, Benson-Carlson-Rickard, [finite groups], Bendel, Pevtsova [infinitesimal group schemes], Friedlander-Pevtsova [finite group schemes]. Will not touch upon this here but the theorem is valid for all modules, not necessarily finite dimensional. This makes it more difficult because the finite dimensional case follows from Theorem 2.4 easily.

Lots of open questions. Finite generation of cohomology for other Hopf algebras and more general finite dimensional algebras; rank varieties and "local" projectivity tests, tensor product property for quantum groups. Calculations of support varieties (beyond Weyl and irreducible modules - see B. Parshall's talk).

3. Modules of constant Jordan type

Time to talk about applications. One can few Theorem 2.4 as an application although it is probably more the fundamentals of the theory. I was choosing between at least four possible topics:

(1) Classification of tensor triangulated subcategories in $\operatorname{stmod} G$.
(2) Non-maximal rank varieties (= sets of π-points α where $\operatorname{rk}\{\alpha(t), M\}$ is not maximal. This explains the name "rank variety").
(3) Modules of constant Jordan type
(4) Vector bundles on projective varieties arising from modular representation theory

And the winner is ...

3.1. Modules of CJT.

We converged to cohomology but now we are going away again. Recall:

The isomorphism class of a $\mathbb{Z}/p$-module $M \leftrightarrow M \simeq \bigoplus_{i=1}^{p} a_i[i] \leftrightarrow \operatorname{JType}(t, M)$, the Jordan type of t as an operator of M.

For a π-point α:

$$\alpha : k[t]/t^p \to kG \rightsquigarrow \alpha^*(M) \simeq a_1[1] + \ldots + a_p[p] \simeq \operatorname{JType}(\alpha(t), M).$$

Revisit projectivity test: M is projective if and only if $\alpha^*(M) = a_p[p]$. In particular, the Jordan type is always the same; it is also *maximal* with respect to the dominance ordering on partitions of $\dim M$.

Definition 3.1. Let G be a finite group scheme, and M be a finite dimensional G-module. Then M is a module of constant Jordan type if $\alpha^*(M)$ has the same Jordan type (same isomorphism class) for any π-point α of G.

Questions: do we have to check at ALL π-points? No.

Theorem 3.2 (Carlson-Friedlander-P.-Suslin)**.** *The property of being of Constant Jordan type does not depend on the choice of a representative of a π-point.*

elem **Example 3.3.** Why do we have to worry about that? The "local" Jordan type can easily change when we change representative of a π-point.

Let $E = \mathbb{Z}/p \times \mathbb{Z}/p$, $p > 2$, $kE = k[x,y]/(x^p, y^p)$. Let $M = k[x,y]/(x^2 - y, x^p)$ be an E-module. Define π-points $\alpha, \beta : k[t]/t^p \to kE$ by

$$\alpha(t) = x^2 - y, \text{ and } \beta(t) = y.$$

Then $\alpha \sim \beta$ because they have proportional "linear" parts. But:

$$\alpha^* M \simeq p[1], \quad \beta^* M \simeq \left[\frac{p+1}{2}\right] \oplus \left[\frac{p-1}{2}\right].$$

Note that this is not the "generic" type. The local Jordan type at most points (of the form $ax + by, a \neq 0$) is projective and does not depend on a representative.

Example 3.4.
- Projective modules; JType $= a[p]$
- The trivial module k; JType $= [1]$
- Endotrivial modules ($\mathrm{End}_k(M) \simeq k + \mathrm{proj}$)

Theorem 3.5 (Carlson-Friedlander-P (local endotriviality test))**.** *M is endo-trivial if and only if M has constant Jordan type $[1] + a[p]$ or $[p-1] + a[p]$.*

Proposition 3.6. *The property of being of constant Jordan type is preserved under*

- *Direct sums*
- *Heller operator*
- *Direct summands*
- *Tensor products*
- *Duals*
- *This is an invariant of a component of a stable Auslander-Reiten quiver of kG*

Remark 3.7. The tensor product property is subtle because

$$\alpha^*(M \otimes N) \not\simeq \alpha^*(M) \otimes \alpha^*(N)$$

as $\mathbb{Z}/p$-modules.

Some fun examples. Let $E = \mathbb{Z}/p \times \mathbb{Z}/p$, $kE = k[x,y]/(x^p, y^p)$.

I. M - an E-module of dimension $2n+1$ ("zigzag module").

$$\begin{tikzcd}
& \bullet \arrow[dl, "y"'] \arrow[dr, "x"] & & \bullet \arrow[dl, "y"'] & \cdots & & \bullet \arrow[dl, "y"'] \arrow[dr, "x"] & \\
\bullet & & \bullet & & \cdots & \bullet & & \bullet
\end{tikzcd}$$

Constant Jordan type $2n+1$.

