KAC WAKIMOTO CONJECTURE FOR LIE SUPERALGEBRAS

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1. Introduction

Definition 1.1. *A $\mathbb{Z}_2$ graded vector space $\mathfrak{g}$ with even bracket $[\bullet, \bullet] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ is a Lie superalgebra iff the following conditions hold*

$$[a, b] = -(-1)^{p(a)p(b)}[b, a];$$

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Example 1.2. *Lie superalgebra $\mathfrak{gl}(m, n)$ is the algebra of matrices of size $m + n$ with $\mathbb{Z}$-grading*

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \left(\begin{array}{c|c} 0 & B \\ \hline 0 & 0 \end{array}\right) \oplus \left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array}\right) \oplus \left(\begin{array}{c|c} 0 & 0 \\ \hline C & 0 \end{array}\right)$$

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Let k be an algebraically closed field of characteristic zero. All simple Lie superalgebras were classified by V. Kac. He divided them in three types

- Contragredient

(a) classical: $(\mathfrak{p})\mathfrak{sl}(m,n)$, $\mathfrak{osp}(m,2n)$;

(b) exceptional: $D(2,1,\alpha)$, G_3 and F_4.

- Strange: $P(n)$ and $Q(n)$.
- Cartan type $W(n)$, $S(n)$, $S'(n)$ and $H(n)$.

In this talk $\mathfrak{g}$ is a contragredient finite-dimensional almost simple superalgebra, i.e. the quotient of $[\mathfrak{g},\mathfrak{g}]$ by the center is simple. Our main example is $\mathfrak{gl}(m,n)$.

As in classical case $\mathfrak{g}$ has a root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta\subset\mathfrak{h}^*} \mathfrak{g}_\alpha,$$

where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | [h, x] = \alpha(h)x\}.$$

For every root $\alpha \in \Delta$ we have dim $\mathfrak{g}_\alpha = 1$. So either $\mathfrak{g}_\alpha \subset \mathfrak{g}_0$ or $\mathfrak{g}_\alpha \subset \mathfrak{g}_1$. Depending on this we call a root α even or odd. We denote by Δ_0 (resp. Δ_1) the set of even (resp. odd roots).

The invariant bilinear symmetric form on $\mathfrak{g}$ induces the form on $(\bullet, \bullet)$ on $\mathfrak{h}^*$ but it is not positive definite on the root lattice. A root α is isotropic if $(\alpha, \alpha) = 0$.

By a choice of a generic hyperplane in $\mathfrak{h}^*$ one fixes the sets positive and negative roots $\Delta = \Delta^+ \cup \Delta^-$ and a triangular decomposition

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Example 1.3. *Let $\mathfrak{g} = \mathfrak{gl}(m, n)$. Then $\mathfrak{h}$ is the subalgebra of diagonal matrices, $\mathfrak{n}^\pm$ is the subalgebra of strictly upper (low) triangular matrices, the roots in the standard basis are*

$$\Delta_1 = \{(\varepsilon_i - \delta_j) | i \leq m, j \leq n\}$$
$$\Delta_0 = \{(\varepsilon_i - \varepsilon_j) | i, j \leq m\} \cup \{(\delta_i - \delta_j) | i, j \leq n\}.$$

The invariant form

$$(\varepsilon_i, \delta_j) = 0,\ (\varepsilon_i, \varepsilon_j) = \delta_{ij},\ (\delta_i, \delta_j) = -\delta_{ij}.$$

2. Highest weight theory

Theorem 2.1. *(Kac) Let $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$, $\lambda \in \mathfrak{h}^*$. A Verma module with highest weight λ*

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} C_\lambda$$

has a unique simple quotient L_λ. Every finite-dimensional simple $\mathfrak{g}$-module is isomorphic to L_λ or $\Pi(L_\lambda)$ for some λ.

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Let $\rho = \frac{1}{2} \sum_{\alpha \in \Delta+} (-1)^{p(\alpha)} \alpha$.

