Deligne's tensor category $\underline{\mathrm{Rep}}\, GL(t)$ and general linear supergroups

Vera Serganova

UC Berkeley

AMS meeting, San Francisco, October 2014

Interplay between supersymmetry and tensor categories

Supergroups $\xrightarrow{\$\$\$\ldots}$ tensor categories.

- Almost all examples of rigid symmetric tensor categories come from representation theory of superalgebras.
- Give rise to universal tensor categories.

Tensor categories $\xrightarrow{\$\$\$\ldots}$ supergroups.

- Classification of representations.
- Calculations of characters and dimensions of some natural representations. Comes–Wilson, Brundan–Stroppel.

Interplay between supersymmetry and tensor categories

Supergroups $\xrightarrow{\$\$\$\ldots}$ tensor categories.

- Almost all examples of rigid symmetric tensor categories come from representation theory of superalgebras.
- Give rise to universal tensor categories.

Tensor categories $\xrightarrow{\$\$\$\ldots}$ supergroups.

- Classification of representations.
- Calculations of characters and dimensions of some natural representations. Comes–Wilson, Brundan–Stroppel.

Formalization of tensor product $\rightsquigarrow$ Tensor categories

- $\mathcal{A}$ is abelian and k-linear (morphisms are k-vector spaces);
- $\mathcal{A}$ is equipped with tensor product $\otimes$, i.e. an exact functor $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, k-linear in both variables, with (functorial) isomorphism $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$;
- $\mathcal{A}$ has a unit object $\mathbf{1}$ such that $\mathrm{End}(\mathbf{1}) = k$;
- Symmetry: functorial isomorphism $\mathbf{s} : X \otimes Y \to Y \otimes X$ such that the composition

$$X \otimes Y \xrightarrow{\mathbf{s}} Y \otimes X \xrightarrow{\mathbf{s}} X \otimes Y$$

is the identity. (Braiding $\rightsquigarrow$ sign rule in supercase.) Warning: no braid groups.

Formalization of tensor product $\rightsquigarrow$ Tensor categories

- $\mathcal{A}$ is abelian and k-linear (morphisms are k-vector spaces);
- $\mathcal{A}$ is equipped with tensor product $\otimes$, i.e. an exact functor $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, k-linear in both variables, with (functorial) isomorphism $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$;
- $\mathcal{A}$ has a unit object $\mathbf{1}$ such that $\mathrm{End}(\mathbf{1}) = k$;
- Symmetry: functorial isomorphism $\mathsf{s} : X \otimes Y \to Y \otimes X$ such that the composition

$$X \otimes Y \xrightarrow{\mathsf{s}} Y \otimes X \xrightarrow{\mathsf{s}} X \otimes Y$$

  is the identity. (Braiding $\rightsquigarrow$ sign rule in supercase.) Warning: no braid groups.

Formalization of tensor product $\rightsquigarrow$ Tensor categories

- $\mathcal{A}$ is abelian and k-linear (morphisms are k-vector spaces);
- $\mathcal{A}$ is equipped with tensor product $\otimes$, i.e. an exact functor $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, k-linear in both variables, with (functorial) isomorphism $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$;
- $\mathcal{A}$ has a unit object $\mathbf{1}$ such that $\mathrm{End}(\mathbf{1}) = k$;
- Symmetry: functorial isomorphism $\mathbf{s} : X \otimes Y \to Y \otimes X$ such that the composition

$$X \otimes Y \xrightarrow{\mathbf{s}} Y \otimes X \xrightarrow{\mathbf{s}} X \otimes Y$$

is the identity. (Braiding $\rightsquigarrow$ sign rule in supercase.) Warning: no braid groups.

Formalization of tensor product $\rightsquigarrow$ Tensor categories

- $\mathcal{A}$ is abelian and k-linear (morphisms are k-vector spaces);
- $\mathcal{A}$ is equipped with tensor product $\otimes$, i.e. an exact functor $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, k-linear in both variables, with (functorial) isomorphism $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$;
- $\mathcal{A}$ has a unit object $\mathbf{1}$ such that $\mathrm{End}(\mathbf{1}) = k$;
- Symmetry: functorial isomorphism $\mathsf{s} : X \otimes Y \to Y \otimes X$ such that the composition

$$X \otimes Y \xrightarrow{\mathsf{s}} Y \otimes X \xrightarrow{\mathsf{s}} X \otimes Y$$

is the identity. (Braiding $\rightsquigarrow$ sign rule in supercase.) Warning: no braid groups.

Formalization of tensor product $\rightsquigarrow$ Tensor categories

- $\mathcal{A}$ is abelian and k-linear (morphisms are k-vector spaces);
- $\mathcal{A}$ is equipped with tensor product $\otimes$, i.e. an exact functor $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, k-linear in both variables, with (functorial) isomorphism $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$;
- $\mathcal{A}$ has a unit object $\mathbf{1}$ such that $\mathrm{End}(\mathbf{1}) = k$;
- Symmetry: functorial isomorphism $\mathbf{s} : X \otimes Y \to Y \otimes X$ such that the composition

$$X \otimes Y \xrightarrow{\mathbf{s}} Y \otimes X \xrightarrow{\mathbf{s}} X \otimes Y$$

is the identity. (Braiding $\rightsquigarrow$ sign rule in supercase.) Warning: no braid groups.

Formalization of tensor product $\rightsquigarrow$ Tensor categories

- $\mathcal{A}$ is abelian and k-linear (morphisms are k-vector spaces);
- $\mathcal{A}$ is equipped with tensor product $\otimes$, i.e. an exact functor $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, k-linear in both variables, with (functorial) isomorphism $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$;
- $\mathcal{A}$ has a unit object $\mathbf{1}$ such that $\mathrm{End}(\mathbf{1}) = k$;
- Symmetry: functorial isomorphism $\mathbf{s} : X \otimes Y \to Y \otimes X$ such that the composition

$$X \otimes Y \xrightarrow{\mathbf{s}} Y \otimes X \xrightarrow{\mathbf{s}} X \otimes Y$$

is the identity. (Braiding $\rightsquigarrow$ sign rule in supercase.) Warning: no braid groups.

Formalization of duality

- Contravariant duality functor $*: \mathcal{A} \to \mathcal{A}$, $X \mapsto X^*$;
- Natural maps: identity $\mathbf{e}: \mathbf{1} \to X \otimes X^*$;
- Contraction (trace) $\mathbf{c}: X^* \otimes X \to \mathbf{1}$;
- compositions

$$X \xrightarrow{\mathbf{e}\otimes 1} X \otimes X^* \otimes X \xrightarrow{1\otimes \mathbf{c}} X$$

and

$$X^* \xrightarrow{1\otimes \mathbf{e}} X^* \otimes X \otimes X^* \xrightarrow{\mathbf{c}\otimes 1} X^*$$

are both equal to the identity 1_X.

A tensor category $\mathcal{A}$ with duality is called rigid.

Formalization of duality

- Contravariant duality functor $*: \mathcal{A} \to \mathcal{A}$, $X \mapsto X^*$;
- Natural maps: identity $\mathbf{e}: \mathbf{1} \to X \otimes X^*$;
- Contraction (trace) $\mathbf{c}: X^* \otimes X \to \mathbf{1}$;
- compositions

$$X \xrightarrow{\mathbf{e}\otimes 1} X \otimes X^* \otimes X \xrightarrow{1\otimes \mathbf{c}} X$$

and

$$X^* \xrightarrow{1\otimes \mathbf{e}} X^* \otimes X \otimes X^* \xrightarrow{\mathbf{c}\otimes 1} X^*$$

are both equal to the identity 1_X.

