KAC WAKIMOTO CONJECTURE FOR LIE SUPERALGEBRAS

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AMS meeting, January 16, 2010
1. Introduction

Definition 1.1. A $\mathbb{Z}_2$ graded vector space $\mathfrak{g}$ with even bracket $[\bullet, \bullet] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ is a Lie superalgebra iff the following conditions hold

\[ [a, b] = -(-1)^{p(a)p(b)}[b, a]; \]

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Example 1.2. Lie superalgebra \( \mathfrak{gl}(m, n) \) is the algebra of matrices of size \( m + n \) with \( \mathbb{Z} \)-grading

\[
g = g_{−1} \oplus g_0 \oplus g_1 = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}
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and the bracket defined by $[X, Y] = XY - (-1)^{p(X)p(Y)}YX$.

Let $k$ be an algebraically closed field of characteristic zero. All simple Lie superalgebras were classified by V. Kac. He divided them in three types

- Contragredient
  (a) classical: $(\mathfrak{p})\mathfrak{sl}(m, n), \mathfrak{osp}(m, 2n);$  
  (b) exceptional: $D(2, 1, \alpha), G_3$ and $F_4$.
- Strange: $P(n)$ and $Q(n)$.
- Cartan type $W(n), S(n), S'(n)$ and $H(n)$.

In this talk $\mathfrak{g}$ is a contragredient finite-dimensional almost simple superalgebra, i.e. the quotient of $[\mathfrak{g}, \mathfrak{g}]$ by the center is simple. Our main example is $\mathfrak{gl}(m, n)$. 
As in classical case $\mathfrak{g}$ has a root decomposition
\[ \mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta \subset \mathfrak{h}^*} \mathfrak{g}_\alpha, \]
where
\[ \mathfrak{g}_\alpha = \{ x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \}. \]
For every root $\alpha \in \Delta$ we have $\dim \mathfrak{g}_\alpha = 1$. So either $\mathfrak{g}_\alpha \subset \mathfrak{g}_0$ or $\mathfrak{g}_\alpha \subset \mathfrak{g}_1$. Depending on this we call a root $\alpha$ even or odd. We denote by $\Delta_0$ (resp. $\Delta_1$) the set of even (resp. odd roots).

The invariant bilinear symmetric form on $\mathfrak{g}$ induces the form on $(\cdot, \cdot)$ on $\mathfrak{h}^*$ but it is not positive definite on the root lattice. A root $\alpha$ is isotropic if $(\alpha, \alpha) = 0$.

By a choice of a generic hyperplane in $\mathfrak{h}^*$ one fixes the sets positive and negative roots $\Delta = \Delta^+ \cup \Delta^-$ and a triangular decomposition
\[ \mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \] where $\mathfrak{n}^\pm = \bigoplus \mathfrak{g}_\alpha$, for all $\alpha \in \Delta^\pm$. 
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\]

**Example 1.3.** Let $\mathfrak{g} = \mathfrak{gl}(m, n)$. Then $\mathfrak{h}$ is the subalgebra of diagonal matrices, $\mathfrak{n}^\pm$ is the subalgebra of strictly upper (low) triangular matrices, the roots in the standard basis are
\[
\Delta_1 = \{ (\varepsilon_i - \delta_j) | i \leq m, j \leq n \}
\]
\[
\Delta_0 = \{ (\varepsilon_i - \varepsilon_j) | i, j \leq m \} \cup \{ (\delta_i - \delta_j) | i, j \leq n \}.
\]
The invariant form
\[
(\varepsilon_i, \delta_j) = 0, \ (\varepsilon_i, \varepsilon_j) = \delta_{ij}, \ (\delta_i, \delta_j) = -\delta_{ij}.
\]
2. **Highest Weight Theory**

**Theorem 2.1. (Kac)** Let $b = \mathfrak{h} \oplus \mathfrak{n}^+$, $\lambda \in \mathfrak{h}^*$. A Verma module with highest weight $\lambda$

$$M(\lambda) = U(g) \otimes_{U(b)} C_\lambda$$

has a unique simple quotient $L_\lambda$. Every finite-dimensional simple $g$-module is isomorphic to $L_\lambda$ or $\Pi(L_\lambda)$ for some $\lambda$. 
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Let $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} (-1)^{p(\alpha)} \alpha$.

**Definition 2.2.** The degree of atypicality of $\lambda$ (at$(\lambda)$) is the maximal number of linearly independent isotropic roots $\alpha$ such that $(\lambda + \rho, \alpha) = 0$.

