Odd structures arising from categorified quantum groups

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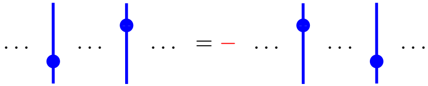

October 25th, 2014

Motivation from Knot theory

Jones polynomial $\longleftrightarrow$ Rep theory of quantum $\mathfrak{sl}_2$

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Khovanov homology $\longleftrightarrow$ Categorified Rep theory of $\mathfrak{sl}_2$

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The discovery of Khovanov homology motivated the study of categorified quantum $\mathfrak{sl}_2$.

This categorification is closely connected to

- The geometry of flag varieties and Grassmannians
- The combinatorics of symmetric functions
- Hecke algebras in type A

Odd Khovanov homology

Khovanov's categorification of the Jones polynomial is not unique.

Ozsváth, Rasmussen, Szabó

Khovanov homology $\neq$ Odd Khovanov homology

Decategorification

Jones polynomial

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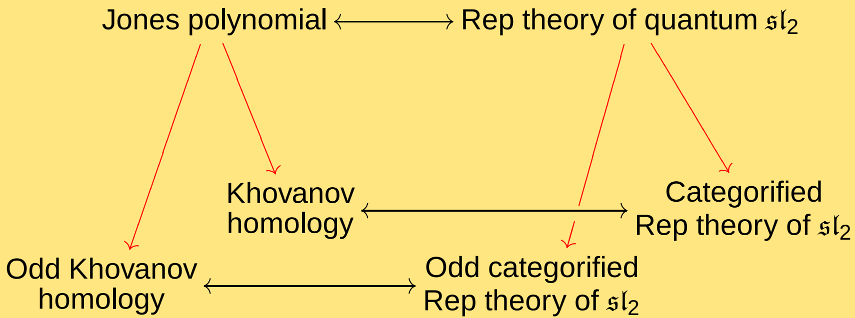
Decategorification: Khovanov homology → Jones polynomial ← Odd Khovanov homology

- Both theories categorify the Jones polynomial
- Both theories agree when coefficients are reduced modulo two
- Shumakovitch showed that both theories are distinct

Idea: Utilize these discoveries in knot theory to discover new structures in geometric representation theory via the connection to quantum groups



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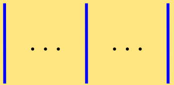
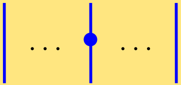
In this story the nilHecke algebra is the star of the show.

Generators for the NilHecke algebra



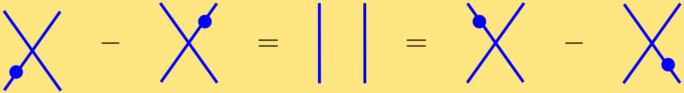
$:= \quad 1 \in \mathcal{NH}_n$



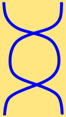
$:= \quad x_r$

$:= \quad \partial_r$

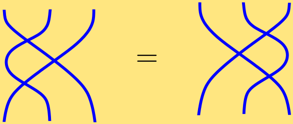
Relations



$$\partial_r x_r - x_{r+1}\partial_r = 1 = x_r\partial_r - \partial_r x_{r+1}$$

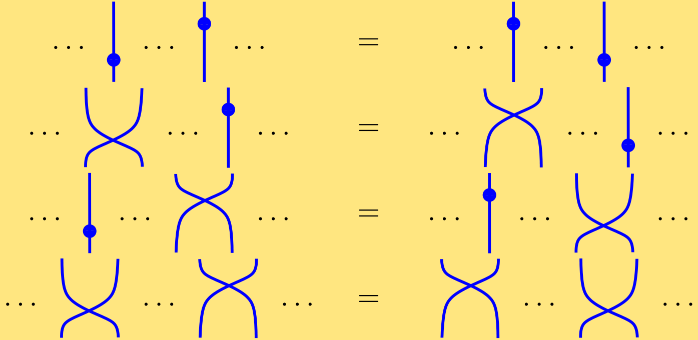


$$\partial_r\partial_r = 0$$



$$\partial_r\partial_{r+1}\partial_r = \partial_{r+1}\partial_r\partial_{r+1}$$

Isotopy relations



Algebraic Isotopy Relations

$$x_i x_j = x_j x_i \quad (i \neq j),$$
$$\partial_i \partial_j = \partial_j \partial_i \quad (|i-j|>1),$$
$$x_i \partial_j = \partial_j x_i \quad (i \neq j, j+1).$$

