A Problem of Kollár and Larsen on Finite Linear Groups and Crepant Resolutions

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Outline

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1. Introduction
   - Definition and Examples
   - Kollár-Larsen Problem
   - Motivation: Algebraic geometry
2. Main Results
   - Groups generated by elements of age ≤ 1
   - Groups generated by elements of age < 1
   - Kollár-Larsen conjecture
3. Age and Deviation
   - Properties of age
   - L^2-deviation
4. Main Ingredients of the Proofs
5. Further Results

Age

$V = \mathbb{C}^n$, $(\cdot,\cdot)$ standard Hermitian form
$||v|| = \sqrt{(v,v)}$
$S^1 = \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$

Definition 1.1 (M. Reid)

Let $g \in GU(V)$ be conjugate to

$$\mathrm{diag}(e^{2\pi i r_1}, \dots, e^{2\pi i r_n}), \quad 0 \le r_j < 1.$$

$\mathrm{age}(g) = \sum_{j=1}^{n} r_j$.

g is **junior** *if* $0 < \mathrm{age}(g) \le 1$.

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Examples of non-scalar junior elements

- Reflections: $g \sim (-1, 1, \ldots, 1)$, age $= 1/2$
- Bireflections: $g \sim (-1, -1, 1, \ldots, 1)$, age $= 1$
- Complex reflections (c.r.):

$$g \sim (\alpha, 1, \ldots, 1), \quad 1 \neq \alpha \in S^1, \quad 0 < \text{age} < 1$$

- Complex bireflections:

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Problem 1.2 (Kollár-Larsen)

Describe finite irreducible subgroups $G < GL(V)$ which are generated, up to scalars, by elements of age < 1 *(resp. of* age ≤ 1*).*

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Crepant resolutions. I

$f : X \to Y$ a resolution $\Longrightarrow K_X = f^* K_Y + \sum_i a_i E_i$
(sum over irreducible exceptional divisors)

$a_i > 0, \quad \forall i \quad \Longrightarrow$ **terminal**

$a_i \geq 0, \quad \forall i \quad \Longrightarrow$ **canonical**

$a_i = 0, \quad \forall i \quad \Longrightarrow$ **crepant**

Criterion 1.3 (Reid-Tai)

Assume $G < GL(V)$ contains no complex reflections. Then the singularity V/G is terminal, resp. canonical, if for all $1 \neq g \in G$, $\mathrm{age}(g) > 1$, *resp.* $\mathrm{age}(g) \geq 1$.

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Crepant resolutions. II

Corollary 1.4 (Ito-Reid)

Assume $G < GL(V)$ is finite and $f : X \to V/G$ is a crepant resolution. Then G contains junior elements.

Crepant resolutions are important in algebraic geometry:

- Minimal models in **Mori**'s program
- Mirror symmetry: Crepant resolutions of X/G, X a Calabi-Yau variety
- String theory:

  If $f : X \to Y$ is a crepant resolution, then the string theories on X and Y are "the same"

  (i.e. same quantum cohomologies: **Ruan, Bryan-Graber**)

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Quotients of Calabi-Yau varieties

Kollár-Larsen: X a smooth Calabi-Yau variety, G finite

Kodaira dimension of X/G is controlled by whether $Stab_x(G)$ contains $g \neq 1$ with age < 1 while acting on T_xX for $x \in X$.

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Theorem 2.1

Let $V = \mathbb{C}^n$ with $n \geq 11$ and let $G < GL(V)$ be a finite irreducible subgroup. Assume that, up to scalars, G is generated by its elements with age ≤ 1. Then G contains a complex bireflection of order 2 or 3, and one of the following statements holds.

(i) $Z(G) \times \mathsf{A}_{n+1} \leq G \leq (Z(G) \times \mathsf{A}_{n+1}) \cdot 2$.

(ii) *G preserves a decomposition $V = V_1 \oplus \ldots \oplus V_n$, with $\dim(V_i) = 1$ and G inducing either S_n or A_n while permuting the n subspaces $V_1, \ldots, V_n$.*

(iii) *$2|n$, and $G = D : \mathsf{S}_{n/2} < GL_2(\mathbb{C}) \wr \mathsf{S}_{n/2}$, a split extension of $D < GL_2(\mathbb{C})^{n/2}$ by $\mathsf{S}_{n/2}$.*

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Let $V = \mathbb{C}^n$ with $n \geq 9$ and let $G < GL(V)$ be a finite irreducible subgroup. Assume that, up to scalars, G is generated by its elements with age < 1. Then G contains a scalar multiple of a complex reflection, and either (i) *or* (ii) *of Theorem 2.1 holds.*

Corollary 2.3

Let $n \geq 11$ and let $G < GL_n(\mathbb{C})$ be a finite irreducible, primitive, tensor indecomposable subgroup. Assume that $\mathbb{C}^n/G$ is not terminal (for instance, it has a crepant resolution). Then one of the following statements holds.

(i) $Z(G) \times \mathsf{A}_{n+1} \leq G \leq (Z(G) \times \mathsf{A}_{n+1}) \cdot 2$.

(ii) *All junior elements of G are central, and $|Z(G)| \geq n$.*

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Let $n \geq 11$ and let $G < GL_n(\mathbb{C})$ be a finite irreducible, primitive, tensor indecomposable subgroup. Assume that $\mathbb{C}^n/G$ is not terminal (for instance, it has a crepant resolution). Then one of the following statements holds.

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Theorem 2.4

Let $n > 4$ and let $G < GL_n(\mathbb{C})$ be a finite irreducible subgroup. Assume that G contains non-central elements g with age < 1, and that $G = \langle g^G \rangle$ for any such g. Then, up to scalars, G is a complex reflection group, and so known by **Shephard-Todd**.

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More about age

- $\mathrm{age}(g) = \mathrm{age}(g|_U) + \mathrm{age}(g|_{V/U})$ if $U \subseteq V$ is g-stable
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Theorem 3.1 (Chen-Ruan)

$$\mathrm{age}(g) + \mathrm{age}(h) - \mathrm{age}(gh) \geq \dim C_V(gh) - \dim C_V(g, h).$$

Proof 1 (Chen-Ruan). Use the existence of a cohomology theory for orbifolds.

Proof 2 (G-T) (following suggestions of **Katz** and **Tao**).
Use interlacing properties of eigenvalues.
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Key remedy

Work with an L^2-deviation instead of age !

$\mathcal{B}$ the collection of all orthonormal bases of V.

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For $T \in GU(V)$,

$$d_2(T) = \inf_{\lambda \in S^1,\ B \in \mathcal{B}} \left(\sum_{v \in B} \| T(b) - \lambda b \|^2 \right)^{1/2}.$$

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  Hence one can invoke character theory.
- $d_2(T)^2 \leq (2.9)\pi \cdot \mathrm{age}(T)$.

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