# IRREDUCIBLE COMPONENTS OF VARIETIES OF REPRESENTATIONS

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$\Lambda$ basic finite dim'l algebra $|_{K=\bar{K}}$

w.l.o.g., $\Lambda = KQ/I$,

where $Q$ is a quiver and $I \leq KQ$

$Q_0 = \{e_1, \dots, e_n\}$ vertices,

$\underline{d} \in (\mathbb{N}_0)^n$ a dim vector.

# PROGRAM.

(I) Determine the irreducible components of the affine variety $\mathrm{Rep}_{\underline{d}}(\Lambda)$ which parametrizes the $\Lambda$-modules with dim vector $\underline{d}$.

(II) For each irreducible component $\mathcal{C} \subseteq \mathrm{Rep}_{\underline{d}}(\Lambda)$, explore the <u>generic properties</u> of the modules "in" $\mathcal{C}$, i.e., the properties shared by all modules in a dense open subset of $\mathcal{C}$.

# INVENTORY

- $\Lambda = KQ$, i.e., $I = 0$ $\Rightarrow \left(\begin{array}{l}\mathrm{Rep}_{\underline{d}}(\Lambda) \text{ is} \\ \text{irreducible}\end{array}\right)$

  Kac, 1982, initiated generic representation theory.

  Schofield, 1992, significantly extended the theory.

- General $\Lambda$

  Crawley-Boevey & Schröer, 2002, generalized a portion of Kac's results.

  Schröer, in 2004, had full success with a class of comm. local tame algebras.

## Examples

a) $\Lambda = KQ$, $Q$ is $1 \overset{\alpha_1}{\underset{\alpha_2}{\rightrightarrows}} 2$,

$\underline{d} = (2,2)$

Generically, the modules in $Rep_{\underline{d}}(\Lambda)$ decompose in the form

$$\begin{matrix} 1 \\ \alpha_1 (\;) \alpha_2 \\ 2 \end{matrix} \oplus \begin{matrix} 1 \\ \alpha_1 (\;) \alpha_2 \\ 2 \end{matrix}$$

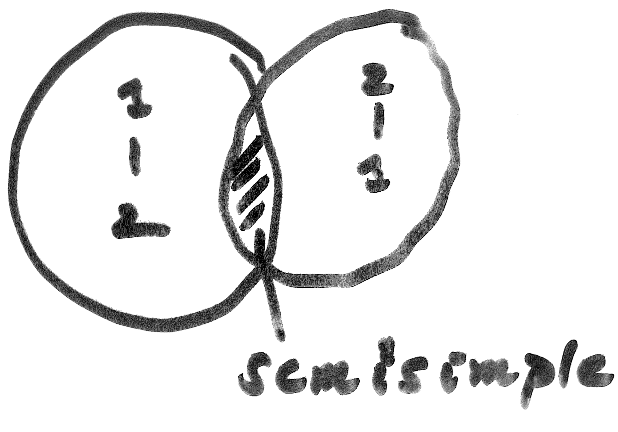
$$M_a \qquad\qquad M_b \qquad\qquad a, b \in \mathbb{P}^1$$

where $M_{\underline{a}} = \Lambda / \Lambda(\underline{a}_1 \alpha_1 + \underline{a}_2 \alpha_2)$

b) $\Lambda = KQ / \langle \alpha_i \alpha_j \mid i \neq j \rangle$, $Q$ is $1 \underset{\alpha_2}{\overset{\alpha_1}{\rightleftarrows}} 2$,

$\underline{d} = (1,1)$

$Rep_{\underline{d}}(\Lambda)$:



STRATEGY: Corner the irred. components of $\mathrm{Rep}_{\underline{d}}(\Lambda)$ via upper semi-continuous maps

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<u>Basic observation</u>. $\mathcal{A}$ a poset, $f: \mathrm{Rep}_{\underline{d}}(\Lambda) \longrightarrow \mathcal{A}$ upper semi-continuous. If $a \in \mathrm{Im}(f)$ is minimal, then

$$\overline{f^{-1}(a)}$$

is a union of irred. comp's of $\mathrm{Rep}_{\underline{d}}(\Lambda)$

<u>Today's main player</u>:

$$\Theta: \mathrm{Rep}_{\underline{d}}(\Lambda) \longrightarrow \mathcal{P} \times \mathcal{P}$$
$$x \longmapsto (\mathbb{S}(M_x), \mathbb{S}^*(M_x)),$$

where $\mathcal{P}$ is a certain set of sequences of semisimple modules, $\mathbb{S}(M_x)$ the radical layering, $\mathbb{S}^*(M_x)$ the socle layering of $M_x$.

## The ingredients of the upper semi-continuous map $\Theta$

$J$ = radical of $\Lambda$, $J^{L+1} = 0$.

$$\mathcal{P} = \left\{ \mathbb{S} = (\mathbb{S}_0, \dots, \mathbb{S}_L) \;\middle|\; \begin{array}{l} \mathbb{S}_l \in \Lambda\text{-mod} \\ \text{semisimple} \\ \sum_0^L \underline{\dim}\, \mathbb{S}_l = \underline{d} \end{array} \right\}$$

Distinguished elements of $\mathcal{P}$:

$$\mathbb{S}(M) = \left( J^l M / J^{l+1} M \right)_{0 \le l \le L},$$

the radical layering of $M \in \Lambda$-mod.

