# Colored tangle invariants and quantum $\mathfrak{sl}_2$ categorification

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# Categorifications of the Reshetikhin-Turaev invariant

The R-T invariant for $\mathcal{U}_q(\mathfrak{sl}_2)$ assigns a $\mathcal{U}_q(\mathfrak{sl}_2)-$ homomorphism to an oriented, framed tangle whose components are labeled by representations.

- Using a certain diagram algebra, Khovanov constructed a categorification when the labels of the tangle are $V_1$.
- Bernstein-Frenkel-Khovanov outlined a categorification for this invariant using category $\mathcal{O}$.
- Stroppel proved the conjectures of [BFK].
- Cautis-Kamnitzer gave a geometric categorification of this invariant.

- For the colored Jones polynomial, Khovanov constructed a categorification using a certain cabling procedure.
- Frenkel-Khovanov-Stroppel categorified $V_{d_1} \otimes \cdots \otimes V_{d_r}$ using categories of Harish-Chandra bimodules.
- Goal: Extend the [FKS] construction to a tangle invariant.
- Webster accomplishes this (and more) using a modification of the Khovanov-Lauda-Rouquier algebra.

# $\mathcal{U}_q(\mathfrak{sl}_2)$

$\mathcal{U}_q(\mathfrak{sl}_2)$ is the $\mathbb{C}(q)$ algebra generated by $E, F, K, K^{-1}$ with relations:

- $KE = q^2 EK$
- $KF = q^{-2} FK$
- $KK^{-1} = 1 = K^{-1}K$
- $EF - FE = \frac{K-K^{-1}}{q-q^{-1}}$.

Let $V_n$ be the $(n+1)$- dimensional irreducible representation with basis $\{v_0, \ldots, v_n\}$ such that

- $Ev_k = [k+1]v_{k+1}$
- $Fv_k = [n-k+1]v_{k-1}$
- $K^{\pm 1}v_k = q^{\pm(2k-n)}v_k$.

# Cup, cap and crossing intertwiners

There is a cap morphism $\cap\colon V_1^{\otimes 2} \to \mathbb{C}(q)$ given by

- $\cap(v_1 \otimes v_1) = \cap(v_0 \otimes v_0) = 0$
- $\cap(v_0 \otimes v_1) = 1$
- $\cap(v_1 \otimes v_0) = -q^{-1}$.

This gives rise to the map $\cap_{i,n}\colon V_1^{\otimes n} \to V_1^{\otimes n-2}$.

There is a cup morphism $\cup\colon \mathbb{C}(q) \to V_1^{\otimes 2}$ given by

- $\cup(1) = v_1 \otimes v_0 - q v_0 \otimes v_1$.

This gives rise to the map $\cup_{i,n}\colon V_1^{\otimes n} \to V_1^{\otimes n+2}$.

# Crossings

There is a crossing map $\Pi\colon V_1^{\otimes 2} \to V_1^{\otimes 2}$ given by

$$\Pi = -q^2 \mathrm{Id} - q(\cup \circ \cap).$$

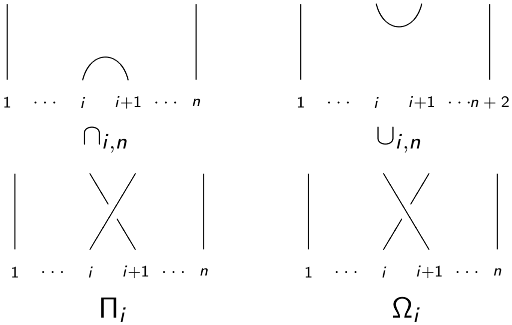
This gives rise to the map $\Pi_{i,n}\colon V_1^{\otimes n} \to V_1^{\otimes n}$.

There is a crossing map $\Omega\colon V_1^{\otimes 2} \to V_1^{\otimes 2}$ given by

$$\Omega = -q^{-2} \mathrm{Id} - q^{-1}(\cup \circ \cap).$$

This gives rise to the map $\Omega_{i,n}\colon V_1^{\otimes n} \to V_1^{\otimes n}$.

