π-POINTS AND APPLICATIONS TO COHOMOLOGY AND REPRESENTATION THEORY
LECTURE I

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Abstract. We discuss the cohomology ring of a finite group G with mod p coefficients. We describe Quillen stratification, support varieties for modules, and their properties. Then we move to the discussion of a more general class of finite group schemes.

1. Cohomology and Support varieties

1.1. **Rappels: generalities.** Let G be a finite group, and k be a field of characteristic p. We asume throughout that k is algebraically closed just for simplicity. If p divides the order of G, then the representation theory of G is not semi-simple (Maschke's theorem does not hold). This is *modular representation theory*, and the object of our study here.

Recall that to a finite group G, we associate the *group algebra*, kG. By definition, kG is generated by $\{g\}_{g\in G}$ as a k-vector space; multiplication is defined via multiplication in G on the basis elements and extended linearly to kG. Recall that kG has a *Hopf* algebra structure:

$$\begin{array}{ll} \text{coproduct} & \nabla : g \mapsto g \otimes g, \\ \text{antipode} & S : g \mapsto g^{-1}. \end{array}$$

We have an equivalence of categories

$$\text{Representations of } G \longleftrightarrow kG-\text{mod}$$

1.2. **Cohomology ring of G.** The category $kG-$mod is an abelian category which has enough projectives.

Remark 1.1. It also has enough injectives, and, moreover, it is a Frobenius category:

$$\text{injectives} = \text{projectives}$$

This follows from the fact that kG is self-injective algebra which means that kG is an injective module over itself.

On an abelian category with enough projectives we can do homological algebra. Let M, N be G-modules, and let $P_\bullet \to M$ be a projective resolution of M. Then

$$\text{Ext}^i(M, N) = \text{H}^i(\text{Hom}_G(P_\bullet, N))$$

In particular,

$$\text{H}^i(G, k) = \text{Ext}^i(k, k)$$

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Moreover,

$$\mathrm{H}^*(G,k) = \bigoplus_{i=0}^{\infty} \mathrm{H}^i(G,k)$$

is a graded commutative algebra.

Two products: $\mathrm{H}^i(G,k) \times \mathrm{H}^j(G,k) \to \mathrm{H}^{i+j}(G,k)$
I. *Cup product* is defined by tensoring projective resolutions and composing with a diagonal approximation map.
II. *Yoneda product* is defined via splicing of long exact sequences. In more detail: $\mathrm{Ext}^i(M,N) \simeq \{$equiv. classes of exact seq. $N \to P_1 \to P_2 \to \ldots \to P_i \to M\}$

$$\mathrm{Ext}^i(M,N) \times \mathrm{Ext}^j(L,M) \to \mathrm{Ext}^{i+j}(L,N)$$

$$\begin{array}{c} N \to \cdots \to M \;\circ\; M \to \cdots \to L \\ \downarrow \\ N \to \cdots \to \boxed{M = M} \to \cdots \to L \\ \downarrow \\ N \to \cdots \to L. \end{array}$$

These two products are nicely *compatible* which leads to graded commutativity almost for free due to the following nice trick:

Eckmann-Hilton argument. Let X be a set with two binary operations, denoted $*$ and $\circ$, and a fixed element e such that

(1) e is the identity for both operations
(2) $(a \circ b) * (c \circ d) = (a * c) \circ (b * d)$

Then these two operations coincide and moreover they are associative and commutative.

Some properties:
I. Kunneth formula. $\mathrm{H}^*(G \times H, k) = \mathrm{H}^*(G,k) \otimes \mathrm{H}^*(H,k)$
II. Finite generation:

Theorem 1.2 (Golod (1959), Venkov (1959), Evens (1961))**.** *The cohomology ring* $\mathrm{H}^\bullet(G,k)$ *of a finite group* G *is a finitely generated* k*-algebra.*

Example 1.3. Let $E = \underbrace{\mathbb{Z}/p \times \cdots \times \mathbb{Z}/p}_{n}$, an *elementary abelian* p*-group of rank* n*.*

Then

$$\mathrm{H}^*(E,k) \simeq k[x_1,\ldots,x_n] \otimes \Lambda^*(y_1,\ldots,y_n)$$

$\deg x_i = 2$, $\deg y_i = 1$.

