The category of elementary subalgebras of a restricted Lie algebra

Jared Warner

University of Southern California

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Some research into the value of pictures

1 picture = 1,000 words[1]

Average speaking rate = 150 words per minute[2]

1 talk = 20 minutes[3]

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$$1 \textit{ picture } = \frac{1}{3} \textit{ talk}$$

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Proof:

$$1 \text{ picture} = 1 \text{ picture} \cdot \frac{1000 \text{ wds}}{1 \text{ picture}} \cdot \frac{1 \text{ min}}{150 \text{ wds}} \cdot \frac{1 \text{ talk}}{20 \text{ min}} = \frac{1}{3} \text{ talk}$$

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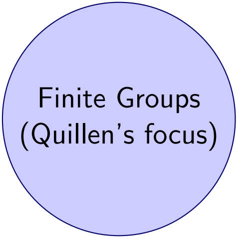
What's the big idea?



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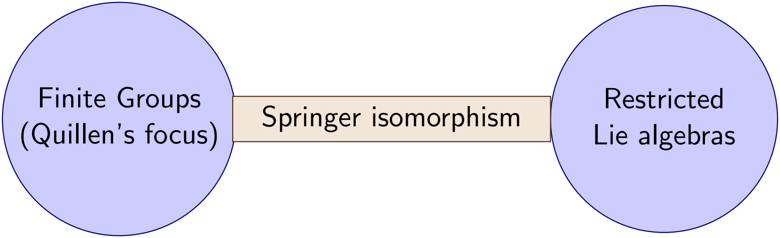

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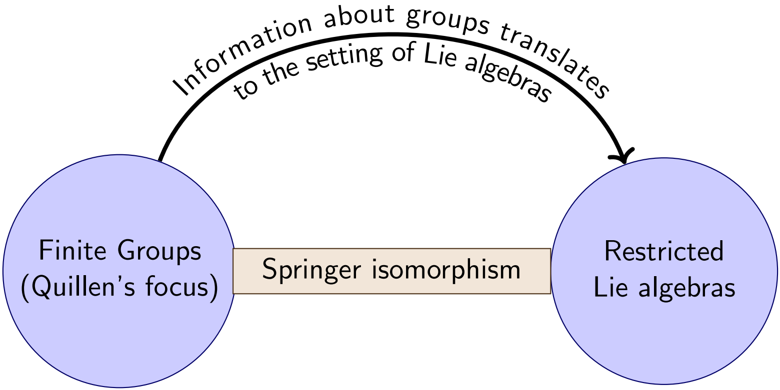

Quillen's category of elementary subgroups

Let Γ be a finite group, and let p be a prime number.

The category of elementary abelian p-subgroups (Quillen, 1971)

Let $\mathcal{E}(\Gamma)$ denote the category whose objects are the elementary abelian p-subgroups of Γ and in which a morphism from E to E' is defined to be a composition of group homomorphisms of the following form:

$$\text{Inclusions: } E \hookrightarrow E' \qquad \text{Conjugations: } E \xrightarrow{\sim} g^{-1}Eg$$

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Note 1: $\mathrm{Hom}_{\mathcal{E}(\Gamma)}(E, E') \neq \varnothing$ if and only if E is conjugate to a subgroup of E'.

Note 2: $\mathrm{Hom}_{\mathcal{E}(\Gamma)}(E, E) \cong N_G(E)/C_G(E)$.

$\mathcal{E}(\Gamma)$ in cohomology

For k an algebraically closed field of characteristic p, let

$$H(\Gamma) := \begin{cases} H^{ev}(\Gamma, k) & p \neq 2 \\ H^*(\Gamma, k) & p = 2 \end{cases} \qquad \text{and} \qquad X_\Gamma := \operatorname{Spec} H(\Gamma)$$

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Inclusions $\iota : E \hookrightarrow \Gamma$ induce continuous maps $\iota_E : X_E \to X_\Gamma$ with the following properties:

- $\iota_E(X_E) \subset \iota_{E'}(X_{E'})$ if and only if $\operatorname{Hom}_{\mathcal{E}(\Gamma)}(E, E') \neq \varnothing$.
- The group $\operatorname{Hom}_{\mathcal{E}(\Gamma)}(E, E)$ determines precisely when two points $\mathfrak{p}, \mathfrak{q} \in X_E$ satisfy $\iota_E(\mathfrak{p}) = \iota_E(\mathfrak{q})$.