# Reshetikhin-Turaev invariant




## Theorem (Reshetikhin-Turaev)

*Let $T$ be an oriented tangle from $n$ points to $m$ points. Let $D_1$ and $D_2$ be two diagrams of the tangle. Then $\phi(D_1), \phi(D_2) \colon V_1^{\otimes n} \to V_1^{\otimes m}$ and $q^{3\gamma(D_1)}\phi(D_1) = q^{3\gamma(D_2)}\phi(D_2)$.*

# Inclusions and projections

Let

- $\mathbf{d} = (d_1, \ldots, d_n)$
- $|\mathbf{d}| = d_1 + \cdots + d_n$
- $l_1(\mathbf{d}) = |\{(\mathbf{i},\mathbf{j}); \mathbf{1} \leq \mathbf{i} < \mathbf{j} \leq \mathbf{n}, \mathbf{d_i} > \mathbf{d_j}\}|$
- $l_2(\mathbf{d}) = |\{(\mathbf{i},\mathbf{j}); \mathbf{1} \leq \mathbf{i} < \mathbf{j} \leq \mathbf{n}, \mathbf{d_i} < \mathbf{d_j}\}|$.

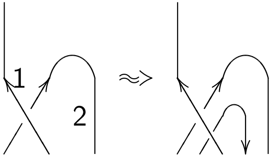
The inclusion map $\iota_n \colon V_n \to V_1^{\otimes n}$ is given by:

$$v_k \mapsto \Sigma_{\mathbf{d},|\mathbf{d}|=k} q^{l_1(\mathbf{d})} v_{d_1} \otimes v_{d_n}.$$

The projection map $\pi_n \colon V_1^{\otimes n} \to V_n$ is given by:

$$v_{d_1} \otimes \cdots \otimes v_{d_n} \mapsto q^{-l_2(\mathbf{d})} \begin{bmatrix} n \\ \mathbf{d} \end{bmatrix}^{-1} v_{|\mathbf{d}|}.$$

# Oriented cabling



# A map for tangles with colors

Let $T$ be an elementary, oriented, framed tangle from $r$ points to $s$ points such that each strand is labeled by a natural number. This naturally gives the $r$ points colors $(d_1, \dots, d_r)$ and the $s$ points colors $(e_1, \dots, e_s)$. We define a map for a diagram $D$ of $T$ :

$$\phi_{\mathrm{col}}(D)\colon V_{d_1} \otimes \cdots \otimes V_{d_r} \to V_{e_1} \otimes \cdots \otimes V_{e_s}$$

$$\phi_{\mathrm{col}}(D) = (\pi_{e_1} \otimes \cdots \otimes \pi_{e_s}) \circ \phi(\mathrm{cab}(D)) \circ (\iota_{d_1} \otimes \cdots \otimes \iota_{d_r})$$

where $\mathrm{cab}(D)$ is an oriented cabling of $D$ and $\phi(\mathrm{cab}(D))\colon V_1^{\otimes(d_1+\cdots+d_r)} \to V_1^{\otimes(e_1+\cdots+e_s)}$.

# Invariant for colored tangles

### Theorem (Reshetikhin-Turaev)

*Let $D_1$ and $D_2$ be two diagrams for an oriented, framed, colored tangle $T$ from $r$ points labeled $(d_1, \dots, d_r)$ to $s$ points labeled $(e_1, \dots, e_s)$. Then*

$$q^{3\gamma(cab(D_1))}\phi_{col}(D_1) = q^{3\gamma(cab(D_2))}\phi_{col}(D_2).$$

# Category $\mathcal{O}$

Let $\mathfrak{b} \subset \mathfrak{gl}_n$ where $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}^+$ is a sum of diagonal and strictly upper triangular matrices.

Let $\mathcal{O}(\mathfrak{gl}_n)$ be the full subcategory of $\mathcal{U}(\mathfrak{gl}_n)-$ modules with objects which are

- Finitely generated
- $\mathfrak{h}$- diagonalizable
- $\mathcal{U}(\mathfrak{b})$- locally finite.

Let $\mathcal{O}_i$ denote the block of $\mathcal{O}$ consisting of modules having a generalized central character corresponding to a weight $\lambda_i$ whose stabilizer under the Weyl group is $S_i \times S_{n-i}$.

There are $\binom{n}{i}$ simple objects in each of these blocks.