A tensor category $\mathcal{A}$ with duality is called rigid.

Formalization of duality

- Contravariant duality functor $* : \mathcal{A} \to \mathcal{A}$, $X \mapsto X^*$;
- Natural maps: identity $\mathbf{e} : \mathbf{1} \to X \otimes X^*$;
- Contraction (trace) $\mathbf{c} : X^* \otimes X \to \mathbf{1}$;
- compositions

$$X \xrightarrow{\mathbf{e} \otimes 1} X \otimes X^* \otimes X \xrightarrow{1 \otimes \mathbf{c}} X$$

and

$$X^* \xrightarrow{1 \otimes \mathbf{e}} X^* \otimes X \otimes X^* \xrightarrow{\mathbf{c} \otimes 1} X^*$$

are both equal to the identity 1_X.

A tensor category $\mathcal{A}$ with duality is called rigid.

Formalization of duality

- Contravariant duality functor $* : \mathcal{A} \to \mathcal{A}$, $X \mapsto X^*$;
- Natural maps: identity $\mathbf{e} : \mathbf{1} \to X \otimes X^*$;
- Contraction (trace) $\mathbf{c} : X^* \otimes X \to \mathbf{1}$;
- compositions

$$X \xrightarrow{\mathbf{e} \otimes 1} X \otimes X^* \otimes X \xrightarrow{1 \otimes \mathbf{c}} X$$

and

$$X^* \xrightarrow{1 \otimes \mathbf{e}} X^* \otimes X \otimes X^* \xrightarrow{\mathbf{c} \otimes 1} X^*$$

are both equal to the identity 1_X.

A tensor category $\mathcal{A}$ with duality is called rigid.

Main Examples

The category Vect of finite-dimensional vector spaces.

- $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z), \quad (x \otimes y) \otimes z \mapsto x \otimes (y \otimes z), \quad \mathbf{1} = k;$
- $\mathrm{s} : X \otimes Y \to Y \otimes X, \quad \mathrm{s}(x \otimes y) := y \otimes x;$
- $\mathbf{c} : X^* \otimes X \to \mathbf{1}, \quad \mathbf{c}(f \otimes x) := f(x);$
- $\mathbf{e} : \mathbf{1} \to X \otimes X^*, \quad \mathbf{e}(1) := \sum e_i \otimes f_i.$

The category SVect of finite-dimensional vector superspaces.

- Objects are $\mathbb{Z}_2$-graded vector spaces $X = X_0 \oplus X_1$.
- The main difference with Vect:

$$\mathrm{s} : X \otimes Y \to Y \otimes X, \quad \mathrm{s}(x \otimes y) := (-1)^{\bar{x}\bar{y}} y \otimes x.$$

- $\mathbf{c}$ and $\mathbf{e}$ are defined by the same formulas as for usual vector spaces.

Main Examples

The category Vect of finite-dimensional vector spaces.

- $(X\otimes Y)\otimes Z \simeq X\otimes(Y\otimes Z), \quad (x\otimes y)\otimes z \mapsto x\otimes(y\otimes z), \quad \mathbf{1} = k;$
- $\mathsf{s} : X \otimes Y \to Y \otimes X, \quad \mathsf{s}(x \otimes y) := y \otimes x;$
- $\mathsf{c} : X^* \otimes X \to \mathbf{1}, \quad \mathsf{c}(f \otimes x) := f(x);$
- $\mathsf{e} : \mathbf{1} \to X \otimes X^*, \quad \mathsf{e}(1) := \sum e_i \otimes f_i.$

The category SVect of finite-dimensional vector superspaces.

- Objects are $\mathbb{Z}_2$-graded vector spaces $X = X_0 \oplus X_1$.
- The main difference with Vect:

$$\mathsf{s} : X \otimes Y \to Y \otimes X, \quad \mathsf{s}(x \otimes y) := (-1)^{\bar{x}\bar{y}} y \otimes x.$$

- c and e are defined by the same formulas as for usual vector spaces.

Main Examples

The category Vect of finite-dimensional vector spaces.

- $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z), \quad (x \otimes y) \otimes z \mapsto x \otimes (y \otimes z), \quad \mathbf{1} = k;$
- $\mathsf{s} : X \otimes Y \to Y \otimes X, \quad \mathsf{s}(x \otimes y) := y \otimes x;$
- $\mathbf{c} : X^* \otimes X \to \mathbf{1}, \quad \mathbf{c}(f \otimes x) := f(x);$
- $\mathbf{e} : \mathbf{1} \to X \otimes X^*, \quad \mathbf{e}(1) := \sum e_i \otimes f_i.$

The category SVect of finite-dimensional vector superspaces.

- Objects are $\mathbb{Z}_2$-graded vector spaces $X = X_0 \oplus X_1$.
- The main difference with Vect:

$$\mathsf{s} : X \otimes Y \to Y \otimes X, \quad \mathsf{s}(x \otimes y) := (-1)^{\bar{x}\bar{y}} y \otimes x.$$

- $\mathbf{c}$ and $\mathbf{e}$ are defined by the same formulas as for usual vector spaces.

Main Examples

The category Vect of finite-dimensional vector spaces.

- $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$, $(x \otimes y) \otimes z \mapsto x \otimes (y \otimes z)$, $\mathbf{1} = k$;
- $\mathsf{s} : X \otimes Y \to Y \otimes X$, $\mathsf{s}(x \otimes y) := y \otimes x$;
- $\mathsf{c} : X^* \otimes X \to \mathbf{1}$, $\mathsf{c}(f \otimes x) := f(x)$;
- $\mathsf{e} : \mathbf{1} \to X \otimes X^*$, $\mathsf{e}(1) := \sum e_i \otimes f_i$.

The category SVect of finite-dimensional vector superspaces.

- Objects are $\mathbb{Z}_2$-graded vector spaces $X = X_0 \oplus X_1$.
- The main difference with Vect:

$$\mathsf{s} : X \otimes Y \to Y \otimes X, \quad \mathsf{s}(x \otimes y) := (-1)^{\bar{x}\bar{y}} y \otimes x.$$

- c and e are defined by the same formulas as for usual vector spaces.

Main Examples

The category Vect of finite-dimensional vector spaces.

- $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$, $\quad (x \otimes y) \otimes z \mapsto x \otimes (y \otimes z)$, $\quad \mathbf{1} = k$;
- $\mathsf{s} : X \otimes Y \to Y \otimes X$, $\quad \mathsf{s}(x \otimes y) := y \otimes x$;
- $\mathbf{c} : X^* \otimes X \to \mathbf{1}$, $\quad \mathbf{c}(f \otimes x) := f(x)$;
- $\mathbf{e} : \mathbf{1} \to X \otimes X^*$, $\quad \mathbf{e}(1) := \sum e_i \otimes f_i$.

The category SVect of finite-dimensional vector superspaces.

- Objects are $\mathbb{Z}_2$-graded vector spaces $X = X_0 \oplus X_1$.
- The main difference with Vect:

$$\mathsf{s} : X \otimes Y \to Y \otimes X, \quad \mathsf{s}(x \otimes y) := (-1)^{\bar{x}\bar{y}} y \otimes x.$$

- $\mathbf{c}$ and $\mathbf{e}$ are defined by the same formulas as for usual vector spaces.