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Theorem (Quillen,1971)

$$X_\Gamma \cong \varinjlim_{E \in \mathcal{E}(\Gamma)} X_E$$

Restricted Lie algebras

Let $(\mathfrak{g}, [-,-], (-)^{[p]})$ be a restricted Lie algebra over k.

Definition - elementary subalgebra

A subalgebra $\epsilon \subset \mathfrak{g}$ is called elementary if

- $[\epsilon, \epsilon] = 0$ and
- $\epsilon^{[p]} = 0$.

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Suppose further that $\mathfrak{g}$ is the Lie algebra of an algebraic group G over k. For any $g \in G$, the derivative of the map

$$\begin{array}{rccc} \mathrm{Int}_g : G & \longrightarrow & G \\ a & \longmapsto & g^{-1}ag \end{array}$$

gives the adjoint action of G on $\mathfrak{g}$: $\mathrm{Ad}_g := d(\mathrm{Int}_g) : \mathfrak{g} \to \mathfrak{g}$.

Category of elementary subalgebras

The category of elementary subalgebras

Let $\mathcal{E}(\mathfrak{g})$ denote the category whose objects are the elementary subalgebras of $\mathfrak{g}$ and in which a morphism from ϵ to ϵ' is defined to be a composition of Lie algebra homomorphisms of the following form:

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Note 1: $\mathrm{Hom}_{\mathcal{E}(\mathfrak{g})}(\epsilon, \epsilon') \neq \varnothing$ if and only if $\mathrm{Ad}_g(\epsilon) \subset \epsilon'$ for some $g \in G$.

Note 2: $\mathrm{Hom}_{\mathcal{E}(\mathfrak{g})}(\epsilon, \epsilon) \cong N_G(\epsilon)/C_G(\epsilon)$.

Category of $\mathbb{F}_q$-expressible subalgebras

Let $q = p^d$ and suppose that G is defined over $\mathbb{F}_q$, so that $G = G_0 \times_{\mathbb{F}_q} k$ for some algebraic group G_0 over $\mathbb{F}_q$ and $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{F}_q} k$ for $\mathfrak{g}_0 := \mathrm{Lie}(G_0)$.

The category of $\mathbb{F}_q$-expressible subalgebras

Let $\mathcal{E}_q(\mathfrak{g})$ be the subcategory of $\mathcal{E}(\mathfrak{g})$ whose objects are subalgebras of the form $\epsilon = \epsilon_0 \otimes_{\mathbb{F}_q} k$ for elementary $\epsilon_0 \subset \mathfrak{g}_0$. The morphisms in $\mathcal{E}_q(\mathfrak{g})$ are inclusion composed with Ad_g for some $g \in G_0(\mathbb{F}_q)$.

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Theorem (W,2014)

Let G be a reductive, connected group defined over $\mathbb{F}_q$. If $p > h(G)$, then the category $\mathcal{E}_q(\mathfrak{g})$ is isomorphic to a full subcategory of $\mathcal{E}(G_0(\mathbb{F}_q))$. If $p = q$, then $\mathcal{E}_p(\mathfrak{g}) \cong \mathcal{E}(G_0(\mathbb{F}_p))$.

Springer isomorphisms

Define

$$\mathcal{N}(\mathfrak{g}) := \{x \in \mathfrak{g} \mid x^{[p]^t} = 0 \text{ for some } t \in \mathbb{Z}^{\geq 0}\}$$
$$\mathcal{U}(G) := \{g \in G \mid g^{p^t} = 1 \text{ for some } t \in \mathbb{Z}^{\geq 0}\}$$

to be the nullcone of $\mathfrak{g}$ and the unipotent variety of G, respectively. Notice that both varieties are equipped with natural G-actions.

Definition - Springer isomorphism

A Springer isomorphism is a G-equivariant isomorphism of varieties $\sigma : \mathcal{N}(\mathfrak{g}) \to \mathcal{U}(G)$.

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Theorem (Springer, 1969)

If p is very good for G, then Springer isomorphisms exist.