# An action of the quantum group

$\mathcal{O}_i \cong \text{mod} - A_i$ (finitely generated) where $A_i$ is the endomorphism algebra of a projective generator. Soergel shows how to equip this algebra with a grading so we may consider the category of finitely generated, graded modules ${}^{\mathbb{Z}}\mathcal{O}_i(\mathfrak{gl}_n)$.

**Proposition**

$\mathbb{C}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} [\oplus_{i=0}^{n} {}^{\mathbb{Z}}\mathcal{O}_i(\mathfrak{gl}_n)] \cong V_1^{\otimes n}$

This follows from the fact that there are $\binom{n}{i}$ simple objects in each category.

## Theorem (Bernstein-Frenkel-Khovanov-Stroppel)

*There exists functors*

- $\mathcal{E}_i \colon {}^{\mathbb{Z}}\mathcal{O}_i(\mathfrak{gl}_n) \to {}^{\mathbb{Z}}\mathcal{O}_{i+1}(\mathfrak{gl}_n)$
- $\mathcal{F}_i \colon {}^{\mathbb{Z}}\mathcal{O}_i(\mathfrak{gl}_n) \to {}^{\mathbb{Z}}\mathcal{O}_{i-1}(\mathfrak{gl}_n)$
- $\mathcal{K}_i, \mathcal{K}_i^{-1} \colon {}^{\mathbb{Z}}\mathcal{O}_i(\mathfrak{gl}_n) \to {}^{\mathbb{Z}}\mathcal{O}_i(\mathfrak{gl}_n)$,

*such that*

- $\mathcal{K}_{i+1}\mathcal{E}_i \cong \mathcal{E}_i\mathcal{K}_i\langle 2\rangle$
- $\mathcal{K}_{i-1}\mathcal{F}_i \cong \mathcal{F}_i\mathcal{K}_i\langle -2\rangle$
- $\mathcal{K}_i\mathcal{K}_i^{-1} \cong Id \cong \mathcal{K}_i^{-1}\mathcal{K}_i$
- $\mathcal{E}_{i-1}\mathcal{F}_i \oplus \oplus_{j=0}^{n-i-1} Id\langle n-2i-1-2j\rangle \cong$
  $\mathcal{F}_{i+1}\mathcal{E}_i \oplus \oplus_{j=0}^{i-1} Id\langle 2i-n-1-2j\rangle$

# Categorification of cups and caps

Theorem (Bernstein-Frenkel-Khovanov-Stroppel)

*There exists functors*

- $\widetilde{\cap}_{j,n} \colon D^b(\oplus_i {}^{\mathbb{Z}}\mathcal{O}_i(\mathfrak{gl}_n)) \to D^b(\oplus_i {}^{\mathbb{Z}}\mathcal{O}_i(\mathfrak{gl}_{n-2}))$
- $\widetilde{\cup}_{j,n} \colon D^b(\oplus_i {}^{\mathbb{Z}}\mathcal{O}_i(\mathfrak{gl}_n)) \to D^b(\oplus_i {}^{\mathbb{Z}}\mathcal{O}_i(\mathfrak{gl}_{n+2}))$

*such that*

- $[\widetilde{\cap}_{j,n}] = \cap_{j,n}$
- $[\widetilde{\cup}_{j,n}] = \cup_{j,n}$.

These functors are compositions of inclusion, Zuckerman, induction, and restriction functors.

# Twisting functors

Arkhipov introduced a $(\mathcal{U}(\mathfrak{gl}_n), \mathcal{U}(\mathfrak{gl}_n))-$ bimodule $S_w$ for every element $w \in S_n$.

Tensoring with this bimodule and twisting by a certain automorphism depending on $w$ is the Arkhipov functor.

Let $T_w \colon {}^{\mathbb{Z}}\mathcal{O}_i(\mathfrak{gl}_n) \to {}^{\mathbb{Z}}\mathcal{O}_i(\mathfrak{gl}_n)$ be the graded version of this functor.

The right adjoint of $T_w$ is the graded Joseph functor $J_w$. For the simple reflection $s_i$ we denote these functors by $T_i$ and $J_i$ respectively.

