Support Varieties and Representation Type

Jörg Feldvoss
University of South Alabama

Sarah Witherspoon
Texas A&M University

Representation Type

$\mathbb{F}$ - algebraically closed field
A - finite dimensional (associative) $\mathbb{F}$-algebra

Definition

(1) A is of finite representation type if there are finitely many iso. classes of indec. (left) A-modules

(2) A is tame if it is not of finite rep. type and if the iso. classes of indec. (left) A-modules in any fixed dimension are almost all contained in a finite number of 1-parameter families

(3) A is wild if the category of fin. dim. (left) A-mods. contains the category of $\mathbb{F}\langle x, y\rangle$-modules

Trichotomy Theorem (Drozd '77, Crawley-Boevey '88)
Every fin. dim. algebra is finite, tame, or wild
(and these three are mutually exclusive)

Example $\mathbb{F}G$, where G is a finite group:
Let $p = \mathrm{char}\,\mathbb{F}$ and $P \in \mathrm{Syl}_p(G)$
(1) (Higman '54) $\mathbb{F}G$ is finite iff P is cyclic
(2) (Bondarenko-Drozd '77) $\mathbb{F}G$ is tame iff $p = 2$ and P is dihedral, semidihedral, or generalized quaternion
(3) Otherwise $\mathbb{F}G$ is wild

Rickard ('90) explained a connection between representation type of $\mathbb{F}G$ and Krull dimension of the cohomology of G, and generalized the theory to self-injective algebras

Determining representation type

There are various tools for finding representation type; one such is the theory of varieties for modules (first used in this context by Farnsteiner '07 for cocommutative Hopf algebras)

We will explain this theory when A is a Hopf algebra (in particular when there is an algebra hom.

$$\Delta : A \rightarrow A \otimes_{\mathbb{F}} A,$$

etc., so that for every two A-modules M, N, their tensor product $M \otimes_{\mathbb{F}} N$ is also an A-module)

Cohomology

From now on, A is a Hopf algebra, and the field $\mathbb{F}$ is an A-module via $\varepsilon : A \to \mathbb{F}$

Notation $\mathrm{H}^*(A) := \mathrm{Ext}^*_A(\mathbb{F}, \mathbb{F})$, itself an algebra with product given by Yoneda composition

Remarks
Since A is a Hopf algebra:
(1) $\mathrm{H}^*(A)$ is (graded) commutative
(2) $\mathrm{Ext}^*_A(M, M)$ is an $\mathrm{H}^*(A)$-module under $-\otimes_{\mathbb{F}} M$

Finite generation of cohomology

We will assume that $\mathrm{H}^*(A)$ is finitely generated and that $\mathrm{Ext}^*_A(M, M)$ is a fin. gen. $\mathrm{H}^*(A)$-mod.

This is known to be true when
(1) $A = \mathbb{F}G$ (Golod '59, Venkov '59, Evens '61)
(2) (More generally) A is cocommutative, i.e. a finite group scheme (Friedlander-Suslin '97)
(3) $A = u_q(\mathfrak{g})$ (Ginzburg-Kumar '93, Bendel-Nakano-Parshall-Pillen)
(4) (More generally) A is "pointed" under some assumptions (Mastnak-Pevtsova-Schauenburg-W)

It is conjectured to be true in general (Etingof-Ostrik '04)

Varieties for modules

Assume $\mathrm{H}^*(A)$ is fin. gen. (graded) commutative, and that $\mathrm{Ext}_A^*(M,M)$ is a fin. gen. $\mathrm{H}^*(A)$-mod.

Definition

$$\mathcal{V}_A(\mathbb{F}) := \mathrm{MaxSpec}(\mathrm{H}^{2*}(A))$$
$$\mathcal{V}_A(M) := \mathrm{MaxSpec}(\mathrm{H}^{2*}(A)/\operatorname{Ann}_{\mathrm{H}^{2*}(A)} \mathrm{Ext}_A^*(M,M))$$

Remark Similar varieties for modules were defined using Hochschild cohomology, under some finiteness assumptions (Snashall-Solberg '04)

Properties of varieties for modules

(1) $\mathcal{V}_A(M) = \{0\}$ iff M is projective
(2) $\dim \mathcal{V}_A(M) = \mathrm{cx}_A(M)$
(3) $\mathcal{V}_A(M \oplus N) = \mathcal{V}_A(M) \cup \mathcal{V}_A(N)$
(4) $\mathcal{V}_A(M \otimes_{\mathbb{F}} N) \subseteq \mathcal{V}_A(M) \cap \mathcal{V}_A(N)$

Remark The containment in (4) is known to be an equality when A is cocommutative (Friedlander-Pevtsova '05)

In general it is only known to be an equality when one of the modules is a special type of module: "Carlson's L_ζ-modules" (Pevtsova-Feldvoss-W)

Wildness criterion

Theorem (Bergh-Solberg, Feldvoss-W)
Let A be a fin. dim. Hopf algebra under finiteness assumptions on cohomology. If $\dim \mathcal{V}_A(\mathbb{F}) \geq 3$, then A is wild.

