# Categorifying the tensor product of a level 1 highest weight and perfect crystal

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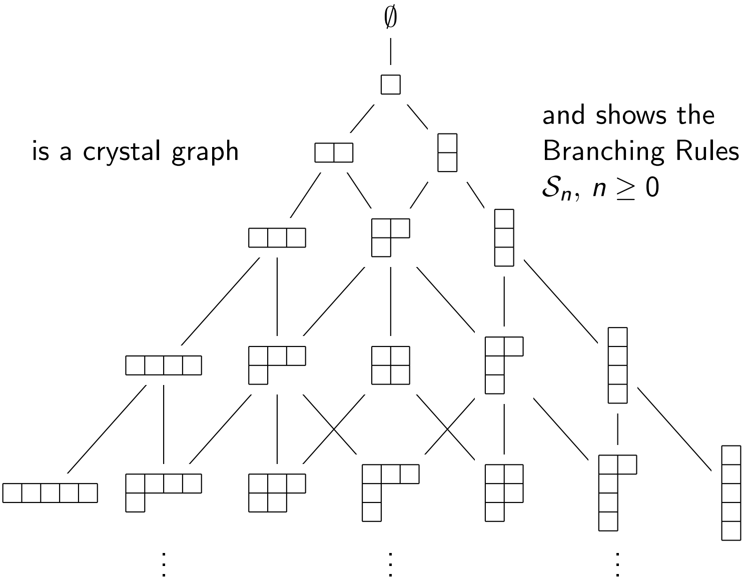
AMS: Categorical Methods in Representation Theory

Joint with Henry Kvinge, UCD

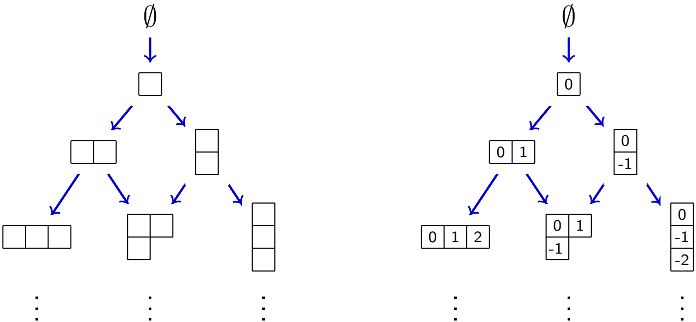
# Young's lattice of partitions

is a crystal graph

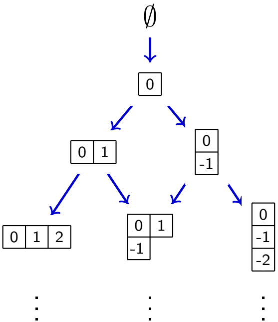


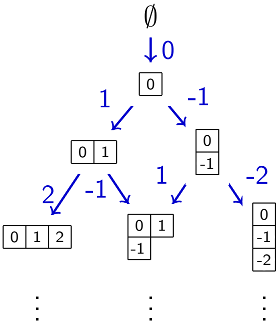
and shows the Branching Rules for $\mathcal{S}_n$, $n \geq 0$

# A crystal graph is directed

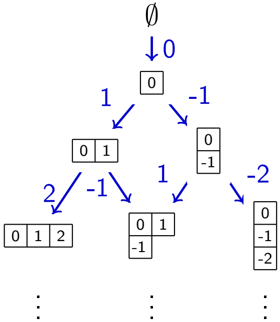


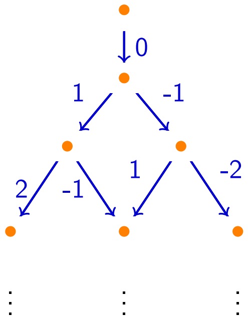
# A crystal graph is directed, $I$-colored $\quad (I = \mathbb{Z})$



