Computing vector bundles for modules of constant Jordan type

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Setup:

- p: prime number,
- k: algebraically closed field of characteristic p,
- $E = \langle g_1, \ldots, g_r \rangle \cong (\mathbb{Z}/p)^r$: elementary abelian p-group of rank r,
- set $X_i = g_i - 1 \in kE$.

Fact: Each $X_i^p = 0$ since k has characteristic p, and we have

$$kE \cong k[t_1, \ldots, t_r]/(t_1^p, \ldots, t_r^p), \qquad X_i \mapsto \overline{t_i}.$$

So kE is a local ring, and the X_i generate $\mathrm{Rad}(kE)$.

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In fact, the X_i are a basis for $\operatorname{Rad}(kE)/\operatorname{Rad}^2(kE)$. We identify the latter with $\mathbb{A}^r(k)$ as a k-vector space via

$$\alpha = (\lambda_1, \dots, \lambda_r) \mapsto \lambda_1 X_1 + \cdots + \lambda_r X_r =: X_\alpha.$$

Fact: Each $X_\alpha^p = 0$, so $(1 + X_\alpha)^p = 1$.

Hence if $\alpha \neq 0$, then $k\langle 1 + X_\alpha \rangle \subseteq kE$ is isomorphic to $k(\mathbb{Z}/p)$. These are called *cyclic shifted subgroups* of kE.

- $E = \langle g_1, \ldots, g_r \rangle, \qquad X_i = g_i - 1,$
- for $\alpha = (\lambda_1, \ldots, \lambda_r) \in \mathbb{A}^r(k)$, $X_\alpha := \lambda_1 X_1 + \cdots + \lambda_r X_r$,
- $X_\alpha^p = 0$.

If M is a finite dimensional kE-module and $\alpha \neq 0$, the Jordan canonical form of X_α acting on M consists of Jordan blocks with eigenvalues zero and lengths at most p.

Let

$$\mathrm{JType}(X_\alpha, M) = [p]^{a_p}[p-1]^{a_{p-1}} \ldots [1]^{a_1},$$

where X_α acts on M with a_j Jordan blocks of length j.

This is a partition of $\dim_k(M)$ and gives the isomorphism type of $M{\downarrow}_{k\langle 1+X_\alpha\rangle}$ as a $k(\mathbb{Z}/p)$-module.

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- For $\alpha = (\lambda_1, \ldots, \lambda_r)$, $X_\alpha := \lambda_1 X_1 + \cdots + \lambda_r X_r$,
- $\mathrm{JType}(X_\alpha, M) = [p]^{a_p}[p-1]^{a_{p-1}} \ldots [1]^{a_1}$.

Carlson and Dade pioneered the study of kE-modules via their restrictions to cyclic shifted subgroups in the 1970s. This led to the theory of support varieties for modular group algebras and finite group schemes.

Definition (Carlson, Friedlander, Pevtsova 2008)

A finite dimensional kE-module M has *constant Jordan type* if the partition $\mathrm{JType}(X_\alpha, M)$ is independent of α.

These form a class of modules closed under direct sums, direct summands, tensor products, k-linear duals and syzygies.

Great news: Modules of constant Jordan type give rise to vector bundles on $\mathbb{P}^{r-1}$ in a natural way!

Let $Y_i = X_i^*$, the element dual to X_i in $(\mathbb{A}^r)^*$.

Then $\mathbb{P}^{r-1} = \operatorname{Proj} k[Y_1, \ldots, Y_r]$.

For $n \in \mathbb{Z}$, Friedlander and Pevtsova define the linear operator

$$\begin{aligned} \theta_M \colon M \otimes_k \mathcal{O}_{\mathbb{P}^{r-1}}(n) &\longrightarrow M \otimes_k \mathcal{O}_{\mathbb{P}^{r-1}}(n+1) \\ m \otimes f &\longmapsto \sum X_i m \otimes Y_i f\,. \end{aligned}$$

For each non-zero $\alpha = (\lambda_1, \ldots, \lambda_r) \in \mathbb{A}^r$, the fibre of θ_M at $\overline{\alpha} = [\lambda_1 : \cdots : \lambda_r] \in \mathbb{P}^{r-1}$ is the linear map $X_\alpha \colon M \to M$.

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Definition (Benson, Pevtsova)

For $1 \leq i \leq p$, define the subquotients

$$\mathcal{F}_i(M) = \frac{\operatorname{Ker}\theta_M \cap \operatorname{Im}\theta_M^{i-1}}{\operatorname{Ker}\theta_M \cap \operatorname{Im}\theta_M^{i}}$$

of $M \otimes_k \mathcal{O}_{\mathbb{P}^{r-1}}$. This assignment is functorial in M.

