LECTURES ON π-POINTS AND APPLICATIONS TO COHOMOLOGY AND REPRESENTATION THEORY LECTURE II

JULIA PEVTSOVA

ABSTRACT. We discuss representation-theoretic approach to support varieties: cyclic shifted subgroups, π-points and rank varieites. We also formulate projectivity tests.

2. π-POINTS AND RANK VARIETIES

First, finishing up from last lecture. Recall that we stopped with a definition of a finite group scheme.

$$\text{Representations of } G \xleftrightarrow{\sim} kG - \text{mod}$$

2.1. Examples.

Example 2.1. (Finite groups).

$$G \rightsquigarrow kG$$

As a functor: for R an integral domain, $\widetilde{G}(R) = G$. In particular, $\widetilde{G}(K) = G$ for any field. In general, $\widetilde{G}(R) = R^{\times|\pi_0(R)|}$, where $\pi_0(R)$ is the set of connected components of $\operatorname{Spec} R$. This is a *constant* finite group scheme.

Exercise. Show that $\widetilde{G} : R \mapsto G$ is not a functor.

Example 2.2. (Frobenius kernels) Define the algebraic group GL_n as

$$\mathrm{GL}_n(R) = \{(a_{ij})_{1\leq i,j\leq n}, \text{invertible over } R\}$$

$k[\mathrm{GL}_n] \simeq k[X_{ij}, \frac{1}{\det}]$, $\nabla : k[\mathrm{GL}_n] \to k[\mathrm{GL}_n] \otimes k[\mathrm{GL}_n]$ sends X_{ij} to $\sum\limits_{\ell=1}^{n} X_{i\ell} \otimes X_{\ell j}$, antipode $S : k[\mathrm{GL}_n] \to k[\mathrm{GL}_n]$ sends X_{ij} to $\frac{\mathrm{Ad}(X_{ij})}{\det}$.

Define Frobenius map:

$$F : \mathrm{GL}_n \xrightarrow{(a_{ij})\to(a_{ij}^p)} \mathrm{GL}_n$$

$$\mathrm{GL}_{n(r)} \stackrel{\text{def}}{=} \operatorname{Ker} F^{(r)}$$

is the *r-th Frobenius kernel* of GL_n. Explicitly,

$$\mathrm{GL}_{n(r)}(R) = \{(a_{ij})_{1\leq i,j\leq n}, a_{ij}^p = \delta_{ij}\}.$$

$k[\mathrm{GL}_{n(r)}] \simeq k[X_{ij}]/(X_{ij}^p - \delta_{ij})$.

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We can define Frobenius map for any group scheme over k (e.g., by embedding into GL_n, but also internally). If $\mathcal{G}$ is any group scheme, then denote by $\mathcal{G}_{(r)} = \mathrm{Ker}\{F^r : \mathcal{G} \to \mathcal{G}\}$.

Definition 2.3. A group scheme is called *infinitesimal* if $k[G]$ is a local algebra.

Frobenius kernels are *infinitesimal* finite group schemes. Geometrically, they only have one point (as Spec of local algebras). Their representation theory, though, is very rich. In particular, it "approximates" representation theory of the big algebraic group for which they are kernels.

Example 2.4. (Restricted Lie algebras) Ask: what Hopf algebras do you know? Expected answer: $U(\mathfrak{g})$.
If G is a group scheme, it has a Lie algebra (analogous to Lie groups and Lie algebras) - the tangent space at the identity, or the space of left invariant derivations. Because of characteristic p it comes naturally with extra structure: the $[p]$-th power (or p-restriction) map:

$$[p] : \mathfrak{g} \to \mathfrak{g}.$$

E.g., $\mathfrak{g} = gl_n$. Then $(a_{ij})^{[p]} = (a_{ij})^p$ (note the difference with the Frobenius map!). Any Lie algebra can be embedded into gl_n and will inherit this pth power. There is also an internal description as usual.

