On p-permutation bimodules and equivalences between blocks of group algebras

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AMS Sectional Meeting
San Francisco State University
October 25–26, 2014

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($\iff$ M is a direct summand of a permutation module
$\iff$ each indecomposable direct summand of M has trivial source.)

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• Here, a *twisted diagonal subgroup* of $G \times H$ is a subgroup of the form

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• $\mathbb{Z}$-basis of $T^\Delta(A,B)$: Isomorphism classes $[M]$ of indecomposable p-permutation (A,B)-bimodules M with twisted diagonal vertices.

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Let $P \leq G$ be a p-subgroup. There exists a functor

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where

$$M(P) := M^P / \sum_{Q<P} \mathrm{tr}_Q^P(M^Q)\,, \quad (\mathrm{tr}_Q^P \colon M^Q \to M^P\,,\; m \mapsto \sum_{x \in P/Q} xm)\,.$$

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If $M = F[X]$ for a G-set X, then

$$ F[X^P] \hookrightarrow M^P \twoheadrightarrow M(P) $$

is an isomorphism. Thus, if M is a p-permutation module then $M(P)$ is a p-permutation module.

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Theorem (B.-Danz, 2012) Let G, H, K be finite groups, let ${}_G X_H$ and ${}_H Y_K$ be bifree bisets, and let $\Delta(U,\varphi,W) \le G \times K$ be a twisted diagonal subgroup. Then the canonical map

$$\coprod_{U \overset{\alpha}{\underset{\sim}{\longleftarrow}} V \overset{\beta}{\underset{\sim}{\longleftarrow}} W} X^{\Delta(U,\alpha,V)} \times_{C_H(V)} Y^{\Delta(V,\beta,W)} \overset{\sim}{\to} (X \times_H Y)^{\Delta(U,\varphi,W)}$$

is a $(C_G(U), C_K(W))$-biset isomorphism. Here, $U \overset{\alpha}{\underset{\sim}{\longleftarrow}} V \overset{\beta}{\underset{\sim}{\longleftarrow}} W$ runs through all factorizations of φ through H, up to H-conjugation.

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Corollary Formula for $(M \otimes_{FH} N)(\Delta(P, \varphi, Q))$, for p-permutation bimodules M and N with twisted diagonal vertices.

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Additionally, let $Y \leq H \times K$, $M \in {}_{FX}\mathrm{mod}$, $N \in {}_{FY}\mathrm{mod}$. Then

$$M \in {}_{F[k_1(X)]}\mathrm{mod}_{F[k_2(X)]} \quad \text{and} \quad N \in {}_{F[k_1(Y)]}\mathrm{mod}_{F[k_2(Y)]}\,,$$

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This module structure has an extension to the group

$$X * Y := \{(g,k) \in G \times K \mid \exists h \in H \colon (g,h) \in X, (h,k) \in Y\} \leq G \times K\,.$$

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Obtain a functor ${}_{FX}\mathrm{mod} \times {}_{FY}\mathrm{mod} \longrightarrow {}_{F[X*Y]}\mathrm{mod}$.

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(a) There exists a unique constituent $[M]$ of γ with vertex of the form $\Delta(D, \varphi, E)$, where D and E are defect groups of A and B. Moreover, M has multiplicity ± 1. We call M the *maximal module* of γ.

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(a) There exists a unique constituent $[M]$ of γ with vertex of the form $\Delta(D, \varphi, E)$, where D and E are defect groups of A and B. Moreover, M has multiplicity ± 1. We call M the *maximal module* of γ.

(b) Every constituent of γ has a vertex contained in $\Delta(D, \varphi, E)$. (Uniformity)

(c) Let (D, e) and (E, f) be maximal Brauer pairs of A and B, respectively, such that

$$e \cdot \gamma(\Delta(D, \varphi, E)) \cdot f \neq 0 \, .$$

Then, $\varphi \colon E \xrightarrow{\sim} D$ is an isomorphism between the associated *fusion systems*.

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(d) The *Brauer correspondents* $a \in \mathrm{Bl}(F[N_G(D)])$ of A and $b \in \mathrm{Bl}(F[N_H(E)])$ of B are Morita equivalent via the p-permutation bimodule

$$\mathrm{Ind}_{\cdots}^{N_G(D) \times N_H(E)}(e \cdot M(\Delta(D, \varphi, E)) \cdot f)$$

(e) If $(P, e_P) \leftrightarrow (Q, f_Q)$ are corresponding Brauer pairs of A and B, then

$$e_P \cdot \gamma(\Delta(P, \varphi, Q)) \cdot f_Q \in T^{\Delta}\big(FC_G(P)e_P, FC_H(Q)f_Q\big)$$

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(f) If $(P, e_P) \leftrightarrow (Q, f_Q)$ are corresponding self-centralizing Brauer pairs of A and B, then the associated *Külshammer-Puig 2-cocycles* on $N_G(P, e_P)/PC_G(P)$ and $N_H(Q, f_Q)/QC_H(Q)$ "coincide via φ".

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(g) The group of p-permutation auto-equivalences of A is finite.

Thank you!