The category of elementary subalgebras of a restricted Lie algebra

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Some research into the value of pictures

1 picture = 1,000 words\(^1\)

Average speaking rate = 150 words per minute\(^2\)

1 talk = 20 minutes\(^3\)

\(^1\)Source: on good authority
\(^2\)Source: the internet
\(^3\)Source: the organizers
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Theorem (W, last week)

\[ 1 \text{ picture} = \frac{1}{3} \text{ talk} \]

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Theorem (W, last week)

\[
1 \text{ picture} = \frac{1}{3} \text{ talk}
\]

Proof:

\[
1 \text{ picture} = 1 \text{ picture} \cdot \frac{1000 \text{ wds}}{1 \text{ picture}} \cdot \frac{1 \text{ min}}{150 \text{ wds}} \cdot \frac{1 \text{ talk}}{20 \text{ min}} = \frac{1}{3} \text{ talk}
\]

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What’s the big idea?

Finite Groups (Quillen’s focus)
What’s the big idea?

The category of elementary subalgebras of a restricted Lie algebra

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Finite groups

Restricted Lie algebras

Springer isomorphisms

An application

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Finite Groups (Quillen’s focus) Springer isomorphism Restricted Lie algebras
What’s the big idea?

Information about groups translates to the setting of Lie algebras.

Finite Groups (Quillen’s focus) \[\Rightarrow\] Springer isomorphism \[\Rightarrow\] Restricted Lie algebras
Let $\Gamma$ be a finite group, and let $p$ be a prime number.

The category of elementary abelian $p$-subgroups (Quillen, 1971)

Let $\mathcal{E}(\Gamma)$ denote the category whose objects are the elementary abelian $p$-subgroups of $\Gamma$ and in which a morphism from $E$ to $E'$ is defined to be a composition of group homomorphisms of the following form:

- Inclusions: $E \hookrightarrow E'$
- Conjugations: $E \xrightarrow{\sim} g^{-1}Eg$
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### The category of elementary abelian $p$-subgroups (Quillen, 1971)

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- **Inclusions:** $E \hookrightarrow E'$
- **Conjugations:** $E \sim \rightarrow g^{-1}Eg$

**Note 1:** $\text{Hom}_{\mathcal{E}(\Gamma)}(E, E') \neq \emptyset$ if and only if $E$ is conjugate to a subgroup of $E'$.

**Note 2:** $\text{Hom}_{\mathcal{E}(\Gamma)}(E, E) \cong N_G(E)/C_G(E)$. 

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**The category of elementary subalgebras of a restricted Lie algebra**

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**Finite groups**

**Restricted Lie algebras**

Springer isomorphisms

**An application**
For $k$ an algebraically closed field of characteristic $p$, let

$$H(\Gamma) := \begin{cases} H^\text{ev}(\Gamma, k) & p \neq 2 \\ H^\ast(\Gamma, k) & p = 2 \end{cases}$$

and $$X_\Gamma := \text{Spec } H(\Gamma)$$
\( \mathcal{E}(\Gamma) \) in cohomology

For \( k \) an algebraically closed field of characteristic \( p \), let

\[
H(\Gamma) := \begin{cases} 
H^{\text{ev}}(\Gamma, k) & p \neq 2 \\
H^*(\Gamma, k) & p = 2
\end{cases}
\]

and \( X_{\Gamma} := \text{Spec} \ H(\Gamma) \).

Inclusions \( \iota : E \hookrightarrow \Gamma \) induce continuous maps \( \iota_E : X_E \to X_{\Gamma} \) with the following properties:

- \( \iota_E(X_E) \subset \iota_{E'}(X_{E'}) \) if and only if \( \text{Hom}_{\mathcal{E}(\Gamma)}(E, E') \neq \emptyset \).
- The group \( \text{Hom}_{\mathcal{E}(\Gamma)}(E, E) \) determines precisely when two points \( p, q \in X_E \) satisfy \( \iota_E(p) = \iota_E(q) \).
\[ H(\Gamma) := \begin{cases} H^{ev}(\Gamma, k) & p \neq 2 \\ H^*(\Gamma, k) & p = 2 \end{cases} \quad \text{and} \quad X_\Gamma := \text{Spec } H(\Gamma) \]

For \( k \) an algebraically closed field of characteristic \( p \), let

\[ X_\Gamma \cong \lim_{E \in \mathcal{E}(\Gamma)} X_E \]

Inclusions \( \iota : E \hookrightarrow \Gamma \) induce continuous maps \( \iota_E : X_E \rightarrow X_\Gamma \) with the following properties:

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Restricted Lie algebras

Let \((\mathfrak{g}, [-, -], (-)^[p])\) be a restricted Lie algebra over \(k\).

