Applications of geometry to modular representation theory

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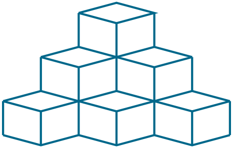


October 25, 2014

G - finite group, k - field.

Study *Representation theory of G over the field k*:

Let M be a vector space over k,
let G act on M via linear transformations:

$$G \times M \longrightarrow M.$$

Equivalently, if $\dim_k M = n$,

$$G \longrightarrow \mathrm{Aut}_k(M) \cong \mathrm{GL}_n(k)$$

$$g \longmapsto (a_{ij}).$$

92 HEINRICH MASCHKE: HIS LIFE AND WORK. [Nov.,

Perhaps of even greater importance is the following theorem * to which Maschke was led in the course of the proof of his cyclotomic theorem: *Every finite group of linear substitutions, all of whose substitutions contain in the same place (not in the principal diagonal) a coefficient equal to zero, is intransitive, i. e.,* it can be so transformed that the new variables fall into a number of sets such that the variables of each set are transformed among themselves. In Burnside's terminology, the essential part of the theorem may be briefly formulated as follows: *Every group of linear substitutions of finite order is completely reducible.*

Bulletin of the AMS, 1908

Heinrich Maschke
1853-1908

Theorem (Maschke, 1898)

Let $G \to \mathrm{GL}_n(\mathbb{C})$ be a complex matrix representation of a finite group G, and assume that all matrices corresponding to the elements of the group have the form $\begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix}$, where the dimension of A_1 is a fixed number $r < n$. Then the representation is equivalent to the one of the same form where all submatrices B are equal to 0.

INDECOMPOSABLE VS. IRREDUCIBLE

A representation is called *reducible* if it has a subrepresentation

$$0 \neq M_1 \subsetneq M.$$

Otherwise, it is *irreducible* or *simple*.
A representation is called *indecomposable* if it does not split as a direct sum of subrepresentations:

$$M \ncong M_1 \oplus M_2.$$

In **char** 0, Maschke's theorem $\Rightarrow$ indecomposable = irreducible.

In **char** p, there are tons of indecomposable modules which are not irreducible: Maschke's theorem fails miserably!

Modular case: $\operatorname{char} k = p$

Table: Indecomposable representations of $G = \mathbb{Z}/p = \langle \sigma \rangle$

name	$[p]$	$[p-1]$	$\dots$	$[3]$	$[2]$	$[1]$
$\sigma - 1$	$\begin{pmatrix} 0 & \mathbf{1} & 0 & & & \\ 0 & 0 & \mathbf{1} & 0 & & \\ & 0 & 0 & \mathbf{1} & 0 & \\ & \vdots & \vdots & \vdots & \vdots & 0 \\ & & & 0 & 0 & \mathbf{1} \\ & & & & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \mathbf{1} & 0 & & \\ 0 & 0 & \mathbf{1} & 0 & \\ & 0 & 0 & \mathbf{1} & 0 \\ & \vdots & \vdots & \vdots & \mathbf{1} \\ & & & 0 & 0 \end{pmatrix}$	$\dots$	$\begin{pmatrix} 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \mathbf{1} \\ 0 & 0 \end{pmatrix}$	(0)

Table: Irreducible representations of $G = \mathbb{Z}/p = \langle \sigma \rangle$

dim	$[1]$
$\sigma - 1$	(0)

$$\text{Representations of } \mathbb{Z}/p \longleftrightarrow \begin{array}{c}\text{Jordan canonical forms of } \sigma, \\ [p]^{a_p} \dots [2]^{a_2} [1]^{a_1}\end{array}$$

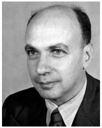
Representations theory

Char 0 → Irreducible characters

Char p → Character theory - not enough!


Frobenius


Burnside


Richard Brauer

C. Curtis, "Pioneers of Representation Theory".

COHOMOLOGY

From now on: $\boxed{k = \overline{\mathbb{F}}_p,\ p \text{ divides } |G|}$.

Representation theory of G (over k) is almost always wild: it is impossible to classify indecomposable modules.

Cyclic group $\mathbb{Z}/p$ is a rare - and useful - exception.