Trace and dimension

Using the rigidity axiom one can construct a canonical isomorphism

$$\delta : \mathrm{End}(X) \xrightarrow{\sim} \mathrm{Hom}(\mathbf{1}, X \otimes X^*)$$

and define the trace:

$$\mathrm{tr} : \mathrm{End}(X) \to \mathrm{End}(\mathbf{1}) = k$$

as the composition

$$\mathbf{1} \xrightarrow{\delta(\varphi)} X \otimes X^* \xrightarrow{s} X^* \otimes X \xrightarrow{c} \mathbf{1}.$$

By definition

$$\dim X = \mathrm{tr}\, 1_X .$$

Trace and dimension

Using the rigidity axiom one can construct a canonical isomorphism

$$\delta : \mathrm{End}(X) \xrightarrow{\sim} \mathrm{Hom}(\mathbf{1}, X \otimes X^*)$$

and define the trace:

$$\mathrm{tr} : \mathrm{End}(X) \to \mathrm{End}(\mathbf{1}) = k$$

as the composition

$$\mathbf{1} \xrightarrow{\delta(\varphi)} X \otimes X^* \xrightarrow{\mathbf{s}} X^* \otimes X \xrightarrow{\mathbf{c}} \mathbf{1}.$$

By definition

$$\dim X = \mathrm{tr}\, 1_X .$$

Trace and dimension

Using the rigidity axiom one can construct a canonical isomorphism

$$\delta : \mathrm{End}(X) \xrightarrow{\sim} \mathrm{Hom}(\mathbf{1}, X \otimes X^*)$$

and define the trace:

$$\mathrm{tr} : \mathrm{End}(X) \to \mathrm{End}(\mathbf{1}) = k$$

as the composition

$$\mathbf{1} \xrightarrow{\delta(\varphi)} X \otimes X^* \xrightarrow{\mathbf{s}} X^* \otimes X \xrightarrow{\mathbf{c}} \mathbf{1}.$$

By definition

$$\dim X = \mathrm{tr}\, 1_X.$$

Trace and dimension

Using the rigidity axiom one can construct a canonical isomorphism

$$\delta : \mathrm{End}(X) \xrightarrow{\sim} \mathrm{Hom}(\mathbf{1}, X \otimes X^*)$$

and define the trace:

$$\mathrm{tr} : \mathrm{End}(X) \to \mathrm{End}(\mathbf{1}) = k$$

as the composition

$$\mathbf{1} \xrightarrow{\delta(\varphi)} X \otimes X^* \xrightarrow{\mathbf{s}} X^* \otimes X \xrightarrow{\mathbf{c}} \mathbf{1}.$$

By definition

$$\dim X = \mathrm{tr}\, 1_X.$$

- Vector spaces: dimension and trace are as usual.
- Vector superspaces: the *(super)trace* of a linear operator is

$$\operatorname{str}\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \operatorname{tr} A - \operatorname{tr} D.$$

and the (super)dimension

$$\operatorname{sdim} X = \dim X_0 - \dim X_1.$$

- In general, the dimension of an object in a symmetric rigid tensor category can be any element of k

- Vector spaces: dimension and trace are as usual.
- Vector superspaces: the *(super)trace* of a linear operator is

$$\operatorname{str}\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \operatorname{tr} A - \operatorname{tr} D.$$

and the (super)dimension

$$\operatorname{sdim} X = \dim X_0 - \dim X_1.$$

- In general, the dimension of an object in a symmetric rigid tensor category can be any element of k

- Vector spaces: dimension and trace are as usual.
- Vector superspaces: the *(super)trace* of a linear operator is

$$\operatorname{str}\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \operatorname{tr} A - \operatorname{tr} D.$$

  and the (super)dimension

$$\operatorname{sdim} X = \dim X_0 - \dim X_1.$$

- In general, the dimension of an object in a symmetric rigid tensor category can be any element of k

- Vector spaces: dimension and trace are as usual.
- Vector superspaces: the *(super)trace* of a linear operator is

$$\operatorname{str}\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \operatorname{tr} A - \operatorname{tr} D.$$

  and the (super)dimension

$$\operatorname{sdim} X = \dim X_0 - \dim X_1.$$

- In general, the dimension of an object in a symmetric rigid tensor category can be any element of k

- G algebraic group (over k), for example, $GL(n)$. The category Rep G of finite-dimensional representations of G is a symmetric rigid tensor category.
- A functor F : Rep $G \to$ Vect (forgetting the G-action). Tensor functor, i.e., preserves all structures of tensor categories, faithful (injective on morphisms), exact.

- An exact faithful tensor functor $F : \mathcal{A} \to$ Vect is called a *fiber functor*.
- A finitely generated symmetric rigid tensor category which has a fiber functor is equivalent to Rep G for some algebraic group G. (Tannakian categories $\rightsquigarrow$ affine group schemes).

- G algebraic group (over k), for example, $GL(n)$. The category Rep G of finite-dimensional representations of G is a symmetric rigid tensor category.
- A functor F : Rep $G \to$ Vect (forgetting the G-action). Tensor functor, i.e., preserves all structures of tensor categories, faithful (injective on morphisms), exact.

- An exact faithful tensor functor $F : \mathcal{A} \to$ Vect is called a *fiber functor*.
- A finitely generated symmetric rigid tensor category which has a fiber functor is equivalent to Rep G for some algebraic group G. (Tannakian categories $\rightsquigarrow$ affine group schemes).

- G algebraic group (over k), for example, $GL(n)$. The category Rep G of finite-dimensional representations of G is a symmetric rigid tensor category.
- A functor F : Rep $G \to$ Vect (forgetting the G-action). Tensor functor, i.e., preserves all structures of tensor categories, faithful (injective on morphisms), exact.

- An exact faithful tensor functor $F : \mathcal{A} \to$ Vect is called a *fiber functor*.
- A finitely generated symmetric rigid tensor category which has a fiber functor is equivalent to Rep G for some algebraic group G. (Tannakian categories $\rightsquigarrow$ affine group schemes).

- G algebraic group (over k), for example, $GL(n)$. The category Rep G of finite-dimensional representations of G is a symmetric rigid tensor category.
- A functor F : Rep $G \to$ Vect (forgetting the G-action). Tensor functor, i.e., preserves all structures of tensor categories, faithful (injective on morphisms), exact.

- An exact faithful tensor functor $F : \mathcal{A} \to$ Vect is called a *fiber functor*.
- A finitely generated symmetric rigid tensor category which has a fiber functor is equivalent to Rep G for some algebraic group G. (Tannakian categories $\rightsquigarrow$ affine group schemes).

Supertannakian formalism

Example. $GL(n)$ may be a group G and may be supergroup G_{super}.
Rep G_{super} has twice as many objects as Rep G:

$$V \longleftrightarrow \Pi V, \quad V_0 \circlearrowright V_1$$

Deligne's trick (Halving the category):

- G is a supergroup, fix $g \in G_0$.
- Rep(G, g) is the subcategory of Rep G, consisting of representations V satisfying $g(v) = (-1)^{\bar{v}} v$.
- Rep(G, g) is a tensor category.

Supertannakian formalism

Example. $GL(n)$ may be a group G and may be supergroup G_{super}. Rep G_{super} has twice as many objects as Rep G:

$$V \longleftrightarrow \Pi V, \quad V_0 \circlearrowright V_1$$

Deligne's trick (Halving the category):

- G is a supergroup, fix $g \in G_0$.
- $\mathrm{Rep}(G, g)$ is the subcategory of Rep G, consisting of representations V satisfying $g(v) = (-1)^{\bar{v}} v$.
- $\mathrm{Rep}(G, g)$ is a tensor category.