$\mathbb{S}^*(M)$ = socle layering of $M$, dual.

Partial order on $\mathcal{P}$: "dominance", inspired by dominance partial order for partitions of pos. integers.

# The varieties parametrizing the modules with fixed radical layering $\mathbb{S}$

$$\mathrm{Rep}_{\underline{d}}(\Lambda) = \bigsqcup_{\substack{\underline{\dim}\,\mathbb{S} = \underline{d} \\ \text{(finite)}}} \mathrm{Rep}\,\mathbb{S},$$

where
$$\mathrm{Rep}\,\mathbb{S} = \{x \in \mathrm{Rep}_{\underline{d}}(\Lambda) \mid \mathbb{S}(M_x) = \mathbb{S}\}.$$

## Key role played by the Rep $\mathbb{S}$

[Babson - HZ - Thomas]

- The irred. components $\mathcal{D}$ of Rep $\mathbb{S}$ can be obtained from $Q$ and $I$, tagged by generic minimal projective presentations of their modules.
- AND: All irred. components of $\mathrm{Rep}_{\underline{d}}(\Lambda)$ are among the $\mathcal{D}$'s ($\mathbb{S}$ runs).

# TRUNCATED PATH ALGEBRAS

= ALGEBRAS OF THE FORM

$$\Lambda = KQ / \langle \text{all paths of length } L+1 \rangle$$

FOR SOME $L \geq 1$.

## Placement in the zoo:

They include

- the hereditary algebras,
- the algebras with $J^2 = 0$, and
- for any basic finite dim'l $\Delta$, $\exists!$ truncated path algebra $\Lambda$ with the same quiver and Loewy length as $\Delta$ such that

$$\Delta = \Lambda / (\text{some ideal}).$$

<u>Note</u>: $\mathrm{Rep}_{\underline{d}}(\Delta) \underset{\text{closed}}{\subseteq} \mathrm{Rep}_{\underline{d}}(\Lambda) \quad \forall \underline{d}$.

# DECISIVE ASSETS

- Theorem [Babson, H-Z, Thomas]

  All subvarieties $\operatorname{Rep} \mathbb{S} \subseteq \operatorname{Rep}_{\underline{d}}(\Lambda)$ are irreducible (rational & smooth). Hence the irreducible components of $\operatorname{Rep}_{\underline{d}}(\Lambda)$ are among the closures

  $$\overline{\operatorname{Rep} \mathbb{S}}.$$

- The realizable $\mathbb{S}$ are recognized by mere inspection of $Q$.

# LOCAL TRUNCATED PATH ALGEBRAS

<u>Theorem 1</u>. $\Lambda = KQ/\langle \text{paths of length } L+1 \rangle$,

where $Q$ is


, $r \geq 2$.

- If $d \leq L+1$, $\mathrm{Rep}_d(\Lambda)$ is irreducible.

- For $d > L+1$ and a s.s. sequence $\mathbb{S} = (\mathbb{S}_0, \dots, \mathbb{S}_L)$ with $\dim \mathbb{S} = d$, TFAE:

① $\overline{\mathrm{Rep}\,\mathbb{S}}$ is an irred. comp. of $\mathrm{Rep}_d(\Lambda)$.

② For $1 \leq \ell \leq L$, $\dim \mathbb{S}_\ell \leq r \cdot \dim \mathbb{S}_{\ell-1}$ and $\dim \mathbb{S}_{\ell-1} \leq r \cdot \dim \mathbb{S}_\ell$.

③ $\mathbb{S}$ is realizable and the generic socle layering of $\mathrm{Rep}\,\mathbb{S}$ is $(\mathbb{S}_L, \dots, \mathbb{S}_0)$.

④ $(\mathbb{S}, \mathbb{S}^*)$ is a minimal element of $\mathrm{Im}\,\Theta$, where $\mathbb{S}^*$ is the generic socle layering of $\mathrm{Rep}\,\mathbb{S}$.

# PRECURSORS

- For $r = 1$, $\Lambda = K[X]/(X^{L+1})$ and all $\mathrm{Rep}_d(\Lambda)$ are irreducible.

- For $r = 2$ and $L = 1$, the irreducible components of $\mathrm{Rep}_d(\Lambda)$ were first determined by Donald-Flanigan, then again by K. Morrison.

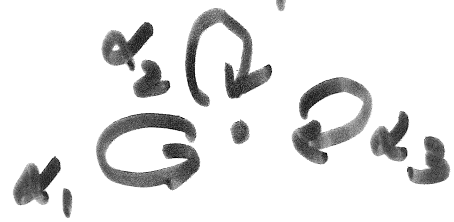
- For arbitrary $r$ and $L = 1$, the components of $\mathrm{Rep}_d(\Lambda)$ were determined by Bleher-Chinburg-HZ. There are

$$d - 2\left\lceil \frac{d}{r+1} \right\rceil + 1$$

of them,
where $\lceil \cdot \rceil$ is the integer ceiling.

# EXAMPLE

$\Lambda = KQ/\langle \text{length } 3 \rangle$, where

$Q$ is 

, $d = 10$.

The irreducible components of $\mathrm{Rep}_d(\Lambda)$ are the $\overline{\mathrm{Rep}\,\mathbb{S}}$ for the following radical layerings $\mathbb{S}$: