Modified Trace and Dimension Functions for Ribbon Categories

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What is a Ribbon Category?

A tensor category $\mathcal{C}$ is a category equipped with:

- a covariant bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called the tensor product;
- a unit object $\mathbb{1}$ - ie. for all V: $V \otimes \mathbb{1} \cong V$ and $\mathbb{1} \otimes V \cong V$;
- associativity - ie. for all U, V, W:

$$(U \otimes V) \otimes W \cong U \otimes (V \otimes W)\,;$$

- The Triangle and Pentagon Axioms hold.

$$
\begin{array}{ccc}
(V \otimes \mathbb{1}) \otimes W & \xrightarrow{\cong} & V \otimes (\mathbb{1} \otimes W) \\
 & \searrow^{\cong} \quad {}^{\cong}\swarrow & \\
 & V \otimes W &
\end{array}
$$

Figure: Triangle Axiom

$$
\begin{array}{ccccc}
 & & ((U \otimes V) \otimes W) \otimes X & & \\
 & \swarrow \cong & & \searrow \cong & \\
(U \otimes (V \otimes W)) \otimes X & & & & (U \otimes V) \otimes (W \otimes X) \\
\downarrow \cong & & & & \downarrow \cong \\
U \otimes ((V \otimes W)) \otimes X & & \xrightarrow{\quad\cong\quad} & & U \otimes (V \otimes (W \otimes X))
\end{array}
$$

Figure: Pentagon Axiom

A duality is a functor $V \mapsto V^*$ such that for all $V \in V$, there is maps

$$b_V : \mathbb{1} \to V \otimes V^* \qquad d_V : V^* \otimes V \to \mathbb{1}.$$

A braiding is isomorphisms for all V, W in $\mathcal{C}$:

$$c_{V,W} : V \otimes W \to W \otimes V$$

A twist is an isomorphism for every V in $\mathcal{C}$:

$$\theta_V : V \to V$$

Definition

A ribbon category is a tensor category $\mathcal{C}$ with *duality*, *braiding*, and *twists*.

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Caveat

Subject to some axioms, of course!

A Baby Example

Finite Dimensional Vector Spaces

Let k be a field, and let $\mathcal{C}$ be the category of finite-dimensional k-vector spaces. Let

$$\begin{aligned} \otimes &= \otimes_k \\ V^* &= \mathrm{Hom}_k(V, k) \\ \mathbb{1} &= k \end{aligned}$$

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Given V in $\mathcal{C}$, let $\{v_i\}$ be a arbitrary fixed basis for V and $\{v_i^*\}$ is the basis for V^* given by $v_i^*(v_j) = \delta_{i,j}$.

$$c_{V,W} : V \otimes W \to W \otimes V$$
$$c_{V,W}(v \otimes w) = w \otimes v$$

$$b_V : \mathbb{1} \to V \otimes V^*$$
$$b_V(1) = \sum v_i \otimes v_i^*.$$

$$d_V : V^* \otimes V \to \mathbb{1}$$
$$d_V(f \otimes v) = f(v).$$

$$\theta_V = \mathrm{Id}_V : V \to V$$

More Interesting Examples

Let k be a fixed field, and let $\mathcal{C}$ be the category of finite dimensional representations for:

- A group;
- A Lie algebra;
- A Lie superalgebra.

Then the same morphisms (or graded versions) make $\mathcal{C}$ a ribbon category.

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Observe, all of these examples have the feature that

$$c_{W,V} \circ c_{V,W} = \mathrm{Id}_{V \otimes W}\,.$$

That is, they are symmetric.

Nonsymmetric Example

If $U_q(\mathfrak{g})$ is a quantum (super)group, then the category $\mathcal{C}$ of finite-dimensional $U_q(\mathfrak{g})$-(super)modules is a ribbon category. But,

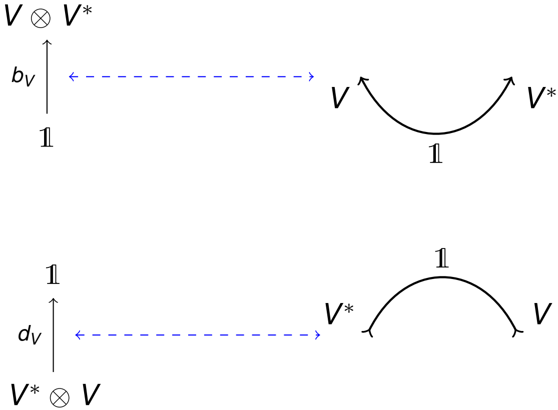
$$c_{W,V} \circ c_{V,W} \neq \mathrm{Id}_{V \otimes W} \,.$$

What about those Axioms?

