Modular representations of bismash products

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Modular group representations: Brauer

G finite group, $\Bbbk$ algebraically closed of char $p > 2$.
Idea: Study G-reps over $\Bbbk$ by using G-reps over $\mathbb{C}$.

$(\mathbb{K}, R, \Bbbk)$ is a p *-modular system* for G if $R \supset \mathbb{Z}$ is a complete DVR with fraction field $\mathbb{K}$, $\mathbb{K}$ splits G, π is a generator for the maximal ideal P of R and $\Bbbk = R/\pi R$.

If $|G| = p^\alpha m$ for $p \nmid m$, then R contains ω, a primitive mth root of 1.

Under the quotient map $f : R \to \mathbb{k} = R/\pi R$, $\bar{\omega} = f(\omega)$ is a primitive mth root of 1 in $\mathbb{k}$.

Let $G_{p'} := \{x \in G \mid p \nmid o(x)\}$. For $x \in G_{p'}$ and any irreducible $\mathbb{k}G$-rep W, the eigenvalues of x on W may be written as $\{\bar{\omega}^{i_1}, \ldots, \bar{\omega}^{i_t}\}$.

The *Brauer character* $\phi : G_{p'} \to \mathbb{K}$ of G afforded by W is given by

$$\phi(x) = \omega^{i_1} + \cdots + \omega^{i_t}.$$

The *decomposition map* is a homomorphism of abelian groups

$$d : G_0(\mathbb{K}G) \to G_0(\Bbbk G)$$

taking a class $[V]$ in $G_0(\mathbb{K}G)$ to $[\overline{M}]$ in $G_0(\Bbbk G)$, where M is any RG-lattice in V with $\overline{M} := M/\pi M$.

Moreover if χ is the $\mathbb{K}$-character for V, then $\chi|_{G_{p'}}$ is the Brauer character of $[\overline{M}]$.

Let $V_1, \ldots, V_n$ be irred $\mathbb{K}$-reps of G with characters χ_i, and $W_1, \ldots, W_d$ the irred $\mathbb{k}$-reps of G with Brauer characters ϕ_j. Then the $\{\phi_j\}$ are a basis for the $\mathbb{K}$-valued class functions on $G_{p'}$. Thus

$$\chi_i|_{G_{p'}} = \sum_j d_{ij}\phi_j, \text{ for } d_{ij} \in \mathbb{Z}.$$

The d_{ij} are the *decomposition numbers*,
$D = [d_{ij}]$ is the *decomposition matrix*, and
$C = D^t D$ is the $d \times d$ *Cartan matrix*.

Theorem(Brauer) $Det(C)$ is a power of p.

Bismash Products

Let $L = FG$ be a factorizable group; that is, $F, G \subset L$, $F \cap G = 1$, and $L = FG$. Then also $L = GF$, and so for any $a \in F$, $x \in G$, there exist unique $a' \in F$, $x' \in G$ such that $xa = a'x' = (x \triangleright a)(x \triangleleft a)$.

Example: For $|G| = n$, $G \hookrightarrow S_n$ by left multiplication on G itself. Also $S_{n-1} \subset S_n$ by fixing the last element of G. Then $G \cap S_{n-1} = 1$ and $S_n = S_{n-1}G$ is a factorization.

F does **not** act as automorphisms of G:
using $S_4 = S_3C_4$, let $x = (1234)^2 \in C_4$, $a = (12) \in S_3$.
Then $xa = (321)(1234)^{-1}$, and so $x \triangleright (12) = (321)$.

For any field $\mathbb{E}$, the function algebra $\mathbb{E}^G = (\mathbb{E}G)^*$ is also a Hopf algebra, as is $\mathbb{E}^F$, using the transpose maps. $\mathbb{E}^G$ has a basis $\{p_g \mid g \in G\}$ dual to the basis of group elements for $\mathbb{E}G$. The actions $\triangleright$ and $\triangleleft$ induce group actions of F on $\mathbb{E}^G$ and of G on $\mathbb{E}^F$.

Let $H := \mathbb{E}^G \rtimes \mathbb{E}F$ be the semidirect product algebra. Similarly $H^* = \mathbb{E}^F \rtimes \mathbb{E}G$ is an algebra.

$H = H(L, F, G) = \mathbb{E}^G \# \mathbb{E}F$ is a Hopf algebra, called the **bismash product**.

Its coalgebra structure is obtained by dualizing the algebra structure of H^*; thus $\Delta_H = (m_{H^*})^*$.

