Deligne’s tensor category $\text{Rep\ } \text{GL}(t)$ and general linear supergroups

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Interplay between supersymmetry and tensor categories

Supergroups \(\rightarrow\) tensor categories.
- Almost all examples of rigid symmetric tensor categories come from representation theory of superalgebras.
- Give rise to universal tensor categories.

Tensor categories \(\rightarrow\) supergroups.
- Classification of representations.
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Formalization of tensor product $\rightsquigarrow$ Tensor categories

- $\mathcal{A}$ is abelian and $k$-linear (morphisms are $k$-vector spaces);
- $\mathcal{A}$ is equipped with tensor product $\otimes$, i.e. an exact functor $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$, $k$-linear in both variables, with (functorial) isomorphism $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$;
- $\mathcal{A}$ has a unit object $1$ such that End($1$) = $k$;
- Symmetry: functorial isomorphism $s : X \otimes Y \to Y \otimes X$ such that the composition

$$X \otimes Y \overset{s}{\to} Y \otimes X \overset{s}{\to} X \otimes Y$$

is the identity. (Braiding $\rightsquigarrow$ sign rule in supercase.) Warning: no braid groups.
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Contravariant duality functor $\ast : \mathcal{A} \to \mathcal{A}$, $X \mapsto X^*$;

Natural maps: identity $e : 1 \to X \otimes X^*$;

Contraction (trace) $c : X^* \otimes X \to 1$;

Compositions

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X \xrightarrow{e \otimes 1} X \otimes X^* \otimes X \xrightarrow{1 \otimes c} X
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and

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are both equal to the identity $1_X$.

A tensor category $\mathcal{A}$ with duality is called rigid.
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Main Examples

The category $\text{Vect}$ of finite-dimensional vector spaces.

- $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$, $(x \otimes y) \otimes z \mapsto x \otimes (y \otimes z)$, $1 = k$;
- $s : X \otimes Y \to Y \otimes X$, $s(x \otimes y) := y \otimes x$;
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- $e : 1 \to X \otimes X^*$, $e(1) := \sum e_i \otimes f_i$.

The category $\text{SVect}$ of finite-dimensional vector superspaces.

- Objects are $\mathbb{Z}_2$-graded vector spaces $X = X_0 \oplus X_1$.
- The main difference with Vect:

$$s : X \otimes Y \to Y \otimes X, \quad s(x \otimes y) := (-1)^{\bar{x}\bar{y}} y \otimes x.$$ 

- $c$ and $e$ are defined by the same formulas as for usual vector spaces.
Symmetric rigid tensor categories
Deligne categories and classical supergroups

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Using the rigidity axiom one can construct a canonical isomorphism

$$\delta : \text{End}(X) \xrightarrow{\sim} \text{Hom}(1, X \otimes X^*)$$

and define the trace:

$$\text{tr} : \text{End}(X) \to \text{End}(1) = k$$

as the composition

$$1 \xrightarrow{\delta(\varphi)} X \otimes X^* \xrightarrow{s} X^* \otimes X \xrightarrow{c} 1.$$ 

By definition

$$\dim X = \text{tr} 1_X.$$
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• Vector spaces: dimension and trace are as usual.
• Vector superspaces: the (super)trace of a linear operator is

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\text{str} \left( \begin{array}{c|c}
A & B \\
C & D
\end{array} \right) = \text{tr} A - \text{tr} D.
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In general, the dimension of an object in a symmetric rigid tensor category can be any element of $k$. 
G algebraic group (over $k$), for example, $GL(n)$. The category Rep $G$ of finite-dimensional representations of $G$ is a symmetric rigid tensor category.

A functor $F : \text{Rep } G \to \text{Vect}$ (forgetting the $G$-action). Tensor functor, i.e., preserves all structures of tensor categories, faithful (injective on morphisms), exact.

An exact faithful tensor functor $F : \mathcal{A} \to \text{Vect}$ is called a fiber functor.