Proposition (Benson, Pevtsova)

M has constant Jordan type $[p]^{a_p} \dots [1]^{a_1}$ if and only if $\mathcal{F}_i(M)$ is a vector bundle of rank a_i on $\mathbb{P}^{r-1}$ for all i.

$$\mathcal{F}_i(M) = \frac{\operatorname{Ker}\theta_M \cap \operatorname{Im}\theta_M^{i-1}}{\operatorname{Ker}\theta_M \cap \operatorname{Im}\theta_M^{i}}$$

Proof idea.

The fibre of $\mathcal{F}_i(M)$ at $\overline{\alpha} \in \mathbb{P}^{r-1}$ is

$$\frac{\operatorname{Ker}(X_\alpha, M) \cap \operatorname{Im}(X_\alpha^{i-1}, M)}{\operatorname{Ker}(X_\alpha, M) \cap \operatorname{Im}(X_\alpha^{i}, M)},$$

the dimension of which is the number of Jordan blocks of length i in the action of X_α on M. □

Fact: Not much is known about which sorts of vector bundles do/don't exist on $\mathbb{P}^n$.

Idea: Try to find relationships between $\mathcal{F}_i(M)$ and the internal structure of M.

Definition (Carlson, Friedlander, Suslin)

A kE-module M has the *equal images property* if $\mathrm{Im}(X_\alpha, M)$ is independent of the choice of non-zero $\alpha \in \mathbb{A}^r(k)$. In this case we have $\mathrm{Im}(X_\alpha, M) = \mathrm{Rad}(M)$ for all α.

Proposition (Carlson, Friendlander, Suslin)

If M has the equal images property, then M has constant Jordan type.

Until further notice, restrict to the case $r = 2$, i.e., $E \cong \mathbb{Z}/p \times \mathbb{Z}/p$.

Definition (Carlson, Friedlander, Suslin)

Let M be a kE-module and S a cofinite subset of $\mathbb{P}^1(k)$. Set

$$_S M = \sum_{\overline{\alpha} \in S} \mathrm{Ker}(X_\alpha, M).$$

The *generic kernel* of M is defined to be the submodule

$$\mathfrak{K}(M) = \bigcap_{S \subseteq \mathbb{P}^1(k) \text{ cofinite}} {}_S M$$

of M.

Remark: There always exists a cofinite $S \subseteq \mathbb{P}^1(k)$ for which $\mathfrak{K}(M) = {}_S M$.

Definition (Carlson, Friedlander, Suslin)

In any rank r, a kE-module M has *constant j-rank* if $\mathrm{rank}(X_\alpha^j, M)$ is independent of the choice of non-zero point $\alpha \in \mathbb{A}^r(k)$.

Theorem (Carlson, Friedlander, Suslin)

Let M be a kE-module in rank two.

1. *The generic kernel $\mathfrak{K}(M)$ has the equal images property.*
2. *If N is any submodule of M having the equal images property, then $N \subseteq \mathfrak{K}(M)$.*
3. *If M has constant 1-rank, then*

$$\mathfrak{K}(M) = {}_{\mathbb{P}^1(k)}M = \sum_{\overline{\alpha} \in \mathbb{P}^1(k)} \mathrm{Ker}(X_\alpha, M).$$

Set $J = \mathrm{Rad}(kE)$ and consider the filtration

$$0 = J^p\mathfrak{K}(M) \subseteq \cdots \subseteq J\mathfrak{K}(M) \subseteq \mathfrak{K}(M) \subseteq J^{-1}\mathfrak{K}(M) \subseteq \cdots$$
$$\cdots \subseteq J^{-p+1}\mathfrak{K}(M) = M.$$

Here, $J^{-i}\mathfrak{K}(M) = \{m \in M \mid J^i m \subseteq \mathfrak{K}(M)\}$.

Proposition (B. 2012)

If M has constant 1-rank, then for any $\alpha \in \mathbb{A}^r(k)$ and $i \leq \min\{j, \ell - 1\}$, the number of Jordan blocks of size i in the action of X_α on $J^{-j}\mathfrak{K}(M)/J^{\ell}\mathfrak{K}(M)$ is equal to that on M.