Fix the basis $\mathcal{B} := \{p_x \# a \mid x \in G, a \in F\}$ of H. On $\mathcal{B}$, the antipode is given by $S(p_x \# a) = p_{(x \triangleleft a)^{-1}} \# (x \triangleright a)^{-1}$, and so $S^2 = id$.

Representations of $H = k^G \# kF$:
(DPR, Mason for $D(G)$ 1990, KMM 02)

For each orbit $\mathcal{O}$ of F on G, fix $x \in \mathcal{O} = \mathcal{O}_x$ and let $F_x =$ stabilizer of x. Let W_x be an irreducible rep of F_x, and define

$$V_x := \mathbb{C}G \otimes_{\mathbb{C}F_x} W_x.$$

V_x is an H-representation, all $y \in G, b \in F, w \in W_x$, via

$$p_y \cdot [b \otimes w] = \delta_{y \triangleleft b, x}(b \otimes w).$$

V_x is irreducible over H and all irreducible modules arise in this way.

Assume as for groups that $(\mathbb{K}, R, \Bbbk)$ is a p-*modular system* for F, and thus for the subgroups F_x. To define Brauer characters for H, we use a subset of $\mathcal{B}$, namely

$$\mathcal{B}_{p'} := \{p_y \# a \mid y \in G, a \in F_{p'} \cap F_y\}$$

This set is closed under S, since if $a \in F_y$, then $S(p_y \# a) = p_{y^{-1}} \# yay^{-1}$. Thus if $a \in F_{p'} \cap F_y$, then also $yay^{-1} \in F_{p'} \cap F_y$.

Recall for $S_4 = S_3 C_4$, can have $x \triangleright (12) = (321) \notin F_{3'}$.

[JM09; N05] Fix a set T_x of right coset reps of F_x in F. If $W = W_x$ is an irred $\Bbbk F_x$-module with character ψ, then the $H_{\Bbbk}$-module $\widehat{W}$ has character $\widehat{\psi}$ given by, for all $y \in G,\ a \in F$,

$$\widehat{\psi}(p_y \# a) = \sum_{t \in T_x,\ t^{-1}at \in F_x} \delta_{y \triangleleft t, x}\ \psi_x(t^{-1}at).$$

[JM 13] If W has classical Brauer character $\phi : \mathbb{K}F_x \to \mathbb{K}$, define the *Brauer character* $\widehat{\phi} : \mathcal{B}_{p'} \to \mathbb{K}$ of W to be

$$\widehat{\phi}(p_y \# a) = \sum_{t \in T_x,\ t^{-1}at \in F_x} \delta_{y \triangleleft t, x}\ \phi(t^{-1}at).$$

Let s be the number of distinct orbits $\mathcal{O}$ of F on G and choose $x_q \in \mathcal{O}_q$, for $q = 1, \ldots, s$. For each F_{x_q}:

Let $\{\chi_{x_q,i}\}$ be the irreducible characters of $\mathbb{K}F_{x_q}$.
Let $\{\psi_{x_q,j}\}$ be the irreducible characters of $\Bbbk F_{x_q}$ and let $\{\phi_{x_q,j}\}$ be their Brauer characters.
Let D_{x_q} be the decomposition matrix for the $\chi_{x_q,i}|_{(F_{x_q})_{p'}}$ in terms of the $\phi_{x_q,j}$.

Lifting $\chi_{x_q,i}$ to $\widehat{\chi}_{x_q,i}$ on $H_{\mathbb{K}}$ and $\phi_{x_q,j}$ to $\widehat{\phi}_{x_q,j}$ on $\mathcal{B}_{p'}$,

$$\widehat{\chi}_{x_q,i}|_{\mathcal{B}_{p'}} = \sum_j d_{\widehat{x_q},ij}\phi_{x_q,j}, \text{ for } d_{\widehat{x_q},ij} \in \mathbb{Z}.$$

Then $\widehat{D}_{x_q} = [\widehat{d_{x_q,ij}}]$ is the decomposition matrix for the $\widehat{\chi}_{x_q,i}|_{\mathcal{B}_{p'}}$ in terms of the $\widehat{\phi}_{x_q,j}$.

Theorem (1) $\widehat{D}_{x_q} = D_{x_q}$
(2) The decomposition matrix for the $\widehat{\chi}_i|_{\mathcal{B}_{p'}}$ with respect to the $\widehat{\phi}_j$ is the block matrix

$$\widehat{D} = \begin{bmatrix} \widehat{D}_{x_1} & 0 & \cdots & 0 \\ 0 & \widehat{D}_{x_2} & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & \widehat{D}_{x_s} \end{bmatrix}$$

where $\widehat{D}_{x_q}$ is the decomposition matrix of $\widehat{\chi}_{x_q,i}|_{\mathcal{B}_{p'}}$ with respect to $\widehat{\phi}_{x_q,j}$.