A finitely generated symmetric rigid tensor category which has a fiber functor is equivalent to Rep $G$ for some algebraic group $G$. (Tannakian categories $\leadsto$ affine group schemes).
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Example. $GL(n)$ may be a group $G$ and may be supergroup $G_{\text{super}}$. $\text{Rep } G_{\text{super}}$ has twice as many objects as $\text{Rep } G$:

$$V \leftrightarrow \Pi V, \quad V_0 \otimes V_1$$

Deligne’s trick (Halving the category):

- $G$ is a supergroup, fix $g \in G_0$.
- $\text{Rep}(G, g)$ is the subcategory of $\text{Rep } G$, consisting of representations $V$ satisfying $g(v) = (-1)^\bar{v}v$.
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Supertannakian formalism

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Theorem (Deligne, 2002)

A tensor category $\mathcal{A}$ is equivalent to some $\text{Rep}(G, g)$ if and only if

- There exists a fiber functor $\mathcal{A} \to \text{SVect}$;
- $\mathcal{A}$ is finitely generated.
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Schur functor

- Consider a tensor category \( \mathcal{A} \).
- For any object \( X \) in \( \mathcal{A} \) consider \( X^{\otimes n} \).
- \( \sigma_{i,i+1} \mapsto 1^{\otimes i-1} \otimes s \otimes 1^{\otimes n-i-1} \mapsto S_n \rightarrow \text{Aut}(X^{\otimes n}) \).

Schur–Weyl duality:

\[
X^{\otimes n} = \bigoplus_{|\lambda|=n} V_\lambda \otimes S_\lambda(X).
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Definition

\( X \mapsto S_\lambda(X) \) is called the Schur functor \( S_\lambda : \mathcal{A} \rightarrow \mathcal{A} \).
Consider a tensor category $\mathcal{A}$.

For any object $X$ in $\mathcal{A}$ consider $X \otimes^n$.

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Theorem (A. Sergeev, 1982)

Let $V$ be an $(m|n)$-dimensional superspace. Then $S_{\lambda}(V) \neq 0$ if and only if $\lambda$ can be covered by an $(m, n)$-hook, or equivalently, $\lambda$ does not contain a rectangular diagram of size $(m + 1) \times (n + 1)$.

Theorem (P. Deligne, 2002)

Let $A$ be a finitely generated rigid symmetric tensor category. The following conditions are equivalent:
(a) Every generator is annihilated by some Schur functor.
(b) $A$ is equivalent to $\text{Rep}(G, g)$ for some algebraic supergroup $G$. 
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Universal tensor categories

- Consider tensor categories generated by one object $X$;
- Want to construct a universal one;
- Analogy: free ring with two generators $X$ and $X^*$;
- Deligne’s category $\text{Rep } GL(t)$, $t = \dim X$.

- Abelianization: a way to extend a category to abelian by forcefully adding kernels and images of all morphisms. There is no such bliss!
- The subtle point. Karoubization: adding kernels and images of all projectors ($p^2 = p$). It exists!
- $\text{Rep } GL(t)$ is a result of Karoubization. Not abelian.
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Deligne’s category $\text{Rep} GL(t)$, $t = \text{dim } X$.

Abelianization: a way to extend a category to abelian by forcefully adding kernels and images of all morphisms. There is no such bliss!

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Universal tensor categories

- Consider tensor categories generated by one object $X$;
- Want to construct a universal one;
- Analogy: free ring with two generators $X$ and $X^*$;
- Deligne’s category $\text{Rep} \ GL(t)$, $t = \dim X$.

- **Abelianization:** a way to extend a category to abelian by forcefully adding kernels and images of all morphisms. There is no such bliss!
- **The subtle point. Karoubization:** adding kernels and images of all projectors ($p^2 = p$). It exists!
- $\text{Rep} \ GL(t)$ is a result of Karoubization. Not abelian.
Ingredients of construction

- Skeleton category.
  - objects: $X^\otimes m \otimes (X^*)^\otimes n$,
  - morphisms: $\text{Hom}(X^\otimes m \otimes (X^*)^\otimes n, X^\otimes k \otimes (X^*)^\otimes l) = \text{walled Brauer diagrams}$.