Polynomial representation

The algebra $\mathcal{NH}_n$ acts on the polynomial ring $\mathrm{Pol}_n := \mathbb{Z}[x_1, x_2, \ldots, x_n]$ with x_i acting by multiplication and ∂_i acting by divided difference operators

$$\partial_i(f) = \frac{f - s_i(f)}{x_i - x_{i+1}} \qquad f \in \mathrm{Pol}_n,$$

$s_i(f)$ is the action of the symmetric group S_n by permuting variables.

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Alternatively, we can define ∂_i by

$$\partial_i(1) = 0, \qquad \partial_i(x_j) = \begin{cases} 1 & \text{if } j = i, \\ -1 & \text{if } j = i+1, \\ 0 & \text{otherwise,} \end{cases}$$

and the "Leibniz rule"

$$\partial_i(fg) = \partial_i(f)g + s_i(f)\partial_i(g) \text{ for all } f, g \in \mathbb{Z}[x_1, \ldots, x_n].$$

Symmetric functions

The ring of symmetric functions has many descriptions

$$\Lambda_n = \mathbb{Z}[x_1, x_2, \dots, x_n]^{S_n} = \bigcap_{i=1}^{n-1} \mathrm{ker}(\partial_i) = \bigcap_{i=1}^{n-1} \mathrm{im}\,(\partial_i).$$

This ring can also be described as $\Lambda_n \cong \mathbb{Z}[\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n]$, where ε_k is the usual elementary symmetric polynomial

$$\varepsilon_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_n}$$

of degree $2k$ (since $\deg(x_i) = 2$).

Example ($n = 3$)

$$\begin{aligned} \varepsilon_1(x_1, x_2, x_3) &= x_1 + x_2 + x_3 \\ \varepsilon_2(x_1, x_2, x_3) &= x_1x_2 + x_2x_3 + x_1x_3 \\ \varepsilon_1(x_1, x_2, x_3) &= x_1x_2x_3 \end{aligned}$$

There are other nice bases for Λ_n such as

- complete symmetric functions

$$h_k(x_1, \dots, x_n) = \sum_{1 \le i_1 \le \cdots \le i_k \le n} x_{i_1} \cdots x_{i_n}$$

  satisfying

$$\sum_{a+b=n} (-1)^b \varepsilon_a h_b = \delta_{n,0}.$$

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- Schur functions

$$s_\lambda = \partial_{w_0}(x_1^{n-1+\lambda_1} x_2^{n-2+\lambda_2} \dots x_n^{\lambda_n})$$

  where w_0 is the longest element of the symmetric group.

The ring of polynomials Pol_n is a free Λ_n-module of rank $n!$. Two natural basis for Pol_n as a free Λ_n module are

- The set $\left\{x_1^{\ell_1} x_2^{\ell_2} \dots x_n^{\ell_n}\right\}$ where $0 \leq \ell_i \leq n - i$.
- The basis of Schubert polynomials

$$\mathfrak{S}_w := \partial_{w_0 w^{-1}}(x_1^{n-1} x_2^{n-2} \dots x_n^0)$$

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From the action of NH_n on Pol_n we get a homomorphism $NH_n \to \mathrm{End}_{\Lambda_n}(\mathrm{Pol}_n) = Mat(n!, \Lambda_n)$.

Theorem (Categorification)

There is an isomorphism (of bialgebras)

$$\bigoplus_{n\in\mathbb{N}} K_0\left(\mathcal{NH}_n - \mathrm{pmod}\right) \longrightarrow \mathbf{U}^+(\mathfrak{sl}_2)$$

$$[\mathcal{NH}_n] \mapsto E^n = [n]E^{(n)}$$

$$[\mathcal{NH}_n e_{1,1}] \mapsto E^{(n)}$$

Cyclotomic quotients (even case)

Given an integer $N \in \mathbb{N}$ we can define the cyclotomic quotient $\mathcal{NH}_n^N$ by quotienting by the ideal $\langle x_1^N \rangle$.

