Deligne's tensor category $\overline{\text{Rep }GL(t)}$ and general linear supergroups

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Interplay between supersymmetry and tensor categories

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- Almost all examples of rigid symmetric tensor categories come from representation theory of superalgebras.
- Give rise to universal tensor categories.

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Tensor categories — \$\$\$... → supergroups.

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- Calculations of characters and dimensions of some natural representations. Comes–Wilson, Brundan–Stroppel.

- A is abelian and k-linear (morphisms are k-vector spaces);
- \mathcal{A} is equipped with tensor product \otimes , i.e. an exact functor $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, k-linear in both variables, with (functorial) isomorphism $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$;
- A has a unit object 1 such that End(1) = k;
- Symmetry: functorial isomorphism $\mathbf{s}: X \otimes Y \to Y \otimes X$ such that the composition

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- Contravariant duality functor $*: A \to A, X \mapsto X^*$;
- Natural maps: identity e : 1 → X ⊗ X*;
- Contraction (trace) $\mathbf{c}: X^* \otimes X \to \mathbf{1}$;
- compositions

$$X \xrightarrow{\mathsf{e} \otimes 1} X \otimes X^* \otimes X \xrightarrow{1 \otimes \mathsf{c}} X$$

and

$$X^* \xrightarrow{1 \otimes \mathbf{e}} X^* \otimes X \otimes X^* \xrightarrow{\mathbf{c} \otimes 1} X^*$$

are both equal to the identity 1_X .

A tensor category ${\cal A}$ with duality is called rigid.

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$$(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$$
, $(x \otimes y) \otimes z \mapsto x \otimes (y \otimes z)$, $1 = k$;

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$$s: X \otimes Y \to Y \otimes X$$
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- Objects are \mathbb{Z}_2 -graded vector spaces $X = X_0 \oplus X_1$.
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$$\delta : \mathsf{End}(X) \xrightarrow{\sim} \mathsf{Hom}(\mathbf{1}, X \otimes X^*)$$

and define the trace:

$$\operatorname{tr}:\operatorname{End}(X)\to\operatorname{End}(\mathbf{1})=k$$

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- G algebraic group (over k), for example, GL(n). The category Rep G of finite-dimensional representations of G is a symmetric rigid tensor category.
- A functor F: Rep G → Vect (forgetting the G-action). Tensor functor, i.e., preserves all structures of tensor categories, faithful (injective on morphisms), exact.
- An exact faithful tensor functor $F: A \rightarrow \text{Vect}$ is called a *fiber* functor.
- A finitely generated symmetric rigid tensor category which has a fiber functor is equivalent to Rep G for some algebraic group G. (Tannakian categories → affine group schemes).

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Supertannakian formalism

Example. GL(n) may be a group G and may be supergroup G_{super} .

Rep G_{super} has twice as many objects as Rep G_{super}

$$V \longleftrightarrow \Pi V$$
, $V_0 \bigcirc V_1$

Deligne's trick (Halving the category):

- G is a supergroup, fix $g \in G_0$.
- Rep(G, g) is the subcategory of Rep G, consisting of representations V satisfying $g(v) = (-1)^{\bar{v}}v$.
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A tensor category A is equivalent to some Rep(G,g) if and only if

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Schur functor

- Consider a tensor category A.
- For any object X in \mathcal{A} consider $X^{\otimes n}$.
- $\sigma_{i,i+1} \mapsto 1^{\otimes i-1} \otimes \mathbf{s} \otimes 1^{\otimes n-i-1} \longrightarrow S_n \to \operatorname{Aut}(X^{\otimes n}).$

Schur-Weyl duality:

$$X^{\otimes n} = \bigoplus_{|\lambda|=n} V_{\lambda} \otimes S_{\lambda}(X).$$

Definition

 $X \mapsto S_{\lambda}(X)$ is called the Schur functor $S_{\lambda} : A \to A$.

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Theorem (A. Sergeev, 1982)

Let V be an (m|n)-dimensional superspace. Then $S_{\lambda}(V) \neq 0$ if and only if λ can be covered by an (m,n)-hook, or equivalently, λ does not contain a rectangular diagram of size $(m+1) \times (n+1)$.