**Definition - elementary subalgebra**

A subalgebra \(\epsilon \subset \mathfrak{g}\) is called **elementary** if

- \([\epsilon, \epsilon] = 0\) and
- \(\epsilon[p] = 0\).
Let \((\mathfrak{g}, [\cdot, \cdot], (-)^{[p]})\) be a restricted Lie algebra over \(k\).

**Definition - elementary subalgebra**

A subalgebra \(\mathfrak{e} \subseteq \mathfrak{g}\) is called **elementary** if

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- \(\mathfrak{e}^{[p]} = 0\).

Suppose further that \(\mathfrak{g}\) is the Lie algebra of an algebraic group \(G\) over \(k\). For any \(g \in G\), the derivative of the map

\[
\text{Int}_g : G \rightarrow G \\
a \mapsto g^{-1}ag
\]

gives the adjoint action of \(G\) on \(\mathfrak{g}\): \(\text{Ad}_g := d(\text{Int}_g) : \mathfrak{g} \rightarrow \mathfrak{g}\).
Category of elementary subalgebras

Let $\mathcal{E}(g)$ denote the category whose objects are the elementary subalgebras of $g$ and in which a morphism from $\epsilon$ to $\epsilon'$ is defined to be a composition of Lie algebra homomorphisms of the following form:

- Inclusions: $\epsilon \hookrightarrow \epsilon'$
- Conjugations: $\epsilon \sim \rightarrow \text{Ad}_g(\epsilon)$

Note 1: $\text{Hom}_{\mathcal{E}(g)}(\epsilon, \epsilon') \neq \emptyset$ if and only if $\text{Ad}_g(\epsilon) \subset \epsilon'$ for some $g \in G$.

Note 2: $\text{Hom}_{\mathcal{E}(g)}(\epsilon, \epsilon') \sim = N_G(\epsilon) / C_G(\epsilon)$. 
Let $\mathcal{E}(\mathfrak{g})$ denote the category whose objects are the elementary subalgebras of $\mathfrak{g}$ and in which a morphism from $\epsilon$ to $\epsilon'$ is defined to be a composition of Lie algebra homomorphisms of the following form:

Inclusions: $\epsilon \hookrightarrow \epsilon'$  
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**Note 1:** $\text{Hom}_{\mathcal{E}(\mathfrak{g})}(\epsilon, \epsilon') \neq \emptyset$ if and only if $\text{Ad}_g(\epsilon) \subset \epsilon'$ for some $g \in G$.

**Note 2:** $\text{Hom}_{\mathcal{E}(\mathfrak{g})}(\epsilon, \epsilon) \cong N_G(\epsilon)/C_G(\epsilon)$. 
Category of $\mathbb{F}_q$-expressible subalgebras

Let $q = p^d$ and suppose that $G$ is defined over $\mathbb{F}_q$, so that $G = G_0 \times \mathbb{F}_q k$ for some algebraic group $G_0$ over $\mathbb{F}_q$ and $\mathfrak{g} = \mathfrak{g}_0 \otimes \mathbb{F}_q k$ for $\mathfrak{g}_0 := \text{Lie}(G_0)$.

The category of $\mathbb{F}_q$-expressible subalgebras

Let $\mathcal{E}_q(\mathfrak{g})$ be the subcategory of $\mathcal{E}(\mathfrak{g})$ whose objects are subalgebras of the form $\epsilon = \epsilon_0 \otimes \mathbb{F}_q k$ for elementary $\epsilon_0 \subset \mathfrak{g}_0$. The morphisms in $\mathcal{E}_q(\mathfrak{g})$ are inclusion composed with $\text{Ad}_g$ for some $g \in G_0(\mathbb{F}_q)$. 
Let $q = p^d$ and suppose that $G$ is defined over $\mathbb{F}_q$, so that $G = G_0 \times_{\mathbb{F}_q} k$ for some algebraic group $G_0$ over $\mathbb{F}_q$ and $\mathfrak{g} = \mathfrak{g}_0 \otimes_{\mathbb{F}_q} k$ for $\mathfrak{g}_0 := \text{Lie}(G_0)$.