Theorem

There is an isomorphism

$$\bigoplus_{n \in \mathbb{N}} K_0\left(\mathcal{NH}_n^N - \mathrm{pmod}\right) \longrightarrow V_N$$

where V_N is the integral version of the irreducible $\mathbf{U}_q(\mathfrak{sl}_2)$-module of highest weight N.

This result relies on the fact that $\mathcal{NH}_n^N$ is Morita equivalent to the cohomology ring of the Grassmannian $Gr(k; N)$ of k-planes in $\mathbb{C}^N$.

Cohomology rings of Grassmannians

Let $\deg(c_i) = 2i$, $\deg(\bar{c}_j) = 2j$. Then there is a graded ring isomorphism

$$H^*(Gr(k,N)) \cong \mathbb{Z}[c_1, \dots, c_k, , \bar{c}_1, \dots, \bar{c}_{N-k}]/I_k$$

where I_k is the ideal generated by equating powers of t in

$$(1 + c_1 t + c_2 t^2 + \cdots + c_k t^k)(1 + \bar{c}_1 t + \cdots + \bar{c}_{N-k} t^{N-k}) = 1.$$

i.e. equating powers of t^n implies

$$\sum_{a+b=n} c_a \bar{c}_b = \delta_{n,0}.$$

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Notice the similarity with symmetric functions

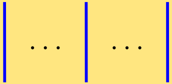
$$(1 + \epsilon_1 t + \dots + \dots \epsilon_k t^k)(1 + (-h_1)t + h_2 t^2 + \dots + (-1)^r h_r t^r + \dots) = 1.$$

We get the ring $H^*(Gr(k,N))$ from Λ_k by imposing the additional relation that $h_j = 0$ for $j > N-k$.

Idea:

Oddify everything we just discussed by finding an “odd” analog of the nilHecke algebra.

Odd NilHecke Generators



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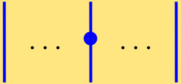

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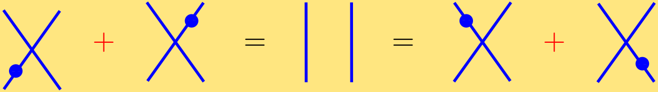

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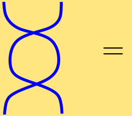

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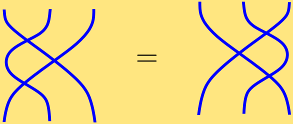

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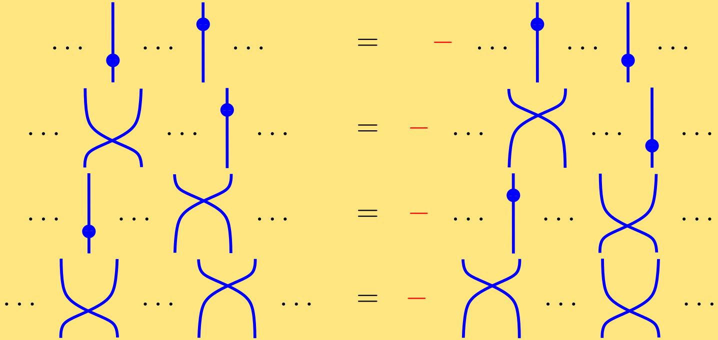
Isotopy relations



Algebraic Isotopy Relations

$$x_i x_j = -x_j x_i \quad (i \neq j),$$
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Skew Polynomial representation

Define the ring of *odd polynomials* to be

$$\mathrm{OPol}_n = \mathbb{Z}\langle x_1, \dots, x_n \rangle / \langle x_i x_j + x_j x_i = 0 \text{ for } i \neq j \rangle.$$

The symmetric group S_n acts as the ring endomorphism

$$s_i(x_j) = \begin{cases} -x_{i+1} & \text{if } j = i, \\ -x_i & \text{if } j = i+1, \\ -x_j & \text{otherwise.} \end{cases}$$

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The *odd divided difference operators* are the linear operators ∂_i defined by

$$\partial_i(1) = 0,$$

$$\partial_i(x_j) = \begin{cases} 1 & \text{if } j = i, i+1, \\ 0 & \text{otherwise,} \end{cases}$$

and the Leibniz rule

$$\partial_i(fg) = \partial_i(f)g + s_i(f)\partial_i(g) \text{ for all } f, g \in \mathbb{Z}\langle x_1, \dots, x_n \rangle.$$