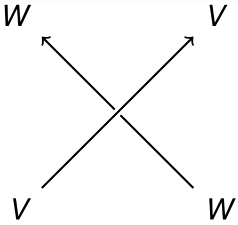
$$
\begin{array}{ccc}
\mathbb{1} \otimes V \cong V & & V \\
d_V \otimes 1 \Big\uparrow & & \Big\uparrow \\
V^* \otimes V \otimes V & & \\
c_{V,V^*} \otimes 1 \Big\uparrow & & \\
V \otimes V^* \otimes V & = & \Big\uparrow \mathrm{Id}_V \\
1 \otimes c_{V,V^*} \Big\uparrow & & \\
V \otimes V \otimes V^* & & \\
1 \otimes b_V \Big\uparrow & & \Big\uparrow \\
V \cong V \otimes \mathbb{1} & & V
\end{array}
$$

Figure: An Axiom

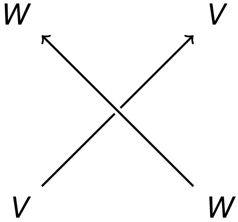
The Diagrammatic Calculus



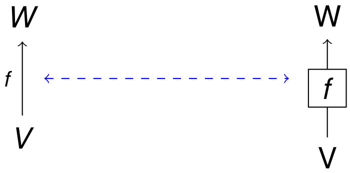
$$\begin{array}{c} W \otimes V \\ c_{V,W} \uparrow \\ V \otimes W \end{array}$$



$$\begin{array}{c} V \otimes W \\ c_{V,W}^{-1} \uparrow \\ W \otimes V \end{array}$$



For a general $f \in \mathrm{Hom}_{\mathcal{C}}(V, W)$:



Rules for Combining Diagrams

- Composition corresponds to vertical concatenation of diagrams (read bottom to top);
- Tensor product corresponds to horizontal concatenation (read left to right).

Algebraically, our axiom was:

$$d'_V \otimes 1 \circ 1 \otimes b'_V = \mathrm{Id}_V \, .$$

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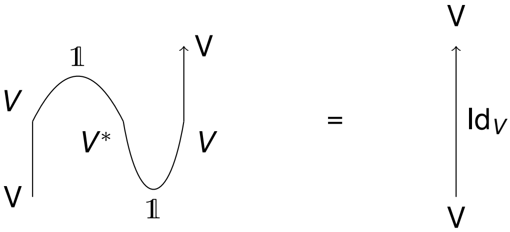
Graphically, our axiom becomes:



Fundamental Observation

The axioms of a Ribbon Category are precisely those required so that isotopic (ie. topologically equivalent) diagrams corresponding to identical morphisms in $\mathcal{C}$!

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White Lie

- Actually, the diagrams should be drawn with ribbons, not lines, and a 360° twist of Id_V corresponds to the twist isomorphism $\theta_V : V \to V$.
- We'll suppress the twists for clarity.

Here and throughout, let $\mathcal{C}$ be a ribbon category and assume

$$K = \mathrm{End}_{\mathcal{C}}(\mathbb{1})$$

is a field.

Example

Define $\mathrm{tr}_{\mathcal{C}} : \mathrm{End}_{\mathcal{C}}(V) \to K$ and $\dim_{\mathcal{C}} : \mathrm{Objects}(\mathcal{C}) \to K$ by:

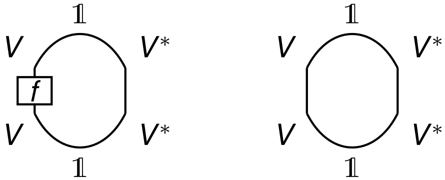

Figure: The categorical trace $\mathrm{tr}_{\mathcal{C}}(f)$ and dimension $\dim_{\mathcal{C}}(V)$.

Algebraically they are given by:

$$\mathrm{tr}_{\mathcal{C}}(f) = d_V \circ c \circ f \otimes 1 \circ b_V$$
$$\dim_{\mathcal{C}}(V) = \mathrm{tr}_{\mathcal{C}}(\mathrm{Id}_V) = d_V \circ c \circ b_V.$$

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For finite dimensional k-vector spaces, and representations of groups and Lie algebras, these are precisely the usual trace and dimension functions with values in

$$K = \mathrm{End}_{\mathcal{C}}(\mathbb{1}) = k.$$

If V is a representation of a Lie superalgebra, then

$$\dim_{\mathcal{C}}(V) = \mathrm{sdim}(V) := \dim(V_{\bar{0}}) - \dim(V_{\bar{1}}),$$

the superdimension of V.