**The category of $\mathbb{F}_q$-expressible subalgebras**

Let $\mathcal{E}_q(\mathfrak{g})$ be the subcategory of $\mathcal{E}(\mathfrak{g})$ whose objects are subalgebras of the form $\epsilon = \epsilon_0 \otimes_{\mathbb{F}_q} k$ for elementary $\epsilon_0 \subset \mathfrak{g}_0$. The morphisms in $\mathcal{E}_q(\mathfrak{g})$ are inclusion composed with $\text{Ad}_g$ for some $g \in G_0(\mathbb{F}_q)$.

**Theorem (W,2014)**

Let $G$ be a reductive, connected group defined over $\mathbb{F}_q$. If $p > h(G)$, then the category $\mathcal{E}_q(\mathfrak{g})$ is isomorphic to a full subcategory of $\mathcal{E}(G_0(\mathbb{F}_q))$. If $p = q$, then $\mathcal{E}_p(\mathfrak{g}) \cong \mathcal{E}(G_0(\mathbb{F}_p))$. 
Define

\[ \mathcal{N}(\mathfrak{g}) := \{ x \in \mathfrak{g} \mid x^{[p]t} = 0 \text{ for some } t \in \mathbb{Z}_{\geq 0} \} \]

\[ \mathcal{U}(G) := \{ g \in G \mid g^{p^t} = 1 \text{ for some } t \in \mathbb{Z}_{\geq 0} \} \]

to be the nullcone of \( \mathfrak{g} \) and the unipotent variety of \( G \), respectively. Notice that both varieties are equipped with natural \( G \)-actions.

**Definition - Springer isomorphism**

A Springer isomorphism is a \( G \)-equivariant isomorphism of varieties \( \sigma : \mathcal{N}(\mathfrak{g}) \to \mathcal{U}(G) \).
Define

\[ \mathcal{N}(g) := \{ x \in g \mid x^{[p]^t} = 0 \text{ for some } t \in \mathbb{Z}^{\geq 0} \} \]
\[ \mathcal{U}(G) := \{ g \in G \mid g^{pt} = 1 \text{ for some } t \in \mathbb{Z}^{\geq 0} \} \]

to be the nullcone of $g$ and the unipotent variety of $G$, respectively. Notice that both varieties are equipped with natural $G$-actions.

**Definition - Springer isomorphism**

A Springer isomorphism is a $G$-equivariant isomorphism of varieties $\sigma : \mathcal{N}(g) \to \mathcal{U}(G)$.

**Theorem (Springer, 1969)**

If $p$ is very good for $G$, then Springer isomorphisms exist.
A canonical Springer isomorphism

Example (Springer isomorphisms are not unique)

Let $G := SL_n$. Then for any $(a_1, \ldots, a_{n-1}) \in k^{n-1}$ with $a_1 \neq 0$ the map

$$\sigma(x) := 1 + a_1 x + a_2 x^2 + \ldots + a_{n-1} x^{n-1}$$

is a Springer isomorphism.
A canonical Springer isomorphism

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$$\sigma(x) := 1 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1}$$

is a Springer isomorphism.

(McNinch, 2005), (Carlson-Lin-Nakano, 2008), (Sobaje, 2014)

If $p > h(G)$, there is a canonical Springer isomorphism $\sigma$, defined over $\mathbb{F}_q$, which satisfies the following properties (among others):

- $[x, y] = 0$ if and only if $(\sigma(x), \sigma(y)) = 1$
- If $[x, y] = 0$, then $\sigma(x + y) = \sigma(x)\sigma(y)$
Proof of theorem

**Theorem (W,2014)**

Let $G$ be a reductive, connected group defined over $\mathbb{F}_p$. If $p > h(G)$, then the category $\mathcal{E}_q(\mathfrak{g})$ is isomorphic to a full subcategory of $\mathcal{E}(G_0(\mathbb{F}_q))$. If $p = q$, then $\mathcal{E}_p(\mathfrak{g}) \cong \mathcal{E}(G_0(\mathbb{F}_p))$. 

**Question:** Which $E \in G_0(\mathbb{F}_q)$ lie in the image of $F$?
Proof of theorem

**Theorem (W, 2014)**

Let \( G \) be a reductive, connected group defined over \( \mathbb{F}_p \). If \( p > h(G) \), then the category \( \mathcal{E}_q(g) \) is isomorphic to a full subcategory of \( \mathcal{E}(G_0(\mathbb{F}_q)) \). If \( p = q \), then \( \mathcal{E}_p(g) \cong \mathcal{E}(G_0(\mathbb{F}_p)) \).