Idea of proof:
Carlson's L_ζ-modules and tensor product property
$\Rightarrow$ in some dimension there are infinitely many iso. classes of indec. A-modules with $\mathrm{cx}_A \geq 2$
By Crawley-Boevey '88, A is wild

Remark Bergh-Solberg proved this more generally for self-injective algebras, using Hochschild cohomology

Application: small quantum groups

$\mathbb{F}$ - characteristic 0

The following was conjectured by Cibils '97 in the simply laced case:

Theorem (Feldvoss-W)
$u_q(\mathfrak{g})$ is wild whenever rank$(\mathfrak{g}) \geq 2$, ℓ is odd,
ℓ is not divisible by 3 if $\mathfrak{g}$ is of type G_2,
$\ell > 3$ if $\mathfrak{g}$ is of type B, C, E, or F,
and $\ell > 5$ if of type E_8

Remark (Suter '94, Xiao '97)
$u_q(\mathfrak{g})$ is tame if rank$(\mathfrak{g}) = 1$

Carlson's L_ζ-modules and tensor product property

Let $\Omega^n_A(\mathbb{F})$ be the nth syzygy of $\mathbb{F}$
Let $\zeta \in \mathrm{H}^n(A) \cong \mathrm{Hom}_A(\Omega^n_A(\mathbb{F}), \mathbb{F})$

Let L_ζ be the kernel of the corresponding map

$$\Omega^n_A(\mathbb{F}) \to \mathbb{F}$$

Theorem (Pevtsova-Feldvoss-W)

$$\mathcal{V}_A(M \otimes L_\zeta) = \mathcal{V}_A(M) \cap \mathcal{V}_A(L_\zeta)$$

and $\mathcal{V}_A(L_\zeta) = Z(\zeta) := \mathrm{MaxSpec}(\mathrm{H}^*(A)/(\zeta))$

Wildness criterion

Theorem (Bergh-Solberg, Feldvoss-W)
Let A be a fin. dim. Hopf algebra under finiteness assumptions on cohomology. If $\dim \mathcal{V}_A(\mathbb{F}) \geq 3$, then A is wild.

Some details of proof: Let $n = \dim \mathcal{V}_A(\mathbb{F}) \geq 3$.

- For some $d > 0$, for all $\alpha \in \mathbb{F}$ there is a nonzero $\zeta_\alpha \in \mathrm{H}^{2d}(A)$ such that
  $\dim Z(\zeta_\alpha) = n - 1$
  $\dim Z(\zeta_\alpha) \cap Z(\zeta_\beta) = n - 2$ for all $\beta \neq \alpha$
- Let X_α be indec. summand of L_{ζ_α} so
  $\mathcal{V}_A(X_\alpha) \subset \mathcal{V}_A(L_{\zeta_\alpha}) = Z(\zeta_\alpha)$,
  $\dim_{\mathbb{F}} \mathcal{V}_A(X_\alpha) = n - 1$
- $X_\alpha \not\cong X_\beta$ for all $\alpha \neq \beta \in \mathbb{F}$ since
  $\mathcal{V}_A(X_\alpha) \cap \mathcal{V}_A(X_\beta)$ has $\dim \leq n - 2$
- $\dim_{\mathbb{F}} X_\alpha \leq \dim_{\mathbb{F}} \Omega^{2d}(\mathbb{F})$ for all α
- $\mathrm{cx}_A(X_\alpha) = \dim_{\mathbb{F}} \mathcal{V}_A(X_\alpha) = n - 1 \geq 2$ for all α

Conclusion: In some dimension there are infinitely many iso. classes of indec. A-modules with $\mathrm{cx}_A \geq 2$
By Crawley-Boevey '88, A is wild

Definition of a small quantum group

$\mathbb{F}$ - characteristic 0
q - primitive ℓth root of 1 in $\mathbb{F}$

$$u_q(\mathfrak{sl}_2) := \mathbb{F}\langle e, f, k \mid e^\ell = 0,\ f^\ell = 0,\ k^\ell = 1, \\ kek^{-1} = q^2 e,\ kfk^{-1} = q^{-2} f, \\ ef - fe = \frac{k - k^{-1}}{q - q^{-1}} \rangle$$

More generally $u_q(\mathfrak{g})$ may be defined for any fin. dim. semisimple Lie algebra $\mathfrak{g}$