

Categorification of quantum groups

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January 16th, 2010

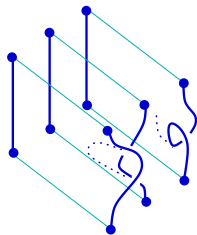
Available at <http://www.math.columbia.edu/~lauda/talks/>

Why categorify quantum groups?

Quantum Group $\xrightarrow{\text{representation category}}$ Braided monoidal category with duals



Categorified Quantum Group $\xrightarrow{\text{representation 2-category}}$ Braided monoidal 2-category with duals



Crane-Frenkel conjectured categorified quantum groups would give 4-dimensional TQFTs

$$U_q^+ \subset U_q(\mathfrak{g})$$

$$\mathfrak{g} = \mathfrak{sl}_n$$

$$E_i = e_{i,i+1} = \begin{matrix} & & & i+1 \\ i & \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 \\ \vdots & \ddots & & \\ 0 & \dots & 0 & \end{pmatrix} \end{matrix}$$

Lie algebra relations:

$$[E_i, E_j] = 0 \quad |i - j| > 1 \qquad [E_i, [E_i, E_j]] = 0 \quad |i - j| = 1$$

Enveloping algebra relations for $U^+(\mathfrak{sl}_n)$

$$E_i E_j = E_j E_i \quad |i - j| > 1$$

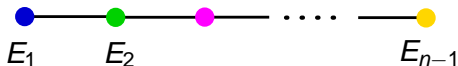
$$2E_i E_j E_i = E_i^2 E_j + E_j E_i^2 \quad j = i \pm 1$$

Quantum enveloping algebra $U_q^+(\mathfrak{sl}_n)$

$$E_i E_j = E_j E_i \quad |i - j| > 1$$

$$\text{quantum 2} \rightarrow (q + q^{-1})E_i E_j E_i = E_i^2 E_j + E_j E_i^2 \quad j = i \pm 1$$

$U_q^+(\mathfrak{sl}_n)$ has a generator E_i for each vertex of the Dynkin graph



U_q^+ for any Γ

Let Γ be an unoriented graph with set of vertices I .

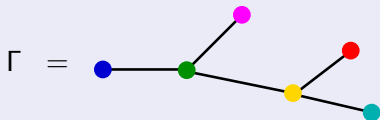
U_q^+ is the $\mathbb{Q}(q)$ -algebra with:

- generators: $E_i \quad i \in I$

- relations: $E_i E_j = E_j E_i$ if $\begin{matrix} i & j \\ \bullet & \bullet \end{matrix}$

- $(q + q^{-1})E_i E_j E_i = E_i^2 E_j + E_j E_i^2$ if $\begin{matrix} i & j \\ \bullet & \bullet \end{matrix}$

U_q^+ is $\mathbb{N}[I]$ graded with $\deg(E_i) = i$.



The goal: categorify $U_q^+(\mathfrak{g})$

The quantum enveloping algebra $U_q(\mathfrak{g})$ of a symmetrizable Kac-Moody Lie algebra \mathfrak{g} has a decomposition

$$U_q(\mathfrak{g}) = U_q^- \otimes U_q(\mathfrak{h}) \otimes U_q^+$$

U_q^+ has the structure of a bialgebra: **try to categorify the bialgebra U_q^+**

The plan: define a new algebra R

$$\begin{array}{c} R\text{-mod} - \left(\begin{array}{l} \text{category of finitely generated} \\ \text{graded projective modules} \end{array} \right) \\ \downarrow \text{Decategorification} \\ \text{(Grothendieck group)} \\ K_0(R\text{-mod}) \cong U_q^+(\mathfrak{g}) \end{array}$$

Grothendieck groups

If R is a graded ring then $K_0(R)$ is the Grothendieck group of the category $R\text{-pmod}$ of graded projective finitely-generated R -modules.

$K_0(R)$ has generators $[M]$ over all objects of $R\text{-pmod}$ and defining relations

$$\begin{aligned} [M] &= [M_1] + [M_2] && \text{if } M \cong M_1 \oplus M_2 \\ [M\{s\}] &= q^s[M] && s \in \mathbb{Z} \end{aligned}$$

$K_0(R)$ is a $\mathbb{Z}[q, q^{-1}]$ -module.

