A Problem of Kollár and Larsen on Finite Linear Groups and Crepant Resolutions

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Outline

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- Definition and Examples
- Kollár-Larsen Problem
- Motivation: Algebraic geometry
- 2 Main Results
 - Groups generated by elements of age \leq 1
 - Groups generated by elements of age < 1
 - Kollár-Larsen conjecture
- 3 Age and Deviation
 - Properties of age
 - L²-deviation



Main Ingredients of the Proofs

Further Results

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Main Ingredients of the Proofs

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Definition and Examples Kollár-Larsen Problem Motivation: Algebraic geometry



$$\begin{split} &V = \mathbb{C}^n, \, (\cdot, \cdot) \text{ standard Hermitian form} \\ &||v|| = \sqrt{(v, v)} \\ &S^1 = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\} \end{split}$$

Definition 1.1 (M. Reid)

Let $g \in GU(V)$ be conjugate to

diag $(e^{2\pi i r_1}, \ldots, e^{2\pi i r_n}), \quad 0 \le r_i < 1.$

 $ext{age}(g) = \sum_{j=1}^n r_j.$ $g ext{ is junior } ext{if } 0 < ext{age}(g) \leq 1.$

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Definition and Examples Kollár-Larsen Problem Motivation: Algebraic geometry

Examples of non-scalar junior elements

- Reflections: $g \sim (-1, 1, \dots, 1)$, age = 1/2
- Bireflections: $g \sim (-1, -1, 1, ..., 1)$, age = 1

• Complex reflections (c.r.):

 $g \sim (\alpha, 1, \dots, 1), \quad 1 \neq \alpha \in S^1, \quad 0 < age < 1$

• Complex bireflections:

 $g \sim (\alpha, \alpha^{-1}, 1, \dots, 1), \quad 1 \neq \alpha \in S^1, \quad \text{age} = 1$

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Problem 1.2 (Kollár-Larsen)

Describe finite irreducible subgroups G < GL(V) which are generated, up to scalars, by elements of age < 1 (resp. of age ≤ 1).

In a sense, the description of finite subgroups G < GL(V)**containing** a non-scalar element of age < 1 (resp. of age \leq 1) reduces to Problem 1.2.

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Definition and Examples Kollár-Larsen Problem Motivation: Algebraic geometry

Crepant resolutions. I

- $f : X \to Y$ a resolution $\Longrightarrow K_X = f^*K_Y + \sum_i a_i E_i$ (sum over irreducible exceptional divisors)
- $a_i > 0, \forall i \implies terminal$
- $a_i \ge 0, \quad \forall i \implies$ canonical
- $a_i = 0, \forall i \implies$ crepant

Criterion 1.3 (Reid-Tai)

Assume G < GL(V) contains no complex reflections. Then the singularity V/G is terminal, resp. canonical, if for all $1 \neq g \in G$, age(g) > 1, resp. age $(g) \ge 1$.

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Crepant resolutions. II

Corollary 1.4 (Ito-Reid)

Assume G < GL(V) is finite and $f : X \rightarrow V/G$ is a crepant resolution. Then G contains junior elements.

Crepant resolutions are important in algebraic geometry:

- Minimal models in Mori's program
- Mirror symmetry: Crepant resolutions of X/G, X a Calabi-Yau variety
- String theory:

If $f : X \rightarrow Y$ is a crepant resolution, then the string theories on X and Y are "the same"

(i.e. same quantum cohomologies: Ruan, Bryan-Graber)

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Definition and Examples Kollár-Larsen Problem Motivation: Algebraic geometry

Quotients of Calabi-Yau varieties

Kollár-Larsen: X a smooth Calabi-Yau variety, G finite

Kodaira dimension of X/G is controlled by whether $Stab_X(G)$ contains $g \neq 1$ with age < 1 while acting on T_XX for $x \in X$.

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Groups generated by elements of age ≤ 1 Groups generated by elements of age < 1Kollár-Larsen conjecture

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Groups generated by elements of age ≤ 1 Groups generated by elements of age < 1Kollár-Larsen conjecture

For simplicity we skip the results for small n.

Theorem 2

Let $V = \mathbb{C}^n$ with $n \ge 11$ and let G < GL(V) be a finite irreducible subgroup. Assume that, up to scalars, G is generated by its elements with age ≤ 1 . Then G contains a complex bireflection of order 2 or 3, and one of the following statements holds.

(i) $Z(G) \times A_{n+1} \leq G \leq (Z(G) \times A_{n+1}) \cdot 2.$

(ii) G preserves a decomposition $V = V_1 \oplus \ldots \oplus V_n$, with

dim(V_i) = 1 and G inducing either S_n or A_n while permuting the n subspaces V_1, \ldots, V_n .

(iii) 2|*n*, and G = D : $S_{n/2} < GL_2(\mathbb{C}) \wr S_{n/2}$, a split extension of D < $GL_2(\mathbb{C})^{n/2}$ by $S_{n/2}$.

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(i) $Z(G) \times A_{n+1} \leq G \leq (Z(G) \times A_{n+1}) \cdot 2$. (ii) *G* preserves a decomposition $V = V_1 \oplus \ldots \oplus V_n$, with dim $(V_i) = 1$ and *G* inducing either S_n or A_n while permuting the *n* subspaces V_1, \ldots, V_n . (iii) 2|n, and $G = D : S_{n/2} < GL_2(\mathbb{C}) \wr S_{n/2}$, a split extension of $D < GL_2(\mathbb{C})^{n/2}$ by $S_{n/2}$.