Restricted enveloping algebra:

$$u(g) = U(g)/\langle x^p - x^{[p]} \rangle$$

This is a fin. dim. cocomm. Hopf algebra. Coproduct: for $x \in \mathfrak{g}$, $\nabla(x) = x \otimes 1 + 1 \otimes x$.

How is $\mathfrak{g}$ a finite group scheme directly? Let $\mathcal{G}$ be a group scheme, and $\mathfrak{g} = \mathrm{Lie}\,\mathcal{G}$. Then $u(\mathfrak{g}) \simeq k[\mathcal{G}_{(1)}]$. Hence,

$$\text{Representations of } \mathcal{G}_{(1)} \xleftrightarrow{\sim} u(\mathfrak{g}) - \text{mod}$$

All cohomological constructions go through for a finite group scheme replacing finite group. The Eckmann-Hilton argument is useful for graded commutativity here. A very important ingredient:

Theorem 2.5 (Friedlander-Suslin, (1997))**.** *Let G be a finite group scheme over a field k of positive characteristic. Then the cohomology algebra* $\mathrm{H}^\bullet(G, k)$ *is finitely generated over k.*

The "geometry" of $\mathrm{Spec}\,\mathrm{H}^\bullet(\mathfrak{g}, k)$ is quite different.

Theorem 2.6 (Friedlander-Parshall, Andersen-Jantzen (1983-84), Suslin-Friedlander-Bendel (1997))**.** *Let $\mathfrak{g}$ be a restricted Lie algebra. Then* $\mathrm{Spec}\,\mathrm{H}^*(\mathfrak{g}, k) \simeq \mathcal{N}^{[p]}(\mathfrak{g})$*, where* $\mathcal{N}^{[p]}(\mathfrak{g}) = \{x \in \mathfrak{g} \mid x^{[p]} = 0\}$.

Remark 2.7. In fact, for $p > h$, $\mathcal{N} = \mathcal{N}^{[p]}$ and
(1) $\mathrm{H}^{\mathrm{ev}}(\mathfrak{g}, k) = k[\mathcal{N}]$
(2) $\mathrm{H}^{\mathrm{ev}}(\mathfrak{g}, k) = 0$

Credit: Friedlander-Parshall, Andersen-Jantzen.
Note the contrast with finite groups, especially in view of Jon Carlson's explanations from yesterday's afternoon.

Remark 2.8. Quite different from the Quillen stratification for finite groups. In particular, for $\mathfrak{g} = \mathrm{Lie}\,\mathcal{G}$ where $\mathcal{G}$ is a classical algebraic group ($\mathrm{GL}_n, \mathrm{SL}_n$, orthogonal, symplectic, E, F, G-type...), this is ?almost? always irreducible. The UGA VIGRE group is a world's expert on that.

Now we temporarily forget all about cohomology. Approach the study of G-modules from a different perspective.

Example 2.9. $E = \mathbb{Z}/p$. $k\mathbb{Z}/p \simeq k[t]/t^p$. There are finitely many indecomposable modules: $[i] \simeq k[t]/t^i$, $1 \leq i \leq p$. In particular, $[p] \simeq k\mathbb{Z}/p$ is the only indecomposable projective and $[1] \simeq k$ is the unique simple module.

The isomorphism class of M, a $\mathbb{Z}/p$-module $\leftrightarrow M \simeq \bigoplus_{i=1}^{p} a_i[i] \leftrightarrow \mathrm{JType}(t, M)$, the Jordan type of t as an operator of M.

"Elementary approach to modular representation theory". For a finite group (scheme) G, consider $k[t]/t^p \subset kG$ (equivalently, p-nilpotent elements $t \in kG$). For a module M, study the family $M\downarrow_{k[t]/t^p}$.

2.2. **Cyclic shifted subgroups.**

Definition 2.10. Let $E = \mathbb{Z}/p^{\times n}$. Let $\{g_1, \dots, g_n\}$ be generators of E, and let $x_i = g_i - 1$ so that $kE \simeq k[x_1, \dots, x_n]/(x_1^p, \dots, t_n^p)$. A *cyclic shifted subgroup* of E is a non-trivial cyclic subgroup of kE generated by $a_1x_1 + \ldots + a_nx_n + 1$. If $\underline{a} = (a_1, \dots, a_n) \in k^n$, denote by $x_{\underline{a}} = a_1x_1 + \ldots + a_nx_n$.

$$\text{Cyclic shifted subgroups } \langle x_{\underline{a}} + 1\rangle \quad \longleftrightarrow \quad \underline{a} \in \mathbb{A}^n \backslash \{0\}$$

.