A useful trick

If e is an idempotent in R , then

$$R \cong Re \oplus R(1 - e).$$

Integral form of U_q^+

Define quantum integers and quantum factorials:

$$[a] := \frac{q^a - q^{-a}}{q - q^{-1}} \qquad [a]! := [a][a-1] \dots [1]$$

Example

- $[1] = 1$
- $[2] = \frac{q^2 - q^{-2}}{q - q^{-1}} = q + q^{-1}$
- $[3] = \frac{q^3 - q^{-3}}{q - q^{-1}} = q^2 + 1 + q^{-2}$

The algebra $U_{\mathbb{Z}}^+$ is the $\mathbb{Z}[q, q^{-1}]$ -subalgebra of U_q^+ generated by all products of quantum divided powers:

$$E_i^{(a)} := \frac{E_i^a}{[a]!}$$

Since

$$E_i^{(2)} = \frac{E_i^2}{q + q^{-1}}$$

we can write the U_q^+ relation

$$(q + q^{-1})E_i E_j E_i = E_i^2 E_j + E_j E_i^2 \quad \text{if } \begin{array}{c} i \quad j \\ \bullet \text{---} \bullet \end{array}$$

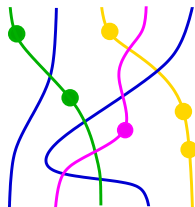
as

$$E_i E_j E_i = E_i^{(2)} E_j + E_j E_i^{(2)} \quad \text{if } \begin{array}{c} i \quad j \\ \bullet \text{---} \bullet \end{array}$$

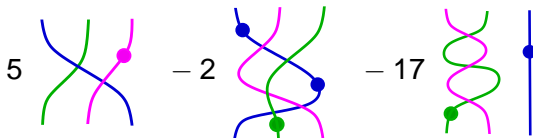
Categorification of U_q^+

Associated to graph Γ consider braid-like diagrams with dots whose strands are labelled by the vertices $i \in I$ of the graph Γ .

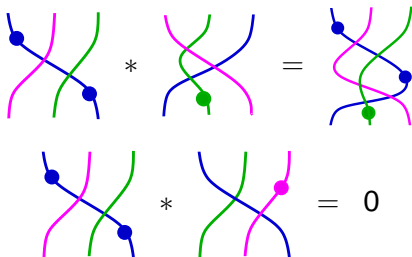
Let $\nu = \sum_{i \in I} \nu_i \cdot i$, for $\nu_i = 0, 1, 2, \dots$
 ν keeps track of how many strands of each color occur in a diagram



Form an abelian group by taking \mathbb{Z} -linear (or \mathbb{k} -linear) combinations of diagrams:



Multiplication is given by stacking diagrams on top of each other when the colors match:



Definition

Given $\nu \in \mathbb{N}[I]$ define the ring $R(\nu)$ as the set of planar diagrams colored by ν , modulo planar braid-like isotopies and the following local relations:

Local relations I

$$= 0$$

$$- \text{ (crossing with dot on lower-right strand) } = \text{ (two parallel vertical strands) }$$

The two parallel vertical strands are labeled i and i at the bottom.

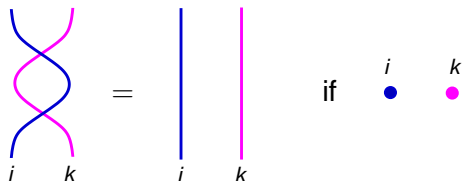
$$- \text{ (crossing with dot on upper-right strand) } = \text{ (two parallel vertical strands) }$$

The two parallel vertical strands are labeled i and i at the bottom.

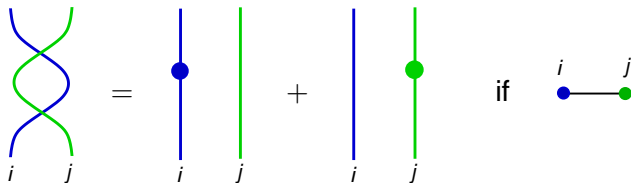
$$= \text{ (full twist of three strands) }$$

The second full twist of three strands is also labeled with i below each strand.

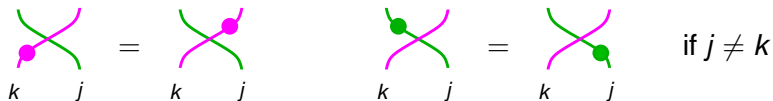
Local relations II



A diagrammatic equation showing a crossing of two strands, labeled i (blue) and k (magenta), equal to two parallel vertical strands, also labeled i and k . To the right, the text "if" is followed by two colored dots: a blue dot labeled i and a magenta dot labeled k .