Odd Symmetric functions

Define the ring of *odd symmetric polynomials* as the subring

$$\mathrm{O}\Lambda_n = \bigcap_{i=1}^{n-1} \ker(\partial_i) = \bigcap_{i=1}^{n-1} \mathrm{im}\,(\partial_i) \quad \subset \mathrm{OPol}_n$$

By analogy with the even case, we introduce the *odd elementary symmetric polynomials*

$$\varepsilon_k(x_1,\dots,x_n) = \sum_{1\leq i_1<\cdots<i_k\leq n} \widetilde{x}_{i_1}\cdots\widetilde{x}_{i_k}, \qquad \textit{where } \widetilde{x}_i = (-1)^{i-1}x_i$$

Example (n=3)

$$\begin{aligned} \varepsilon_1 &= x_1 - x_2 + x_3 \\ \varepsilon_2 &= -x_1x_2 + x_2x_3 - x_2x_3 \\ \varepsilon_3 &= -x_1x_2x_3 \end{aligned}$$

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- The polynomials ε_r are odd symmetric.

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Proposition

The ring of odd polynomials OPol_n is a free left (resp. right $\mathrm{O}\Lambda_n$) module with basis given by odd Schubert polynomials

$$\mathfrak{S}_w := \partial_{w_0 w^{-1}}(x_1^{n-1} x_2^{n-2} \dots x_n^0)$$

This allows us to identify the endomorphism ring $\mathrm{End}_{\mathrm{O}\Lambda_n}(\mathrm{OPol}_n)$ as a matrix ring $Mat(n!, \mathrm{O}\Lambda_n)$. The action of $\mathcal{ONH}_n$ on odd polynomials gives rise to

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The algebras $\mathcal{ONH}_n$ were discovered independently by Kang-Kashiwara-Tshuchioka and are closely related to earlier work of Khongsap-Wang.

Odd cyclotomic quotients

Odd cyclotomic quotients $\mathcal{ONH}_n^N$ can be defined in the same way as ordinary cyclotomic quotients by quotienting by x_1^N.

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Proposition (Ellis, Khovanov, L)

One can show that $\mathcal{ONH}_n^N$ is Morita equivalent to a noncommutative ring $OH^*(Gr(k;N))$ called the *odd cohomology of the Grassmannian.*

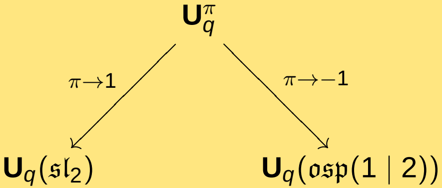
- Just like the usual case, $OH^*(Gr(k;N)) \cong \mathrm{O}\Lambda_k/\langle h_r \mid r > N-k\rangle$.
- The ring $OH^*(Gr(k;N))$ has the same graded rank as $H^*(Gr(k;N))$ and these rings become isomorphic when coefficients are reduced modulo two.
- The ring $OH^*(Gr(k;N))$ has a basis of appropriate odd Schur functions.

Covering Kac-Moody algebras

The existence of the even and the odd theories has a representation theoretic explanation via the work of Hill-Wang and Clark-Wang.

Introduce a parameter π with $\pi^2 = 1$.

Covering Kac-Moody algebras



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$$\mathbf{U}_q^{\pi} \xrightarrow{\pi \to 1} \mathbf{U}_q(\mathfrak{sl}_2)$$

$$\mathbf{U}_q^{\pi} \xrightarrow{\pi \to -1} \mathbf{U}_q(\mathfrak{osp}(1 \mid 2))$$

- There is a novel new bar involution $\overline{q} = \pi q^{-1}$.

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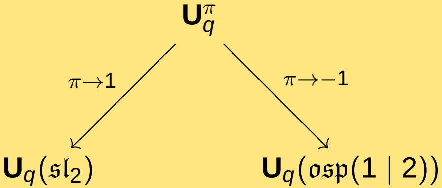

- There is a novel new bar involution $\overline{q} = \pi q^{-1}$.
- This leads to the first construction of canonical bases for super Lie algebras! (Positive parts for super Lie algebras Hill-Wang, entire quantum group in rank 1 by Clark-Wang.)