- Additive closure (adding direct sums).
- Karoubian envelope (adding kernels of all projectors).
Properties of Deligne’s categories

- \( \text{Rep } GL(t) \) is **semisimple** and therefore abelian if and only if \( t \notin \mathbb{Z} \).
- Indecomposable objects in \( \text{Rep } GL(t) \) are enumerated by pairs of partitions: \( (\lambda, \mu) \leftrightarrow Y_{\lambda, \mu} \).
- \( Y_{\lambda, \mu}^* \cong Y_{\mu, \lambda} \).
- \( \dim Y_{\lambda, \mu} \) is a polynomial in \( t \).
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If $t = n \in \mathbb{Z}$, the Deligne categories have canonical semisimple quotients.

Semisimple quotients of $\text{Rep} \ GL(t)$:
- $\text{Rep}(GL(n), 1)$ for $n > 0$;
- $\text{Rep}(GL(-n), -1)$ for $n < 0$.

Universality

**Theorem (P. Deligne, 2002)**

Let $A$ be a symmetric rigid $k$-linear category, and $V$ be an object of dimension $t$. There exists a unique (up to isomorphism) tensor functor $F : \text{Rep} \ GL(t) \to A$ such that $F(X) = V$. 
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Towards abelianization: building blocks

\[ t = m - n = m' - n' \]

\[ \text{Rep } GL(t) \xrightarrow{F_{m,n}} \text{Rep } GL(m, n) \]

\[ \xrightarrow{F_{m',n'}} \text{Rep } GL(m', n') \]

Tensor functor \( E_{m,n} : \text{Rep } GL(m, n) \rightarrow \text{Rep } GL(m - 1, n - 1) \).
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\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
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Tensor functor $F_x$ (Duflo, V.S.)

- $g$ Lie superalgebra;
- $x \in g_1$ such that $[x, x] = 2x^2 = 0$;
- If $M$ is a representation of $g$, then $x^2 M = 0$;
- Set $M_x = \ker x/\im x$, $g_x = \ker \text{ad}_x / \im \text{ad}_x$ (cohomology);
- $M_x$ is a representation of $g_x$;
- $F_x : \text{Rep } g \rightarrow \text{Rep } g_x$ ($F_x(M) := M_x$) is a tensor functor.

Our case: $g = gl(m, n)$, $x$ is an odd matrix of rank 1, $g_x = gl(m-1, n-1)$. Notation $F_x = E_{m,n}$
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Our case: $\mathfrak{g} = \mathfrak{gl}(m, n)$, $x$ is an odd matrix of rank 1, $\mathfrak{g}_x = \mathfrak{gl}(m - 1, n - 1)$. Notation $F_x = E_{m,n}$
Towards abelianization: filtration

Lemma

A filtration on $\text{Rep} \, GL(m, n)$ (tensor rank $\geq k$)

$$\mathcal{F}^1 \text{Rep} \, GL(m, n) \subset \cdots \subset \mathcal{F}^k \text{Rep} \, GL(m, n) \subset \cdots,$$

- $\mathcal{F}^k \text{Rep} \, GL(m, n) \otimes \mathcal{F}^l \text{Rep} \, GL(m, n) \to \mathcal{F}^{k+l} \text{Rep} \, GL(m, n)$;
- $E_{m,n}$ preserves this filtration;
- If $m, n \gg k$ then

$$E_{m,n} : \mathcal{F}^k \text{Rep} \, GL(m, n) \to \mathcal{F}^k \text{Rep} \, GL(m - 1, n - 1)$$

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is an equivalence of abelian categories.
First, horizontal inverse limit, then vertical direct limit.
Let $t \in \mathbb{Z}$ be fixed and $m - n = t$. For every $k > 0$ one can define an abelian category

$$
\text{Rep}_k \ GL(t) := \lim \leftarrow F^k \text{Rep} \ GL(m, n).
$$

Lemma (Abelianization)

(a) \quad \text{Rep} \ GL(t) := \lim \rightarrow \text{Rep}_k \ GL(t)

is a symmetric rigid tensor category (abelian!).