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(iii) 2|n, and G = D: $S_{n/2} < GL_2(\mathbb{C}) \wr S_{n/2}$, a split extension of $D < GL_2(\mathbb{C})^{n/2}$ by $S_{n/2}$.

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Groups generated by elements of age ≤ 1 Groups generated by elements of age < 1Kollár-Larsen conjecture

Outline

Kollár-Larsen Problem Motivation: Algebraic geometry Main Results Groups generated by elements of age < 1 Groups generated by elements of age < 1 Kollár-Larsen conjecture Properties of age L²-deviation

Groups generated by elements of age ≤ 1 Groups generated by elements of age < 1Kollár-Larsen conjecture

Theorem 2.2

Let $V = \mathbb{C}^n$ with $n \ge 9$ and let G < GL(V) be a finite irreducible subgroup. Assume that, up to scalars, G is generated by its elements with age < 1. Then G contains a scalar multiple of a complex reflection, and either (i) or (ii) of Theorem 2.1 holds.

Corollary 2.3

Let $n \ge 11$ and let $G < GL_n(\mathbb{C})$ be a finite irreducible, primitive, tensor indecomposable subgroup. Assume that \mathbb{C}^n/G is not terminal (for instance, it has a crepant resolution). Then one of the following statements holds. (i) $Z(G) \times A_{n+1} \le G \le (Z(G) \times A_{n+1}) \cdot 2$. (ii) All junior elements of G are central, and $|Z(G)| \ge n$.

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Groups generated by elements of age ≤ 1 Groups generated by elements of age < 1 Kollár-Larsen conjecture

Outline

Definition and Examples Kollár-Larsen Problem Motivation: Algebraic geometry Main Results Groups generated by elements of age < 1 Kollár-Larsen conjecture Properties of age L²-deviation

- Main Ingredients of the Proof
- Further Results

Groups generated by elements of age ≤ 1 Groups generated by elements of age < 1Kollár-Larsen conjecture

Theorem 2.4

Let n > 4 and let $G < GL_n(\mathbb{C})$ be a finite irreducible subgroup. Assume that G contains non-central elements g with age < 1, and that $G = \langle g^G \rangle$ for any such g. Then, up to scalars, G is a complex reflection group, and so known by **Shephard-Todd**.

Fails for n = 4: $C_3 \times 2A_m < GL_4(\mathbb{C})$ with m = 6, 7.

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Properties of age L²-deviation

Outline

- Kollár-Larsen Problem Motivation: Algebraic geometry Kollár-Larsen conjecture Age and Deviation Properties of age
 - L²-deviation
 - Main Ingredients of the Proof
 - Further Results

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Properties of age L²-deviation

More about age

• $age(g) = age(g|_U) + age(g|_{V/U})$ if $U \subseteq V$ is g-stable

- age(diag(g, h)) = age(g) + age(h)
- $age(gh) \le age(g) + age(h)$ if gh = hg
- But: $age(g^{-1})
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Properties of age L²-deviation

Chen-Ruan inequality

Theorem 3.1 (Chen-Ruan)

 $\operatorname{age}(g) + \operatorname{age}(h) - \operatorname{age}(gh) \ge \dim C_V(gh) - \dim C_V(g,h).$

Proof 1 (Chen-Ruan). Use the existence of a cohomology theory for orbifolds.

Proof 2 (G-T) (following suggestions of **Katz** and **Tao**). Use interlacing properties of eigenvalues. Write *g* as a product of *m* commuting c.r.'s and induct on *m*

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Properties of age L^2 -deviation

Outline

Kollár-Larsen Problem Motivation: Algebraic geometry Kollár-Larsen conjecture Age and Deviation Properties of age

- L²-deviation
- 4

Main Ingredients of the Proofs

Further Results

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Properties of age L^2 -deviation

Key remedy

Work with an L^2 -deviation instead of age !

 \mathcal{B} the collection of all orthonormal bases of V.

Definition 3.2

For $T \in GU(V)$

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Properties of age L^2 -deviation



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Properties of age L^2 -deviation



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Definition 3.2

For $T \in GU(V)$,

$$d_2(T) = \inf_{\lambda \in S^1, \ B \in \mathcal{B}} \left(\sum_{v \in B} ||T(b) - \lambda b||^2 \right)^{1/2}.$$

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Properties of age L^2 -deviation

Properties of Deviation

• $d_2(\alpha T) = d_2(T)$ for $\alpha \in S^1$

- $d_2(ATA^{-1}) = d_2(T)$ for $A \in GU(V)$
- $d_2(T^{-1}) = d_2(T)$
- More importantly, $d_2(T)^2 = 2(\dim V |\operatorname{Tr}(T)|)$

Hence one can invoke character theory.

• $d_2(T)^2 \le (2.9)\pi \cdot \text{age}(T)$. 9 can be attained. If $\text{age}(T) \le 1$ then $d_2(T)^2 \le 9.111$

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- Aschbacher's Theorem: to reduce to the almost-quasi-simple case
- Bounds on character ratios:
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