Theorem 2.11 (Dade, 1978)**.** *A finite-dimensional kE-module M is projective (=free) if and only if the restriction of M to every cyclic shifted subgroup is projective (=free).*

Generalization:

Theorem 2.12 (Benson-Carlson-Rickard, 1996)**.** *A kE-module M (can be infinite-dimensional) is projective (=free) if and only if the restriction of M to every cyclic shifted subgroup is projective (=free).*

Hence, the "elementary approach" detects projectivity.

Jon Carlson introduced the following construction:

Definition 2.13 (Rank variety)**.**

$$V_E(M) = \{\underline{\alpha} \in \mathbb{A}^n_k : \text{ such that } M\downarrow_{\langle x_{\underline{a}}+1\rangle} \text{ is not free } \} \cup \{0\}$$

Reformulation of "Dade's lemma": M is projective if and only if $V_E(M) = 0$. Carlson conjectured there was a close relationship between this "elementary approach" and cohomology; the conjecture was proved by Avrunin and Scott; several other proofs appeared later; perhaps 4 or 5 due to Jon Carlson:).

isom **Theorem 2.14** (Avrunin-Scott, 1982)**.**

$$V_E(M) \simeq |G|_M.$$

Remark 2.15. Carlson proved the "tensor product property" (property N4 form last time) for the rank variety side. Using Theorem 2.14, Avrunin-Scott generalized it to support varieties for all finite groups.

2.3. **Rank varieties for Lie algebras.** Recall the restricted Nullcone $\mathcal{N}^{[p]} = \{x \in \mathfrak{g} \mid x^{[p]} = 0\}$.

Definition 2.16. Let M be a restricted g-module ($=u(\mathfrak{g})$–module). Then $V_{\mathfrak{g}}(M) = \{x \in \mathcal{N}^{[p]} :$ such that $M \downarrow_{\langle x \rangle}$ is not free $\} \cup \{0\}$

Theorem 2.17 (Friedlander-Parshall(1986), Suslin-Friedlander-Bendel(1997))**.**

$$|\mathfrak{g}|_M \simeq V_{\mathfrak{g}}(M).$$

Remark 2.18. There is also a theory of "rank" varieties for Frobenius kernels of arbitrary height (due to Suslin, Friedlnader, Bendel). The role of shifted subgroups is played by *one parameter subgroups.*

2.4. π**-points.** Disclosure: I'll actually talk about p-points for simplicity. π-points show up when we allow field extensions which makes tremendous sense geometrically.

A finite group scheme U is unipotent abelian if kU is a commutative local algebra. E.g., group algebra of an abelian p-group.

Definition 2.19. Let G be a finite group scheme. A π**-point** α of G is a flat map of algebras

$$\begin{array}{ccc} k[t]/t^p & \overset{\alpha}{\dashrightarrow} & kG \\ & \searrow \quad \nearrow & \\ & kA & \end{array}$$

which factors through some unipotent abelian subgroup scheme $A \subset G$.

Examples of π-points:

- Cyclic shifted subgroups for E,
- p-nilpotent elements in restricted Lie algebras
- one parameter subgroups for Frobenius kernels.

Connection to cohomology:

Proposition 2.20. *A π-point $\alpha : k[t]/t^p \to kG$ induces a non-trivial map in cohomology:* $\alpha^* : \mathrm{H}^\bullet(G,k) \to \mathrm{H}^\bullet(k[t]/t^p, k) \simeq k[x]$.