II.
(3.7.1)

$$\begin{tikzcd}
 & \langle v_1 \rangle \arrow[dl, "y"'] \arrow[dr, "x"] & & \langle v_2 \rangle \arrow[dl, "y"'] \arrow[dr, "x"] & & \langle v_3 \rangle \arrow[dl, "y"'] \arrow[dr, "x"] & & \langle v_4 \rangle \arrow[dl, "y"'] \arrow[dr, "x"] & \\
\bullet \arrow[dr, "x"'] & & \bullet \arrow[dl, "y"] \arrow[dr, "x"'] & & \bullet \arrow[dl, "y"] \arrow[dr, "x"'] & & \bullet \arrow[dl, "y"] \arrow[dr, "x"'] & & \bullet \arrow[dl, "y"] \\
 & \bullet & & \bullet & & \bullet & & \bullet &
\end{tikzcd}$$

For $p = 5$, this module has constant Jordan type $3[3] + 2[2]$. If $p > 5$, then it does not have constant Jordan type. Namely, the Jordan type for both x and y is $3[3] + 2[2]$, whereas the Jordan type of $x + y$ is $4[3] + 1[1]$.

More generally, the module above for $p = 5$ is a special case of $M = I^{p-2}/I^{p+1}$, where I is the augmentation ideal of kE. It is of constant Jordan type $(p-2)[3]+2[2]$.

Exercise. Classify modules of constant Jordan type for $\mathrm{SL}_{2(1)}$. Find all types that occur.

General question: Which Jordan types can occur as types of modules of Constant Jordan type.

Theorem 3.8. *Assume* $\dim \Pi(G) \geq 1$. *Then for any* n *there exists an indecomposable* G*-module of constant Jordan type* $n[1] + [\mathrm{proj}]$.

Theorem 3.9 (Benson, 2008)**.** *Assume* $\dim \Pi(G) \geq 1$ *for a finite group* G*, and assume* $p \geq 5$. *There does not exist a module of constant Jordan type* $[a] + m[p]$ *where* $1 < a < p - 1$.

Question: Which types are realizable?

3.2. **Non-maximal rank varieties.** Why "rank" variety? Here is an alternative description:
$\Pi(G)_M = \{[\alpha] \mid \mathrm{JType}(\alpha(t), M) = m[p]\} = \{[\alpha] \mid \mathrm{rk}\{\alpha(t), M\} = \dim M - \frac{\dim M}{p}\}$
Hence, $\Pi(G)_M$ consists of all π-points α such that the rank of $\alpha(t)$ is the maximal possible.

max **Proposition 3.10.** *Let* M *be a finite-dimensional* G*-module, and let* $\alpha : k[t]/t^p \to kG$ *br a* π*-point such that* $\mathrm{rk}\{\alpha(t), M\}$ *is maximal among all* π*-points of* G. *Then for any* π*-point* β*, we have* $\mathrm{rk}\{\beta(t), M\} = \mathrm{rk}\{\alpha(t), M\}$.

The "maximal rank" is well-defined on equivalence classes of π-points.
We revisit here Example 3.3.

Definition 3.11. The non-maximal rank variety $\Gamma(G)_M \subset \Pi(G)$ of M is defined as

$$\Gamma(G)_M = \{[\alpha] \mid \mathrm{rk}\{\alpha(t), M\} \text{ is not maximal } \}$$

Remark 3.12. It is indeed a projective variety - a closed subset in $\Pi(G)$.

In fact, we get a whole string of new invariants of a module M, invariants which are finer than $\Pi(G)_M$:

Definition 3.13. The non-maximal ith rank variety $\Gamma^i(G)_M \subset \Pi(G)$ of M is defined as

$$\Gamma^i(G)_M = \{[\alpha] \mid \mathrm{rk}\{\alpha(t^i), M\} \text{ is not maximal } \}$$

for $1 \leq i \leq p-1$

Remarks:
1. $\Gamma^i(G)_M$ is ALWAYS a proper subvariety.
2. If $\Pi(G)_M$ is a proper subvariety of $\Pi(G)$ then $\Gamma^i(G)_M = \Pi(G)_M$ for all i. So this is interesting precisely for modules for which support varieties give no information.
3. M is of constant Jordan type if and only if $\Gamma^1(G)_M = \ldots = \Gamma^{p-1}(G)_M = \emptyset$.

Definition 3.14. M is a module of constant rank if $\mathrm{rk}\{\alpha(t), M\}$ is independent of a π-point α. Equivalently, $\Gamma^1(G)_M = \emptyset$.

And now we cycle our path back in to cohomology.

Definition 3.15. For M a module of constant rank, and $\zeta \in \mathrm{H}^1(G, M)$, we define

one

$$Z(\zeta) \;\equiv\; \{[\alpha_K] \mid \alpha_K^*(\zeta) = 0\} \;\subset\; \Pi(G). \tag{3.15.1}$$

Suppose $\zeta \in \mathrm{H}^1(G, M) = \mathrm{Ext}^1_G(k, M)$ is represented by an extension $0 \to M \to E_\zeta \to k \to 0$.

Proposition 3.16. *Then*

$$Z(\zeta) = \begin{cases} \Pi(G), & \text{if } \zeta \text{ splits at every } \pi\text{-point} \\ \Gamma^1(G)_{E_\zeta}, & \text{otherwise.} \end{cases}$$

In particular, $Z(\zeta) \;\subset\; \Pi(G)$ is closed.

If $M = \Omega^{1-2n}k$, and $\zeta \in \mathrm{H}^1(G, M) = \mathrm{H}^{2n}(G, k)$, then $E_\zeta = \Omega^{2n}L_\zeta$, and

$$\Gamma^1(G)_{E_\zeta} = \Gamma^1(G)_{L_\zeta} = \langle \zeta \rangle.$$