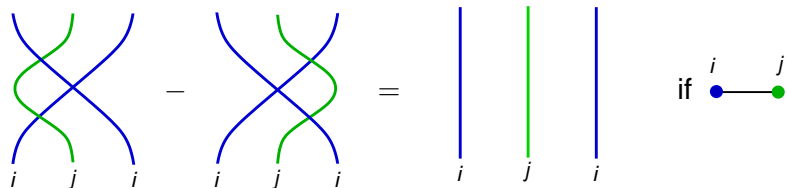


A diagrammatic equation showing a crossing of two strands, labeled i (blue) and j (green), equal to the sum of two configurations: two parallel vertical strands with a blue dot on the i strand, and two parallel vertical strands with a green dot on the j strand. To the right, the text "if" is followed by a horizontal line connecting a blue dot labeled i and a green dot labeled j .

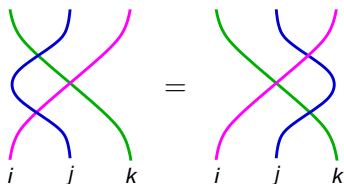


Two diagrammatic equations. The first shows a crossing of strands k (magenta) and j (green) with a magenta dot at the crossing, equal to a crossing of strands k and j with a magenta dot on the k strand. The second shows a crossing of strands k and j with a green dot at the crossing, equal to a crossing of strands k and j with a green dot on the j strand. To the right, the text "if $j \neq k$ " is displayed.

Local relations III



A diagrammatic equation showing the resolution of a crossing between two strands. On the left, two strands, one blue and one green, cross. The blue strand starts at the bottom left labeled i , goes up and right, then down and left to the top right labeled i . The green strand starts at the bottom middle labeled j , goes up and left, then down and right to the top middle labeled j . This is followed by a minus sign and another crossing where the blue strand starts at the bottom left labeled i , goes up and left, then down and right to the top left labeled i . The green strand starts at the bottom middle labeled j , goes up and right, then down and left to the top middle labeled j . This is followed by an equals sign and three vertical strands: blue on the left labeled i , green in the middle labeled j , and blue on the right labeled i . To the right of this is the word "if" followed by a diagram of two dots, a blue one on the left labeled i and a green one on the right labeled j , connected by a horizontal line.



A diagrammatic equation showing the resolution of a crossing between three strands. On the left, three strands, blue, green, and pink, cross. The blue strand starts at the bottom left labeled i , goes up and right, then down and left to the top middle labeled j . The green strand starts at the bottom middle labeled j , goes up and left, then down and right to the top right labeled k . The pink strand starts at the bottom right labeled k , goes up and left, then down and right to the top left labeled i . This is followed by an equals sign and another crossing where the blue strand starts at the bottom left labeled i , goes up and left, then down and right to the top middle labeled j . The green strand starts at the bottom middle labeled j , goes up and right, then down and left to the top right labeled k . The pink strand starts at the bottom right labeled k , goes up and right, then down and left to the top left labeled i .

otherwise,

some of i, j, k may be equal

Grading

$q \rightarrow$ grading shift

$$\deg \left(\begin{array}{c} | \\ \bullet \\ | \end{array} \right) = 2$$
$$\deg \left(\begin{array}{cc} & \\ i & j \\ & \end{array} \right) = \begin{cases} -2 & \text{if } i = j \\ 0 & \text{if } \begin{array}{cc} & \\ \bullet & \bullet \end{array} \\ 1 & \text{if } \begin{array}{cc} & \\ \bullet & \text{---} \bullet \end{array} \end{cases}$$

The $R(\nu)$ relations are homogeneous with respect to this grading.

R_ν is the associative, F -algebra on generators $1_{\underline{i}}$, $x_{a,\underline{i}}$; $\psi_{b,\underline{i}}$ for $1 \leq a \leq m$, $1 \leq b \leq m-1$ and $\underline{i} \in \text{Seq}(\nu)$ subject to the following relations for $\underline{i}, \underline{j} \in \text{Seq}(\nu)$:

$$1_{\underline{i}}1_{\underline{j}} = \delta_{\underline{i},\underline{j}}1_{\underline{i}},$$

$$x_{a,\underline{i}} = 1_{\underline{i}}x_{a,\underline{i}}1_{\underline{i}},$$

$$\psi_{a,\underline{i}} = 1_{s_a(\underline{i})}\psi_{a,\underline{i}}1_{\underline{i}},$$

$$x_{a,\underline{i}}x_{b,\underline{i}} = x_{b,\underline{i}}x_{a,\underline{i}},$$

$$\psi_{a,s_a(\underline{i})}\psi_{a,\underline{i}} = \begin{cases} 0 & \text{if } i_r = i_{r+1} \\ 1_{\underline{i}} & \text{if } (\alpha_{i_a}, \alpha_{i_{a+1}}) = 0 \\ \left(x_{a,\underline{i}}^{-\langle i_a, i_{a+1} \rangle} + x_{a+1,\underline{i}}^{-\langle i_{a+1}, i_a \rangle}\right) 1_{\underline{i}} & \text{if } (\alpha_{i_a}, \alpha_{i_{a+1}}) \neq 0 \text{ and } i_a \neq i_{a+1} \end{cases},$$