If V is a representation of a quantum group $U_q(\mathfrak{g})$, then

$$\dim_{\mathcal{C}}(V)$$

is the *quantum dimension*.

Low Dimensional Topology

Main Idea (Reshetikhin and Turaev)

Given a ribbon category $\mathcal{C}$, you can construct a knot invariant as follows:

$$\text{A knot } K \rightsquigarrow \text{ A knot diagram for } K \rightsquigarrow \text{End}_{\mathcal{C}}(\mathbb{1}).$$

The second step is by labelling the diagram of K with objects of $\mathcal{C}$ and using the diagrammatic calculus to interpret as a morphism in $\mathcal{C}$.

Problems

1. Many categories arising in algebra are symmetric (i.e. the square of the braiding is the identity) and, hence, yield only trivial invariants.
2. Many objects in these categories have categorical dimension zero and, again, necessarily yield only trival invariants.

Solutions

1. Use quantum (super)groups.

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2. Define modified dimension functions:
   - Typical representations for Type I quantum supergroups (Geer - Patureau-Mirand);
   - Nilpotent representations of $U_q(\mathfrak{sl}_2)$ at a root of unity (Geer - Patureau-Mirand - Turaev);
   - General ribbon category (Geer - Kujawa - Patureau-Mirand).

Generalities

Definition

Given J in $\mathcal{C}$, let $\mathcal{I}_J$ be the full subcategory of all objects V in $\mathcal{C}$ for which there exist an object X and morphisms

$$\alpha : V \to J \otimes X \qquad\qquad \beta : J \otimes X \to V,$$

with

$$\beta \circ \alpha = \mathrm{Id}_V .$$

Definition

A *trace* on $\mathcal{I}_J$ is a family of K-linear functions $\{t_V\}_{V\in\mathcal{I}_J}$,

$$t_V : \mathrm{End}_{\mathcal{C}}(V) \to K,$$

which satisfy natural generalizations of properties of the ordinary trace function:

1. If $U, V \in \mathcal{I}_J$ then for any morphisms $f : V \to U$ and $g : U \to V$ in $\mathcal{C}$ we have
$$t_V(g \circ f) = t_U(f \circ g).$$
2. If $U \in \mathcal{I}_J$ and $W \in \mathrm{Objects}(\mathcal{C})$ then for any $f \in \mathrm{End}_{\mathcal{C}}(U \otimes W)$ we have
$$t_{U\otimes W}(f) = t_U(\mathrm{Tr}_R(f)).$$

Definition

A K-linear function $t : \mathrm{End}_{\mathcal{C}}(J) \to K$ is an *ambidextrous trace on J* if for all $h \in \mathrm{End}_{\mathcal{C}}(J \otimes J)$,

$$t\left(\mathrm{Tr}_L(h)\right) = t\left(\mathrm{Tr}_R(h)\right).$$

Where:

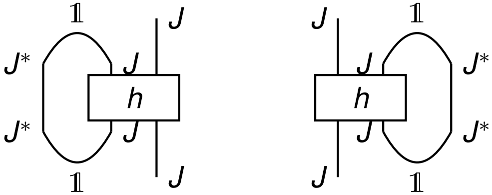

Figure: $\mathrm{Tr}_L(h)$ and $\mathrm{Tr}_R(h)$.

Theorem

- If $\mathcal{I}_J$ is an ideal of a ribbon category $\mathcal{C}$ and $\{t_V\}_{V\in\mathcal{I}_J}$ is a trace on this ideal, then each t_V is an ambidextrous trace on V.
- Conversely, if J in $\mathcal{C}$ admits an ambidextrous trace, then there is a unique trace on $\mathcal{I}_J$ determined by that ambidextrous trace.

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Definition

Given a trace $\{t_V\}_{V\in\mathcal{I}_J}$, the *modified dimension function*

$$\mathrm{d}_J : \mathrm{Objects}\,(\mathcal{I}_J) \to K$$

is defined by:

$$\mathrm{d}_J(V) = t_V(\mathrm{Id}_V).$$

A Baby Example

If $J = \mathbb{1}$:

- Then $\mathcal{I}_{\mathbb{1}} = \mathcal{C}$ and the categorical trace defines a trace on $\mathcal{I}_{\mathbb{1}}$.
- It's easy to see that the identity map $\text{End}_{\mathcal{C}}(\mathbb{1}) \to K$ defines an ambidextrous trace on $\mathbb{1}$.
- In this way we recover the categorical trace and dimension functions.