Definition 2.2. *The degree of atypicality of λ ($at(\lambda)$) is the maximal number of linearly independent isotropic roots α such that $(\lambda + \rho, \alpha) = 0$.*

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For a $\mathbb{Z}_2$-graded space $\mathrm{sdim}V = \mathrm{dim}V_0 - \mathrm{dim}V_1$.

Conjecture 2.3. *(Kac Wakimoto). $sdim L(\lambda) \neq 0$ if and only if $at(\lambda) = def(\mathfrak{g})$.*

Example 2.4. $\mathfrak{g} = \mathfrak{sl}(1,1)$. *The basis*

$$X = \left(\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}\right), Y = \left(\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \end{array}\right), C = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & 1 \end{array}\right).$$

Then dim $M(\lambda) = (1,1)$, *the action is given by*

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Theorem 2.5. *(a)(Duflo, S.) If* $at(\lambda) < def(\mathfrak{g})$, *then* $sdim L(\lambda) = 0$;
(b) (S.) Kac-Wakimoto conjecture holds for $\mathfrak{gl}(m,n)$ *and* $\mathfrak{osp}(m,2n)$.

Geometric methods

- Associated variety
- Geometric induction.

3. Center of universal enveloping algebra

Let $Z(\mathfrak{g})$ denote the center of universal enveloping algebra. As in classical case we have the Harish-Chandra homomorphism $Z(\mathfrak{g}) \to U(\mathfrak{h}) = \mathrm{Pol}(\mathfrak{h}^*)$, which induces a homomorphism

$$\Phi : \mathfrak{h} \to \mathrm{Hom}(Z(\mathfrak{g}), k), \; \Phi(\lambda) = \chi_\lambda.$$

The center $Z(\mathfrak{g})$ was described by Kac and Sergeev independently. We need only the following

Corollary 3.1. *If $\chi_\lambda = \chi_\mu$, then $at(\lambda) = at(\mu)$. Hence $at(\chi)$ is well defined.*

Let $\mathcal{F}$ be the category of all finite-dimensional $\mathfrak{g}$-modules semisimple over $\mathfrak{g}_0$, and $\mathcal{F}^\chi$ be the subcategory of modules which admit generalized central character χ. We have a block decomposition $\mathcal{F} = \bigoplus \mathcal{F}^\chi$.

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Theorem 3.2. *If k =at(χ). Then the block $\mathcal{F}^\chi$ is equivalent to the block $\mathcal{F}_0$ containing a trivial module for one of the superalgebras $\mathfrak{gl}(k,k)$, $\mathfrak{osp}(2k+1,2k)$, $\mathfrak{osp}(2k,2k)$, $\mathfrak{osp}(2k+2,2k)$.*

4. Associated variety

Definition 4.1. *Let*

$$X = \{x \in \mathfrak{g}_1 | [x, x] = 0\}.$$

For any $\mathfrak{g}$-module M and $x \in X$ define

$$M_x = \mathbf{ker} x / \mathbf{im} x$$

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$$X_M = \{x \in X | M_x \neq 0)\}.$$

X_M is called the associated variety of M.

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Properties of X_M

- sdim$M$ =sdimM_x for any $x \in X$
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For any $\mathfrak{g}$ satisfying our assumptions X has finitely many G_0-orbits. One can introduce rk: $X \to \mathbb{N}$ such that rk(x)=rk(y) implies dim $G_0 x$ = dim $G_0 y$, and rk(x) is maximal iff $G_0 x$ is open.

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If $\mathfrak{g} = \mathfrak{gl}(m, n)$, then $\mathfrak{g}_x = \mathfrak{gl}(m-k, n-k)$ and $\bar{\mathfrak{g}} = \mathfrak{gl}(k, k)$ if $k = \mathrm{rk}(x)$.