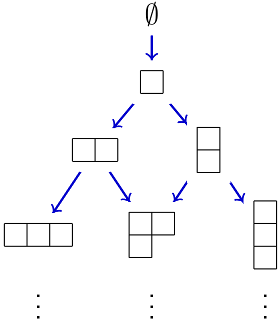



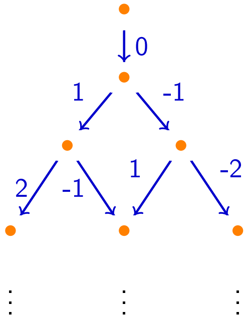
# A crystal graph is directed, $I$-colored $\quad (I = \mathbb{Z}, \mathfrak{g} = \mathfrak{sl}_\infty)$



# A crystal graph is directed, $I$-colored $\quad (I = \mathbb{Z})$




with extra data, satisfying various axioms

This is $B(\Lambda_0)$, the crystal of the basic representation of $\mathfrak{g} = \mathfrak{sl}_\infty$.

# Categorification crystal operator = soc $i$-Res

$\mathrm{Rep}(S_n)_{n \geq 0}$ categorifies $B(\Lambda_0)$

or $V(\Lambda_0)$, the basic representation of $\mathfrak{sl}_\infty$

simple $S^\lambda$ $\longleftrightarrow$ node $\lambda$

$\mathrm{Hom}_{S_{n-1}}(S^\mu, \mathrm{Res}^n_{n-1} S^\lambda) \neq 0$ $\longleftrightarrow$ edge $\mu \overset{i}{\longleftarrow} \lambda$

in $i$-block

corresponding to $Z(\mathbb{Q}S_{n-1})$-action more generally KLR algebras

Rep$(R(\nu))$ categorifies $B(\infty)$ for $U_q(\mathfrak{g})$

$\nu \in Q^+$

crystal of $U_q^-$

Rep$(R^\Lambda(\nu))$ $\quad B(\Lambda)$

$\nu \in Q^+$

crystal of $V(\Lambda)$

simple $\longleftrightarrow$ node

$\mathrm{Hom}_{R(\nu-\alpha_i)}(S, \mathrm{Res}^{\nu}_{\nu-\alpha_i} S) \neq 0$ edge

# type $A$: $I = \mathbb{Z}$ $\quad (\mathfrak{sl}_\infty)$ or $I = \mathbb{Z}/p\mathbb{Z}$ $\quad (\widehat{\mathfrak{sl}}(p))$

History

type A
affine Hecke algebra $H_n$ $\quad \leftrightarrow \quad B(\infty)$
(Rep$^t$ subcategory)
cyclotomic Hecke algebra $H_n^\Lambda$ $\quad \leftrightarrow \quad B(\Lambda)$

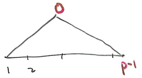
"level" 1:

$$\mathrm{Rep}\, H_n^{\Lambda_i} \longleftrightarrow \begin{cases} \mathrm{Rep}\ \mathbb{Q}\mathfrak{S}_n, & t \text{ generic} \\ \mathrm{Rep}\ \mathbb{F}_p\mathfrak{S}_n, & t^p = 1 \end{cases} \longleftrightarrow B(\Lambda_i)$$

$$\begin{cases} \mathcal{U}(\mathfrak{sl}(\infty)), & t \text{ generic} \\ \mathcal{U}(\widehat{\mathfrak{sl}(p)}), & t^p = 1 \end{cases}$$

