Exponential Maps in Characteristic \( p \) (featuring: One-Parameter Subgroups of Reductive Groups)

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### Preliminaries:

- **$k$** - algebraically closed field.
- **$G$** - affine algebraic group over $k$.
- **$\mathbb{G}_a$** - $k$ as an algebraic group under addition.
- One-parameter subgroup of $G$ is a homomorphism from $\mathbb{G}_a$ to $G$.
- **$\mathfrak{g}$** - Lie algebra of $G$.
- **$\mathcal{N}$** - nilpotent variety of $\mathfrak{g}$.
- **$\mathcal{U}$** - unipotent variety of $G$.
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### In characteristic $p > 0$

There is a $p$-mapping on $\mathfrak{g}$, $X \mapsto X^{[p]}$. We set $\mathcal{N}_p \subseteq \mathcal{N}$ to be $\{X : X^{[p]} = 0\}$.

Similarly, let $\mathcal{U}_p \subseteq \mathcal{U}$ be $\{u : u^p = 1\}$. 
More On Nilpotent and Unipotent Elements

In any characteristic, fix a closed embedding $\rho : G \rightarrow GL_n$.

$X \in g$ is **nilpotent** if $d\rho(X)$ is a nilpotent matrix.

$u \in G$ is **unipotent** if $\rho(u) - I_n$ is a nilpotent matrix.
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In char. $p > 0$, $d\rho(X^p) = d\rho(X)^p$. 
For $GL_n$, clear that $\exists$ a $GL_n$-equivariant isomorphism $N \xrightarrow{\sim} U$ given by
$$X \mapsto I_n + X,$$
however, in characteristic 0...

The Exponential Map is Better
$$X \mapsto \exp(X) = I_n + X + \frac{X^2}{2} + \cdots,$$ better respects group structure of $GL_n$:

- For all $c \in G$ a, the map $c \mapsto \exp(cX)$ defines one-parameter subgroup of $GL_n$.
- If $G$ a closed subgroup, $X \in g \subseteq \mathfrak{gl}_n$, then $\exp(X) \in G$.
- If $X, Y \in N$ in same Borel subalgebra, then $\log(\exp(X)\exp(Y)) = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \cdots$ (Baker-Campbell-Hausdorff formula)

This formulation doesn’t work in positive characteristic.
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Let characteristic $k = p > 0$.

**Springer (1969)**

If $G$ is semisimple, simply-connected, and char. is good for $G$, then there exists a $G$-equivariant isomorphism $\mathcal{N} \sim \rightarrow \mathcal{U}$. Such a map is called a Springer isomorphism. One application is that there is a bijection between nilpotent and unipotent $G$-orbits. In fact:

Serre (1999)

Every Springer isomorphism for $G$ determines the same bijection between nilpotent and unipotent orbits.

Moral: for some applications, any two Springer isomorphisms are equally useful. For others, we'd like one which is “more similar” to the exponential map (i.e. respecting group properties).
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**Moral:** for some applications, any two Springer isomorphisms are equally useful. For others, we’d like one which is “more similar” to the exponential map (i.e. respecting group properties).
More precisely, if $\sigma$ is to fill the role of the exponential map in characteristic $p$, it should have the following properties:
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### Property 1: A Good Restriction to Certain Parabolic Subgroups

Serre proved that if $P \leq G$ parabolic with $U = R_u(P)$ having nilpotence class less than $p$, then $\exists$ a $P$-equivariant isomorphism

$$\varepsilon_P : \text{Lie}(U) \to U$$

which essentially comes from base-changing exponential map in characteristic 0. We require that $\sigma$ restricts on $U$ to $\varepsilon_P$ for all such $P$. 
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**Carlson-Lin-Nakano (2008), McNinch (2005)**

If $p \geq h$, the Coxeter number of $G$, then there is precisely one Springer isomorphism $\sigma$ for $G$ satisfying Property 1.
Property 2: Obtaining Embeddings of Witt Groups:

In characteristic $p$, every $e \neq g \in \mathbb{G}_a$ has order $p$. However, when $p < h$ there are unipotent elements in $G$ of order $p^r$, $r > 1$ (for example, if $p = 2$ then $SL_3$ has elements of order 4), so we can’t expect every unipotent element to lie inside closed group isomorphic to $\mathbb{G}_a$. 

Let $W_m$ be the group of truncated Witt vectors. As a variety, $W_m \cong A_m$. It is an abelian unipotent group, and has elements of maximal order $p^m$. We require:

If $X \neq 0$, and $m$ is the least integer such that $X[p^m] = 0$, then $\sigma$ defines an embedding $A_m \to G$ given by $(a_0, a_1, \ldots, a_{m-1}) \mapsto \sigma(a_0 X) \sigma(a_1 X[p]) \cdots \sigma(a_{m-1} X[p^{m-1}])$, the image of which is a closed subgroup of $G$ isomorphic to $W_m$. 