$$\psi_{b,s_a(\underline{i})}\psi_{a,\underline{i}} = \psi_{a,s_b(\underline{i})}\psi_{b,\underline{i}} \quad \text{if } |a-b| > 1,$$

$$\psi_{a,s_{a+1}s_a(\underline{i})}\psi_{a+1,s_a(\underline{i})}\psi_{a,\underline{i}} - \psi_{a+1,s_a s_{a+1}(\underline{i})}\psi_{a,s_{a+1}(\underline{i})}\psi_{a+1,\underline{i}} =$$

$$= \begin{cases} \sum_{r=0}^{-\langle i_a, i_{a+1} \rangle - 1} x_{a,\underline{i}}^r x_{a+2,\underline{i}}^{-\langle i_a, i_{a+1} \rangle - 1 - r} & \text{if } i_a = i_{a+2} \text{ and } (\alpha_{i_a}, \alpha_{i_{a+1}}) \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$\psi_{a,\underline{i}}x_{b,\underline{i}} - x_{s_a(b),s_a(\underline{i})}\psi_{a,\underline{i}} = \begin{cases} 1_{\underline{i}} & \text{if } a = b \text{ and } i_a = i_{a+1} \\ -1_{\underline{i}} & \text{if } a = b + 1 \text{ and } i_a = i_{a+1} \\ 0 & \text{otherwise.} \end{cases}$$

Projective modules from idempotents

Let $R = \bigoplus_{\nu} R(\nu)$. For each product of E_i 's in U_q^+ we have an idempotent in R :

$$E_i E_j E_k E_i E_j E_\ell \quad \mapsto \quad 1_{ijkij\ell} := \begin{array}{cccccc} | & | & | & | & | & | \\ i & j & k & i & j & \ell \end{array}$$

This gives rise to a projective module

$$\mathcal{E}_i \mathcal{E}_j \mathcal{E}_k \mathcal{E}_i \mathcal{E}_j \mathcal{E}_\ell := R 1_{ijkij\ell} = R(2i + 2j + k + \ell) 1_{ijkij\ell}$$

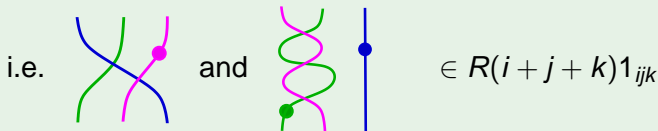
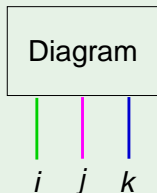
corresponding to the idempotent $1_{ijkij\ell}$ above.

Example

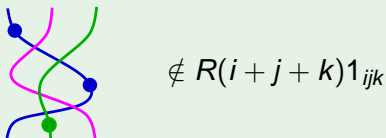
Consider

$$R1_{ijk} = R(i+j+k)1_{ijk}$$

The projective module $\mathcal{E}_i\mathcal{E}_j\mathcal{E}_k := R(i+j+k)1_{ijk}$ consists of linear combinations of diagrams that have the sequence ijk at the bottom



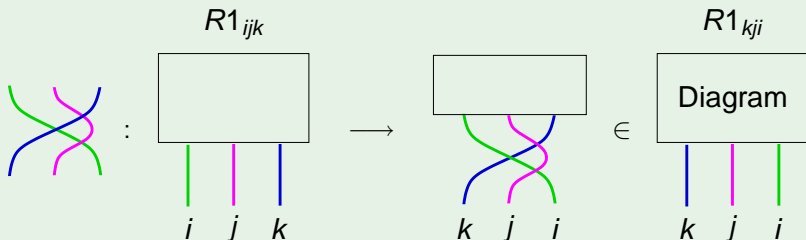
But



We can construct maps between projective modules by adding diagrams at the *bottom*

Example

We get a module map from $\mathcal{E}_i \mathcal{E}_j \mathcal{E}_k := R(i+j+k)1_{ijk}$ to $\mathcal{E}_k \mathcal{E}_j \mathcal{E}_i := R(i+j+k)1_{kji}$ as follows:



Given a graded module M and a Laurent polynomial $f = \sum f_a q^a \in \mathbb{Z}[q, q^{-1}]$ write

$$M^{\oplus f} \quad \text{or} \quad \bigoplus_f M$$

to denote the direct sum over $a \in \mathbb{Z}$ of f_a copies of $M\{a\}$

Example

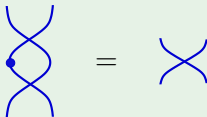
Since $[3] = q^2 + 1 + q^{-2} \in \mathbb{Z}[q, q^{-1}]$, for a graded module M

$$\bigoplus_{[3]} M = M\{2\} \oplus M\{0\} \oplus M\{-2\}$$

Example ($n = 2$)

$$E_i^{(2)} = \frac{E_i^2}{q+q^{-1}} \quad \text{or} \quad E_i^2 = (q + q^{-1})E_i^{(2)}$$

Recall that



so that $e_2 =$  is an idempotent.