To navigate this wild territory, find useful invariants.
(1) Irreducible $\neq$ indecomposable $\Rightarrow$ lots of non-split extensions

$$0 \to M_1 \to M \to M_2 \to 0$$

(2) The functor $M \mapsto M^G$ is not exact $\Rightarrow$ study its derived functors

$$(1) + (2) \Rightarrow \text{cohomology } H^*(G, M).$$

Origins - topology, Eilenberg-Steenrod.

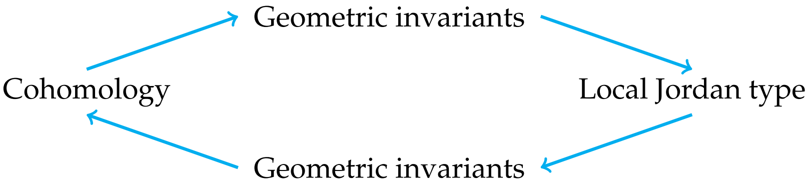
D. Quillen, *"The spectrum of an equivariant cohomology ring* I, II,*"* Ann. Math. 94 (1971)

$\Longrightarrow$ new chapter in modular representation theory.

$$G \rightsquigarrow \operatorname{Spec} H^*(G,k) = |G|$$

an affine algebraic variety.

Plan of the talk:



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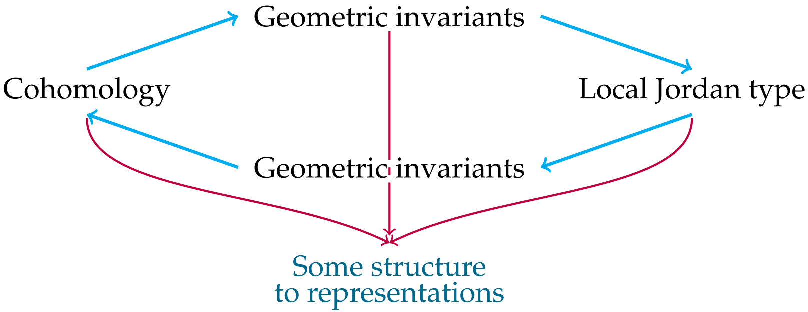


QUILLEN STRATIFICATION

$E = (\mathbb{Z}/p)^{\times n}$ - an elementary abelian p-group of rank n.

$$H^*(E,k) = k[Y_1,\ldots,Y_n] \otimes \underbrace{\Lambda^*(s_1,\ldots,s_n)}_{\text{nilpotents}}$$

$|E| = \operatorname{Spec} H^*(E,k) = \operatorname{Spec} k[Y_1,\ldots,Y_n] \simeq \mathbb{A}^n$
$\operatorname{Proj} H^*(E,k) = \mathbb{P}^{n-1}$

Theorem (Quillen, 1971)

$|G| = \operatorname{Spec} H^*(G,k)$ is stratified by $|E| \subset |G|$, where $E \subset G$ runs over all elementary abelian p-subgroups of G.

Corollary (Atiyah-Swan conjecture)

$$\operatorname{Krull\,dim} H^*(G,k) = \max_{E\subset G} \operatorname{rk} E$$

SUPPORT VARIETIES FOR MODULES

Alperin-Evens, Carlson:

$$M \longmapsto \operatorname{supp} M \subset \operatorname{Proj} H^*(G,k)$$

$\operatorname{supp} M$ - an algebraic variety defined in terms of the action of $H^*(G,k)$ on $\operatorname{Ext}^*(M,M)$.

- Quillen stratification theorem for $\operatorname{supp} M$
- Realization (modules are not only “wild” but ubiquitous): For any closed subvariety $X \subset \operatorname{Proj} H^*(G,k)$, there exists a finite dimensional representation M such that $\operatorname{supp} M = X$
- Tensor product theorem: $\operatorname{supp} M \cap \operatorname{supp} N = \operatorname{supp} M \otimes N$

LOCAL JORDAN TYPE

Carlson: $\operatorname{supp} M$ can be described in an "elementary" way.

Need notation:

$E = (\mathbb{Z}/p)^{\times n}$. Choose generators $\sigma_1, \ldots, \sigma_n$.

Let $x_i = \sigma_i - 1$.