Supertannakian formalism

Example. $GL(n)$ may be a group G and may be supergroup G_{super}. Rep G_{super} has twice as many objects as Rep G:

$$V \longleftrightarrow \Pi V, \quad V_0 \circlearrowright V_1$$

Deligne's trick (Halving the category):

- G is a supergroup, fix $g \in G_0$.
- $\operatorname{Rep}(G, g)$ is the subcategory of Rep G, consisting of representations V satisfying $g(v) = (-1)^{\bar{v}} v$.
- $\operatorname{Rep}(G, g)$ is a tensor category.

Theorem (Deligne, 2002)

A tensor category $\mathcal{A}$ is equivalent to some $\mathrm{Rep}(G, g)$ *if and only if*

- *There exists a fiber functor* $\mathcal{A} \to \mathrm{SVect}$*;*
- $\mathcal{A}$ *is finitely generated.*

Theorem (Deligne, 2002)

A tensor category $\mathcal{A}$ is equivalent to some $\mathrm{Rep}(G, g)$ *if and only if*

- *There exists a fiber functor* $\mathcal{A} \to \mathrm{SVect}$*;*
- $\mathcal{A}$ *is finitely generated.*

Schur functor

- Consider a tensor category $\mathcal{A}$.
- For any object X in $\mathcal{A}$ consider $X^{\otimes n}$.
- $\sigma_{i,i+1} \mapsto 1^{\otimes i-1} \otimes \mathbf{s} \otimes 1^{\otimes n-i-1} \quad \rightsquigarrow \quad S_n \to \mathrm{Aut}(X^{\otimes n})$.

Schur–Weyl duality:

$$X^{\otimes n} = \bigoplus_{|\lambda|=n} V_\lambda \otimes S_\lambda(X).$$

Definition

$X \mapsto S_\lambda(X)$ is called the Schur functor $S_\lambda : \mathcal{A} \to \mathcal{A}$.

Schur functor

- Consider a tensor category $\mathcal{A}$.
- For any object X in $\mathcal{A}$ consider $X^{\otimes n}$.
- $\sigma_{i,i+1} \mapsto 1^{\otimes i-1} \otimes \mathbf{s} \otimes 1^{\otimes n-i-1} \quad \rightsquigarrow \quad S_n \to \mathrm{Aut}(X^{\otimes n})$.

Schur–Weyl duality:

$$X^{\otimes n} = \bigoplus_{|\lambda|=n} V_\lambda \otimes S_\lambda(X).$$

Definition

$X \mapsto S_\lambda(X)$ is called the Schur functor $S_\lambda : \mathcal{A} \to \mathcal{A}$.

Schur functor

- Consider a tensor category $\mathcal{A}$.
- For any object X in $\mathcal{A}$ consider $X^{\otimes n}$.
- $\sigma_{i,i+1} \mapsto 1^{\otimes i-1} \otimes \mathbf{s} \otimes 1^{\otimes n-i-1} \quad \rightsquigarrow \quad S_n \to \mathrm{Aut}(X^{\otimes n})$.

Schur–Weyl duality:

$$X^{\otimes n} = \bigoplus_{|\lambda|=n} V_\lambda \otimes S_\lambda(X).$$

Definition

$X \mapsto S_\lambda(X)$ is called the Schur functor $S_\lambda : \mathcal{A} \to \mathcal{A}$.

Schur functor

- Consider a tensor category $\mathcal{A}$.
- For any object X in $\mathcal{A}$ consider $X^{\otimes n}$.
- $\sigma_{i,i+1} \mapsto 1^{\otimes i-1} \otimes \mathbf{s} \otimes 1^{\otimes n-i-1} \quad \rightsquigarrow \quad S_n \to \mathrm{Aut}(X^{\otimes n})$.

Schur–Weyl duality:

$$X^{\otimes n} = \bigoplus_{|\lambda|=n} V_\lambda \otimes S_\lambda(X).$$

Definition

$X \mapsto S_\lambda(X)$ is called the Schur functor $S_\lambda : \mathcal{A} \to \mathcal{A}$.

Schur functor

- Consider a tensor category $\mathcal{A}$.
- For any object X in $\mathcal{A}$ consider $X^{\otimes n}$.
- $\sigma_{i,i+1} \mapsto 1^{\otimes i-1} \otimes \mathbf{s} \otimes 1^{\otimes n-i-1} \quad \rightsquigarrow \quad S_n \to \mathrm{Aut}(X^{\otimes n})$.

Schur–Weyl duality:

$$X^{\otimes n} = \bigoplus_{|\lambda|=n} V_\lambda \otimes S_\lambda(X).$$

Definition

$X \mapsto S_\lambda(X)$ is called the Schur functor $S_\lambda : \mathcal{A} \to \mathcal{A}$.

Theorem (A. Sergeev,1982)

Let V be an $(m|n)$-dimensional superspace. Then $S_\lambda(V) \neq 0$ if and only if λ can be covered by an (m,n)-hook, or equivalently, λ does not contain a rectangular diagram of size $(m+1) \times (n+1)$.

Theorem (P. Deligne, 2002)

Let $\mathcal{A}$ be a finitely generated rigid symmetric tensor category. The following conditions are equivalent:
(a) Every generator is annihilated by some Schur functor.
(b) $\mathcal{A}$ is equivalent to $\mathrm{Rep}(G, g)$ for some algebraic supergroup G.

Theorem (A. Sergeev,1982)

Let V be an $(m|n)$-dimensional superspace. Then $S_\lambda(V) \neq 0$ if and only if λ can be covered by an (m,n)-hook, or equivalently, λ does not contain a rectangular diagram of size $(m+1)\times(n+1)$.

Theorem (P. Deligne, 2002)

Let $\mathcal{A}$ be a finitely generated rigid symmetric tensor category. The following conditions are equivalent:
(a) Every generator is annihilated by some Schur functor.
(b) $\mathcal{A}$ is equivalent to $\mathrm{Rep}(G,g)$ *for some algebraic supergroup G.*

Universal tensor categories

- Consider tensor categories generated by one object X;
- Want to construct a universal one;
- Analogy: free ring with two generators X and X^*;
- Deligne's category $\underline{\text{Rep}}\, GL(t)$, $t = \dim X$.

- Abelianization: a way to extend a category to abelian by forcefully adding kernels and images of all morphisms. There is no such bliss!
- The subtle point. Karoubization: adding kernels and images of all projectors ($p^2 = p$). It exists!
- $\underline{\text{Rep}}\, GL(t)$ is a result of Karoubization. Not abelian.

Universal tensor categories

- Consider tensor categories generated by one object X;
- Want to construct a universal one;
- Analogy: free ring with two generators X and X^*;
- Deligne's category $\underline{\text{Rep}}\, GL(t)$, $t = \dim X$.

- Abelianization: a way to extend a category to abelian by forcefully adding kernels and images of all morphisms. There is no such bliss!
- The subtle point. Karoubization: adding kernels and images of all projectors ($p^2 = p$). It exists!
- $\underline{\text{Rep}}\, GL(t)$ is a result of Karoubization. Not abelian.