# Functors for crossings

Proposition (Khomenko-Mazorchuk-Ovsienko-Stroppel)

*There are distinguished triangles of functors:*

- $LT_i \to Id\langle -2 \rangle \to \widetilde{\cup}_{i,n-2} \circ \widetilde{\cap}_{i,n}\langle -1 \rangle[[1]]$
- $\widetilde{\cup}_{i,n-2} \circ \widetilde{\cap}_{i,n}\langle 1 \rangle[[-1]] \to Id\langle 2 \rangle \to RJ_i.$

Corollary

- $[LT_{i,n}[[1]]] = \Omega_{i,n}$
- $[RJ_{i,n}[[-1]]] = \Pi_{i,n}.$

Now to any elementary tangle diagram $T$ we may associate a functor $\widetilde{\phi}(T)$.

# Categorification of Jones polynomial

**Theorem (Stroppel)**

*Let $T$ be an oriented tangle from $n$ points to $m$ points. Let $D_1$ and $D_2$ be two diagrams of $T$. Let*

$$\widetilde{\phi}(D_1), \widetilde{\phi}(D_2) \colon D^b(\oplus_i{}^{\mathbb{Z}}\mathcal{O}_i(\mathfrak{gl}_n)) \to D^b(\oplus_i{}^{\mathbb{Z}}\mathcal{O}_i(\mathfrak{gl}_m))$$

*be the corresponding functors associated to the unoriented tangles. Then*

$$\widetilde{\phi}(D_1)\langle 3\gamma(D_1)\rangle \cong \widetilde{\phi}(D_2)\langle 3\gamma(D_2)\rangle.$$

# Harish-Chandra bimodules

Let $\mathcal{H}(\mathfrak{gl}_n)$ be the full subcategory of finitely generated, finite length $(\mathcal{U}(\mathfrak{gl}_n),\mathcal{U}(\mathfrak{gl}_n))$-bimodules which are locally finite with respect to the adjoint action of $\mathcal{U}(\mathfrak{gl}_n)$.

Let ${}_i^{\mathbb{Z}}\mathcal{H}_{\mathbf{d}}^1(\mathfrak{gl}_n)$ denote the graded version of the full subcategory of modules which with respect to the left action has a generalized central character corresponding to an integral dominant weight $\lambda_i$ and with respect to the right action has a true central character corresponding to $\lambda_{\mathbf{d}}$ where

- stabilizer of $\lambda_i$ is $S_i \times S_{n-i}$
- stabilizer of $\lambda_{\mathbf{d}}$ is $S_{d_1} \times \cdots \times S_{d_r}$.

# Categorification of tensor products

Theorem (Frenkel-Khovanov-Stroppel)

- $\oplus_{i=0}^{n} \mathbb{C}(q) \otimes_{\mathbb{Z}[q,q^{-1}]} [{}_i^{\mathbb{Z}}\mathcal{H}_{\mathbf{d}}^{1}(\mathfrak{gl}_n)] \cong V_{d_1} \otimes \cdots \otimes V_{d_r}$
- *There are functors $\mathcal{E}_i, \mathcal{F}_i, \mathcal{K}_i^{\pm 1}$ on this category which satisfy the functorial isomorphisms from earlier.*

# Bernstein-Gelfand functors

Let $M(\lambda_{\mathbf{d}})$ be the Verma module whose highest weight is $\lambda_{\mathbf{d}}$.

There is a projection functor:

$${}_i\widetilde{\pi}_{\mathbf{d}} \colon {}^{\mathbb{Z}}\mathcal{O}_i(\mathfrak{gl}_n) \to {}^{\mathbb{Z}}_i\mathcal{H}^1_{\mathbf{d}}(\mathfrak{gl}_n)$$

which is a graded version of the functor defined by:

$$M \mapsto \mathrm{Hom}_{\mathbb{C}}(M(\lambda_{\mathbf{d}}), M)^{\mathrm{fin}}.$$

There is an inclusion functor:

$${}_i\widetilde{\iota}_{\mathbf{d}} \colon {}^{\mathbb{Z}}_i\mathcal{H}^1_{\mathbf{d}}(\mathfrak{gl}_n) \to {}^{\mathbb{Z}}\mathcal{O}_i(\mathfrak{gl}_n)$$

which is a graded version of the functor defined by:

$$M \mapsto M \otimes_{\mathcal{U}(\mathfrak{gl}_n)} M(\lambda_{\mathbf{d}}).$$

# Projectively presented $\mathcal{O}$

Let ${}^{\mathbb{Z}}\mathcal{O}_{i,\mathbf{d}}(\mathfrak{gl}_n)$ denote the graded version of the full subcategory of $\mathcal{O}_i(\mathfrak{gl}_n)$ of modules $M$ which have projective presentations by projectives indexed by longest double coset representatives in $S_{\mathbf{d}} \backslash S_n / S_i \times S_{n-i}$.