Geometrically:
A π-point $k[t]/t^p \to kG \quad \rightsquigarrow \quad \mathrm{H}^\bullet(G,k) \to \mathrm{H}^\bullet(k[t]/t^p, k) \simeq k[x] \quad \rightsquigarrow$
$\mathbb{A}^1 = \operatorname{Spec} k[x] \to \operatorname{Spec} \mathrm{H}^\bullet(G,k) = |G|$.
Projectivize (factor out the scalar action of k^*): pt $\in$ Proj |G|. Hence,

$$\pi\text{-point} \rightsquigarrow \text{a point on } \operatorname{Proj} \mathrm{H}^\bullet(G,k)$$

.

But too many π-points collapse.

2.5. Π**-space.** Let M be a G-module, $\alpha : k[t]/t^p \to kG$ be a π-point. Denote by $\alpha^* M$ the restriction of M to $k[t]/t^p$ via α.

Definition 2.21. Let α, β be two π-points of G.
$\alpha \sim \beta \iff$ for any finite-dimensional G–module M, $\alpha^* M$ is free if and only if $\beta^* M$ is free.

Remark 2.22. What is behind the equivalence relation? Let's bisect it in the case of elementary abelian p-group. Let $kE = k[x_1, \ldots, x_n]/(x_i^p)$. Cyclic shifted subgroups and Rank varieties were defined in terms of these generators x_i. What if we change generators? Let $kE = k[x_1', \ldots, x_n']/((x_i')^p)$. We need to compare restrictions of a module M to $\langle x_{\underline{a}}+1 \rangle$ and $\langle x_{\underline{a}}'+1 \rangle$ where $x_{\underline{a}} = a_1x_1 + \ldots + a_nx_n + 1$, $x_{\underline{a}}' = a_1x_1' + \ldots + a_nx_n' + 1$.

Exercise. $x_{\underline{a}} - x_{\underline{a}}' \in I^2 = (x_1, \ldots, x_n)^2$.

The equivalence relation in this case says the following:

$$M \downarrow_{\langle x_{\underline{a}}+1 \rangle} \text{ is projective if and only if } M \downarrow_{\langle x_{\underline{a}}+p(x_1,\ldots,x_n)+1 \rangle} \text{ is projective}$$

where $p(x_1, \ldots, x_n)$ is any polynomial without constant or linear term.

Definition 2.23. Support space of a finite group scheme G:

$$\Pi(G) = < \pi\text{-points} > / \sim$$

Support space of a G-module M:

$$\Pi(G)_M = < [\alpha] : k[t]/t^p \to kG : \alpha^* M \text{ is not free} >$$

Topology: closed sets are $\Pi(G)_M$ for finite dimensional G-modules M.

This specializes to

- $\operatorname{Proj} V_E$ and $\operatorname{Proj} V_E(M)$ for $G = E$, an elementary abelian p-group.
- $\operatorname{Proj} \mathcal{N}^{[p]}$ and $V_{\mathfrak{g}}(M)$ for a restricted Lie algebra $\mathfrak{g}$
- SFB theory of varieties of one-parameter subgroups for Frobenius kernels

main **Theorem 2.24** (Friedlander-P.).

$$\Pi(G) \simeq \operatorname{Proj} |G|$$

$$\underbrace{\Pi(G)_M}_{local\ prop} \simeq \underbrace{\operatorname{Proj} |G|_M}_{cohomology}$$

for any finite dimensional G-module M

$\Pi(G)$ has an intrinsic topology and a scheme structure. It's isomorphic to $\operatorname{Proj} |G|$ with respect to both of these structures.

Theorem 2.25 (Detection of projectivity $\sim$ Dade's lemma). *M is projective* $\Leftrightarrow$ $\Pi(G)_M = \emptyset$ $\Leftrightarrow$ *M is free when restricted to any subalgebra $k[t]/t^p \to kG$.*

Credit: Dade, Chouinard, Benson-Carlson-Rickard, [finite groups], Bendel, Pevtsova [infinitesimal group schemes], Friedlander-Pevtsova [finite group schemes]. Will not touch upon this here but the theorem is valid for all modules, not necessarily finite dimensional. This makes it more difficult because the finite dimensional case follows from Theorem 2.24 easily.