A canonical Springer isomorphism

Example (Springer isomorphisms are not unique)

Let $G := SL_n$. Then for any $(a_1, \ldots, a_{n-1}) \in k^{n-1}$ with $a_1 \neq 0$ the map

$$\sigma(x) := 1 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1}$$

is a Springer isomorphism.

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(McNinch, 2005), (Carlson-Lin-Nakano, 2008), (Sobaje, 2014)

If $p > h(G)$, there is a canonical Springer isomorphism σ, defined over $\mathbb{F}_q$, which satisfies the following properties (among others):

- $[x, y] = 0$ if and only if $(\sigma(x), \sigma(y)) = 1$
- If $[x, y] = 0$, then $\sigma(x + y) = \sigma(x)\sigma(y)$

Proof of theorem

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Proof: Define $\mathcal{F} : \mathcal{E}_q(\mathfrak{g}) \to \mathcal{E}(G_0(\mathbb{F}_q))$ by

$$\mathcal{F}(\epsilon) := \sigma(\epsilon_0)$$

$$\mathcal{F}(\mathrm{Ad}_g) := \mathrm{Int}_g$$

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Question: Which $E \in G_0(\mathbb{F}_q)$ lie in the image of $\mathcal{F}$?

$\mathbb{F}_q$-linear subgroups

For any $\lambda \in k$, $g \in \mathcal{U}(G)$, define $g^\lambda := \sigma(\lambda\sigma^{-1}(g))$.

Definition - $\mathbb{F}_q$-linear subgroup

An elementary abelian subgroup $E \subset G$ is $\mathbb{F}_q$-linear if $g^\lambda \in E$ for all $g \in E$, $\lambda \in \mathbb{F}_q$.

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Proposition (W,2014)

- *All $E \subset G$ are $\mathbb{F}_p$-linear.*
- *Any $E \subset G$ is contained in a canonical $\mathbb{F}_q$-linear subgroup*
- *The rank of all finite $\mathbb{F}_q$-linear subgroups is divisible by d.*
- *The image of $\mathcal{F}$ is exactly the set of $\mathbb{F}_q$-linear elementary abelian subgroups of $G_0(\mathbb{F}_q)$.*

A non-example

Example of a subgroup that is not $\mathbb{F}_q$-linear

Let $G = \mathrm{SL}_3$, let $d = 2$, and let $\lambda \in \mathbb{F}_q \setminus \mathbb{F}_p$. In this case, we have $\sigma(X) = I + X + \frac{1}{2}X^2$. The elementary abelian subgroup of rank 2 defined as follows:

$$E = \left\langle g = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\rangle$$

is not $\mathbb{F}_q$-linear.

Application

Let $\mathbb{E}(r, \mathfrak{g})$ denote the set of all r-dimensional elementary subalgebras of $\mathfrak{g}$.

Theorem (Carlson-Friedlander-Pevtsova, 2012)

The natural embedding $\mathbb{E}(r, \mathfrak{g}) \hookrightarrow \mathrm{Grass}(r, \mathfrak{g})$ *is a closed embedding. If* $\mathfrak{g} = \mathrm{Lie}(G)$*, then* $\mathbb{E}(r, \mathfrak{g})$ *is a* G*-variety under* Ad.

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Theorem (W,2014)

Let $\mathfrak{g} = \mathrm{Lie}(G)$ for G connected and reductive, let $p > h(G)$, and let $R = R(\mathfrak{g})$ be the largest integer such that $\mathbb{E}(R, \mathfrak{g}) \neq \varnothing$. If the simple factors of (G, G) are of classical type, then $\mathbb{E}(R, \mathrm{Lie}(G))$ is a union of finitely many G-orbits.

Remark: Verifying the theorem for all G would require knowledge of elementary abelian subgroups of the $\mathbb{F}_q$-rational points of the exceptional groups.

Questions

- What role does the category $\mathcal{E}(\mathfrak{g})$ play in restricted Lie algebra cohomology à la Quillen?
- What is the cohomological significance of $R = R(\mathfrak{g})$?
- What are the closed subsets of $\mathbb{E}(r, \mathfrak{g})$? When is $\mathbb{E}(r, \mathfrak{g})$ irreducible?

Thank you for listening!