Universal tensor categories

- Consider tensor categories generated by one object X;
- Want to construct a universal one;
- Analogy: free ring with two generators X and X^*;
- Deligne's category $\underline{\text{Rep}}\, GL(t)$, $t = \dim X$.

- Abelianization: a way to extend a category to abelian by forcefully adding kernels and images of all morphisms. There is no such bliss!
- The subtle point. Karoubization: adding kernels and images of all projectors ($p^2 = p$). It exists!
- $\underline{\text{Rep}}\, GL(t)$ is a result of Karoubization. Not abelian.

Universal tensor categories

- Consider tensor categories generated by one object X;
- Want to construct a universal one;
- Analogy: free ring with two generators X and X^*;
- Deligne's category $\underline{\mathrm{Rep}}\, GL(t)$, $t = \dim X$.

- Abelianization: a way to extend a category to abelian by forcefully adding kernels and images of all morphisms. There is no such bliss!
- The subtle point. Karoubization: adding kernels and images of all projectors ($p^2 = p$). It exists!
- $\underline{\mathrm{Rep}}\, GL(t)$ is a result of Karoubization. Not abelian.

Universal tensor categories

- Consider tensor categories generated by one object X;
- Want to construct a universal one;
- Analogy: free ring with two generators X and X^*;
- Deligne's category $\underline{\text{Rep}}\, GL(t)$, $t = \dim X$.

- Abelianization: a way to extend a category to abelian by forcefully adding kernels and images of all morphisms. There is no such bliss!
- The subtle point. Karoubization: adding kernels and images of all projectors ($p^2 = p$). It exists!
- $\underline{\text{Rep}}\, GL(t)$ is a result of Karoubization. Not abelian.

Universal tensor categories

- Consider tensor categories generated by one object X;
- Want to construct a universal one;
- Analogy: free ring with two generators X and X^*;
- Deligne's category $\underline{\text{Rep}}\, GL(t)$, $t = \dim X$.

- Abelianization: a way to extend a category to abelian by forcefully adding kernels and images of all morphisms. There is no such bliss!
- The subtle point. Karoubization: adding kernels and images of all projectors $(p^2 = p)$. It exists!
- $\underline{\text{Rep}}\, GL(t)$ is a result of Karoubization. Not abelian.

Ingredients of construction

- Skeleton category.
  - objects: $X^{\otimes m} \otimes (X^*)^{\otimes n}$,
  - morphisms: $\mathrm{Hom}(X^{\otimes m} \otimes (X^*)^{\otimes n}, X^{\otimes k} \otimes (X^*)^{\otimes l})$ = walled Brauer diagrams.
- Additive closure (adding direct sums).
- Karoubian envelope (adding kernels of all projectors).

Properties of Deligne's categories

- $\underline{\mathrm{Rep}}\, GL(t)$ is semisimple and therefore abelian if and only if $t \notin \mathbb{Z}$.
- Indecomposable objects in $\underline{\mathrm{Rep}}\, GL(t)$ are enumerated by pairs of partitions: $(\lambda, \mu) \leftrightarrow Y_{\lambda,\mu}$.
- $Y^*_{\lambda,\mu} \simeq Y_{\mu,\lambda}$.
- $\dim Y_{\lambda,\mu}$ is a polynomial in t.

Properties of Deligne's categories

- $\underline{\mathrm{Rep}}\, GL(t)$ is semisimple and therefore abelian if and only if $t \notin \mathbb{Z}$.
- Indecomposable objects in $\underline{\mathrm{Rep}}\, GL(t)$ are enumerated by pairs of partitions: $(\lambda, \mu) \leftrightarrow Y_{\lambda,\mu}$.
- $Y^*_{\lambda,\mu} \simeq Y_{\mu,\lambda}$.
- $\dim Y_{\lambda,\mu}$ is a polynomial in t.

Properties of Deligne's categories

- $\underline{\mathrm{Rep}}\, GL(t)$ is semisimple and therefore abelian if and only if $t \notin \mathbb{Z}$.
- Indecomposable objects in $\underline{\mathrm{Rep}}\, GL(t)$ are enumerated by pairs of partitions: $(\lambda, \mu) \leftrightarrow Y_{\lambda,\mu}$.
- $Y^*_{\lambda,\mu} \simeq Y_{\mu,\lambda}$.
- $\dim Y_{\lambda,\mu}$ is a polynomial in t.

Properties of Deligne's categories

- $\underline{\mathrm{Rep}}\, GL(t)$ is semisimple and therefore abelian if and only if $t \notin \mathbb{Z}$.
- Indecomposable objects in $\underline{\mathrm{Rep}}\, GL(t)$ are enumerated by pairs of partitions: $(\lambda, \mu) \leftrightarrow Y_{\lambda,\mu}$.
- $Y^*_{\lambda,\mu} \simeq Y_{\mu,\lambda}$.
- $\dim Y_{\lambda,\mu}$ is a polynomial in t.

Semisimple quotient

If $t = n \in \mathbb{Z}$, the Deligne categories have canonical semisimple quotients.

Semisimple quotients of $\underline{\text{Rep}}\, GL(t)$:

- $\text{Rep}(GL(n), 1)$ for $n > 0$;
- $\text{Rep}(GL(-n), -1)$ for $n < 0$.

Universality

Theorem (P. Deligne, 2002)

Let $\mathcal{A}$ be a symmetric rigid k-linear category, and V be an object of dimension t. There exists a unique (up to isomorphism) tensor functor $F : \underline{\text{Rep}}\, GL(t) \to \mathcal{A}$ such that $F(X) = V$.

Semisimple quotient

If $t = n \in \mathbb{Z}$, the Deligne categories have canonical semisimple quotients.

Semisimple quotients of $\underline{\text{Rep}}\, GL(t)$:

- $\text{Rep}(GL(n), 1)$ for $n > 0$;
- $\text{Rep}(GL(-n), -1)$ for $n < 0$.

Universality

Theorem (P. Deligne, 2002)

Let $\mathcal{A}$ be a symmetric rigid k-linear category, and V be an object of dimension t. There exists a unique (up to isomorphism) tensor functor $F : \underline{\text{Rep}}\, GL(t) \to \mathcal{A}$ such that $F(X) = V$.

Semisimple quotient

If $t = n \in \mathbb{Z}$, the Deligne categories have canonical semisimple quotients.

Semisimple quotients of $\underline{\text{Rep}}\, GL(t)$:

- $\text{Rep}(GL(n), 1)$ for $n > 0$;
- $\text{Rep}(GL(-n), -1)$ for $n < 0$.

Universality

Theorem (P. Deligne, 2002)

Let $\mathcal{A}$ be a symmetric rigid k-linear category, and V be an object of dimension t. There exists a unique (up to isomorphism) tensor functor $F : \underline{\text{Rep}}\, GL(t) \to \mathcal{A}$ such that $F(X) = V$.