The group algebra

$$kE = \frac{k[\sigma_1, \ldots, \sigma_n]}{(\sigma_1^p - 1, \ldots, \sigma_n^p - 1)} \simeq \frac{k[x_1, \ldots, x_n]}{(x_1^p, \ldots, x_n^p)}$$

thanks to the "freshman calculus rule": $\sigma_i^p - 1 = (\sigma_i - 1)^p$.

$$\{\text{Representations of } E\} \overset{\sim}{\longleftrightarrow} \{kE \text{ - modules}\}$$

Local approach:

$$\lambda = (\lambda_1, \ldots, \lambda_n) \longmapsto X_\lambda = \lambda_1 x_1 + \ldots + \lambda_n x_n \in kE$$

For M a kE-module,

$$M \longmapsto \{ \quad \mathrm{JType}(X_\lambda, M) \mid \lambda \in k^n \quad \}$$

Dade, Carlson: $\langle X_\lambda + 1 \rangle \simeq \mathbb{Z}/p \subset kE$ - cyclic shifted subgroup.

Table: Indecomposable Jordan blocks for $\mathrm{JType}(X_\lambda, M)$

name	$[p]$	$[p-1]$	$\ldots$	$[3]$	$[2]$	$[1]$
block	$\begin{pmatrix} 0 & 1 & 0 & & & \\ 0 & 0 & 1 & 0 & & \\ & 0 & 0 & 1 & 0 & \\ & \vdots & \vdots & \vdots & \vdots & 0 \\ & & & 0 & 0 & 1 \\ & & & & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & 0 & \\ & 0 & 0 & 1 & 0 \\ & \vdots & \vdots & \vdots & 1 \\ & & & 0 & 0 \end{pmatrix}$	$\ldots$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	(0)

$$\mathrm{JType}\{X_\lambda, M\} \quad \longleftrightarrow \quad [p]^{a_p} \ldots [2]^{a_2} [1]^{a_1}$$

WHAT CAN BE DETECTED LOCALLY?

Theorem (Dade, 1978,"Dade's Lemma")

Let $E = (\mathbb{Z}/p)^{\times n}$, and let M be a finite dimensional kE-module. Then M is free if and only if

$$\mathrm{JType}(X_\lambda, M) = [p]^m$$

for every $\lambda \in k^n \backslash \{0\}$.

Equivalently, the restriction of M to every cyclic shifted subgroup $\langle X_\lambda + 1 \rangle$ is free.

Theorem (Avrunin-Scott, 1982)

$$\underbrace{\operatorname{supp} M}_{\text{cohomology}} = \underbrace{\{\lambda = [\lambda_1 : \ldots : \lambda_n] \in \mathbb{P}^{n-1} \mid \mathrm{JType}(\lambda, M) \neq [p]^m\}}_{\text{local approach}}$$

Corollary $\operatorname{supp} M \cap \operatorname{supp} N = \operatorname{supp} M \otimes N$

In a joint work with E. Friedlander, the *local approach* has been generalized to any *finite group (scheme)* via the notion of π-points. We proved

- Avrunin-Scott's theorem (local approach to supports = cohomological approach)
- Appropriate analogue of Quillen stratification
- Tensor product theorem for supports

The theory of π-points led to discovery of a new class of modules which turned out to be very interesting even for elementary abelian p-groups: modules of constant Jordan type.

MODULES OF CONSTANT JORDAN TYPE

$E = (\mathbb{Z}/p)^{\times n}, kE = k[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p),$
$\lambda = (\lambda_1, \ldots, \lambda_n), X_\lambda = \lambda_1 x_1 + \ldots + \lambda_n x_n.$

Definition [Carlson-Friedlander-P., 2008]

M is a module of constant Jordan type if $\mathrm{JType}(X_\lambda, M)$ is independent of $\lambda \neq 0$.

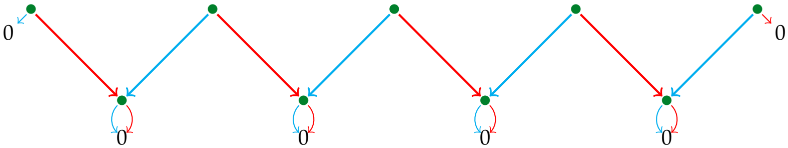
Friedlander-P-Suslin, 2007: the property of constant Jordan type is independent of the choice of generators of E.

Special case of a much more general theorem: maximal Jordan type of a module for any finite group scheme is well-defined.