$t^{\mathbb{Z}}$


$t^{\mathbb{Z}}$



$\otimes$ of crystals - Categorify

Highest weight crystals - Webster 1001.2020

Losev-Webster 1303.1336

There are other crystals, e.g. perfect crystals

$B$ = perfect level $\ell$, $\Lambda \in P^+$ level $\ell$

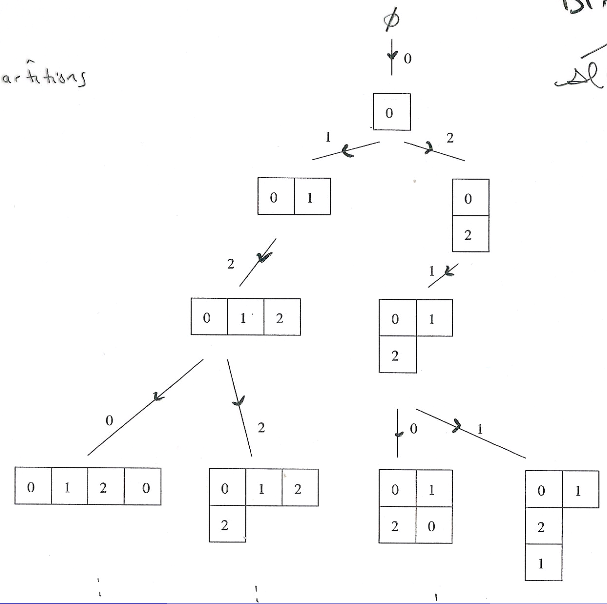
$B \otimes B(\Lambda) \simeq B(\sigma(\Lambda))$ some $\sigma(\Lambda) \in P^+$ level $\ell$

Categorify for $\ell = 1$, types ABCD (affine)

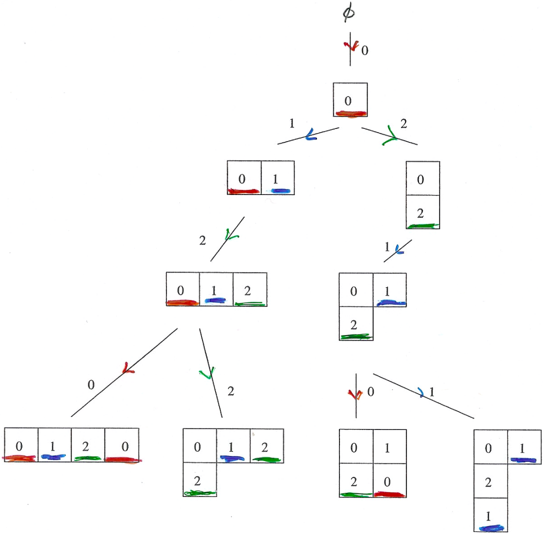
$B(\Lambda_0)$

$\widehat{\mathfrak{sl}}(3)$

3-regular partitions



# Branching = soc Res

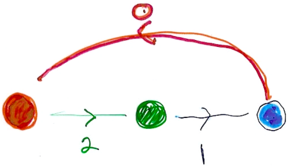



$B(\Lambda_0)$

Kleshchev

# level 1 perfect crystal $B$

$B \otimes B(\Lambda_i) \simeq B(\Lambda_{i-1})$

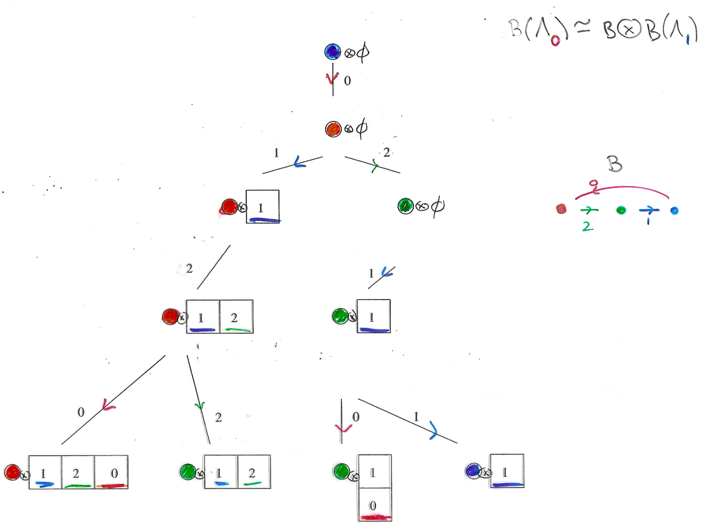



perfect $B$

level 1

$\widehat{\mathfrak{sl}}(3)$

# Combinatorial description of isomorphism



# Combinatorial description of isomorphism

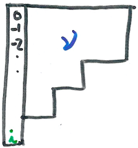
## $B(\Lambda_0) \simeq B \otimes B(\Lambda_1)$