The defect of $g$ (def$(g)$) is the maximal number of linearly independent isotropic roots.
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$$\text{def}(\mathfrak{gl}(m, n)) = m - n, \text{ for } m \geq n.$$ For a $\mathbb{Z}_2$-graded space $s\text{dim}V = \dim V_0 - \dim V_1$. 

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$$\text{def}(gl(m, n)) = m - n, \text{ for } m \geq n.$$  

For a $\mathbb{Z}_2$-graded space $\text{sdim} V = \dim V_0 - \dim V_1$.

Conjecture 2.3. (Kac Wakimoto). $\text{sdim} L(\lambda) \neq 0$ if and only if $\text{at}(\lambda) = \text{def}(g)$.
Example 2.4. $g = \mathfrak{sl}(1, 1)$. The basis

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

Then $\dim M(\lambda) = (1, 1)$, the action is given by

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}, C = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}.$$ 

For $c \neq 0$, $M(\lambda)$ is simple. If $c = 0$, then $at(\lambda) = 1$ and $s \dim L(\lambda) = \pm 1$. 
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Theorem 2.5. (a) (Duflo, S.) If $at(\lambda) < \text{def}(g)$, then $sdim L(\lambda) = 0$;

(b) (S.) Kac-Wakimoto conjecture holds for $gl(m, n)$ and $osp(m, 2n)$.

Geometric methods

- Associated variety
- Geometric induction.
3. CENTER OF UNIVERSAL ENVELOPING ALGEBRA

Let $Z(g)$ denote the center of universal enveloping algebra. As in classical case we have the Harish-Chandra homomorphism $Z(g) \to U(h) = \text{Pol}(h^*)$, which induces a homomorphism

$$
\Phi : h \to \text{Hom}(Z(g), k), \quad \Phi(\lambda) = \chi_\lambda.
$$

The center $Z(g)$ was described by Kac and Sergeev independently. We need only the following

**Corollary 3.1.** If $\chi_\lambda = \chi_\mu$, then $at(\lambda) = at(\mu)$. Hence $at(\chi)$ is well defined.

Let $\mathcal{F}$ be the category of all finite-dimensional $g$-modules semisimple over $g_0$, and $\mathcal{F}^\chi$ be the subcategory of modules which admit generalized central character $\chi$. We have a block decomposition $\mathcal{F} = \bigoplus \mathcal{F}^\chi$. 
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Let \( Z(\mathfrak{g}) \) denote the center of universal enveloping algebra. As in classical case we have the Harish-Chandra homomorphism \( Z(\mathfrak{g}) \rightarrow U(\mathfrak{h}) = \text{Pol}(\mathfrak{h}^*) \), which induces a homomorphism
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**Theorem 3.2.** If \( k = \text{at}(\chi) \). Then the block \( \mathcal{F}^\chi \) is equivalent to the block \( \mathcal{F}_0 \) containing a trivial module for one of the superalgebras \( \mathfrak{gl}(k, k), \mathfrak{osp}(2k+1, 2k), \mathfrak{osp}(2k, 2k), \mathfrak{osp}(2k+2, 2k) \).
4. ASSOCIATED VARIETY

Definition 4.1. Let

\[ X = \{ x \in \mathfrak{g}_1 \mid [x, x] = 0 \} \]

For any \( \mathfrak{g} \)-module \( M \) and \( x \in X \) define

\[ M_x = \ker x / \text{im} x \]

and

\[ X_M = \{ x \in X \mid M_x \neq 0 \} \]

\( X_M \) is called the associated variety of \( M \).

If \( M \) is finite-dimensional, \( X_M \) is a \( G_0 \)-invariant closed subvariety in \( X \).
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**Properties of \( X_M \)**

- \( \text{sdim} M = \text{sdim} M_x \) for any \( x \in X \)
- \( X_M \oplus N = X_M \cup X_N \)
- \( X_M \otimes N = X_M \cap X_N \)
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For any \( \mathfrak{g} \) satisfying our assumptions \( X \) has finitely many \( G_0 \)-orbits. One can introduce \( \text{rk}: X \to \mathbb{N} \) such that \( \text{rk}(x) = \text{rk}(y) \) implies \( \dim G_0 x = \dim G_0 y \), and \( \text{rk}(x) \) is maximal iff \( G_0 x \) is open.
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Let