Note that $\mathrm{H}^*(E,k)$ has lots of nilpotent elements.

$$\mathrm{H}^\bullet(G,k) = \begin{cases} \mathrm{H}^{\mathrm{ev}}(G,k), & \text{if } \; p \neq 2 \\ \mathrm{H}^*(G,k), & \text{if } \; p = 2 \end{cases}$$

This algebra is (honestly) commutative.
Observations:

- $\mathrm{H}^\bullet(E,k)_{\mathrm{red}} = k[x_1,\ldots,x_n]$;

- Krull dimension of $\mathrm{H}^\bullet(E,k)$ is n.

Question. (Atiyah–Swan, Segal). What is the Krull dimension of $\mathrm{H}^\bullet(G,k)$?

Answer. D. Quillen, "The spectrum of an equivariant cohomology ring, I, II", *Annals of Math*, 94, no.3, p. 71 (1971).

1.3. **Quillen stratification theorem.** Krull dimension of $\mathrm{H}^\bullet(G,k) = \dim \mathrm{Specm}\ \mathrm{H}^\bullet(G,k)$. We replace the study of $\mathrm{H}^\bullet(G,k)$ with

$$\mathrm{Specm}\ \mathrm{H}^\bullet(G,k) = \{\text{maximal ideals in } \mathrm{H}^\bullet(G,k)\} \text{ with Zariski topology.}$$

Quillen showed that $\mathrm{Spec}\,\mathrm{H}^\bullet(G,k)$ is "determined" by $E \subset G$, where E runs over all elementary abelian p-subgroups of G. (The prime ideal spectrum $\mathrm{Spec}\,\mathrm{H}^*(G,k)$ can, and probably should, be considered here instead of Specm).

Notation: $\boxed{|G| = \mathrm{Specm}\ \mathrm{H}^\bullet(G,k)}$

Remark 1.4.

$$|E| = \mathrm{Specm}\, k[x_1,\dots,x_r] \simeq \mathbb{A}^r$$

$$\underbrace{(x_1-\lambda_1,\dots,x_r-\lambda_r)}_{\text{max ideal}} \leftrightarrow \underbrace{(\lambda_1,\dots\lambda_r)}_{\text{point on } \mathbb{A}^r}$$

Naturality: $E \subset G \quad \rightsquigarrow \quad \mathrm{H}^\bullet(G,k) \to \mathrm{H}^\bullet(E,k) \quad \rightsquigarrow \quad \mathrm{res}_{G,E} : |E| \to |G|.$

Theorem 1.5 (Quillen)**.** *(weak form)*

- $\mathrm{res}_{G,E} : |E| \to |G|$ *is a finite map*
- $|G| = \bigcup\limits_{E \subset G} \mathrm{res}_{G,E}\,|E|$

Theorem 1.6 (Quillen)**.** *(strong form)*

$$|G| = \operatorname*{colim}_{E \subset G} |E|$$

Corollary 1.7 (Atiyah-Swan conjecture)**.** $\mathrm{Krull}\dim \mathrm{H}^\bullet(\mathrm{G},\mathrm{k}) = \max\limits_{\mathrm{E}\subset\mathrm{G}} \mathrm{rk}\,\mathrm{E}$

Corollary 1.8. *Irreducible components of* $|G|$ $\leftrightarrow$ *conjugacy classes of maximal elementary abelian subgroups of* G*.*

Remark 1.9. This approach tells us nothing about cohomology in any particular degree. Later today you would learn about calculations in low degree cohomology.

1.4. **Examples.**

Example 1.10. Let $p=2$. $D_4 = \langle \sigma, \tau \mid \sigma^4 = \tau^2 = 1, \tau\sigma\tau = \sigma^{-1}\rangle$

Elementary abelian p-subgroups in D_4:

$$\langle \tau, \sigma^2\tau\rangle = (\mathbb{Z}/2)^2 \qquad\qquad \langle \sigma\tau, \sigma^3\tau\rangle = (\mathbb{Z}/2)^2$$

$$\nwarrow \qquad \nearrow$$

$$\langle \sigma^2 \rangle = \mathbb{Z}/2$$

Specm $\mathrm{H}^\bullet(D_8, k)$

$$\mathbb{A}^2 \longleftarrow \mathbb{A}^1 \longrightarrow \mathbb{A}^2$$

$|D_4| \simeq \mathbb{A}^2 \times_{\mathbb{A}^1} \mathbb{A}^2$

In this case we can compare the answer to the explicit calculation of $\mathrm{H}^*(D_4, k)$ which is known:

$$\mathrm{H}^*(D_4, k) = k[x_1, x_2, z]/(x_1x_2)$$

Example 1.11. $\mathrm{GL}_3(\mathbb{F}_p)$, $p > 3$. Exercise.