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Remark 4.3. *If rk(x) is maximal, then rk$(x) =$def$(\mathfrak{g})$, and $\mathfrak{g}_x$ is a reductive Lie algebra or a Lie superalgebra with semisimple category of finite-dimensional modules.*

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Theorem 4.4. *Fix $x \in X$. If $M \in \mathcal{F}^\chi$, then M_x is a $\mathfrak{g}_x$-module with central character whose degree of atypicality is $at(\chi) - rk(x)$.*

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Corollary 4.5. *Let $\mathfrak{g} \neq \mathfrak{osp}(2m, 2n)$ with $m > n$. If at$(\chi) =$def$(\mathfrak{g})$ and x belongs to an open G_0-orbit, then there exists a simple finite-dimensional $\mathfrak{g}_x$-module V such that M_x is a direct sum of several copies of V and $\Pi(V)$ for any $M \in \mathcal{F}^{\chi}$. If $\mathfrak{g} = \mathfrak{osp}(2m, 2n)$ with $m > n$, then $\mathfrak{g}_x = o(2m - 2n)$ and M_x is a direct sum of several copies of V, V^{σ}, $\Pi(V)$ and $\Pi(V^{\sigma})$, where σ is an involution induced by the symmetry of Dynkin diagram of $o(2m - 2n)$.*

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Corollary 4.6. *If M is a simple $\mathfrak{g}$-module and $X_M = X$ then the degree of atypicality of M equals def$(\mathfrak{g})$.*

Conjecture 4.7. *If M is simple, then M_x is either a sum of several copies of V or a sum of several copies of $\Pi(V)$.*

5. GEOMETRIC INDUCTION

Geometric induction

Fix a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$. We construct a functor Γ from the category of finite-dimensional $\mathfrak{p}$-modules to the category of $\mathfrak{g}$-modules by putting $\Gamma(V)$ to be the maximal finite-dimensional quotient of the induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V$. It is easy to see that Γ is not exact and one can define a derived functor Γ_i.

If $\mathcal{L}(V) = G \times_P V$, then

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Theorem 5.2. *(Penkov)*

$$\sum(-1)^i ch(\Gamma_i(V)) = \sum_{w \in W} (-1)^w w\left(\frac{chV e^{\rho}}{\prod_{\alpha \in \Delta_1(\mathfrak{g}/\mathfrak{p})} 1 + e^{\alpha}}\right),$$

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The proof is based on supergeometry: one can consider a filtration of $\mathcal{L}(V)$ by vector bundles on the underlying variety G_0/P_0 and use the usual Borel-Weil-Bott theorem.

Theorem 5.3. *Here $\mathfrak{g} = \mathfrak{gl}(m,n)$ or $\mathfrak{osp}(2m+1,2n)$. Let at $\chi =$def$(\mathfrak{g})$ and $x \in X$ belong to an open G_0-orbit, then $\mathcal{F}$ is equivalent to the most atypical block $\mathcal{F}_0$ of $\bar{\mathfrak{g}}$, and $M_x = V \otimes \bar{M}_x$ where $\bar{M}$ is the image of M under the functor $\mathcal{F}^\chi \to \mathcal{F}_0$.*

Idea of the proof. Chose $\mathfrak{p}$ with reductive part $\mathfrak{g}_x \oplus \bar{\mathfrak{g}}$. Using translation functor $M \to (M \otimes E)^\eta$ which establish an equivalence between $\mathcal{F}^\chi$ and $\mathcal{F}^\eta$ several times, make M stable. Then set $\bar{M} = M^{\mathfrak{n}}$ where $\mathfrak{n}$ is the nil-radical of $\mathfrak{p}$.

A simple observation that sdim$V \neq 0$ implies the following

Theorem 5.3. *Here* $\mathfrak{g} = \mathfrak{gl}(m,n)$ *or* $\mathfrak{osp}(2m+1,2n)$. *Let at* $\chi =$*def*$(\mathfrak{g})$ *and* $x \in X$ *belong to an open* G_0*-orbit, then* $\mathcal{F}$ *is equivalent to the most atypical block* $\mathcal{F}_0$ *of* $\bar{\mathfrak{g}}$, *and* $M_x = V \otimes \bar{M}_x$ *where* $\bar{M}$ *is the image of* M *under the functor* $\mathcal{F}^\chi \to \mathcal{F}_0$.