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- Yes! Typical reps of Type I quantum supergroups, quantum $\mathfrak{sl}_2$ at a root of unity, ...
- But first some general results.

Theorem

Let V be an object in $\mathcal{I}_J$, assume J is irreducible and $\mathcal{I}_J$ admits a nontrivial trace. Then:

1. Let $U \in \mathcal{I}_V \subseteq \mathcal{I}_J$. If $\mathrm{d}_J(V) = 0$, then $\mathrm{d}_J(U) = 0$.
2. If V is irreducible, then the epimorphism $d_V \otimes \mathrm{Id}_J : V^* \otimes V \otimes J \to J$ splits if and only if $\mathrm{d}_J(V) \neq 0$.
3. If J is not projective in $\mathcal{C}$ and P is projective in $\mathcal{C}$, then P is an object of $\mathcal{I}_J$ and $\mathrm{d}_J(P) = 0$.

When $J = \mathbb{1}$

One Specialization

Let $\mathcal{C} = \mathcal{I}_{\mathbb{1}}$ be the category of finite dimensional G-modules over an algebraically closed field k of characteristic p. Then:

1. If p divides $\dim_k(V)$, then p divides $\dim_k(U)$ for any direct summand U of any $V \otimes X$.
2. If V is irreducible, then k is a direct summand of $V \otimes V^*$ if and only if p does not divide $\dim_k(V)$.
3. If p divides the order of G and P is projective in $\mathcal{C}$, then p divides $\dim_k(P)$.

(Landrock, Benson - Carlson)

When $J = \mathbb{1}$, cont.

Another Specialization

Let $\mathcal{C} = \mathcal{I}_{\mathbb{1}}$ be the finite dimensional representations of a complex Lie superalgebra. Then:

1. If $\mathrm{sdim}(V) = 0$, then $\mathrm{sdim}(U) = 0$ for any direct summand U of any $V \otimes X$.
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Yet More Specializations

Let $\mathcal{C} = \mathcal{I}_{\mathbb{1}}$ be the finite dimensional representations of a quantum group (Andersen), quantum supergroups, ...

A Real Example (finally!)

Let's consider an application of this framework to representation theory.

Lie Superalgebras

- Say $\mathfrak{g}$ is a complex basic Lie superalgebra and $\mathcal{C}$ is its finite dimensional reps.
- Let atypicality and defect be as in Vera's talk.
- In particular,

$$\mathrm{def}(\mathfrak{g}) = \mathrm{atyp}(\mathbb{C}).$$

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Conjecture

Let L be a simple object in $\mathcal{C}$. Then,

$$\mathrm{atyp}(L) = \mathrm{def}(\mathfrak{g}) \text{ if and only if } \mathrm{sdim}(L) \neq 0.$$

(Kac-Wakimoto 1996, Serganova 2009)

In our language:

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Generalized KW Conjecture

Let $\mathfrak{g}$ be a basic classical Lie superalgebra, let J be a simple $\mathfrak{g}$-supermodule such that $\mathcal{I}_J$ admits a nonzero trace and $L \in \mathcal{I}_J$ be a simple $\mathfrak{g}$-supermodule. Then,

$$\operatorname{atyp}(L) = \operatorname{atyp}(J) \text{ if and only if } d_J(L) \neq 0.$$

Type A

Theorem

Let $\mathfrak{g} = \mathfrak{gl}(m|n)$, let J be a simple $\mathfrak{g}$-supermodule which admits an ambidextrous trace, and let $L \in \mathcal{I}_J$ be a simple $\mathfrak{g}$-supermodule. Then the following are true.

1. One always has $\mathrm{atyp}(L) \leq \mathrm{atyp}(J)$.
2. If $\mathrm{d}_J(L) \neq 0$, then $\mathrm{atyp}(L) = \mathrm{atyp}(J)$.
3. If $\mathrm{atyp}(J) = 0$, then $\mathrm{atyp}(L) = 0$ and $\mathrm{d}_J(L) \neq 0$.
4. If J and L are polynomial, then J necessarily admits an ambidextrous trace (i.e. it does not have to be assumed), and $\mathrm{d}_J(L) \neq 0$ if and only if $\mathrm{atyp}(L) = \mathrm{atyp}(J)$.

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That is, for $\mathfrak{gl}(m|n)$ we proved one direction of the generalized KW conjecture in general. Both directions for atypicality zero and polynomial representations.