**Theorem (Bernstein-Gelfand)**

$$ {}_i\widetilde{\iota}_{\mathbf{d}} \colon {}_i^{\mathbb{Z}}\mathcal{H}^1_{\mathbf{d}}(\mathfrak{gl}_n) \to {}^{\mathbb{Z}}\mathcal{O}_{i,\mathbf{d}}(\mathfrak{gl}_n) $$

*is an equivalence of categories with inverse functor* ${}_i\widetilde{\pi}_{\mathbf{d}}$.

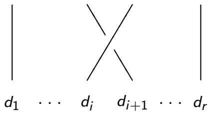
Frenkel-Khovanov-Stroppel show that on standard objects, these functors categorify inclusion and projection.

# Twisting functors on projectively presented $\mathcal{O}$

Let

- $\mathbf{d} = (d_1, \ldots, d_j, d_{j+1}, \ldots, d_r)$
- $\mathbf{d}' = (d_1, \ldots, d_{j+1}, d_j, \ldots, d_r)$

Let $w_{\mathbf{d},\mathbf{d}'}$ be the element in the symmetric group associated to the cabling of the braid given below.




$w_{\mathbf{d},\mathbf{d}'}$

### Proposition

*$LT_{w_{\mathbf{d},\mathbf{d}'}}$ maps a standard object of ${}^{\mathbb{Z}}\mathcal{O}_{i,\mathbf{d}}(\mathfrak{gl}_n)$ to an object of ${}^{\mathbb{Z}}\mathcal{O}_{i,\mathbf{d}'}(\mathfrak{gl}_n)$*

### Proposition

- *$LT_{w_{\mathbf{d},\mathbf{d}'}}$ restricts to a functor $D^{<}({}^{\mathbb{Z}}\mathcal{O}_{i,\mathbf{d}}(\mathfrak{gl}_n)) \to D^{<}({}^{\mathbb{Z}}\mathcal{O}_{i,\mathbf{d}'}(\mathfrak{gl}_n))$*
- *$RJ_{w_{\mathbf{d},\mathbf{d}'}}$ restricts to a functor $D^{<}({}^{\mathbb{Z}}\mathcal{O}_{i,\mathbf{d}}(\mathfrak{gl}_n)) \to D^{<}({}^{\mathbb{Z}}\mathcal{O}_{i,\mathbf{d}'}(\mathfrak{gl}_n))$*

# Functors for elementary colored tangles

Let $D$ be any elementary colored tangle diagram. We may associate a functor on Harish-Chandra categories, $\widetilde{\phi}_{\text{col}}(D)$, by assigning to it the inclusion B-G functor composed with $\widetilde{\phi}(\text{cab}(D))$ composed with the projection B-G functor.

## Theorem

*Let $T$ be an oriented, framed, tangle from $r$ points labeled by $\mathbf{d} = (d_1, \ldots, d_r)$ to $s$ points labeled by $\mathbf{e} = (e_1, \ldots, e_s)$. Let $D_1$ and $D_2$ be two tangle diagrams for $T$. Then,*

$$\widetilde{\phi}_{col}(D_1)\langle 3\gamma(cab(D_1))\rangle \cong \widetilde{\phi}_{col}(D_2)\langle 3\gamma(cab(D_2))\rangle :$$

$$\oplus_{i=0}^{|\mathbf{d}|} D^{<}({}_{i}^{\mathbb{Z}}\mathcal{H}_{\mathbf{d}}^{1}(\mathfrak{gl}_{|\mathbf{d}|})) \to \oplus_{i=0}^{|\mathbf{e}|} D^{<}({}_{i}^{\mathbb{Z}}\mathcal{H}_{\mathbf{e}}^{1}(\mathfrak{gl}_{|\mathbf{e}|})).$$