AMS Meeting

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Special Session on Categorical and Algebraic Methods in Representation Theory

Blocks in Deligne's category $\underline{\mathbf{Rep}}(S_t)$

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(joint work with Jonathan Comes)

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F – field of characteristic zero

S_N – symmetric group

$V = \langle e_1, \ldots e_N \rangle_F \in \mathrm{Rep}_F(S_N)$ – standard N–dimensional representation

Reminder: Any irreducible representation of S_N appears in $V^{\otimes a}$ for some a.

Partition of a finite set $\Omega = \sqcup \pi_i$

Example: $\{1, 2, 3, 4, 5\} = \{1, 3, 4\} \sqcup \{2, 5\}$

Observation: For large N, $\mathrm{Hom}_{S_N}(F, V^{\otimes a})$ has a basis labeled by partitions of $[1, ..., a]$

Example: $\sum_{i,j} e_i \otimes e_j \otimes e_i \otimes e_i \otimes e_j$

Corollary: For large N, $\mathrm{Hom}_{S_N}(V^{\otimes a}, V^{\otimes b})$ has a basis $\{t_\pi^{(N)}\}$ labeled by partitions π of $[1, ..., a+b]$

Computation: For $t_\pi^{(N)} \in \mathrm{Hom}_{S_N}(V^{\otimes a}, V^{\otimes b})$ and $t_\nu^{(N)} \in \mathrm{Hom}_{S_N}(V^{\otimes b}, V^{\otimes c})$, we have

$$t_\nu^{(N)} \circ t_\pi^{(N)} = N^{l(\pi,\nu)} t_{\mu(\pi,\nu)}^{(N)}$$

The category $\underline{\mathbf{Rep}}_0(S_t)$:
Pick $t \in F$

Objects: $[I]$ where I is a finite sets

Morphisms: $\mathrm{Hom}([I],[J]) = F-$vector space with basis $\{t_\pi\}$ labeled by partitions π of $I \sqcup J$.

Composition of morphisms: same as above

$$t_\nu \circ t_\pi = t^{l(\pi,\nu)} t_{\mu(\pi,\nu)}$$

Example: End([I]) – *partition algebras* studied by P. Martin, W. Doran, D. Wales, T. Halverson, A. Ram et al

Additional structures on $\underline{\mathbf{Rep}}_0(S_t)$:
Define $[I] \otimes [J] := [I \sqcup J]$
Then $\underline{\mathrm{Rep}}_0(S_t)$ has a natural structure of *symmetric rigid tensor* category

The category $\underline{\mathbf{Rep}}(S_t)$:

$\underline{\mathrm{Rep}}(S_t) :=$ *Karoubian* (or *pseudo-abelian*) *envelope* of $\underline{\mathrm{Rep}}_0(S_t)$

Karoubian envelope:

Stage 1: add formal direct sums $[I] \oplus [J]$

Stage 2: add new objects (A, e) where A is an object and $e = e^2 \in \mathrm{Hom}(A, A)$

$$\mathrm{Hom}((A, e), (B, f)) := e\mathrm{Hom}(A, B)f$$

with obvious composition

The category $\underline{\mathrm{Rep}}(S_t)$ is still symmetric rigid tensor category:

$$(A, e) \otimes (B, f) = (A \otimes B, e \otimes f)$$

The category $\underline{\mathrm{Rep}}(S_t)$ is *Karoubian*

Krull-Schmidt Theorem holds in $\underline{\mathrm{Rep}}(S_t)$

Questions:

(1) What are indecomposable objects?

(2) What are blocks?

(3) What is a structure of blocks?

Indecomposable objects of $\underline{\mathbf{Rep}}(S_t)$:

Let $\bullet$ denote a one element set
Symmetric group S_I acts on $[I]$ since

$$[I] = [\bullet] \otimes \ldots \otimes [\bullet]$$

Let Y be a Young diagram of size $|I|$
E_Y – irreducible representation of S_I
$[I]_Y := ([I] \otimes E_Y)^{S_I}$ – E_Y-isotypic component of $[I]$

Theorem. (J. Comes, V. O.)
(a) There is exactly one indecomposable object T_Y which is a direct summand of $[I]_Y$ and which is not a direct summand of $[J]$ with $|J| < |I|$.
(b) The objects T_Y are pairwise non-isomorphic and exhaust all indecomposable objects of $\underline{\mathrm{Rep}}(S_t)$.

Blocks:

$\mathcal{A}$ – (nice) additive category

Ind($\mathcal{A}$) – set of isomorphism classes of indecomposable objects

$A, B \in \mathrm{Ind}(\mathcal{A})$ are *linked* if

$$\mathrm{Hom}(A, B) \neq 0$$

Generate weakest equivalence relation on Ind($\mathcal{A}$) such that linked objects are equivalent

Equivalence classes are called *blocks*

Y – Young diagram represented as

$$\lambda_1 \geq \lambda_2 \geq \ldots$$

$t_Y := t - |Y|, \lambda_1 - 1, \lambda_2 - 2, \ldots$

Theorem. (J. Comes, V. O.)
T_Y is in the same block as T_Z if and only if t_Y is a permutation of t_Z.

Structure of blocks:
Infinitely many blocks of size 1
Finitely many blocks of infinite size

Theorem. (J. Comes, V. O.)
(a) All blocks of size 1 are semisimple (i.e. $\mathrm{End}(T) = F$)
(b) Infinite blocks appear only for $t \in \mathbb{Z}_{\geq 0}$. All infinite blocks are equivalent to each other as additive categories

Moreover a quiver of *the* infinite block is explicitly computed

Corollary. (P. Deligne)
$\underline{\mathrm{Rep}}(S_t)$ is semisimple if and only if $t \notin \mathbb{Z}_{\geq 0}$.

In particular for $t \notin \mathbb{Z}_{\geq 0}$, the category $\underline{\mathrm{Rep}}(S_t)$ is *abelian*

Specializations:

Question: What are *abelian* tensor categories $\mathcal{A}$ and tensor functors $\underline{\mathrm{Rep}}(S_t) \to \mathcal{A}$?

Example: Assume $t = d \in \mathbb{Z}_{\geq 0}$
$\underline{\mathrm{Rep}}(S_d) \to \mathrm{Rep}(S_d)$, $[\bullet] \mapsto V$; surjective (but not injective) on Hom's

Deligne's construction:
abelian tensor category $\underline{\mathrm{Rep}}^{ab}(S_d)$ and fully faithful functor

$$\underline{\mathrm{Rep}}(S_d) \to \underline{\mathrm{Rep}}^{ab}(S_d)$$

Theorem. (J. Comes, V.O., conjectured by P. Deligne)
Let $\mathcal{A}$ be an abelian tensor category and let $F : \underline{\mathrm{Rep}}(S_d) \to \mathcal{A}$ be a tensor functor. Then F factors through either

$$\underline{\mathrm{Rep}}(S_d) \to \mathrm{Rep}(S_d)$$

or

$$\underline{\mathrm{Rep}}(S_d) \to \underline{\mathrm{Rep}}^{ab}(S_d).$$