Pictorial examples

$E = \mathbb{Z}/p \times \mathbb{Z}/p$, $kE = k[x_1, x_2]/(x_1^p, x_2^p)$
M is a kE-module, $\dim M = 9$
Basis of M: green dots •. Action of E: x_1 ↘ x_2 ↙

"Picture" of M:



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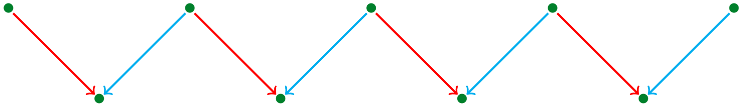

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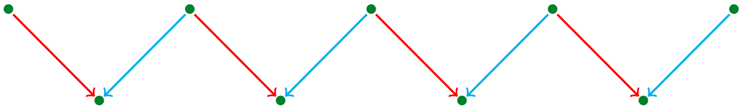

$\lambda = (1, 0)$
$X_\lambda = \lambda_1 x_1 + \lambda_2 x_2 = x_1$
$\mathrm{JType}(x_1, M) = ?$

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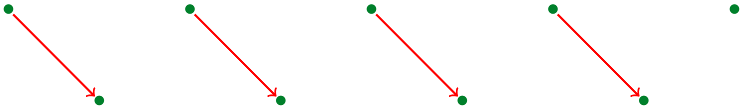

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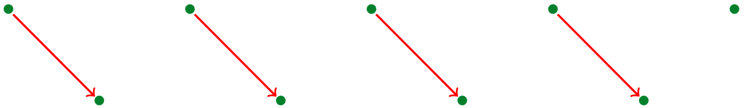

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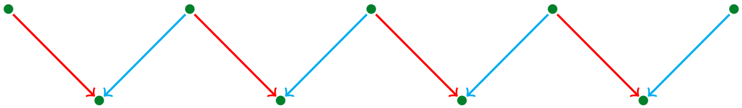

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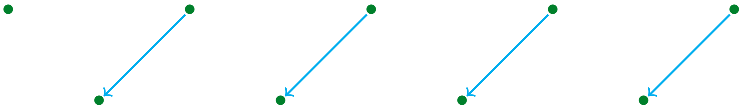

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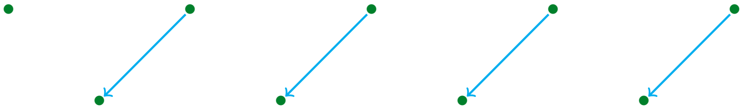

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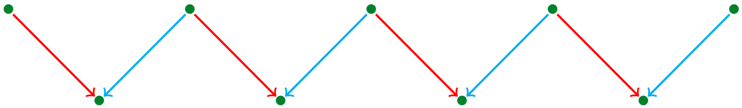

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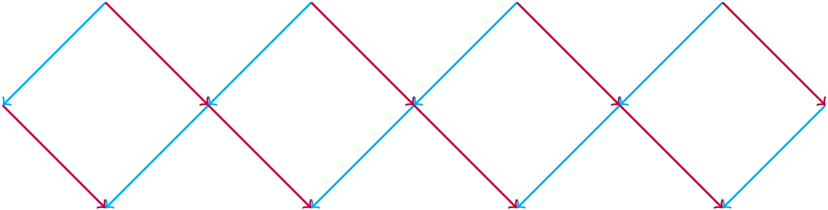
Indeed, M is a module of Constant Jordan type $[2]^4[1]$.

$E = \mathbb{Z}/p \times \mathbb{Z}/p, kE = k[x_1, x_2]/(x_1^p, x_2^p)$
$\dim M = 13$



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M is a module of constant Jordan type only for p=5.
For $p > 5$,

$$\mathrm{JType}(x_1 + x_2, M) = [3]^4[1]$$

Realization of Jordan types

Question

Which Jordan types can be realized with modules of constant Jordan type?

Theorem (Benson, 2010)

Assume $\operatorname{rk} E \geq 2$, $p \geq 5$. *There does not exist a module of constant Jordan type*

$$[p]^a [2]$$

Conjectures of Suslin, Rickard restricting possible Jordan types - wide open.

Most recent progress - Benson, Baland, using geometric methods.

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$$[p]^a[j]$$

$2 \leq j \leq p-2$.