**Proof:** Define \( \mathcal{F} : \mathcal{E}_q(g) \to \mathcal{E}(G_0(\mathbb{F}_q)) \) by

\[
\mathcal{F}(\epsilon) := \sigma(\epsilon_0)
\]

\[
\mathcal{F}(\text{Ad}_g) := \text{Int}_g
\]
Proof of theorem

Theorem (W,2014)

Let $G$ be a reductive, connected group defined over $\mathbb{F}_p$. If $p > h(G)$, then the category $\mathcal{E}_q(g)$ is isomorphic to a full subcategory of $\mathcal{E}(G_0(\mathbb{F}_q))$. If $p = q$, then $\mathcal{E}_p(g) \cong \mathcal{E}(G_0(\mathbb{F}_p))$.

Proof: Define $\mathcal{F} : \mathcal{E}_q(g) \to \mathcal{E}(G_0(\mathbb{F}_q))$ by

$$\mathcal{F}(\epsilon) := \sigma(\epsilon_0)$$

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Question: Which $E \in G_0(\mathbb{F}_q)$ lie in the image of $\mathcal{F}$?
For any $\lambda \in k$, $g \in \mathcal{U}(G)$, define $g^\lambda := \sigma(\lambda \sigma^{-1}(g))$.

**Definition - $\mathbb{F}_q$-linear subgroup**

An elementary abelian subgroup $E \subset G$ is $\mathbb{F}_q$-linear if $g^\lambda \in E$ for all $g \in E$, $\lambda \in \mathbb{F}_q$. 
$\mathbb{F}_q$-linear subgroups

For any $\lambda \in k$, $g \in U(G)$, define $g^\lambda := \sigma(\lambda \sigma^{-1}(g))$.

**Definition - $\mathbb{F}_q$-linear subgroup**

An elementary abelian subgroup $E \subset G$ is $\mathbb{F}_q$-linear if $g^\lambda \in E$ for all $g \in E$, $\lambda \in \mathbb{F}_q$.

**Proposition (W,2014)**

- All $E \subset G$ are $\mathbb{F}_p$-linear.
- Any $E \subset G$ is contained in a canonical $\mathbb{F}_q$-linear subgroup.
- The rank of all finite $\mathbb{F}_q$-linear subgroups is divisible by $d$.
- The image of $F$ is exactly the set of $\mathbb{F}_q$-linear elementary abelian subgroups of $G_0(\mathbb{F}_q)$. 
A non-example

Example of a subgroup that is not $\mathbb{F}_q$-linear

Let $G = \text{SL}_3$, let $d = 2$, and let $\lambda \in \mathbb{F}_q \setminus \mathbb{F}_p$. In this case, we have $\sigma(X) = I + X + \frac{1}{2}X^2$. The elementary abelian subgroup of rank 2 defined as follows:

$$E = \langle g = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rangle$$

is not $\mathbb{F}_q$-linear.
Application

Let $\mathbb{E}(r, \mathfrak{g})$ denote the set of all $r$-dimensional elementary subalgebras of $\mathfrak{g}$.

Theorem (Carlson-Friedlander-Pevtsova, 2012)

The natural embedding $\mathbb{E}(r, \mathfrak{g}) \hookrightarrow \text{Grass}(r, \mathfrak{g})$ is a closed embedding. If $\mathfrak{g} = \text{Lie}(G)$, then $\mathbb{E}(r, \mathfrak{g})$ is a $G$-variety under $\text{Ad}$. 

Remark: Verifying the theorem for all $G$ would require knowledge of elementary abelian subgroups of the $\mathbb{F}_q$-rational points of the exceptional groups.
Application

Let $E(r, g)$ denote the set of all $r$-dimensional elementary subalgebras of $g$.

**Theorem (Carlson-Friedlander-Pevtsova, 2012)**

The natural embedding $E(r, g) \hookrightarrow \text{Grass}(r, g)$ is a closed embedding. If $g = \text{Lie}(G)$, then $E(r, g)$ is a $G$-variety under $\text{Ad}$.

**Theorem (W, 2014)**

Let $g = \text{Lie}(G)$ for $G$ connected and reductive, let $p > h(G)$, and let $R = R(g)$ be the largest integer such that $E(R, g) \neq \emptyset$. If the simple factors of $(G, G)$ are of classical type, then $E(R, \text{Lie}(G))$ is a union of finitely many $G$-orbits.

Remark: Verifying the theorem for all $G$ would require knowledge of elementary abelian subgroups of the $\mathbb{F}_q$-rational points of the exceptional groups.
Questions

- What role does the category $E(g)$ play in restricted Lie algebra cohomology à la Quillen?
- What is the cohomological significance of $R = R(g)$?
- What are the closed subsets of $E(r, g)$? When is $E(r, g)$ irreducible?