$$g_x = C_g(x) / [x, \mathfrak{g}], \bar{g} = C_g(g_x).$$

If $g = \mathfrak{gl}(m, n)$, then $g_x = \mathfrak{gl}(m - k, n - k)$ and $\bar{g} = \mathfrak{gl}(k, k)$ if $k = \text{rk}(x)$. 
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Remark 4.3. If $\text{rk}(x)$ is maximal, then $\text{rk}(x) = \text{def}(\mathfrak{g})$, and $\mathfrak{g}_x$ is a reductive Lie algebra or a Lie superalgebra with semisimple category of finite-dimensional modules.
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Corollary 4.5. Let $\mathfrak{g} \neq \mathfrak{osp}(2m, 2n)$ with $m > n$. If $at(\chi) = \text{def}(\mathfrak{g})$ and $x$ belongs to an open $G_0$-orbit, then there exists a simple finite-dimensional $\mathfrak{g}_x$-module $V$ such that $M_x$ is a direct sum of several copies of $V$ and $\Pi(V)$ for any $M \in \mathcal{F}^{\chi}$. If $\mathfrak{g} = \mathfrak{osp}(2m, 2n)$ with $m > n$, then $\mathfrak{g}_x = \mathfrak{o}(2m - 2n)$ and $M_x$ is a direct sum of several copies of $V$, $V^\sigma$, $\Pi(V)$ and $\Pi(V^\sigma)$, where $\sigma$ is an involution induced by the symmetry of Dynkin diagram of $\mathfrak{o}(2m - 2n)$.
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Corollary 4.6. If $M$ is a simple $\mathfrak{g}$-module and $X_M = X$ then the degree of atypicality of $M$ equals $\text{def}(\mathfrak{g})$.

Conjecture 4.7. If $M$ is simple, then $M_x$ is either a sum of several copies of $V$ or a sum of several copies of $\Pi(V)$. 
5. GEOMETRIC INDUCTION

Geometric induction

Fix a parabolic subalgebra $p \subset g$. We construct a functor $\Gamma$ from the category of finite-dimensional $p$-modules to the category of $g$-modules by putting $\Gamma(V)$ to be the maximal finite-dimensional quotient of the induced module $U(g) \otimes_{U(p)} V$. It is easy to see that $\Gamma$ is not exact and one can define a derived functor $\Gamma_i$.

If $\mathcal{L}(V) = G \times_P V$, then

$$\Gamma_i(V) = H^i(G/P, \mathcal{L}(V^*))^*.$$
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Theorem 5.2. (Penkov)
\[
\sum (-1)^i \text{ch}(\Gamma_i(V)) = \sum_{w \in W} (-1)^w w \left( \frac{\text{ch}V e^\rho}{\prod_{\alpha \in \Delta_1(g/p)} 1 + e^\alpha} \right),
\]
where
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D = \frac{\prod_{\alpha \in \Delta_1^+} e^{\alpha/2} + e^{-\alpha/2}}{\prod_{\alpha \in \Delta_0^+} e^{\alpha/2} - e^{-\alpha/2}}.
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If $\mathcal{L}(V) = G \times_P V$, then

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Definition 5.1. $V$ is stable if $\Gamma_i(V) = 0$ for $i > 0$.

Theorem 5.2. (Penkov)

$$\sum (-1)^i ch(\Gamma_i(V)) = \sum_{w \in W} (-1)^w w \left( \frac{chVE^\rho}{\prod_{\alpha \in \Delta_{1}(\mathfrak{g}/\mathfrak{p})} 1 + e^\alpha} \right),$$

where

$$D = \frac{\prod_{\alpha \in \Delta^+_1} e^{\alpha/2} + e^{-\alpha/2}}{\prod_{\alpha \in \Delta^+_0} e^{\alpha/2} - e^{-\alpha/2}}.$$  

The proof is based on supergeometry: one can consider a filtration of $\mathcal{L}(V)$ by vector bundles on the underlying variety $G_0/P_0$ and use the usual Borel-Weil-Bott theorem.
Theorem 5.3. Here $\mathfrak{g} = \mathfrak{gl}(m, n)$ or $\mathfrak{osp}(2m + 1, 2n)$. Let at $\chi = \text{def}(\mathfrak{g})$ and $x \in X$ belong to an open $G_0$-orbit, then $\mathcal{F}$ is equivalent to the most atypical block $\mathcal{F}_0$ of $\bar{\mathfrak{g}}$, and $M_x = V \otimes \bar{M}_x$ where $\bar{M}$ is the image of $M$ under the functor $\mathcal{F}^\chi \to \mathcal{F}_0$.