Theorem (B., K. Chan, Pevtsova)

Let $r = 2$. If M is a kE-module of constant Jordan type and $i \leq \min\{j, \ell - 1\}$, then $\mathcal{F}_i(J^{-j}\mathfrak{K}(M)/J^{\ell}\mathfrak{K}(M))$ is a vector bundle on $\mathbb{P}^1(k)$, and we have

$$\mathcal{F}_i(J^{-j}\mathfrak{K}(M)/J^{\ell}\mathfrak{K}(M)) \cong \mathcal{F}_i(M).$$

Example

For $i = 1$, this shows that $\mathcal{F}_1(J^{-1}\mathfrak{K}(M)/J^2\mathfrak{K}(M)) \cong \mathcal{F}_1(M)$. The former has Loewy length three, regardless of the Loewy length of M.

Definition (Carlson, Friedlander, Suslin)

Fix $n > 0$. The *nth power generic kernel* is the submodule

$$\mathfrak{K}^n(M) = \bigcap_{S \subseteq \mathbb{P}^1(k) \text{ cofinite}} \sum_{\overline{\alpha} \in S} \operatorname{Ker}(X_\alpha^n, M).$$

Again, there always exists a cofinite $S \subseteq \mathbb{P}^1(k)$ satisfying $\mathfrak{K}^n(M) = \sum_{\overline{\alpha} \in S} \operatorname{Ker}(X_\alpha^n, M)$.

If M has constant 1-rank, then $\mathfrak{K}^n(M)$ is contained in $J^{-n+1}\mathfrak{K}(M)$, so we have inclusions

$$\begin{array}{ccccccccc} \mathfrak{K}(M) & \subseteq & J^{-1}\mathfrak{K}(M) & \subseteq & J^{-2}\mathfrak{K}(M) & \subseteq \cdots \subseteq & J^{-p+1}\mathfrak{K}(M) & & \\ \| & & \cup\!| & & \cup\!| & & \| & & \\ \mathfrak{K}^1(M) & \subseteq & \mathfrak{K}^2(M) & \subseteq & \mathfrak{K}^3(M) & \subseteq \cdots \subseteq & \mathfrak{K}^p(M) & = & M \end{array}.$$

Dually, one can define the *nth power generic image* of M to be the submodule

$$\mathfrak{I}^n(M) = \sum_{S \subseteq \mathbb{P}^1(k) \text{ cofinite}} \bigcap_{\overline{\alpha} \in S} \operatorname{Im}(X_\alpha^n, M).$$

Theorem (B., K. Chan, Pevtsova)

If M has constant Jordan type and $i \leq \min\{n-1, m-1\}$, then

$$\mathcal{F}_i(\mathfrak{K}^n(M)/\mathfrak{I}^m\mathfrak{K}^n(M)) \cong \mathcal{F}_i(M)$$

and

$$\mathcal{F}_i(\mathfrak{K}^n(M/\mathfrak{I}^m(M))) \cong \mathcal{F}_i(M).$$

Example

For $i = 1$, we obtain $\mathcal{F}_1(\mathfrak{K}^2(M)/\mathfrak{I}^2\mathfrak{K}^2(M)) \cong \mathcal{F}_1(M)$.

All of this works for arbitrary rank r!

Naively define $\mathfrak{K}^n(M) = \sum_{\overline{\alpha} \in \mathbb{P}^{r-1}(k)} \operatorname{Ker}(X_\alpha^n, M)$.

Then for $i \leq n-1$ and all $\overline{\alpha} \in \mathbb{P}^{r-1}(k)$ we have

$$\frac{\operatorname{Ker}(X_\alpha, \mathfrak{K}^n(M)) \cap \operatorname{Im}(X_\alpha^{i-1}, \mathfrak{K}^n(M))}{\operatorname{Ker}(X_\alpha, \mathfrak{K}^n(M)) \cap \operatorname{Im}(X_\alpha^{i}, \mathfrak{K}^n(M))} = \frac{\operatorname{Ker}(X_\alpha, M) \cap \operatorname{Im}(X_\alpha^{i-1}, M)}{\operatorname{Ker}(X_\alpha, M) \cap \operatorname{Im}(X_\alpha^{i}, M)}.$$

via the inclusion $\mathfrak{K}^n(M) \subseteq M$.

This is used to show that if M has constant Jordan type, then $\mathfrak{K}^n(M)$ has constant j-rank for all $j \leq n$.

The above two facts imply that $\mathcal{F}_i(\mathfrak{K}^n(M))$ is a vector bundle on $\mathbb{P}^{r-1}(k)$ and that $\mathcal{F}_i(\mathfrak{K}^n(M)) = \mathcal{F}_i(M)$.

Thank you for your time.