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We still know little about the wild representation theory of E.
J. Carlson, E. Friedlander, A. Suslin, "Modules for $\mathbb{Z}/p \times \mathbb{Z}/p$", Comment. Math. Helv. 86 (2011).
What can we do? Compare kE-modules to another category we know little about!

"Globalize" the action of X_λ on a kE-module M.

$kE = k[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$

$k[Y_1, \ldots, Y_n]$ - homogeneous coordinate ring of $\mathbb{P}^{n-1}$

$$\Theta = x_1 \otimes Y_1 + \cdots + x_n \otimes Y_n \in kE \otimes k[Y_1, \ldots, Y_n]$$

$X_\lambda = \lambda_1 x_1 + \ldots + \lambda_n x_n$ - specialization of Θ under $(Y_1, \ldots, Y_n) \mapsto (\lambda_1, \ldots, \lambda_n)$

$$\Theta_M : M \otimes \mathcal{O}_{\mathbb{P}^{n-1}} \longrightarrow M \otimes \mathcal{O}_{\mathbb{P}^{n-1}}(1).$$

Specializing at $\lambda = [\lambda_1 : \ldots : \lambda_n] \quad \rightsquigarrow \quad$ action of X_λ on M.

FROM MODULES OF CJT TO VECTOR BUNDLES

$$\Theta_M : M \otimes \mathcal{O}_{\mathbb{P}^{n-1}} \longrightarrow M \otimes \mathcal{O}_{\mathbb{P}^{n-1}}(1).$$

$$\Theta_M(m \otimes f) = \sum x_i m \otimes Y_i f$$

Theorem (Friedlander-P., 2008)

From modules of CJT to vector bundles

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Theorem (Friedlander-P., 2008)

If M is a module of constant Jordan type for an elementary abelian p-group E of rank n, then

$$\operatorname{Ker} \Theta_M, \operatorname{Im} \Theta_M, \operatorname{Coker} \Theta_M$$

are algebraic vector bundles on $\mathbb{P}^{n-1}$.

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Theorem (Friedlander-P., 2008)

If M is a module of constant Jordan type for a restricted Lie algebra $\mathfrak{g}$, then

$$\operatorname{Ker}\Theta_M, \operatorname{Im}\Theta_M, \operatorname{Coker}\Theta_M$$

are algebraic vector bundles on the projectivization of the nilpotent cone $\mathcal{N}(\mathfrak{g})$.

From modules of CJT to vector bundles

$$\Theta_M : M \otimes \mathcal{O}_{\mathbb{P}^{n-1}} \longrightarrow M \otimes \mathcal{O}_{\mathbb{P}^{n-1}}(1).$$

$$\Theta_M(m \otimes f) = \sum x_i m \otimes Y_i f$$

Theorem (Friedlander-P., 2008)

If M is a module of constant Jordan type for an infinitesimal group scheme G, then

$$\operatorname{Ker} \Theta_M, \operatorname{Im} \Theta_M, \operatorname{Coker} \Theta_M$$

are algebraic vector bundles on $\operatorname{Proj} H^*(G, k)$.

FROM MODULES OF CJT TO VECTOR BUNDLES

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are algebraic vector bundles on $\mathbb{P}^{n-1}$.

BUNDLES ON $\mathbb{P}^n$

Horrocks-Mumford bundle, 1972, an indecomposable rank 2 bundle on $\mathbb{P}^4$ with 15000 symmetries.

Reconstructed by D. Benson from a $(\mathbb{Z}/p)^5$-module of dim 30 via the correspondence given by Θ.

Open Question

Does there exist an indecomposable rank 2 algebraic vector bundle on $\mathbb{P}^n$, $n \geq 6$?

Hartshorne's conjecture: NO.

For $p = 2$, the Tango[1] bundle of rank 2 on $\mathbb{P}^5$ is an indecomposable bundle of rank 2.

[1] Tango, Hiroshi - Japanese mathematician

Realization for vector bundles

$$\Theta_M : M \otimes \mathcal{O}_{\mathbb{P}^{n-1}} \longrightarrow M \otimes \mathcal{O}_{\mathbb{P}^{n-1}}(1).$$

$$\mathcal{F}_i(M) := \frac{\operatorname{Ker} \Theta_M \cap \operatorname{Im} \Theta_M^{i-1}}{\operatorname{Ker} \Theta_M \cap \operatorname{Im} \Theta_M^{i}}$$

M - module of CJT $[p]^{a_p} \dots [1]^{a_1} \quad \Rightarrow \quad \dim \mathcal{F}_i(M) = a_i$.