Given $p$-regular partition $\nu$

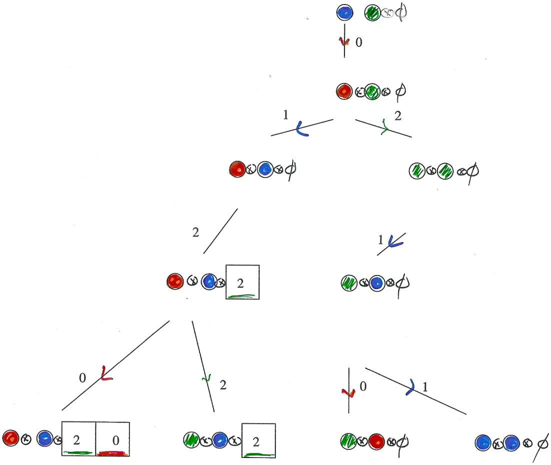
and $i \bmod p$

$\exists!$ $p$-regular partition $\lambda$

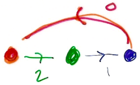
with



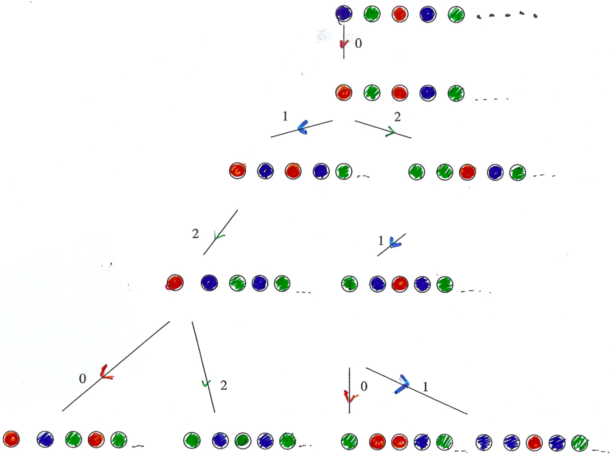
$B(\Lambda_0) \cong B \otimes B \otimes B(\Lambda_2)$



# Kyoto "Path" model; recover crystal rule




$B(\Lambda_0)$



# Categorify $B \otimes B(\Lambda_i) \simeq B(\Lambda_0)$

Ind  $\boxtimes D^{\nu} \xrightarrow{f} D^{\lambda}$


Thm
① [V] f exists in type $A^{(1)}_{p-1}$

[Kvinge-V] " " type $B^{(1)}, C^{(1)}, D^{(1)}, A^{(2)}, D^{(2)}$

② [V] f commutes with the crystal operators $\overset{\mu}{\underset{\lambda}{\phi}}{}_i$ in type $A$.

[Kvinge-V] " " type $B^{(1)}, C^{(1)}, D^{(1)}, A^{(2)}, D^{(2)}$

In type A $\begin{array}{|c|}\hline 0\\ -1\\ -2\\ \vdots\\ i\\ \hline\end{array}$ is the 1-dim sign module.

In other types, replace sgn with modules as in [V- 0511221].

Conjecture Uniqueness

$$pr^*_{\Lambda_0} \operatorname{cosoc} \operatorname{Ind} \begin{array}{|c|}\hline 0\\ \vdots\\ i\\ \hline\end{array} \boxtimes D^{\nu} = D^{\lambda}$$

where $pr_{\Lambda_0} : H_n \twoheadrightarrow H_n^{\Lambda_0}$ i.e. $H_n^{\Lambda_0} = H_n / I$

# Other directions

- level $\ell$ - conversations with Tingley suggest analogue of sgn modules is an obstacle beyond level 1.

- other perfect crystals - cuspidals ?