The classification of thick tensor ideals for Lie Superalgebras

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Tensor Ideals

Say $\mathcal{F}$ is a (k-linear additive) tensor category. That is:

- $\mathcal{F}$ has a direct sum and a tensor product.
- We have a unit object: $X \otimes \mathbb{1} \cong X \cong \mathbb{1} \otimes X$.
- We have commutativity: $X \otimes Y \cong Y \otimes X$.
- Various axioms (exactness of $\otimes$, associativity, etc.)

Examples:

- Category of *all* G-modules.
- Category of finite dimensional G-modules.
- Category of *all* $\mathfrak{g}$-modules.
- Category of finite dimensional $\mathfrak{g}$-modules.

A (thick) tensor ideal $\mathcal{I} \subseteq \mathcal{F}$ is a subcategory which satisfies:

- $X, Y \in \mathcal{I}$, the $X \oplus Y \in \mathcal{I}$.
- $X \in \mathcal{I}$, $Y \in \mathcal{F}$, then $X \otimes Y \in \mathcal{I}$.
- $X \oplus Y \in \mathcal{I}$, then $X, Y \in \mathcal{I}$.

Examples:

- $\mathcal{F}$.
- $\{0\}$.
- $Proj := \{P \in \mathcal{F} \mid P \text{ Projective}\}$.
  Note that $Proj \subseteq \mathcal{I}$ for all nonzero ideals $\mathcal{I}$.

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Question:

Given an interesting $\mathcal{F}$, can we classify the thick tensor ideals?

Given $\mathcal{F}$, let **K** be the stable module category for $\mathcal{F}$. The objects are the same as in $\mathcal{F}$ but

$$\mathrm{Hom}_{\mathrm{Stab}(\mathcal{F})}(M, N) = \mathrm{Hom}_{\mathcal{F}}(M, N)/\sim$$

where $f \sim g$ if and only if $f - g$ factors through a projective.

Effects:

- in **K** the projectives are isomorphic to 0.
- In fact, M and N are isomorphic in **K** if there are projectives P_1 and P_2 so that in $\mathcal{F}$

$$M \oplus P_1 \cong N \oplus P_2.$$

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Good News:

But **K** is a triangulated category for nice $\mathcal{F}$.

Tensor Triangulated Categories

Let **K** be a Tensor Triangulated Category:

- **K** is a triangulated category;
- There is a tensor product functor $- \otimes - : \mathbf{K} \times \mathbf{K} \to \mathbf{K}$;
- We have $M \otimes N \cong N \otimes M$ for all M, N;
- And a unit object $\mathbb{1}$.

An object $C \in \mathbf{K}$ is compact if

$$\mathrm{Hom}_{\mathbf{K}}(C, \oplus_{i \in I} M_i) \cong \oplus_{i \in I} \mathrm{Hom}_K(C, M_i)$$

K is compactly generated if

$$\mathrm{Hom}_{\mathbf{K}}(C, M) = 0$$

for all compact C implies $M = 0$.

Standing Assumptions

K is a tensor triangulated category which is compactly generated, $\mathbb{1}$ is compact, and all compact objects are dualizable. We'll write $\mathbf{K}^c$ for the full subcategory of compact objects.

Example

$$\mathbf{K} = \text{Stab}\,(\text{all } G\text{-modules})$$
$$\mathbf{K}^c = \text{Stab}\,(\text{finite dimensional } G\text{-modules})$$

Tensor Triangular Geometry

An ideal $\mathcal{P}$ is a prime ideal if it is proper and if

$$X \otimes Y \in \mathcal{P} \text{ implies } X \in \mathcal{P} \text{ or } Y \in \mathcal{P}.$$

The Spectrum of $\mathbf{K}^c$ is

$$\mathrm{Spc}(\mathbf{K}^c) = \{\mathcal{P} \mid \mathcal{P} \text{ a prime ideal of } \mathbf{K}\}$$

with the Zariski topology.

Why is this interesting?

- $\mathbf{K} = \text{Stab}\,(G - \text{Mod})$, $\mathbf{K}^c = \text{Stab}\,(G - \text{mod})$:[a]

$$\text{Spc}\,(\mathbf{K}^c) \cong \text{Spec}\,(H^\bullet(G, k))\,.$$

- R a commutative Noetherian ring:

$\mathbf{K} = D(R - \text{Mod})$, $\mathbf{K}^c = D^b_{\text{perf}}(R - \text{mod})$:[b]

$$\text{Spc}\,(\mathbf{K}^c) \cong \text{Spec}(R).$$

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[a] Benson-Carlson-Rickard, Friedlander-Pevtsova, Benson-Iyengar-Krause
[b] Hopkins, Neeman

Theorem (Balmer)

We can define a "support variety" theory on **K** by:

$$\text{supp} : \mathbf{K}^c \to \text{Spc}(\mathbf{K}^c)$$

by

$$\text{supp}(M) = \{\mathcal{P} \in \text{Spc}(\mathbf{K}^c) \mid M \notin \mathcal{P}\}\,.$$

This has all the properties desirable of a support variety theory and is "universal" among such theories.

Theorem (Balmer)

The map

$$\{\text{tensor ideals}\} \to \{\text{specialization closed subsets of } \mathrm{Spc}(\mathbf{K}^c)\}$$

given by

$$\mathcal{I} \mapsto \bigcup_{M \in \mathcal{I}} \mathrm{supp}(M)$$

is a bijection.

