Detecting Projectivity in Sheaves Associated to Representations of Restricted Lie Algebras

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Outline

p-Lie Algebras

Let k be an algebraically closed field of positive characteristic p.

Definition

A *restricted Lie algebra* over k is a Lie algebra $\mathfrak{g}$ together with a map

$$(-)^{[p]}\colon \mathfrak{g} \to \mathfrak{g}$$

satisfying some conditions.

The conditions on $(-)^{[p]}$ are meant to force it to behave like a p^{th} power map $X \mapsto X^p$.

p-LIE ALGEBRAS

Example

All of the classical Lie algebras

$$\begin{aligned} \mathfrak{gl}_n &= \{n \times n \text{ matrices}\} \\ \mathfrak{sl}_n &= \{X \in \mathfrak{gl}_n \mid \text{trace}\, X = 0\} \\ \mathfrak{sp}_n &= \{X \in \mathfrak{gl}_n \mid X \text{ is Hamiltonian}\} \\ \mathfrak{o}_n &= \{X \in \mathfrak{gl}_n \mid X \text{ is skew-symmetric}\} \\ \mathfrak{so}_n &= \mathfrak{o}_n \cap \mathfrak{sl}_n \end{aligned}$$

are restricted Lie algebras via

$$[XY] = XY - YX \qquad \text{and} \qquad X^{[p]} = X^p.$$

p-LIE ALGEBRAS

Definition

Let $\mathfrak{g}$ be a restricted Lie algebra and M a $\mathfrak{g}$-module. Then M is called *restricted* if

$$X^{[p]} \cdot m = \overbrace{X \cdots X}^{p \text{ times}} \cdot m \quad \forall X \in \mathfrak{g}, m \in M$$

THE SUPPORT VARIETY

The restricted nullcone

$$\mathcal{N}_p = \left\{ x \in \mathfrak{g} \mid x^{[p]} = 0 \right\}$$

is conical and Zariski closed.

Definition

The *support variety* of $\mathfrak{g}$ is the projectivization $\mathbb{P}(\mathcal{N}_p)$ of the restricted nullcone.

p-PARTITIONS

Definition

A *p-restricted partition* is a finite sequence of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ such that:

$$p \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m.$$

Notation

Let $\mathscr{P}_p$ be the set of all p-restricted partitions.

p-Partitions

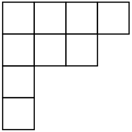
Two convenient ways of writing the partition $(4, 3, 1, 1)$

- Exponential notation.

$$(4, 3, 1, 1) \quad \longrightarrow \quad [4][3][1]^2$$

- Young diagram.

$$(4, 3, 1, 1) \quad \longrightarrow$$



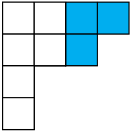
p-PARTITIONS

Definition

The *j-rank* of a partition is the number of boxes in columns $j+1, j+2, \cdots$ of the Young diagram.

Example

The partition



has 2-rank 3.

Local Jordan Type

Fix a finitely generated $\mathfrak{g}$-module M and take $v \in \mathbb{P}(\mathcal{N}_p)$.

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$$v \xrightarrow[X \text{ action}]{\text{matrix of}} A$$

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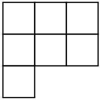
This is the *Jordan type* of M at v:

$$\mathrm{JType}(M, v) =$$



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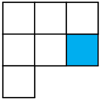

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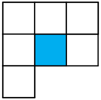

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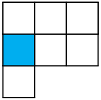

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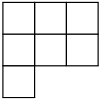

LOCAL JORDAN TYPE

Definition

The *local Jordan type* of M is the function

$$\mathrm{JType}(M, -) \colon \mathbb{P}(\mathcal{N}_p) \to \mathscr{P}_p.$$

Local Jordan Type

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The *local Jordan type* of M is the function

$$\mathrm{JType}(M,-)\colon \mathbb{P}(\mathcal{N}_p) \to \mathscr{P}_p.$$

Theorem (Suslin, Friedlander, Bendel)

A restricted $\mathfrak{g}$-module M is projective if and only if its local Jordan type is the constant function

$$v \mapsto [p]^{\frac{\dim M}{p}}.$$

Local Jordan Type

Definition

- If $\mathrm{JType}(M, v)$ does not depend on v then we say M has *constant Jordan type*.
- If the j-rank of $\mathrm{JType}(M, v)$ does not depend on v then we say M has *constant j-rank*.