Idea of the proof. Chose $\mathfrak{p}$ with reductive part $\mathfrak{g}_x \oplus \bar{\mathfrak{g}}$. Using translation functor $M \to (M \otimes E)^\eta$ which establish an equivalence between $\mathcal{F}^\chi$ and $\mathcal{F}^\eta$ several times, make M stable. Then set $\bar{M} = M^{\mathfrak{n}}$ where $\mathfrak{n}$ is the nil-radical of $\mathfrak{p}$.

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Corollary 5.4. *If Kac-Wakimoto conjecture holds for $\bar{\mathfrak{g}}$, it holds for $\mathfrak{g}$.*

Let $\bar{\mathfrak{g}} = \mathfrak{gl}(n,n)$. Choose now the maximal parabolic subalgebra $\mathfrak{q}$ with reductive part $\mathfrak{gl}(1) \oplus \mathfrak{gl}(n-1,n)$.

Lemma 5.5.

$$\sum (-1)^i sdim \Gamma_i(V) = 0.$$

Assume that $\mathrm{at}(\chi) > 0$, then we may assume without loss of generality that a highest weight λ of $L(\lambda) \in \mathcal{F}^\chi$ is integral. On the lattice Λ of all integral weights one can introduce the parity $p : \Lambda \to \mathbb{Z}_2$. We assume that the parity of the highest vector in $L(\lambda)$ equals $p(\lambda)$.

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Theorem 5.6. *If $i > 0$, then*

$$\Gamma_i(L_{\mathfrak{q}}(\lambda)) = \bigoplus (L(\nu)),$$

and $p(\lambda + \nu) = p(i + 1)$ for all ν appearing in the direct sum.

If $i = 0$, then we have the following exact sequence

$$0 \to \bigoplus L(\mu) \to \Gamma_0(L_{\mathfrak{q}}(\lambda)) \to L(\lambda) \to 0,$$

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Corollary 5.7. *There is some linear order on Λ such that*

$$sdim L(\lambda) = \sum_{\mu < \lambda} (-1)^{p(\mu+\lambda)} sdim L(\mu).$$

This implies sdim $L(\lambda)$ is positive for even λ and negative for odd λ! Hence Kac-Wakimoto conjecture.

Now let M be a finite-dimensional $\mathfrak{g}$-module and $h \in \mathfrak{h}$. Write

$$\mathrm{ch}_M(h) = \mathrm{str}_M\left(e^h\right).$$

Obviously, ch_M is W-invariant analytic function on $\mathfrak{h}$. We can write Taylor series for ch_M at $h = 0$

$$\mathrm{ch}_M(h) = \sum_{i=0}^{\infty} p_i(h),$$

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Theorem 5.8. *(Duflo, S.) Assume that all odd roots of $\mathfrak{g}$ are isotropic. Let M be a finite-dimensional $\mathfrak{g}$-module, s be the codimension of X_M in X. The order of ch_M at zero is greater or equal than s. Moreover, the polynomial $p_s(h)$ in Taylor series for ch_M is determined uniquely up to proportionality.*

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Remark 5.9. *In fact, the above theorem gives a hint how to define the "superdimension" for a block which is not maximal atypical. One can try to define it as a coefficient of p_s for some suitable normalization of p_s. We conjecture that it is possible to find a normalization so that the superdimension of every simple object is integral and non-zero.*

Open questions and remarks

- Check the conjecture for exceptional cases (defect is always 1)
- As it follows from the proof sdim is the same for almost all simple objects in a block
- Get a formula for superdimension
- Modules of superdimension zero form an ideal in the tensor category $\mathcal{F}$. Study the quotient.