Exponential Maps In Characteristic p (featuring: One-Parameter Subgroups of Reductive Groups)

Paul Sobaje

University of Southern California

October 25, 2014

Preliminaries:

- k - algebraically closed field.
- G - affine algebraic group over k.
- $\mathbb{G}_a$ - k as an algebraic group under addition.
- one-parameter subgroup of G is a homomorphism from $\mathbb{G}_a$ to G.
- $\mathfrak{g}$ - Lie algebra of G.
- $\mathcal{N}$ - nilpotent variety of $\mathfrak{g}$.
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In characteristic $p > 0$

There is a p-mapping on $\mathfrak{g}$, $X \mapsto X^{[p]}$. We set $\mathcal{N}_p \subseteq \mathcal{N}$ to be $\{X : X^{[p]} = 0\}$.

Similarly, let $\mathcal{U}_p \subseteq \mathcal{U}$ be $\{u : u^p = 1\}$.

More On Nilpotent and Unipotent Elements

In any characteristic, fix a closed embedding $\rho : G \to GL_n$.

$X \in \mathfrak{g}$ is **nilpotent** if $d\rho(X)$ is a nilpotent matrix.

$u \in G$ is **unipotent** if $\rho(u) - I_n$ is a nilpotent matrix.

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In char. $p > 0$, $d\rho(X^{[p]}) = d\rho(X)^p$.

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The Exponential Map is Better

$X \mapsto \exp(X) = I_n + X + X^2/2 + \cdots X^{n-1}/(n-1)!$ better respects group structure of GL_n:

- For all $c \in \mathbb{G}_a$, the map $c \mapsto \exp(cX)$ defines one-parameter subgroup of GL_n.
- If G closed subgroup, $X \in \mathfrak{g} \subseteq \mathfrak{gl}_n$, then $\exp(X) \in G$.
- If $X, Y \in \mathcal{N}$ in same Borel subalgebra, then $\log(\exp(X)\exp(Y)) =$

$$X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] + [Y, [Y, X]]) + \cdots$$

(Baker-Campbell-Hausdorff formula)

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This formulation doesn't work in positive characteristic.

Let characteristic $k = p > 0$.

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Moral: for some applications, any two Springer isomorphisms are equally useful. For others, we'd like one which is "more similar" to the exponential map (i.e. respecting group properties).

More precisely, if σ is to fill the role of the exponential map in characteristic p, it should have the following properties:

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Property 1: A Good Restriction to Certain Parabolic Subgroups

Serre proved that if $P \leq G$ parabolic with $U = R_u(P)$ having nilpotence class less than p, then $\exists$ a P-equivariant isomorphism

$$\varepsilon_P : \mathrm{Lie}(U) \to U$$

which essentially comes from base-changing exponential map in characteristic 0. We require that σ restricts on U to ε_P for all such P.

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Carlson-Lin-Nakano (2008), McNinch (2005)

If $p \geq h$, the Coxeter number of G, then there is precisely one Springer isomorphism σ for G satisfying Property 1.

Property 2: Obtaining Embeddings of Witt Groups:

In characteristic p, every $e \neq g \in \mathbb{G}_a$ has order p. However, when $p < h$ there are unipotent elements in G of order $p^r, r > 1$ (for example, if $p = 2$ then SL_3 has elements of order 4), so we can't expect every unipotent element to lie inside closed group isomorphic to $\mathbb{G}_a$.

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We require: If $X \neq 0$, and m is the least integer such that $X^{[p^m]} = 0$, then σ defines an embedding $\mathbb{A}^m \to G$ given by

$$(a_0, a_1, \ldots, a_{m-1}) \mapsto \sigma(a_0 X)\sigma(a_1 X^{[p]}) \cdots \sigma(a_{m-1} X^{[p^{m-1}]}),$$

the image of which is a closed subgroup of G isomorphic to $\mathcal{W}_m$.