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One-Parameter Subgroups of Reductive Groups
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the image of which is a closed subgroup of $G$ isomorphic to $\mathcal{W}_m$. 
Let $G$ be a semisimple simply-connected group, and suppose that $p$ is good for $G$. Then $\exists$ a Springer isomorphism $\sigma : \mathcal{N} \sim \rightarrow \mathcal{U}$ satisfying Properties 1 and 2.

These properties do not uniquely specify an isomorphism, but every Springer isomorphism satisfying Property 1 restricts to the same isomorphism $\exp : \mathcal{N}_p \sim \rightarrow \mathcal{U}_p$. 

**Ingredient and Application: Abelian Unipotent Overgroups**

Let $u \in \mathcal{U}$. Question: what is minimal connected subgroup containing it?

Studied extensively by Testerman, Seitz, McNinch, and Proud, an application given by Serre.

Our proof relies in particular on result of Seitz: take $X$ a regular nilpotent element, $T$ the image of an associated cocharacter of $X$, and consider $T$-decomposition of $C_G(X)_0$. 

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One-Parameter Subgroups of Reductive Groups
Main Result

Theorem (S., 2014)

Let $G$ be a semisimple simply-connected group, and suppose that $p$ is good for $G$. Then $\exists$ a Springer isomorphism $\sigma : \mathcal{N} \sim \mathcal{U}$ satisfying Properties 1 and 2.

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Artin-Hasse Exponential

The Artin-Hasse exponential is the power series

$$E_p(t) = \exp\left(t + \frac{t^p}{p} + \frac{t^{p^2}}{p^2} + \frac{t^{p^3}}{p^3} + \cdots\right)$$

One can show that $E_p(t) \in \mathbb{Z}_p[t] \subseteq \mathbb{Q}[t]$. 
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One can show that $E_p(t) \in \mathbb{Z}_p[t] \subseteq \mathbb{Q}[t]$.

If $G$ is a classical matrix group ($GL_n, SO_n, Sp_n$), then one choice of $\sigma$ is given by

$$\sigma(X) = E_p(X)$$

This does not work for arbitrary embeddings of $G$ semisimple into $GL_n$. 
Applications - the map

$$\exp : \mathcal{N}_p \sim \rightarrow \mathcal{U}_p$$

has been useful in support variety theory, and problems related to support varieties. One application will be seen tomorrow in Jared Warner’s talk.
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Comparing Support Varieties over \( G(\mathbb{F}_p) \) and \( g \)

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Comparing Support Varieties over $G(\mathbb{F}_p)$ and $\mathfrak{g}$

Carlson-Lin-Nakano used the existence of $\exp$ ($p \geq h$) to compare the support varieties of a rational $G$-module $M$ over $G(\mathbb{F}_p)$ and $\mathfrak{g}$. 
Suslin-Friedlander-Bendel (1997)

Let $\mathcal{G}$ be an infinitesimal group scheme over $k$ of height $r$, $\mathbb{H}^\bullet(\mathcal{G}, k)$ its cohomology ring. Then the variety corresponding to $\mathbb{H}^\bullet(\mathcal{G}, k)$ is homeomorphic to the variety of group scheme homomorphisms from $\text{Hom}_{gs/k}(\mathbb{G}_a(r), \mathcal{G})$. 
Let $G$ be an infinitesimal group scheme over $k$ of height $r$, $H^\bullet(G, k)$ its cohomology ring. Then the variety corresponding to $H^\bullet(G, k)$ is homeomorphic to the variety of group scheme homomorphisms from $\text{Hom}_{\text{gs}/k}(\mathbb{G}_a(r), G)$.


If $G$ is semisimple, simply-connected, and $p$ good for $G$, then $\text{Hom}_{\text{gs}/k}(\mathbb{G}_a(r), G(r))$ identifies canonically with commuting $r$-tuples of elements in $\mathcal{N}_p$. 
Support varieties for rational $G$-modules

In recent work, Eric Friedlander has studied support varieties for rational $G$-modules, where $G$ is a linear algebraic group, via the space

$$\text{Hom}_{\text{gs}/k}(\mathbb{G}_a, G).$$

The group $G$ must be assumed to have a structure of exponential type. For $G$ semisimple, simply-connected, and $p \geq h$ (probably $p$ good), such a structure can be given by $\exp$. 
An interesting and (seemingly) related question:
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**Exponentiating Representations**

If $G$ semisimple, when does a representation for $g$ extend to one for $G$?