Constant Jordan type = constant j-rank for all j.

THE GLOBAL OPERATOR

Let $\{x_1, \ldots, x_n\}$ be a basis of $\mathfrak{g}$ with dual basis $\{\widehat{x_1}, \ldots, \widehat{x_n}\}$.

Definition

The *global operator* is the sheaf map

$$\Theta \colon M \otimes_k \mathcal{O}_{\mathbb{P}(\mathcal{N}_p)} \to M \otimes_k \mathcal{O}_{\mathbb{P}(\mathcal{N}_p)}(1)$$

defined by the action of

$$\sum_{i=1}^{n} x_i \otimes \widehat{x_i}.$$

The Global Operator

Theorem (Friedlander, Pevtsova)

Given any point $v \in \mathbb{P}(\mathcal{N}_p)$, the Jordan normal form of the specialization

$$\Theta \otimes k(v) \colon M \to M$$

gives the partition $\mathrm{JType}(M, v)$*.*

ASSOCIATED SHEAVES

Note that Θ can be composed with itself by shifting further copies.

$$M \otimes_k \mathcal{O}_{\mathbb{P}(\mathcal{N}_p)} \xrightarrow{\Theta} M \otimes_k \mathcal{O}_{\mathbb{P}(\mathcal{N}_p)}(1) \xrightarrow{\Theta(1)} M \otimes_k \mathcal{O}_{\mathbb{P}(\mathcal{N}_p)}(2)$$

We choose the shifts such that constructions take place in degree 0. For example:

$$\ker\left\{\Theta^j, M\right\} = \ker\left[\Theta^j \colon M \otimes_k \mathcal{O}_{\mathbb{P}(\mathcal{N}_p)} \to M \otimes_k \mathcal{O}_{\mathbb{P}(\mathcal{N}_p)}(j)\right]$$

$$\operatorname{im}\left\{\Theta^j, M\right\} = \operatorname{im}\left[\Theta^j \colon M \otimes_k \mathcal{O}_{\mathbb{P}(\mathcal{N}_p)}(-j) \to M \otimes_k \mathcal{O}_{\mathbb{P}(\mathcal{N}_p)}\right]$$

$$\operatorname{coker}\left\{\Theta^j, M\right\} = \operatorname{coker}\left[\Theta^j \colon M \otimes_k \mathcal{O}_{\mathbb{P}(\mathcal{N}_p)}(-j) \to M \otimes_k \mathcal{O}_{\mathbb{P}(\mathcal{N}_p)}\right]$$

ASSOCIATED SHEAVES

These sheaves tell us about the local Jordan type of M.

Theorem (Friedlander, Pevtsova, S.)

Let $U \subseteq \mathbb{P}(\mathcal{N}_p)$ be open and connected. Given the statements:

(1) *M has constant j-rank on U.*

(2) *The restriction of* $\operatorname{coker}\{\Theta^j, M\}$ *to U is a locally free sheaf.*

(3) *The restriction of* $\operatorname{im}\{\Theta^j, M\}$ *to U is a locally free sheaf.*

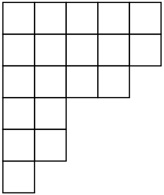
(4) *The restriction of* $\ker\{\Theta^j, M\}$ *to U is a locally free sheaf.*

we have

$$(1) \Leftrightarrow (2) \Rightarrow (3) \Rightarrow (4).$$

ASSOCIATED SHEAVES

Let $p = 5$ and $\mathrm{JType}(A) = [5]^2[4][2]^2[1]$.