The Lie Superalgebra $\mathfrak{gl}(m|n)$

Let $\mathfrak{g} = \mathfrak{gl}(m|n)$ be the Lie superalgebra of $(m+n) \times (m+n)$ matrices. The $\mathbb{Z}_2$-grading is given by

$$\mathfrak{g}_{\bar{0}} = \left\{ \begin{pmatrix} W & 0 \\ 0 & Z \end{pmatrix} \right\} \qquad \mathfrak{g}_{\bar{1}} = \left\{ \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix} \right\}$$

and with bracket

$$[A, B] = AB - (-1)^{\overline{A} \cdot \overline{B}} BA.$$

Note: $\mathfrak{g}_{\bar{0}} \cong \mathfrak{gl}(m) \oplus \mathfrak{gl}(n)$.

Representations

Let $\mathcal{C}$ be the category of $\mathfrak{g}$-supermodules which are finitely semisimple over $\mathfrak{g}_{\bar{0}}$.

Let $\mathcal{F}$ be the category of f.d. $\mathfrak{g}$-supermodules which are finitely semisimple over $\mathfrak{g}_{\bar{0}}$.

We then form:

$$\mathbf{K} = \mathrm{Stab}\,(\mathcal{C})$$
$$\mathbf{K}^c = \mathrm{Stab}\,(\mathcal{F})$$

Question:

What is $\operatorname{Spc}(\mathbf{K}^c)$?

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"In all applications though, the crucial anchor point is the computation of the triangular spectrum $\mathrm{Spc}(\mathbf{K}^c)$ in the first place. Without this knowledge, abstract results of tensor triangular geometry are difficult to translate into concrete terms. It is therefore a major challenge to compute the spectrum $\mathrm{Spc}(\mathbf{K}^c)$ in as many examples as possible..."

— Balmer, "Spectra, spectra, spectra"

The Detecting Subalgebra

Let $\mathfrak{f} \subseteq \mathfrak{g} = \mathfrak{gl}(n|n)$ be the subalgebra

$$\mathfrak{f} = \left\{ \begin{pmatrix} W & X \\ Y & Z \end{pmatrix} \in \mathfrak{g} \mid W, X, Y, Z \text{ are diagonal matrices} \right\}.$$

Let $G_0 = GL(n) \times GL(n)$ with the adjoint action on $\mathfrak{g}$ and let

$$N = \mathrm{Norm}_{G_0}(\mathfrak{f}_{\bar{1}}) \cong T \rtimes \Sigma_n.$$

Since N acts on $\mathfrak{f}_{\bar{1}}$, it acts on $S^\bullet(\mathfrak{f}_{\bar{1}}^*)$ by ring automorphisms.

Theorem (BKN)

$$\text{res} : \text{Ext}^\bullet_{\mathcal{F}(\mathfrak{g})}(\mathbb{C}, \mathbb{C}) \xrightarrow{\cong} \text{Ext}^\bullet_{\mathcal{F}(\mathfrak{f})}(\mathbb{C}, \mathbb{C})^N \cong S^\bullet(\mathfrak{f}_{\bar{1}}^*)^N$$

Theorem (Lehrer-Nakano-Zhang)

For any $M, N \in \mathcal{F}(\mathfrak{g})$:

$$\text{res} : \text{Ext}^\bullet_{\mathcal{F}(\mathfrak{g})}(M, N) \hookrightarrow \text{Ext}^\bullet_{\mathcal{F}(\mathfrak{f})}(\mathbb{C}, \mathbb{C}).$$

Theorem (BKN)

Let $\mathbf{K}^c = \mathrm{Stab}\,(\mathcal{F})$ for $\mathfrak{gl}(m|n)$. Then there is an explicit homeomorphism

$$\mathrm{Proj}\left(N - \mathrm{Spec}\left(S^\bullet\left(\mathfrak{f}_{\bar{1}}^*\right)\right)\right) \xrightarrow{\cong} \mathrm{Spc}\,(\mathbf{K}^c)$$

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If N is a group acting on a commutative ring R, then:

What is $N - \mathrm{Spec}(R)$?

Since N acts on R, N also acts on $X := \mathrm{Spec}(R)$. We then have three natural spaces and maps between them:

$$X \to X/N := \{\text{N-orbits in } X\} \to X//N := \mathrm{Spec}\left(R^N\right)$$

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Lemma

$$X \to X/N \to N - \text{Spec}(R) \to X//N$$

Theorem

There is a bijection between the closed sets of $N - \text{Spec}(R)$ and the N-stable closed sets of $\text{Spec}(R)$.

The proof

Key Theorem (Dell'Ambrogio, Pevtsova-Smith, BKN)

Let **K** be a compactly generated TTC with set-indexed coproducts. Let X be a Zariski space and let

$$V : \mathbf{K} \to \text{Subsets of } X \text{ such that:}$$

1. $V(0) = \varnothing$, $V(\mathbb{1}) = X$;
2. $V(\bigoplus_{i \in I} M_i) = \bigcup_{i \in I} V(M_i)$;
3. $V(\Sigma M) = V(M)$;
4. for any distinguished triangle $M \to N \to Q \to \Sigma M$ we have
$$V(N) \subseteq V(M) \cup V(Q);$$
5. $V(M \otimes N) = V(M) \cap V(N)$;
6. $V(M) = V(M^*)$ is closed for $M \in \mathbf{K}^c$.

Key Theorem, cont.

7. $V(M) = \varnothing$ if and only if $M = 0$;
   (Projectivity Testing Property)
8. for any closed $W \subseteq X$ there exists an $M \in \mathbf{K}$ such that $V(M) = W$.
   (Realization Property).

Then $f : X \to \mathrm{Spc}\,(\mathbf{K}^c)$ given by the universal property:

$$f(x) = \{M \in \mathbf{K}^c \mid x \notin V(M)\}$$

is a homeomorphism.

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For $\mathbf{K}^c = \mathrm{Stab}(\mathcal{F}(\mathfrak{g}))$:

$$V(M) := \mathcal{V}_{\mathfrak{f}}(M),$$

the $\mathfrak{f}$-support variety.