As for groups, $\widehat{C} = \widehat{D}^t \widehat{D}$ is called the *Cartan matrix*.

Theorem: $Det(\widehat{C})$ is a power of p, so is odd.

Proof: Apply Brauer's theorem to each block $\widehat{D}_{x_q}$.

Frobenius-Schur indicators

A bilinear form is H-$invariant$ if
$\sum \langle h_1 \cdot v, h_2 \cdot w \rangle = \varepsilon(h) 1_{\mathbb{E}}$ for all $h \in H,\ v, w \in V$.

Frobenius-Schur (1906) did case of $\mathbb{C}G$. Thompson (1986) case of char $p > 0$.

Theorem [GM09] Let H be fin dim over $\mathbb{E}$, split by $\mathbb{E}$, with $S^2 = id$, and let V be an irred H-module.
Then V has a well-defined Frobenius-Schur indicator $\nu(V) \in \{0, 1, -1\}$. Moreover
(1) $\nu(V) = 0 \iff V^* \ncong V$.
(2) $\nu(V) = 1$ (respectively -1) $\iff$ V admits a non-degenerate H-invariant symmetric (resp., alternating) bilinear form.

We prove an analog for bismash products of a theorem of J. Thompson (1986).

Theorem: (Jedwab-M 13) Let $\Bbbk$ be algebraically closed of characteristic $p > 2$, $L = FG$ a factorizable group, and $(\mathbb{K}, R, \Bbbk)$ a p *-modular system* for F. Consider the bismash products $H_{\mathbb{C}} = \mathbb{C}^G \# \mathbb{C}F$ and $H_{\Bbbk} = \Bbbk^G \# \Bbbk F$.

If $\psi = \psi^*$ is an irreducible $H_{\Bbbk}$-character with Brauer character ϕ, then there is an irreducible $\mathbb{K}$-character $\chi = \chi^*$ of $H_{\mathbb{K}}$ such that $d(\chi|_{\mathcal{B}_{p'}}, \phi)$ is odd.
Moreover for such a χ, $\nu(\chi) = \nu(\psi)$.

Corollary: (Jedwab-M 13) If all irreducible $H_{\mathbb{C}} = \mathbb{C}^G \# \mathbb{C}F$ modules have Schur indicator $+1$ (respectively ± 1), the same is true for all irreducible $H_{\Bbbk}$-modules.

Theorem: (Timmer 2014) For G of order n, consider $S_n = S_{n-1}G$ as above and let $H = \mathbb{C}^G \# \mathbb{C}S_{n-1}$. Then H is totally orthogonal for all n.

(J-M 09) case of $S_n = S_{n-1}C_n$.

Some ingredients in the proof:
Recall the p-modular system $(\mathbb{K}, R, \mathbb{k})$, where $\mathbb{Q} \subset \mathbb{K}$ and $\mathbb{K}$ splits $H_{\mathbb{Q}}$.

1. For V an $H_{\mathbb{K}}$-module, an $R\mathcal{B}$-*lattice* in V is a fin gen $R\mathcal{B}$-submodule L of V such that $\mathbb{K}L = V$.

2. For M a fin gen $H_{\mathbb{E}}$-module with a non-degen bilinear $H_{\mathbb{E}}$-invariant symmetric or skew form, let M_1 be max in M with $\langle M_1, M_1 \rangle = 0$, and let ${M_1}^{\perp} = \{m \in M | \langle m, M_1 \rangle = 0\}$.
The *Witt kernel* of M is $\mathcal{W}(M) := {M_1}^{\perp}/M_1$.

The blocks in $H_{\Bbbk}$ correspond to the blocks in $R\mathcal{B}$, by using the decomposition map and the remark about indecomposable modules.

Problem: Unfortunately $\phi^* = \phi$ does NOT imply $\widehat{\phi}^* = \widehat{\phi}$, nor conversely.

However if $\phi^* = \phi$, then the block B of $R\mathcal{B}$ containing $\widehat{\phi}$ satisfies $B^* = B$. Thus in the decomposition of any $\widehat{\chi}$ in B, the $\widehat{\phi}$ and $\widehat{\phi}^*$ appear in pairs, unless they are self-dual.
Get that the Cartan matrix for the self-dual $\widehat{\chi}_i$'s has odd determinant.