Towards abelianization: building blocks

$t = m - n = m' - n'$

$$\begin{array}{ccc} \underline{\mathrm{Rep}}\, GL(t) & \xrightarrow{F_{m,n}} & \mathrm{Rep}\, GL(m,n) \\ & \searrow^{F_{m',n'}} & \\ & & \mathrm{Rep}\, GL(m',n') \end{array}$$

Tensor functor $E_{m,n} : \mathrm{Rep}\, GL(m,n) \to \mathrm{Rep}\, GL(m-1,n-1)$.

$$\begin{array}{ccc} \underline{\mathrm{Rep}}\, GL(t) & \xrightarrow{F_{m,n}} & \mathrm{Rep}\, GL(m,n) \\ & \searrow^{F_{m-1,n-1}} & \downarrow E_{m,n} \\ & & \mathrm{Rep}\, GL(m-1,n-1) \end{array}$$

Towards abelianization: building blocks

$t = m - n = m' - n'$

$$\begin{array}{ccc} \underline{\mathrm{Rep}}\, GL(t) & \xrightarrow{F_{m,n}} & \mathrm{Rep}\, GL(m,n) \\ & \searrow^{F_{m',n'}} & \\ & & \mathrm{Rep}\, GL(m',n') \end{array}$$

Tensor functor $E_{m,n} : \mathrm{Rep}\, GL(m,n) \to \mathrm{Rep}\, GL(m-1,n-1)$.

$$\begin{array}{ccc} \underline{\mathrm{Rep}}\, GL(t) & \xrightarrow{F_{m,n}} & \mathrm{Rep}\, GL(m,n) \\ & \searrow^{F_{m-1,n-1}} & \downarrow E_{m,n} \\ & & \mathrm{Rep}\, GL(m-1,n-1) \end{array}$$

Tensor functor F_x (Duflo, V.S.)

- $\mathfrak{g}$ Lie superalgebra;
- $x \in \mathfrak{g}_1$ such that $[x, x] = 2x^2 = 0$;
- If M is a representation of $\mathfrak{g}$, then $x^2 M = 0$;
- Set $M_x = \mathbf{ker} x / \mathbf{im} x$, $\mathfrak{g}_x = \mathbf{ker}\, \mathrm{ad}_x / \mathbf{im}\, \mathrm{ad}_x$ (cohomology);
- M_x is a representation of $\mathfrak{g}_x$;
- $F_x : \mathrm{Rep}\, \mathfrak{g} \to \mathrm{Rep}\, \mathfrak{g}_x$ ($F_x(M) := M_x$) is a tensor functor.

Our case: $\mathfrak{g} = \mathfrak{gl}(m, n)$, x is an odd matrix of rank 1, $\mathfrak{g}_x = \mathfrak{gl}(m-1, n-1)$. Notation $F_x = E_{m,n}$

Tensor functor F_x (Duflo, V.S.)

- $\mathfrak{g}$ Lie superalgebra;
- $x \in \mathfrak{g}_1$ such that $[x, x] = 2x^2 = 0$;
- If M is a representation of $\mathfrak{g}$, then $x^2 M = 0$;
- Set $M_x = \mathbf{ker} x / \mathbf{im} x$, $\mathfrak{g}_x = \mathbf{ker}\, \mathrm{ad}_x / \mathbf{im}\, \mathrm{ad}_x$ (cohomology);
- M_x is a representation of $\mathfrak{g}_x$;
- $F_x : \operatorname{Rep} \mathfrak{g} \to \operatorname{Rep} \mathfrak{g}_x$ ($F_x(M) := M_x$) is a tensor functor.

Our case: $\mathfrak{g} = \mathfrak{gl}(m, n)$, x is an odd matrix of rank 1, $\mathfrak{g}_x = \mathfrak{gl}(m-1, n-1)$. Notation $F_x = E_{m,n}$

Towards abelianization: filtration

Lemma

A filtration on Rep $GL(m,n)$ *(tensor rank* $\geq k$*)*

$$\mathcal{F}^1 \operatorname{Rep} GL(m,n) \subset \cdots \subset \mathcal{F}^k \operatorname{Rep} GL(m,n) \subset \ldots,$$

- $\mathcal{F}^k \operatorname{Rep} GL(m,n) \otimes \mathcal{F}^l \operatorname{Rep} GL(m,n) \to \mathcal{F}^{k+l} \operatorname{Rep} GL(m,n)$;
- $E_{m,n}$ *preserves this filtration;*
- *If* $m, n \gg k$ *then*

$$E_{m,n} : \mathcal{F}^k \operatorname{Rep} GL(m,n) \to \mathcal{F}^k \operatorname{Rep} GL(m-1, n-1)$$

is an equivalence of abelian categories.

Towards abelianization: filtration

Lemma

A filtration on Rep $GL(m,n)$ *(tensor rank* $\geq k$*)*

$$\mathcal{F}^1 \operatorname{Rep} GL(m,n) \subset \cdots \subset \mathcal{F}^k \operatorname{Rep} GL(m,n) \subset \ldots,$$

- $\mathcal{F}^k \operatorname{Rep} GL(m,n) \otimes \mathcal{F}^l \operatorname{Rep} GL(m,n) \to \mathcal{F}^{k+l} \operatorname{Rep} GL(m,n)$;
- $E_{m,n}$ *preserves this filtration;*
- *If* $m, n \gg k$ *then*

$$E_{m,n} : \mathcal{F}^k \operatorname{Rep} GL(m,n) \to \mathcal{F}^k \operatorname{Rep} GL(m-1,n-1)$$

is an equivalence of abelian categories.

Towards abelianization: filtration

Lemma

A filtration on Rep $GL(m,n)$ *(tensor rank* $\geq k$*)*

$$\mathcal{F}^1 \operatorname{Rep} GL(m,n) \subset \cdots \subset \mathcal{F}^k \operatorname{Rep} GL(m,n) \subset \dots,$$

- $\mathcal{F}^k \operatorname{Rep} GL(m,n) \otimes \mathcal{F}^l \operatorname{Rep} GL(m,n) \to \mathcal{F}^{k+l} \operatorname{Rep} GL(m,n)$;
- $E_{m,n}$ *preserves this filtration;*
- *If* $m, n \gg k$ *then*

$$E_{m,n} : \mathcal{F}^k \operatorname{Rep} GL(m,n) \to \mathcal{F}^k \operatorname{Rep} GL(m-1,n-1)$$

is an equivalence of abelian categories.

$$
\begin{array}{ccccccc}
 & & \dots & & \dots & & \\
 & & \downarrow & & \downarrow & & \\
\xleftarrow{E_{m,n}} & & \mathcal{F}^k \operatorname{Rep} GL(m,n) & \xleftarrow{E_{m+1,n+1}} & \mathcal{F}^k \operatorname{Rep} GL(m+1,n+1) & \xleftarrow{E_{m+2,n+2}} & \\
 & & \downarrow & & \downarrow & & \\
\xleftarrow{E_{m,n}} & & \mathcal{F}^{k+1} \operatorname{Rep} GL(m,n) & \xleftarrow{E_{m+1,n+1}} & \mathcal{F}^{k+1} \operatorname{Rep} GL(m+1,n+1) & \xleftarrow{E_{m+2,n+2}} & \\
 & & \downarrow & & \downarrow & & \\
 & & \dots & & \dots & &
\end{array}
$$

First, horizontal inverse limit, then vertical direct limit.

Let $t \in \mathbb{Z}$ be fixed and $m - n = t$.
For every $k > 0$ one can define an abelian category

$$\overline{\mathrm{Rep}}_k\, GL(t) := \varprojlim \mathcal{F}^k\, \mathrm{Rep}\, GL(m,n).$$

Lemma (Abelianization)

(a)

$$\overline{\mathrm{Rep}}\, GL(t) := \varinjlim \overline{\mathrm{Rep}}_k\, GL(t)$$

is a symmetric rigid tensor category (abelian!).
(b) There exists a fully faithful functor $H : \underline{\mathrm{Rep}}\, GL(t) \to \overline{\mathrm{Rep}}\, GL(t)$.