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- (a) Every generator is annihilated by some Schur functor
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- Want to construct a universal one;
- Analogy: free ring with two generators X and X*;
- Deligne's category Rep GL(t), $t = \dim X$.
- Abelianization: a way to extend a category to abelian by forcefully adding kernels and images of all morphisms. There is no such bliss!
- The subtle point. Karoubization: adding kernels and images of all projectors ($p^2 = p$). It exists!
- Rep GL(t) is a result of Karoubization. Not abelian.

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Ingredients of construction

- Skeleton category.
 - objects: $X^{\otimes m} \otimes (X^*)^{\otimes n}$,
 - morphisms: $\operatorname{Hom}(X^{\otimes m} \otimes (X^*)^{\otimes n}, X^{\otimes k} \otimes (X^*)^{\otimes l}) = \operatorname{walled}$ Brauer diagrams.
- Additive closure (adding direct sums).
- Karoubian envelope (adding kernels of all projectors).

- Rep GL(t) is semisimple and therefore abelian if and only if $t \notin \mathbb{Z}$.
- Indecomposable objects in Rep GL(t) are enumerated by pairs of partitions: $(\lambda, \mu) \leftrightarrow Y_{\lambda, \mu}$.
- $Y_{\lambda,\mu}^* \simeq Y_{\mu,\lambda}$.
- dim $Y_{\lambda,\mu}$ is a polynomial in t.

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Semisimple quotient

If $t = n \in \mathbb{Z}$, the Deligne categories have canonical semisimple quotients.

Semisimple quotients of Rep GL(t):

- Rep(GL(n), 1) for n > 0;
- Rep(GL(-n), -1) for n < 0.

Universality

Theorem (P. Deligne, 2002)

Let \mathcal{A} be a symmetric rigid k-linear category, and V be an object of dimension \mathbf{t} . There exists a unique (up to isomorphism) tensor functor $F: \operatorname{Rep} \operatorname{GL}(t) \to \mathcal{A}$ such that F(X) = V.

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Universality

Theorem (P. Deligne, 2002)

Let $\mathcal A$ be a symmetric rigid k-linear category, and V be an object of dimension $\mathbf t$. There exists a unique (up to isomorphism) tensor functor $F: \operatorname{Rep} GL(t) \to \mathcal A$ such that F(X) = V.

Towards abelianization: building blocks

$$t=m-n=m'-n'$$

$$\underbrace{\operatorname{Rep} GL(t) \xrightarrow{F_{m,n}} \operatorname{Rep} GL(m,n)}_{F_{m',n'}} \operatorname{Rep} GL(m',n')$$

Tensor functor $E_{m,n}$: Rep $GL(m,n) \to \text{Rep } GL(m-1,n-1)$.

$$\frac{\operatorname{Rep} GL(t) \xrightarrow{F_{m,n}} \operatorname{Rep} GL(m,n)}{F_{m-1,n-1}} \xrightarrow{E_{m,n}} \operatorname{Rep} GL(m-1,n-1)$$

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Tensor functor F_x (Duflo, V.S.)

- g Lie superalgebra;
- $x \in \mathfrak{g}_1$ such that $[x, x] = 2x^2 = 0$;
- If M is a representation of \mathfrak{g} , then $x^2M=0$;
- Set $M_x = \frac{\text{ker}x}{\text{im}x}$, $\mathfrak{g}_x = \frac{\text{ker} \text{ad}_x}{\text{im} \text{ad}_x}$ (cohomology);
- M_X is a representation of \mathfrak{g}_X ;
- F_x : Rep $\mathfrak{g} \to \operatorname{\mathsf{Rep}} \mathfrak{g}_x \ (F_x(M) := M_x)$ is a tensor functor.