Some open questions.

1. Number of irreducible components in $|\,\mathrm{GL}_n(\mathbb{F}_p)|$
2. Dimension of the *minimal* irreducible component in $|\,\mathrm{GL}_n(\mathbb{F}_p)|$?

1.5. **Support varieties for modules.**

Definition 1.12. Let $I_M = \mathrm{Ker}\{\ \mathrm{H}^\bullet(G, k) = \mathrm{Ext}^\bullet_G(k, k) \xrightarrow{\otimes M} \mathrm{Ext}^*_G(M, M)\ \}$. The support variety of M, $|G|_M \subset |G|$, is the subvariety of Specm $\mathrm{H}^\bullet(G, k)$ defined by the ideal I_M. (Equiv., $|G|_M \simeq$ Specm $\mathrm{H}^\bullet(G, k)/I_M$).

Can also be define in terms of Yoneda product: $\mathrm{Ext}^*_G(k, k)$ acts on $\mathrm{Ext}^*_G(M, M)$ via Yoneda product:

$$\mathrm{Ext}^i(k, k) \times \mathrm{Ext}^j(M, M) \to \mathrm{Ext}^{i+j}(M, M)$$

$$\begin{array}{c}
[k \to \cdots \to k] \ \times \ [M \to \cdots \to M] \\
\Big\downarrow {\scriptstyle (\otimes M, \mathrm{id})} \\
[M \to \cdots \to M] \ \times \ [M \to \cdots \to M] \\
\Big\downarrow \\
M \to \cdots \to \boxed{M = M} \to \cdots \to M \\
\Big\downarrow \\
M \to \cdots \to M.
\end{array}$$

"Support variety" = "where representation theory meets cohomology".

Properties.

1. (Avrunin-Scott) Quillen stratification for $|G|_M$.
2. (Alperin-Evens) $\mathrm{cx}\, M = \dim |G|_M$
   ($\mathrm{cx}\, M = \min\{s \mid \dim P_n \le cn^{s-1}\}$ where $P_\bullet \to M$ runs over all proj. resolutions of M).
3. $|G|_{M\oplus N} = |G|_M \cup |G|_N$
4. $|G|_{M\otimes N} = |G|_M \cap |G|_N$
5. $|G| = |G|_k$

(6) $0 \to M_1 \to M_2 \to M_3 \to 0$. Then $|G|_{M_i} \subset |G|_{M_j} \cup |G|_{M_\ell}$

1.6. **Finite group schemes.** We now extend these cohomological constructions to a more general class of finite group schemes - with sometimes strikingly different results.

1.7. **Definitions.**

Definition 1.13. An affine (algebraic) scheme X is a representable functor

$$X : \text{fin. gen. comm. } k\text{-algebras} \longrightarrow \text{sets}$$

We denote by $k[X]$ the coordinate algebra of X (the commutative k-algebra (of finite type), representing X)

G is represented by $k[X] \sim X(R) = \mathrm{Hom}_{k-alg}(k[X], R)$.

Definition 1.14. An (affine algebraic) group scheme G is a representable functor

$$G : \text{fin. gen. comm. } k\text{-algebras} \longrightarrow \text{groups}$$

We denote by $k[G]$ the coordinate algebra of G (the commutative k-algebra (of finite type), representing G)

Remark 1.15. $k[G]$ is a commutative Hopf algebra.

Definition 1.16. G is a finite group scheme if $k[G]$ is a finite k-algebra (finite dimensional as a vector space over k).

Let G be a finite group scheme. Let

$$kG \overset{def}{=} k[G]^{\#},$$

a linear dual to $k[G]$. Then kG is a finite-dimensional CO-commutative Hopf k-algebra, called the **group algebra**. Also known as: algebra of distributions.

We have equivalences of categories:

$$\text{Finite group schemes over } k \overset{\sim}{\longleftrightarrow} \text{fin. dim. commutative Hopf } k-\text{algebras}$$

$$\text{Representations of } G \overset{\sim}{\longleftrightarrow} k[G]-\text{comod} \overset{\sim}{\longleftrightarrow} kG-\text{mod}$$