Let $t \in \mathbb{Z}$ be fixed and $m - n = t$.
For every $k > 0$ one can define an abelian category

$$\overline{\mathrm{Rep}}_k\, GL(t) := \lim_{\leftarrow} \mathcal{F}^k\, \mathrm{Rep}\, GL(m,n).$$

Lemma (Abelianization)

(a)

$$\overline{\mathrm{Rep}}\, GL(t) := \lim_{\rightarrow} \overline{\mathrm{Rep}}_k\, GL(t)$$

is a symmetric rigid tensor category (abelian!).
(b) There exists a fully faithful functor $H : \underline{\mathrm{Rep}}\, GL(t) \to \overline{\mathrm{Rep}}\, GL(t)$.

Theorem

Let $\mathcal{A}$ be a symmetric rigid tensor category and $F : \underline{\mathrm{Rep}}\, GL(t) \to \mathcal{A}$ be a tensor functor.

(a) $V = F(X)$ is annihilated by some Schur functor $\Rightarrow \exists \Phi$

$$\begin{array}{ccc} \underline{\mathrm{Rep}}\, GL(t) & \xrightarrow{F_{m,n}} & \mathrm{Rep}\, GL(m,n) \\ & {}_{F}\searrow & \downarrow{\scriptstyle \Phi} \\ & & \mathcal{A} \end{array}$$

(b) $V = F(X)$ is not annihilated by any Schur functor $\Rightarrow \exists \Phi$

$$\begin{array}{ccc} \underline{\mathrm{Rep}}\, GL(t) & \xrightarrow{H} & \overline{\mathrm{Rep}}\, GL(t) \\ & {}_{F}\searrow & \downarrow{\scriptstyle \Phi} \\ & & \mathcal{A} \end{array}$$

The category $\operatorname{Rep}\mathfrak{gl}(\infty)$

$$\mathfrak{g} = \mathfrak{gl}(\infty) = \varinjlim \mathfrak{gl}(n).$$

Let V and V_* be natural and conatural modules.

Definition

A subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is a finite corank subalgebra if there exist finite dimensional subspaces $W \subset V$ and $W' \subset V_*$ such that $\mathfrak{k}$ annihilates every vector in W, W'.

We define $\operatorname{Rep}\mathfrak{gl}(\infty)$ as a full subcategory of $\mathfrak{g}$-modules whose objects M satisfy the following conditions

- M is integrable;
- For every $m \in M$ the annihilator of m in $\mathfrak{g}$ has finite corank;
- M has finite length.

The category $\operatorname{Rep}\mathfrak{gl}(\infty)$

$$\mathfrak{g} = \mathfrak{gl}(\infty) = \lim_{\rightarrow} \mathfrak{gl}(n).$$

Let V and V_* be natural and conatural modules.

Definition

A subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is a finite corank subalgebra if there exist finite dimensional subspaces $W \subset V$ and $W' \subset V_*$ such that $\mathfrak{k}$ annihilates every vector in W, W'.

We define $\operatorname{Rep}\mathfrak{gl}(\infty)$ as a full subcategory of $\mathfrak{g}$-modules whose objects M satisfy the following conditions

- M is integrable;
- For every $m \in M$ the annihilator of m in $\mathfrak{g}$ has finite corank;
- M has finite length.

$\operatorname{Rep}\mathfrak{gl}(\infty)$ is symmetric, monoidal and universal in this class. (Penkov–Dan-Cohen –S., Sam–Snowden)

There exists a faithful tensor functor $K : \operatorname{Rep}\mathfrak{gl}(\infty) \to \overline{\operatorname{Rep}}\, GL(t)$.

$$\underline{\operatorname{Rep}}\, GL(t) \xrightarrow{H} \overline{\operatorname{Rep}}\, GL(t) \xleftarrow{K} \operatorname{Rep}\mathfrak{gl}(\infty).$$

- $\underline{\operatorname{Rep}}\, GL(t)$ is symmetric rigid, but not abelian.
- $\operatorname{Rep}\mathfrak{gl}(\infty)$ is symmetric abelian but not rigid.
- $\overline{\operatorname{Rep}}\, GL(t)$ is a locally highest weight category
- Simple objects in $\operatorname{Rep}\mathfrak{gl}(\infty)$ $\mapsto$standard objects in $\overline{\operatorname{Rep}}\, GL(t)$.
- Indecomposable objects in $\underline{\operatorname{Rep}}\, GL(t) \mapsto$ tilting objects in $\overline{\operatorname{Rep}}\, GL(t)$.

$\operatorname{Rep}\mathfrak{gl}(\infty)$ is symmetric, monoidal and universal in this class.
(Penkov–Dan-Cohen –S., Sam–Snowden)
There exists a faithful tensor functor $K : \operatorname{Rep}\mathfrak{gl}(\infty) \to \overline{\operatorname{Rep}}\, GL(t)$.

$$\underline{\operatorname{Rep}}\, GL(t) \xrightarrow{H} \overline{\operatorname{Rep}}\, GL(t) \xleftarrow{K} \operatorname{Rep}\mathfrak{gl}(\infty).$$

- $\underline{\operatorname{Rep}}\, GL(t)$ is symmetric rigid, but not abelian.
- $\operatorname{Rep}\mathfrak{gl}(\infty)$ is symmetric abelian but not rigid.
- $\overline{\operatorname{Rep}}\, GL(t)$ is a locally highest weight category
- Simple objects in $\operatorname{Rep}\mathfrak{gl}(\infty) \mapsto$standard objects in $\overline{\operatorname{Rep}}\, GL(t)$.
- Indecomposable objects in $\underline{\operatorname{Rep}}\, GL(t) \mapsto$ tilting objects in $\overline{\operatorname{Rep}}\, GL(t)$.

$\operatorname{Rep}\mathfrak{gl}(\infty)$ is symmetric, monoidal and universal in this class. (Penkov–Dan-Cohen –S., Sam–Snowden)
There exists a faithful tensor functor $K : \operatorname{Rep}\mathfrak{gl}(\infty) \to \overline{\operatorname{Rep}}\, GL(t)$.

$$\underline{\operatorname{Rep}}\, GL(t) \xrightarrow{H} \overline{\operatorname{Rep}}\, GL(t) \xleftarrow{K} \operatorname{Rep}\mathfrak{gl}(\infty).$$

- $\underline{\operatorname{Rep}}\, GL(t)$ is symmetric rigid, but not abelian.
- $\operatorname{Rep}\mathfrak{gl}(\infty)$ is symmetric abelian but not rigid.
- $\overline{\operatorname{Rep}}\, GL(t)$ is a locally highest weight category
- Simple objects in $\operatorname{Rep}\mathfrak{gl}(\infty)$ $\mapsto$standard objects in $\overline{\operatorname{Rep}}\, GL(t)$.
- Indecomposable objects in $\underline{\operatorname{Rep}}\, GL(t)$ $\mapsto$ tilting objects in $\overline{\operatorname{Rep}}\, GL(t)$.