$\mathcal{E}_i^{(2)}$ is the projective module for this idempotent

$$\mathcal{E}_i^{(2)} := R(2i)e_2\{1\}$$

$$\mathcal{E}_i^2 \cong \mathcal{E}_i^{(2)}\{1\} \oplus \mathcal{E}_i^{(2)}\{-1\}$$

Categorification of $E_i E_j E_i = E_i^{(2)} E_j + E_j E_i^{(2)}$

The relation

$$\begin{array}{c} \text{if } \begin{array}{c} i \quad j \\ \bullet \text{---} \bullet \end{array} \end{array}$$

together with the other relations imply

$$\mathcal{E}_i \mathcal{E}_j \mathcal{E}_i \cong \mathcal{E}_j \mathcal{E}_i^{(2)} \oplus \mathcal{E}_i^{(2)} \mathcal{E}_j$$

We have shown that all the quantum Serre relations hold as isomorphisms between projective modules. This implies that they also hold in the Grothendieck ring.

Work over a field \mathbb{k} .

Theorem (M.Khovanov, A. L. arXiv:0803.4121)

There is an isomorphism of twisted bialgebras:

$$\begin{array}{ccc} \gamma: U_{\mathbb{Z}}^+ & \longrightarrow & K_0(R) \\ E_{i_1}^{(a_1)} E_{i_2}^{(a_2)} \cdots E_{i_k}^{(a_k)} & \mapsto & \left[\mathcal{E}_{i_1}^{(a_1)} \mathcal{E}_{i_2}^{(a_2)} \cdots \mathcal{E}_{i_k}^{(a_k)} \right] \end{array}$$

multiplication \mapsto multiplication given by [Ind]

comultiplication \mapsto comultiplication given by [Res]

The semilinear form on $U_{\mathbb{Z}}^+$ maps to the HOM form on $K_0(R)$

$$(x, y) = (\gamma(x), \gamma(y))$$

Theorem (simply-laced)

$$U_{\mathbb{Z}}^+ \xrightarrow{\sim} K_0(R)$$

Lusztig-Kashiwara canonical basis \longleftrightarrow indecomposable projective $[P]$

arXiv:0901.4450

Brundan and Kleshchev gave an algebraic proof when Γ is a chain or a cycle.

arXiv:0901.3992

The general case (over \mathbb{C}) was proven by Varagnolo and Vasserot who showed that rings $R(\nu)$ in the simply-laced case were isomorphic to certain Ext-algebras of Perverse sheaves on Lusztig quiver varieties.

Cyclotomic quotients

For a given weight $\lambda = \sum_{i \in I} \lambda_i \cdot \Lambda_i$ define the cyclotomic quotient R_ν^λ of $R(\nu)$ by imposing the additional relations: for any sequence $i_1 i_2 \cdots i_m$ of vertices of Γ

λ_{i_1} dots on the first strand of any sequence is zero

$$\longrightarrow \begin{array}{ccccccc} \lambda_{i_1} & & & & & & \\ | & | & | & \cdots & | & & \\ \bullet & & & & & & \\ | & | & | & & | & & \\ i_1 & i_2 & i_3 & & i_m & & \end{array} = 0$$

This is analogous to taking the Ariki-Koike cyclotomic quotient of the affine Hecke algebra:

$$H_d^\lambda := H_d / \left\langle \prod_{i \in I} (X_1 - q^i)^{\lambda_i} \right\rangle$$

Cyclotomic quotient conjecture

The category of finitely-generated graded modules over the ring

$$R^\lambda = \bigoplus_{\nu \in \mathbb{N}[I]} R_\nu^\lambda$$

categorifies the integrable version of the representation V_λ of $U_q(\mathfrak{g})$ of highest weight λ .

$$\begin{array}{ccc} V(\lambda) & \xrightarrow{\sim} & K_0(R^\lambda) \\ \text{Lusztig-Kashiwara} & \xrightarrow{\quad} & \text{indecomposable} \\ \text{canonical basis} & & \text{projective } [P] \end{array}$$