Idea of the proof. Chose $\mathfrak{p}$ with reductive part $\mathfrak{g}_x \oplus \bar{\mathfrak{g}}$. Using translation functor $M \to (M \otimes E)^n$ which establish an equivalence between $\mathcal{F}^\chi$ and $\mathcal{F}^n$ several times, make $M$ stable. Then set $\bar{M} = M^n$ where $n$ is the nil-radical of $\mathfrak{p}$.

A simple observation that $\text{sdim} V \neq 0$ implies the following
Theorem 5.3. Here \( g = \mathfrak{gl}(m, n) \) or \( \mathfrak{osp}(2m + 1, 2n) \). Let at \( \chi = \text{def}(g) \) and \( x \in X \) belong to an open \( G_0 \)-orbit, then \( F \) is equivalent to the most atypical block \( F_0 \) of \( \bar{g} \), and \( M_x = V \otimes \bar{M}_x \) where \( \bar{M} \) is the image of \( M \) under the functor \( F^\chi \rightarrow F_0 \).

Idea of the proof. Chose \( p \) with reductive part \( g_x \oplus \bar{g} \). Using translation functor \( M \rightarrow (M \otimes E)^n \) which establish an equivalence between \( F^\chi \) and \( F^n \) several times, make \( M \) stable. Then set \( \bar{M} = M^n \) where \( n \) is the nil-radical of \( p \).

A simple observation that \( \text{sdim} V \neq 0 \) implies the following

Corollary 5.4. If Kac-Wakimoto conjecture holds for \( \bar{g} \), it holds for \( g \).
Theorem 5.3. Here $\mathfrak{g} = \mathfrak{gl}(m, n)$ or $\mathfrak{osp}(2m + 1, 2n)$. Let at $\chi = \text{def}(\mathfrak{g})$ and $x \in X$ belong to an open $G_0$-orbit, then $\mathcal{F}$ is equivalent to the most atypical block $\mathcal{F}_0$ of $\bar{\mathfrak{g}}$, and $M_x = V \otimes \bar{M}_x$ where $\bar{M}$ is the image of $M$ under the functor $\mathcal{F}^\chi \to \mathcal{F}_0$.

Idea of the proof. Chose $\mathfrak{p}$ with reductive part $\mathfrak{g}_x \oplus \bar{\mathfrak{g}}$. Using translation functor $M \to (M \otimes E)^n$ which establish an equivalence between $\mathcal{F}^\chi$ and $\mathcal{F}^\eta$ several times, make $M$ stable. Then set $\bar{M} = M^n$ where $n$ is the nil-radical of $\mathfrak{p}$.

A simple observation that $\text{sdim} V \neq 0$ implies the following

Corollary 5.4. If Kac-Wakimoto conjecture holds for $\bar{\mathfrak{g}}$, it holds for $\mathfrak{g}$.

Let $\bar{\mathfrak{g}} = \mathfrak{gl}(n, n)$. Choose now the maximal parabolic subalgebra $\mathfrak{q}$ with reductive part $\mathfrak{gl}(1) \oplus \mathfrak{gl}(n - 1, n)$.

Lemma 5.5.

$$\sum (-1)^i \text{sdim} \Gamma_i(V) = 0.$$
Assume that $a_t(\chi) > 0$, then we may assume without loss of generality that a highest weight $\lambda$ of $L(\lambda) \in \mathcal{F}^X$ is integral. On the lattice $\Lambda$ of all integral weights one can introduce the parity $p : \Lambda \to \mathbb{Z}_2$. We assume that the parity of the highest vector in $L(\lambda)$ equals $p(\lambda)$. 
Assume that $a(\chi) > 0$, then we may assume without loss of generality that a highest weight $\lambda$ of $L(\lambda) \in \mathcal{F}^\chi$ is integral. On the lattice $\Lambda$ of all integral weights one can introduce the parity $p : \Lambda \to \mathbb{Z}_2$. We assume that the parity of the highest vector in $L(\lambda)$ equals $p(\lambda)$.

**Theorem 5.6.** If $i > 0$, then

$$\Gamma_i(L_q(\lambda)) = \bigoplus (L(\nu)),$$

and $p(\lambda + \nu) = p(i + 1)$ for all $\nu$ appearing in the direct sum.

If $i = 0$, then we have the following exact sequence

$$0 \to \bigoplus L(\mu) \to \Gamma_0(L_q(\lambda)) \to L(\lambda) \to 0,$$

and $p(\mu) = p(\lambda) + 1$ for all $\mu$ appearing in the left term.
Assume that \( \chi > 0 \), then we may assume without loss of generality that a highest weight \( \lambda \) of \( L(\lambda) \in \mathcal{F} \) is integral. On the lattice \( \Lambda \) of all integral weights one can introduce the parity \( p : \Lambda \to \mathbb{Z}_2 \). We assume that the parity of the highest vector in \( L(\lambda) \) equals \( p(\lambda) \).