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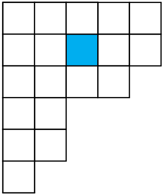

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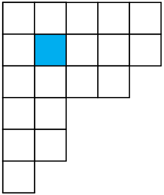

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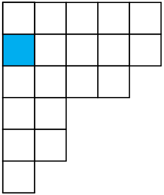

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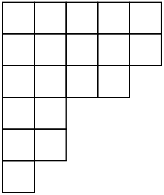

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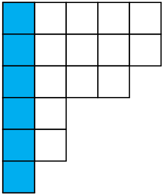

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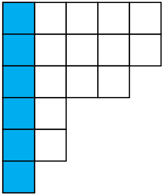
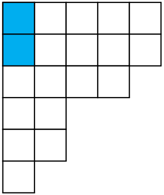


$\ker A$



$\operatorname{im} A^4$

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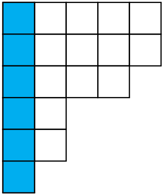

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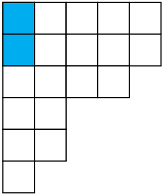

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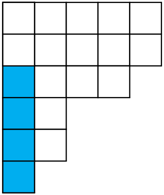

$\dfrac{\ker A}{\operatorname{im} A^4}$

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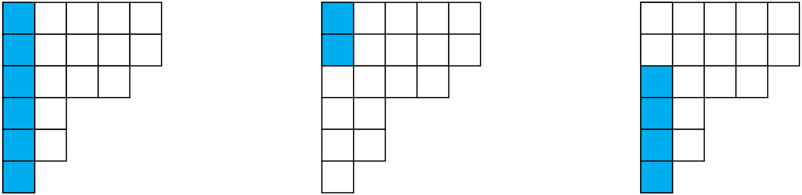

$\ker A$ $\qquad$ $\operatorname{im} A^4$ $\qquad$ $\frac{\ker A}{\operatorname{im} A^4}$

So $\dim\left(\frac{\ker A}{\operatorname{im} A^{p-1}}\right)$ gives the number of blocks of size $< p$.

Associated Sheaves

Definition

For $1 \leq i < p$ let

$$\mathcal{H}^{[1]}(M) = \frac{\ker\{\Theta, M\}}{\mathrm{im}\{\Theta^{p-1}, M\}}.$$

ASSOCIATED SHEAVES

Definition

For $1 \leq i < p$ let

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Question

Is it true that M is projective if and only if $\mathcal{H}^{[1]}(M) = 0$?

Projectivity and $\mathcal{H}^{[1]}$

Theorem (Friedlander, Pevtsova, 2011)

A G-module M is projective if and only if $\mathcal{H}^{[1]}(M) = 0$ *and M has constant* 1 *and* $(p-1)$*-rank.*

PROJECTIVITY AND $\mathcal{H}^{[1]}$

Theorem (S.)

If $\mathcal{H}^{[1]}(M) = 0$ *then the following conditions are equivalent:*

- *M is projective,*
- *M has constant 1-rank,*
- *M has constant $(p-1)$-rank,*
- $\operatorname{coker}\{\Theta, M\}$ *is locally free,*
- $\operatorname{coker}\{\Theta^{p-1}, M\}$ *is locally free,*
- $\operatorname{im}\{\Theta, M\}$ *is locally free,*
- $\operatorname{im}\{\Theta^{p-1}, M\}$ *is locally free,*
- $\ker\{\Theta, M\}$ *is locally free,*
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PROJECTIVITY AND $\mathcal{H}^{[1]}$

Corollary

If $\mathbb{P}(\mathcal{N}_p)$ is a nonsingular curve then M is projective if and only if $\mathcal{H}^{[1]}(M) = 0$.

PROJECTIVITY AND $\mathcal{H}^{[1]}$

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Example

If $p = 3$ then the standard representation $M = k^3$ of $\mathfrak{sl}_3$ does not have constant Jordan type but $\mathcal{H}^{[1]}(M) = 0$.

Projectivity and $\mathcal{H}^{[1]}$

So the question becomes, for which $\mathfrak{g}$ does the correspondence

$$\mathcal{H}^{[1]}(M) = 0 \qquad \Leftrightarrow \qquad M \text{ projective}$$

hold?

PROJECTIVITY AND $\mathcal{H}^{[1]}$

Theorem (S.)

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PROJECTIVITY AND $\mathcal{H}^{[1]}$

Theorem (S.)

If $\mathbb{P}(\mathcal{N}_p)$ is nonsingular then a $\mathfrak{g}$-module M is projective if and only if $\mathcal{H}^{[1]}(M) = 0$.

The only known examples of nonprojective M such that $\mathcal{H}^{[1]}(M) = 0$ occur for $\mathfrak{sl}_n$ when $p \mid n$. So the nonsingular case remains open.

The End

Thanks for listening :)