Theorem (Benson-P., 2012)

For any vector bundle $\mathcal{F}$ on $\mathbb{P}^{n-1}$, there exists a kE-module M of constant Jordan type such that

(i) *if $p = 2$, then $\mathcal{F}_1(M) \cong \mathcal{F}$.*

(ii) *if p is odd, then $\mathcal{F}_1(M) \cong F^*(\mathcal{F})$, where $F\colon \mathbb{P}^n_k \to \mathbb{P}^n_k$ is the Frobenius morphism.*

FINITE GROUP SCHEMES

$$\left\{ \begin{array}{c} \text{finite group} \\ \text{schemes} \\ G \end{array} \right\} \sim \left\{ \begin{array}{c} \text{finite dimensional} \\ \text{cocommutative} \\ \text{Hopf algebras} \\ kG \end{array} \right\}$$

For geometrically minded: $kG = k[G]^* = \mathrm{Hom}_k(k[G], k)$.

$$\{\text{Representations of } G \text{ over } k\} \quad \longleftrightarrow \quad \{kG\text{-modules }\}$$

Examples:

- Finite groups. kG is the group algebra
- Restricted Lie algebras. For $\mathcal{G}$ - algebraic group (GL_n, SL_n, Sp_{2n}, SO_n), $\mathfrak{g} = \mathrm{Lie}\,\mathcal{G}$
- Frobenius kernels $\mathcal{G}_{(r)} = \mathrm{Ker}\, F^{(r)} : \mathcal{G} \to \mathcal{G}$

Local approach for a finite group (scheme) G:

Replace $X_\lambda = \lambda_1 x_1 + \ldots + \lambda_n x_n \in kE$ with π-points

$$\alpha : k[t]/t^p \to kG$$

Theorem (Dade's lemma revisited)

Let G be a finite group scheme, and M be a kG-module. Then M is projective if and only if for every field extension K/k and every flat algebra map $\alpha : K[t]/t^p \to KG_K$, the $K[t]/t^p$-module $\alpha^*(M_K)$ is projective.

Benson-Carlson-Rickard, Bendel, Pevtsova,
Benson-Iyengar-Krause-Pevtsova

Important: holds for infinite-dimensional modules.

It is impossible to classify indecomposable modules for kG, but we can classify equivalence classes of modules "up to extensions".

$$G \rightsquigarrow \operatorname{stmod} kG$$

Applying the most general version of Dade's lemma, the theory of support varieties and π-points, and ideas from topology (Bousfield localization), one can "stratify" $\operatorname{stmod} kG$ with $\operatorname{Proj} H^*(G, k)$ for *any finite group scheme* G:

$$\left\{ \begin{array}{c} \text{Thick tensor ideal} \\ \text{subcategories} \\ \text{of } \operatorname{stmod} kG \end{array} \right\} \sim \left\{ \begin{array}{c} \text{Subsets of } \operatorname{Proj} H^*(G, k) \\ \text{closed under} \\ \text{specialization} \end{array} \right\}$$

Precursors/motivation: Devinatz-Hopkins-Smith (stable homotopy theory), Neeman, Thomason (AG).

BIG StMod G CATEGORY

D. Benson, S. Iyengar, H. Krause, *"Stratifying modular representations of finite groups"*, Ann. of Math. 174 (2011): StMod kG for a finite group is "stratified" by Proj $H^*(G,k)$:

$$\left\{ \begin{array}{c} \text{Localizing tensor} \\ \text{ideal subcategories} \\ \text{of StMod}\, kG \end{array} \right\} \sim \left\{ \begin{array}{c} \text{Subsets of} \\ \text{Proj}\, H^*(G,k) \end{array} \right\}$$

Techniques above (local Jordan type and π-points), combined with Benson-Iyengar-Krause theory of local cohomology functors and support, yielded a new, much shorter proof of more topological flavor of this classification (Benson-Iyengar-Krause-P., in progress).

QUIZ!

$E = (\mathbb{Z}/2)^{\times 3}$

$= \quad \Omega^3 k$

THANK YOU