Theorem (S., 2014)

Let G be a semisimple simply-connected group, and suppose that p is good for G. Then $\exists$ a Springer isomorphism $\sigma : \mathcal{N} \xrightarrow{\sim} \mathcal{U}$ satisfying Properties 1 and 2.

These properties do not uniquely specify an isomorphism, but every Springer isomorphism satisfying Property 1 restricts to the same isomorphism $\overline{\exp} : \mathcal{N}_p \xrightarrow{\sim} \mathcal{U}_p$.

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Ingredient and Application: Abelian Unipotent Overgroups

Let $u \in \mathcal{U}$. **Question:** what is minimal connected subgroup containing it? Studied extensively by Testerman, Seitz, McNinch, and Proud, an application given by Serre.

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Our proof relies in particular on result of Seitz: take X a regular nilpotent element, T the image of an associated cocharacter of X, and consider T-decomposition of $C_G(X)^0$.

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Artin-Hasse Exponential

The Artin-Hasse exponential is the power series

$$E_p(t) = \exp\left(t + \frac{t^p}{p} + \frac{t^{p^2}}{p^2} + \frac{t^{p^3}}{p^3} + \cdots\right)$$

One can show that $E_p(t) \in \mathbb{Z}_{(p)}[\![t]\!] \subseteq \mathbb{Q}[\![t]\!]$.

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One can show that $E_p(t) \in \mathbb{Z}_{(p)}[\![t]\!] \subseteq \mathbb{Q}[\![t]\!]$.

If G is a classical matrix group (GL_n, SO_n, Sp_n), then one choice of σ is given by

$$\sigma(X) = E_p(X)$$

This **does not work** for arbitrary embeddings of G semisimple into GL_n.

Applications - the map

$$\overline{\exp} : \mathcal{N}_p \xrightarrow{\sim} \mathcal{U}_p$$

has been useful in support variety theory, and problems related to support varieties. One application will be seen tomorrow in **Jared Warner's** talk.

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Comparing Support Varieties over $G(\mathbb{F}_p)$ and $\mathfrak{g}$

$$\begin{array}{ccc} & G\text{-mod} & \\ \swarrow & & \searrow \\ \mathfrak{g}\text{-mod} & & G(\mathbb{F}_p)\text{-mod} \end{array}$$

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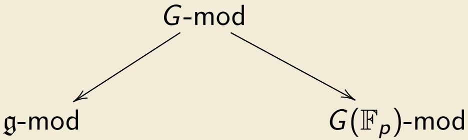

Carlson-Lin-Nakano used the existence of $\overline{\exp}$ ($p \geq h$) to compare the support varieties of a rational G-module M over $G(\mathbb{F}_p)$ and $\mathfrak{g}$.

Suslin-Friedlander-Bendel (1997)

Let $\mathcal{G}$ be an infinitesimal group scheme over k of height r, $\mathrm{H}^\bullet(\mathcal{G}, k)$ its cohomology ring. Then the variety corresponding to $\mathrm{H}^\bullet(\mathcal{G}, k)$ is homeomorphic to the variety of group scheme homomorphisms from $\mathrm{Hom}_{gs/k}(\mathbb{G}_{a(r)}, \mathcal{G})$.

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Suslin-Friedlander-Bendel (1997), McNinch (2001), S. (2014)

If G is semisimple, simply-connected, and p good for G, then $\mathrm{Hom}_{gs/k}(\mathbb{G}_{a(r)}, G_{(r)})$ identifies canonically with commuting r-tuples of elements in $\mathcal{N}_p$.

Support varieties for rational G-modules

In recent work, Eric Friedlander has studied support varieties for rational G-modules, where G is a linear algebraic group, via the space

$$\mathrm{Hom}_{\mathrm{gs}/k}(\mathbb{G}_a, G).$$

The group G must be assumed to have a structure of **exponential type**. For G semisimple, simply-connected, and $p \geq h$ (probably p good), such a structure can be given by $\overline{\exp}$.

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Exponentiating Representations

If G semisimple, when does a representation for $\mathfrak{g}$ extend to one for G?