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Our case: $\mathfrak{g} = \mathfrak{gl}(m,n)$, x is an odd matrix of rank 1, $\mathfrak{g}_x = \mathfrak{gl}(m-1,n-1)$. Notation $F_x = E_{m,n}$

Towards abelianization: filtration

Lemma

A filtration on Rep GL(m, n) (tensor rank $\geq k$)

$$\mathcal{F}^1$$
 Rep $GL(m, n) \subset \cdots \subset \mathcal{F}^k$ Rep $GL(m, n) \subset \cdots$

- $\mathcal{F}^k \operatorname{\mathsf{Rep}} \mathsf{GL}(m,n) \otimes \mathcal{F}^l \operatorname{\mathsf{Rep}} \mathsf{GL}(m,n) \to \mathcal{F}^{k+l} \operatorname{\mathsf{Rep}} \mathsf{GL}(m,n)$;
- $E_{m,n}$ preserves this filtration;
- If $m, n \gg k$ then

$$E_{m,n}: \mathcal{F}^k \operatorname{\mathsf{Rep}} \mathsf{GL}(m,n) o \mathcal{F}^k \operatorname{\mathsf{Rep}} \mathsf{GL}(m-1,n-1)$$

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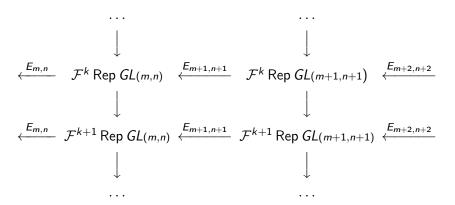
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First, horizontal inverse limit, then vertical direct limit.

Let $t \in \mathbb{Z}$ be fixed and m - n = t. For every k > 0 one can define an abelian category

$$\overline{\operatorname{\mathsf{Rep}}}_k \ \mathit{GL}(t) := \lim_{\leftarrow} \mathcal{F}^k \ \operatorname{\mathsf{Rep}} \ \mathit{GL}(m,n).$$

Lemma (Abelianization)

$$\overline{\operatorname{\mathsf{Rep}}}\ \mathit{GL}(t) := \lim_{\stackrel{}{ o}} \overline{\operatorname{\mathsf{Rep}}}_k\ \mathit{GL}(t)$$

is a symmetric rigid tensor category (<mark>abelian!</mark>).

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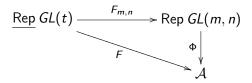
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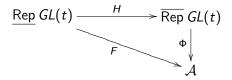
Theorem

Let $\mathcal A$ be a symmetric rigid tensor category and $F: \underline{\mathsf{Rep}}\ \mathit{GL}(t) \to \mathcal A$ be a tensor functor.

(a) V = F(X) is annihilated by some Schur functor $\Rightarrow \exists \Phi$



(b) V = F(X) is not annihilated by any Schur functor $\Rightarrow \exists \Phi$



The category $\operatorname{Rep}\mathfrak{gl}(\infty)$

$$\mathfrak{g}=\mathfrak{gl}(\infty)=\lim_{\longrightarrow}\mathfrak{gl}(n).$$

Let V and V_* be natural and conatural modules.

Definition

A subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is a finite corank subalgebra if there exist finite dimensional subspaces $W \subset V$ and $W' \subset V_*$ such that \mathfrak{k} annihilates every vector in W, W'.

We define $\operatorname{Rep}\mathfrak{gl}(\infty)$ as a full subcategory of \mathfrak{g} -modules whose objects M satisfy the following conditions

- M is integrable;
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Rep $\mathfrak{gl}(\infty)$ is symmetric, monoidal and universal in this class. (Penkov–Dan-Cohen –S., Sam–Snowden)

There exists a faithful tensor functor K : $\mathsf{Rep}\, \mathfrak{gl}(\infty) o \overline{\mathsf{Rep}}\, \mathit{GL}(t)$.

$$\operatorname{Rep} \operatorname{GL}(t) \xrightarrow{H} \operatorname{\overline{Rep}} \operatorname{GL}(t) \xleftarrow{K} \operatorname{Rep} \operatorname{\mathfrak{gl}}(\infty).$$

- Rep GL(t) is symmetric rigid, but not abelian.
- Rep $\mathfrak{gl}(\infty)$ is symmetric abelian but not rigid.
- $\overline{\text{Rep}} GL(t)$ is a locally highest weight category
- Simple objects in Rep $\mathfrak{gl}(\infty) \mapsto$ standard objects in Rep GL(t).
- Indecomposable objects in Rep $GL(t) \mapsto$ tilting objects in Rep GL(t).

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