$\operatorname{Rep}\mathfrak{gl}(\infty)$ is symmetric, monoidal and universal in this class. (Penkov–Dan-Cohen –S., Sam–Snowden)
There exists a faithful tensor functor $K : \operatorname{Rep}\mathfrak{gl}(\infty) \to \overline{\operatorname{Rep}}\, GL(t)$.

$$\underline{\operatorname{Rep}}\, GL(t) \xrightarrow{H} \overline{\operatorname{Rep}}\, GL(t) \xleftarrow{K} \operatorname{Rep}\mathfrak{gl}(\infty).$$

- $\underline{\operatorname{Rep}}\, GL(t)$ is symmetric rigid, but not abelian.
- $\operatorname{Rep}\mathfrak{gl}(\infty)$ is symmetric abelian but not rigid.
- $\overline{\operatorname{Rep}}\, GL(t)$ is a locally highest weight category
- Simple objects in $\operatorname{Rep}\mathfrak{gl}(\infty)$ $\mapsto$standard objects in $\overline{\operatorname{Rep}}\, GL(t)$.
- Indecomposable objects in $\underline{\operatorname{Rep}}\, GL(t) \mapsto$ tilting objects in $\overline{\operatorname{Rep}}\, GL(t)$.

$\mathrm{Rep}\,\mathfrak{gl}(\infty)$ is symmetric, monoidal and universal in this class. (Penkov–Dan-Cohen –S., Sam–Snowden)
There exists a faithful tensor functor $K : \mathrm{Rep}\,\mathfrak{gl}(\infty) \to \overline{\mathrm{Rep}}\,GL(t)$.

$$\underline{\mathrm{Rep}}\,GL(t) \xrightarrow{H} \overline{\mathrm{Rep}}\,GL(t) \xleftarrow{K} \mathrm{Rep}\,\mathfrak{gl}(\infty).$$

- $\underline{\mathrm{Rep}}\,GL(t)$ is symmetric rigid, but not abelian.
- $\mathrm{Rep}\,\mathfrak{gl}(\infty)$ is symmetric abelian but not rigid.
- $\overline{\mathrm{Rep}}\,GL(t)$ is a locally highest weight category
- Simple objects in $\mathrm{Rep}\,\mathfrak{gl}(\infty) \mapsto$standard objects in $\overline{\mathrm{Rep}}\,GL(t)$.
- Indecomposable objects in $\underline{\mathrm{Rep}}\,GL(t) \mapsto$ tilting objects in $\overline{\mathrm{Rep}}\,GL(t)$.

Rep $\mathfrak{gl}(\infty)$ is symmetric, monoidal and universal in this class. (Penkov–Dan-Cohen –S., Sam–Snowden)
There exists a faithful tensor functor $K : \operatorname{Rep}\mathfrak{gl}(\infty) \to \overline{\operatorname{Rep}}\, GL(t)$.

$$\underline{\operatorname{Rep}}\, GL(t) \xrightarrow{H} \overline{\operatorname{Rep}}\, GL(t) \xleftarrow{K} \operatorname{Rep}\mathfrak{gl}(\infty).$$

- $\underline{\operatorname{Rep}}\, GL(t)$ is symmetric rigid, but not abelian.
- Rep $\mathfrak{gl}(\infty)$ is symmetric abelian but not rigid.
- $\overline{\operatorname{Rep}}\, GL(t)$ is a locally highest weight category
- Simple objects in Rep $\mathfrak{gl}(\infty)$ $\mapsto$ standard objects in $\overline{\operatorname{Rep}}\, GL(t)$.
- Indecomposable objects in $\underline{\operatorname{Rep}}\, GL(t) \mapsto$ tilting objects in $\overline{\operatorname{Rep}}\, GL(t)$.

$\operatorname{Rep}\mathfrak{gl}(\infty)$ is symmetric, monoidal and universal in this class. (Penkov–Dan-Cohen –S., Sam–Snowden)
There exists a faithful tensor functor $K : \operatorname{Rep}\mathfrak{gl}(\infty) \to \overline{\operatorname{Rep}}\, GL(t)$.

$$\underline{\operatorname{Rep}}\, GL(t) \xrightarrow{H} \overline{\operatorname{Rep}}\, GL(t) \xleftarrow{K} \operatorname{Rep}\mathfrak{gl}(\infty).$$

- $\underline{\operatorname{Rep}}\, GL(t)$ is symmetric rigid, but not abelian.
- $\operatorname{Rep}\mathfrak{gl}(\infty)$ is symmetric abelian but not rigid.
- $\overline{\operatorname{Rep}}\, GL(t)$ is a locally highest weight category
- Simple objects in $\operatorname{Rep}\mathfrak{gl}(\infty) \mapsto$standard objects in $\overline{\operatorname{Rep}}\, GL(t)$.
- Indecomposable objects in $\underline{\operatorname{Rep}}\, GL(t) \mapsto$ tilting objects in $\overline{\operatorname{Rep}}\, GL(t)$.

$\operatorname{Rep}\mathfrak{gl}(\infty)$ is symmetric, monoidal and universal in this class. (Penkov–Dan-Cohen –S., Sam–Snowden)
There exists a faithful tensor functor $K : \operatorname{Rep}\mathfrak{gl}(\infty) \to \overline{\operatorname{Rep}}\, GL(t)$.

$$\underline{\operatorname{Rep}}\, GL(t) \xrightarrow{H} \overline{\operatorname{Rep}}\, GL(t) \xleftarrow{K} \operatorname{Rep}\mathfrak{gl}(\infty).$$

- $\underline{\operatorname{Rep}}\, GL(t)$ is symmetric rigid, but not abelian.
- $\operatorname{Rep}\mathfrak{gl}(\infty)$ is symmetric abelian but not rigid.
- $\overline{\operatorname{Rep}}\, GL(t)$ is a locally highest weight category
- Simple objects in $\operatorname{Rep}\mathfrak{gl}(\infty)$ $\mapsto$standard objects in $\overline{\operatorname{Rep}}\, GL(t)$.
- Indecomposable objects in $\underline{\operatorname{Rep}}\, GL(t)$ $\mapsto$ tilting objects in $\overline{\operatorname{Rep}}\, GL(t)$.

Open problems

- Ideals in the Deligne category $\underline{\text{Rep}}\, GL(t)$.
  J. Comes (2012): thick ideals come from $\mathfrak{gl}(m,n)$.
- Deligne conjecture.
- Kazhdan–Lusztig theory for $\overline{\text{Rep}}$.
- Conjecture: F_x is semisimple.

Open problems

- Ideals in the Deligne category $\underline{\mathrm{Rep}}\, GL(t)$.
  J. Comes (2012): thick ideals come from $\mathfrak{gl}(m,n)$.
- Deligne conjecture.
- Kazhdan–Lusztig theory for $\overline{\mathrm{Rep}}$.
- Conjecture: F_x is semisimple.

Open problems

- Ideals in the Deligne category $\underline{\text{Rep}}\, GL(t)$.
  J. Comes (2012): thick ideals come from $\mathfrak{gl}(m,n)$.
- Deligne conjecture.
- Kazhdan–Lusztig theory for $\overline{\text{Rep}}$.
- Conjecture: F_x is semisimple.

Open problems

- Ideals in the Deligne category $\underline{\text{Rep}}\, GL(t)$.
  J. Comes (2012): thick ideals come from $\mathfrak{gl}(m,n)$.
- Deligne conjecture.
- Kazhdan–Lusztig theory for $\overline{\text{Rep}}$.
- Conjecture: F_x is semisimple.

Open problems

- Ideals in the Deligne category $\underline{\text{Rep}}\, GL(t)$.
  J. Comes (2012): thick ideals come from $\mathfrak{gl}(m,n)$.
- Deligne conjecture.
- Kazhdan–Lusztig theory for $\overline{\text{Rep}}$.
- Conjecture: F_x is semisimple.