(b) There exists a fully faithful functor $H : \text{Rep} \ GL(t) \rightarrow \text{Rep} \ GL(t)$. 
Let $t \in \mathbb{Z}$ be fixed and $m - n = t$. For every $k > 0$ one can define an abelian category

$$\overline{\text{Rep}}_k \ GL(t) := \lim \mathcal{F}^k \text{Rep} \ GL(m, n).$$

**Lemma (Abelianization)**

(a) $\overline{\text{Rep}} GL(t) := \lim \overline{\text{Rep}}_k \ GL(t)$

is a symmetric rigid tensor category (**abelian!**).

(b) There exists a fully faithful functor $H : \overline{\text{Rep}} GL(t) \to \overline{\text{Rep}} GL(t)$. 
Theorem

Let $\mathcal{A}$ be a symmetric rigid tensor category and $F : \text{Rep } GL(t) \to \mathcal{A}$ be a tensor functor.

(a) $V = F(X)$ is annihilated by some Schur functor $\Rightarrow \exists \Phi$

(b) $V = F(X)$ is not annihilated by any Schur functor $\Rightarrow \exists \Phi$
The category \( \text{Rep} \, gl(\infty) \)

\[ g = gl(\infty) = \lim_{\to} gl(n). \]

Let \( V \) and \( V_\ast \) be natural and conatural modules.

**Definition**

A subalgebra \( \mathfrak{k} \subset g \) is a finite corank subalgebra if there exist finite dimensional subspaces \( W \subset V \) and \( W' \subset V_\ast \) such that \( \mathfrak{k} \) annihilates every vector in \( W, W' \).

We define \( \text{Rep} \, gl(\infty) \) as a full subcategory of \( g \)-modules whose objects \( M \) satisfy the following conditions

- \( M \) is integrable;
- For every \( m \in M \) the annihilator of \( m \) in \( g \) has finite corank;
- \( M \) has finite length.
The category $\text{Rep} \, \mathfrak{gl}(\infty)$

$$\mathfrak{g} = \mathfrak{gl}(\infty) = \lim_{\to} \mathfrak{gl}(n).$$

Let $V$ and $V_*$ be natural and conatural modules.

**Definition**

A subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is a finite corank subalgebra if there exist finite dimensional subspaces $W \subset V$ and $W' \subset V_*$ such that $\mathfrak{k}$ annihilates every vector in $W, W'$.

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Rep $\mathfrak{gl}(\infty)$ is symmetric, monoidal and universal in this class. (Penkov–Dan-Cohen –S., Sam–Snowden)

There exists a faithful tensor functor $K : \text{Rep } \mathfrak{gl}(\infty) \to \text{Rep } \text{GL}(t)$.

$$\text{Rep } \text{GL}(t) \xrightarrow{H} \text{Rep } \text{GL}(t) \xleftarrow{K} \text{Rep } \mathfrak{gl}(\infty).$$

- Rep $\text{GL}(t)$ is symmetric rigid, but not abelian.
- Rep $\mathfrak{gl}(\infty)$ is symmetric abelian but not rigid.
- Rep $\text{GL}(t)$ is a locally highest weight category.
- Simple objects in $\text{Rep } \mathfrak{gl}(\infty)$ correspond to standard objects in $\text{Rep } \text{GL}(t)$.
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Open problems

- Ideals in the Deligne category $\text{Rep } GL(t)$.
  J. Comes (2012): thick ideals come from $\mathfrak{gl}(m, n)$.
- Deligne conjecture.
- Kazhdan–Lusztig theory for $\text{Rep}$.
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