**Theorem 5.6.** If \( i > 0 \), then
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0 \to \bigoplus L(\mu) \to \Gamma_0(L_q(\lambda)) \to L(\lambda) \to 0,
\]
and \( p(\mu) = p(\lambda) + 1 \) for all \( \mu \) appearing in the left term.

**Corollary 5.7.** There is some linear order on \( \Lambda \) such that
\[
sdimL(\lambda) = \sum_{\mu < \lambda} (-1)^{p(\mu + \lambda)} sdimL(\mu).
\]

This implies \( sdim \ L(\lambda) \) is positive for even \( \lambda \) and negative for odd \( \lambda \)! Hence Kac-Wakimoto conjecture.
Now let $M$ be a finite-dimensional $\mathfrak{g}$-module and $h \in \mathfrak{h}$. Write

$$\text{ch}_M (h) = \text{str}_M (e^h).$$

Obviously, $\text{ch}_M$ is $W$-invariant analytic function on $\mathfrak{h}$. We can write Taylor series for $\text{ch}_M$ at $h = 0$

$$\text{ch}_M (h) = \sum_{i=0}^{\infty} p_i (h),$$

where $p_i (h)$ is a homogeneous polynomial of degree $i$ on $\mathfrak{h}$. The order of $\text{ch}_M$ at zero is by definition the minimal $i$ such that $p_i \neq 0$. 
Now let $M$ be a finite-dimensional $\mathfrak{g}$-module and $h \in \mathfrak{h}$. Write
\[ \text{ch}_M(h) = \text{str}_M(e^h). \]
Obviously, $\text{ch}_M$ is $W$-invariant analytic function on $\mathfrak{h}$. We can write Taylor series for $\text{ch}_M$ at $h = 0$
\[ \text{ch}_M(h) = \sum_{i=0}^{\infty} p_i(h), \]
where $p_i(h)$ is a homogeneous polynomial of degree $i$ on $\mathfrak{h}$. The order of $\text{ch}_M$ at zero is by definition the minimal $i$ such that $p_i \not\equiv 0$.

**Theorem 5.8.** (Duflo, S.) Assume that all odd roots of $\mathfrak{g}$ are isotropic. Let $M$ be a finite-dimensional $\mathfrak{g}$-module, $s$ be the codimension of $X_M$ in $X$. The order of $\text{ch}_M$ at zero is greater or equal than $s$. Moreover, the polynomial $p_s(h)$ in Taylor series for $\text{ch}_M$ is determined uniquely up to proportionality.
Now let $M$ be a finite-dimensional $\mathfrak{g}$-module and $h \in \mathfrak{h}$. Write
\[
\text{ch}_M (h) = \text{str}_M (e^h) .
\]
Obviously, $\text{ch}_M$ is $W$-invariant analytic function on $\mathfrak{h}$. We can write Taylor series for $\text{ch}_M$ at $h = 0$
\[
\text{ch}_M (h) = \sum_{i=0}^{\infty} p_i (h) ,
\]
where $p_i (h)$ is a homogeneous polynomial of degree $i$ on $\mathfrak{h}$. The order of $\text{ch}_M$ at zero is by definition the minimal $i$ such that $p_i \not\equiv 0$.

**Theorem 5.8.** (Duflo, S.) Assume that all odd roots of $\mathfrak{g}$ are isotropic. Let $M$ be a finite-dimensional $\mathfrak{g}$-module, $s$ be the codimension of $X_M$ in $X$. The order of $\text{ch}_M$ at zero is greater or equal than $s$. Moreover, the polynomial $p_s (h)$ in Taylor series for $\text{ch}_M$ is determined uniquely up to proportionality.

**Remark 5.9.** In fact, the above theorem gives a hint how to define the “superdimension” for a block which is not maximal atypical. One can try to define it as a coefficient of $p_s$ for some suitable normalization of $p_s$. We conjecture that it is possible to find a normalization so that the superdimension of every simple object is integral and non-zero.
Open questions and remarks

- Check the conjecture for exceptional cases (defect is always 1)
- As it follows from the proof sdim is the same for almost all simple objects in a block
- Get a formula for superdimension
- Modules of superdimension zero form an ideal in the tensor category $\mathcal{F}$. Study the quotient.